Benjamini-Schramm convergence and limiting eigenvalue density of random matrices

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Introduction

We review the application of the notion of local convergence on infinite rooted graphs, known as Benjamini-Schramm convergence, to the calculation of the global eigenvalue density of random matrices from the β-Gaussian ensemble. By regarding a random matrix as the weighted adjacency matrix of a graph, and choosing the root of such a graph with uniform probability, one can use the Benjamini-Schramm convergence to produce the probability distribution of the infinite graph which results when the size of the matrix tends to infinity. We illustrate how the Wigner semicircle law is obtained from the distribution of the limiting graph.

The Benjamini-Schramm convergence

Following [1], we consider the set of connected graphs $G = (V, E)$, and we define rooted graphs as ordered pairs (G, o) where the vertex $o \in V$ is the root. We define the space of isomorphism classes of rooted, connected, and locally finite graphs (that is, graphs with a finite number of edges connected to any vertex) by $\mathcal{X}$. Consider the locally finite rooted graphs (G, o) and (G', o'). Then, we can define the metric

$$d[(G, o), (G', o')] := 2^{-k},$$

where

$$k[(G, o), (G', o')] := \sup\{r \in \mathbb{N}_0 : B_r(G, o) \simeq B_r(G', o')\}$$

and $B_r(G, o)$ is the subgraph of radius r around o of G, with the same root, o. In addition, if we define by $\mathcal{X}_M \subset \mathcal{X}$ the class of graphs of maximum degree M, we see that it is compact under the metric $d[(G, o), (G', o')]$.

Now, suppose that the unrooted finite graph H is given a root with uniform probability among its vertices. With this, one has a probability distribution $\mu_H(\mathcal{A})$, where $\mathcal{A}$ is a Borel subset of $\mathcal{X}$, which gives the probability that (H, o) is an element of $\mathcal{A}$. With these definitions in place we state the following.

Definition 1 (Benjamini-Schramm convergence)**.** Consider the sequence of rooted graphs $\{(G_j, o_j)\}_{j=0}^{\infty}$, with roots chosen randomly with uniform probability. The rooted graph (G, o) is the **distributional limit** of the sequence if for every $r > 0$ and every finite rooted graph (H, o'),

$$\mathbb{P}[(H, o') \simeq (B_r(G_j, o_j), o_j)] \overset{j\to\infty}{\longrightarrow} \mathbb{P}[(H, o') \simeq (B_r(G, o), o)].$$

In other words, the probability law of (G_j, o_j) tends weakly to the law of (G, o) as $j \to \infty$.

Random rooted weighted graphs

A sparse symmetric matrix can be viewed as the adjacency matrix of a finite graph. In particular, consider random matrices from the β-Gaussian ensemble [2]. These matrices are tridiagonal, so they represent a graph in which each vertex is connected to two neighbors through edges with random weight $b_{j,j+1} \sim \chi_{\beta(j-1)}$, and to itself through an edge with weight $a_i \sim N(0, 2)$. The Benjamini-Schramm convergence can be extended directly to weighted graphs, and following [3], one can show that the adjacency operator A is bounded on the space $\mathscr{L}^2(G)$ of square summable functions on the vertex set V of $G = (V, E)$; the action of the adjacency operator on a function $f \in \mathscr{L}^2(G)$ is given by

$$[Af](x) = \sum_{(x,y)\in E} l((x, y))f(y),$$

where $l((x, y))$ denotes the weight of the edge connecting the vertices x and y. The importance of the adjacency operator stems from its close relationship with the expected spectral measure of G, which is given by

$$\mu_{G,o}(X) = \langle P_X \chi_o, \chi_o\rangle,$$

where X is a Borel set of $\mathbb{R}$, the inner product of $f, g \in \mathscr{L}^2(G)$ is given by

$$\langle f, g\rangle = \sum_{x\in V} \bar{f}(x)g(x),$$

P_X is the orthogonal projection to the linear envelope of the eigenfunctions of A which have eigenvalue $\lambda \in X$, and $\chi_o(x)$ is equal to 1 if $x = o \in E$ and zero otherwise. The relationship between the spectral measure and the adjacency operator is given by the following.

Proposition 2. *Denote by $\{e_i\}_i$ the set of eigenfunctions on $V(G)$ of the adjacency operator A, with eigenvalues $\{\lambda_i\}_i$. Then,*

$$\mu_{G,o}(X) = \sum_i \mathbf{1}_{\{\lambda_i\in X\}} e_i^2(o).$$

That is, the spectral measure of (G, o) is given by the eigenfunctions of A evaluated at the root.

Expected spectral measure and eigenvalue density of large random matrices

From the previous considerations, one can calculate the eigenvalue density from the spectral density by computing the expected spectral density w.r.t. the distribution of the roots. It can be shown that for the distributional limit (G, o) of the sequence $\{(G_j, o_j)\}_j$, the spectral density of the adjacent operator A of the limiting graph is given by

$$\mu_G(X) = \mathbb{E}[\mu_{G,o}(X)],$$

where the expectation is taken over the root distribution on $V(G)$. In this talk, we will review this machinery in detail, and we will illustrate how it can be applied to show that, for example, the limiting spectral density that corresponds to the β-Gaussian ensemble with $\beta = 1$ is given by

$$\mu_u(dx) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{1}_{\{2\sqrt{u}\cos(\omega) \in [x, x+dx)\}} d\omega,$$

where u is a uniformly-distributed random variable in the interval $(0, 1)$, and its expected spectral density (or eigenvalue density) is given by the well-known Wigner semicircle law,

$$\mu(dx) = \frac{1}{2\pi}\sqrt{4 - x^2}\, dx.$$

Bibliography

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