GLOBAL SOLUTION OF THE COUPLED KPZ EQUATIONS

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1. Introduction: the KPZ equation

The KPZ equation

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}(\partial_x h)^2 + \xi, \quad t > 0,\ x \in \mathbb{T}, \tag{1}$$

where ξ is a space-time white noise, appears as a space-time scaling limit of the fluctuations of weakly asymmetric microscopic models. Since the solution h of the equation (1) is expected to have a regularity $(\frac{1}{2})^-$, i.e. $\frac{1}{2}-\kappa$ for every $\kappa > 0$ in spatial variable, the square term is ill-posed. Instead, the *Cole-Hopf solution* of the KPZ equation is defined by $h_{\mathrm{CH}} = \log Z$, where Z is the solution of the multiplicative stochastic heat equation

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + Z\xi.$$

At formal level, Itô's formula yields that $h = h_{\mathrm{CH}}$ solves the equation

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}\{(\partial_x h)^2 - \infty\} + \xi.$$

This heuristic equation should be reformulated into the approximation

$$\partial_t h^\epsilon = \tfrac{1}{2}\partial_x^2 h^\epsilon + \tfrac{1}{2}\{(\partial_x h^\epsilon)^2 - c_\eta^\epsilon\} + \xi^\epsilon, \tag{2}$$

where $\xi^\epsilon(t,x) = (\xi(t) * \eta^\epsilon)(x)$ is a smeared noise with an even mollifier $\eta^\epsilon(x) = \epsilon^{-1}\eta(\epsilon^{-1}x)$ $(\epsilon > 0)$, and $c_\eta^\epsilon = \|\eta^\epsilon\|_{L^2}^2 = \epsilon^{-1}\|\eta\|_{L^2}^2$.

Recently developed theories of *regularity structures* [5], or *paracontrolled calculus* [2] constructed the well-posedness theory for the KPZ equation independent to the probability space. Let $\mathcal{C}^\theta$ be the completion of the set of smooth functions on $\mathbb{T}$ under the $\mathcal{B}^\theta_{\infty,\infty}(\mathbb{T})$ norm and $\bar{\mathcal{C}}^\theta = \mathcal{C}^\theta \cup \{\Delta\}$ be the extended space with a "death point" Δ.

Theorem 1.1 ([4, 3, 7])**.** *Let $\theta \in (0, \frac{1}{2})$. There exist a Polish space $\mathcal{M}$, a lower semicontinuous map $T_* : \mathcal{C}^\theta \times \mathcal{M} \to (0,\infty]$, and a map*

$$S : \mathcal{C}^\theta \times \mathcal{M} \ni (h_0, \Xi) \mapsto h \in C([0,\infty), \bar{\mathcal{C}}^\theta)$$

such that, $h|_{[0,T_*)} \in C([0,T_*), \mathcal{C}^\theta)$, $h|_{[T_*,\infty)} \equiv \Delta$, the map S is continuous with respect to the $C([0,T],\mathcal{C}^\theta)$-norm on the set $\{(h_0,\Xi)\,; T_*(h_0,\Xi) > T\}$ for every $T > 0$, and for every probability space (Ω, P) (which admits a space-time white noise) there exists a measurable map $\Xi : \Omega \to \mathcal{M}$ such that

$$h_{\mathrm{CH}}(h_0,\omega) = S(h_0, \Xi(\omega)) \quad P\text{-a.s. } \omega,$$

where $h_{\mathrm{CH}}(h_0,\omega)$ is the Cole-Hopf solution with initial value $h_0 \in \mathcal{C}^\theta$. Moreover, there exists a measurable map $\Xi^\epsilon : \Omega \to \mathcal{M}$ such that $\lim_{\epsilon\downarrow 0} \Xi^\epsilon = \Xi$ in probability, and $h^\epsilon(h_0,\omega) = S(h_0,\Xi^\epsilon(\omega))$ solves (2) with initial value $h_0 \in \mathcal{C}^\theta$.

Theorem 1.1 and the properties of the Cole-Hopf solution imply that $T_*(h_0, \Xi(\omega)) = \infty$, P-a.s. ω. On the other hand, the fact that $T_*(h_0, \Xi) = \infty$ for every $(h_0, \Xi) \in \mathcal{C}^\theta \times \mathcal{M}$ was shown by Gubinelli and Perkowski [3] by using the Cole-Hopf transform again.

2. Main result: the coupled KPZ equations

Let $d \in \mathbb{N}$. For given constants $\{\Gamma^\alpha_{\beta\gamma}\}_{1\le\alpha,\beta,\gamma\le d}$ and the independent space-time white noises $\{\xi^\alpha\}_{1\le\alpha\le d}$, we consider the coupled KPZ equations

$$\partial_t h^\alpha = \tfrac{1}{2}\partial_x^2 h^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \xi^\alpha, \quad 1 \le \alpha \le d,\ t > 0,\ x \in \mathbb{T}, \tag{3}$$

where the summation symbol $\sum$ over (β, γ) is omitted. Such system naturally appears as a scaling limit of microscopic systems with d (local) conserved quantities. As with (1), the ill-posed equation (3) should be reformulated into the approximation

$$\partial_t h^{\epsilon,\alpha} = \tfrac{1}{2}\partial_x^2 h^{\epsilon,\alpha} + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^{\epsilon,\beta}\partial_x h^{\epsilon,\gamma} - c^\epsilon_\eta \delta^{\beta\gamma} - C^{\epsilon,\beta\gamma}) + \xi^{\epsilon,\alpha}, \tag{4}$$

where $C^\epsilon = (C^{\epsilon,\beta\gamma})_{\beta,\gamma}$ is a matrix behaving as $O(|\log \epsilon|)$ in general. It is not difficult to show the similar well-posedness result to Theorem 1.1 for the coupled equations, except for the existence of global-in-time solution like the Cole-Hopf solution.

In order to obtain the global existence, we assume the symmetry condition

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} = \Gamma^\beta_{\alpha\gamma}. \tag{5}$$

Then indeed we can choose $C^\epsilon = 0$. Under the condition (5), the distribution μ of $(\partial_x B^\alpha)_\alpha$, where $(B^\alpha)_\alpha$ is the d-tuple of independent Brownian bridges on $\mathbb{T}$, is invariant under the process $(\partial_x h^\alpha)_\alpha$, where h is the limit point of the sequence (h^ϵ) defined by (4). This implies that for μ-a.e. $u_0 \in (\mathcal{C}^{\theta-1})^d = \mathcal{C}^{\theta-1}(\mathbb{T}, \mathbb{R}^d)$, it holds that

$$T_*(h_0, \Xi(\omega)) = \infty, \quad P\text{-a.s. } \omega \tag{6}$$

for every h_0 such that $\partial_x h_0 = u_0$ ([1]). By using the fact that the limit process h is a strong Feller process on the space $(\bar{\mathcal{C}}^\theta)^d$ ([6]), the global existence (6) can be shown for every initial value.

Theorem 2.1 ([1, 6])**.** *Let* $\theta \in (0, \frac{1}{2})$*. Under the symmetry condition* (5)*, we have* (6) *for every* $h_0 \in (\mathcal{C}^\theta)^d$*.*

References

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