On mean-field approximation of particle systems with annihilation and spikes

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On a filtered probability space let us consider the following interactions of $N(\ge 2)$ Brownian particles each of which diffuses on the nonnegative half line $\mathbb{R}_+$ and is attracted towards the average position of all the particles. When a particle i attains the boundary 0, it is annihilated (default) and a new particle (also called i) spikes immediately in the middle of particles. More precisely, let us denote by $X_t := (X_t^1, \ldots, X_t^N)$ the positions of these particles, where $X_t^i(\ge 0)$ is the position of particle i at time $t \ge 0$ for $i = 1, \ldots, N$. With the average $\overline{X}_t := (X_t^1 + \cdots + X_t^N) / N$ the dynamics of the system is determined by

$$
\begin{aligned}
X_t^i &= X_0^i + \int_0^t b(X_s^i, \overline{X}_s) \mathrm{d}s + W_t^i + \int_0^t \overline{X}_{s-} \Big(\mathrm{d}M_s^i - \frac{1}{N} \sum_{j \neq i} \mathrm{d}M_s^j \Big) ; \quad t \ge 0 , \\
M_t^i &:= \sum_{k=1}^{\infty} \mathbf{1}_{\{\tau_k^i \le t\}} , \quad \tau_k^i := \inf \Big\{ s > \tau_{k-1}^i : X_{s-}^i - \frac{\overline{X}_{s-}}{N} \sum_{j \neq i} (M_s^j - M_{s-}^j) \le 0 \Big\} ,
\end{aligned}
\tag{1}
$$

for $i = 1, \ldots, N$, $k \in \mathbb{N}$, where $W_t := (W_t^1, \ldots, W_t^N)$, $t \ge 0$ is an N-dimensional Brownian motion, M_t^i is the cumulative number of defaults by time $t \ge 0$, τ_k^i is the k-th default time with $\tau_0^i = 0$ of particle i. Here we assume that $b : \mathbb{R}_+^2 \to \mathbb{R}$ is (globally) Lipschitz continuous, i.e., there exists a constant $\kappa > 0$ such that

$$
|b(x_1, m_1) - b(x_2, m_2)| \le \kappa \big(|x_1 - x_2| + |m_1 - m_2| \big) \tag{2}
$$

for all $x_1, x_2, m_1, m_2 \in \mathbb{R}_+$, and we also impose the condition

$$
\sum_{i=1}^{N} b(x^i, \overline{x}) \equiv 0 \tag{3}
$$

for every $x := (x^1, \ldots, x^N) \in \mathbb{R}_+^N$ and $\overline{x} := (x^1 + \cdots + x^N) / N$ on the drift function $b(\cdot, \cdot)$.

Given a standard Brownian motion $W.$ we shall consider a system $X. := (X_\cdot^1, \ldots, X_\cdot^N)$, $M. := (M_\cdot^1, \ldots, M_\cdot^N))$ described by (1) with (2)-(3) on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with filtration $\mathbb{F} := (\mathcal{F}_t, t \ge 0)$. In particular, we are concerned with (1) that there might be multiple defaults at the same time with positive probability, i.e.,

$$
\mathbb{P}\big(\exists (i,j) \ \exists t \in [0, \infty) \ \text{ such that } \ X_t^i = X_t^j = 0 \big) > 0 .
$$

We shall construct a solution to (1) with a specific boundary behavior of defaults until the time $\overline{\tau}_0 := \inf\{ s > 0 : \max_{1 \le i \le N} X_s^i = 0 \}$. Let us define the following map $\Phi(x) := (\Phi^1(x), \ldots, \Phi^N(x)) : [0, \infty)^N \mapsto [0, \infty)^N$ and set-valued function $\Gamma : \mathbb{R}_+^N \to \{1, \ldots, N\}$ defined by $\Gamma_0(x) := \{ i \in \{1, \ldots, N\} : x^i = 0 \}$,

$$
\Gamma_{k+1}(x) := \Big\{ i \in \{1, \ldots, N\} \setminus \bigcup_{\ell=1}^{k} \Gamma_\ell(x) \, : \, x^i - \frac{\overline{x}}{N} \cdot \Big| \bigcup_{\ell=1}^{k} \Gamma_\ell(x) \Big| \le 0 \Big\} ; \quad k = 0, 1, 2, \ldots, N-3
$$

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$$\Gamma(x) := \bigcup_{k=0}^{N-2} \Gamma_k(x)\,, \quad \Phi^i(x) := x^i + \overline{x}\Big(\Big(1+\frac{1}{N}\Big)\cdot \mathbf{1}_{\{i\in\Gamma(x)\}} - \frac{1}{N}\cdot|\Gamma(x)|\Big) \tag{4}$$

for $x = (x^1, \ldots, x^N) \in \mathbb{R}_+^N$, $i = 1, \ldots, N$ with $\overline{x} := (x^1 + \cdots + x_N)\,/\,N \geq 0$. Note that $\Phi([0,\infty)^N \setminus \{\mathbf{0}\}) \subseteq [0,\infty)^N \setminus \{\mathbf{0}\}$ and $\Phi(\mathbf{0}) = \mathbf{0} = (0, \ldots, 0)$.

Lemma 1. *Given a standard Brownian motion $W.$ and the initial configuration $X_0 \in (0,\infty)^N$ one can construct the process $(X., M.)$ which is the unique, strong solution to (1) with (2), (3) on $[0, \overline{\tau}_0]$, such that if there is a default, i.e., $|\Gamma(X_{t-})| \geq 1$ at time t, then the post-default behavior is determined by the process with $X_t^i = \Phi^i(X_{t-})$ for $i = 1, \ldots, N$.*

Now let us discuss the system (1) with (2)-(3) as a mean-field approximation for nonlinear equation of McKean-Vlasov type. For the sake of concreteness, let us assume $b(x,m) = -a(x-m)$, $x, m \in [0,\infty)$ for some $a > 0$. By the theory of propagation of chaos (e.g., Tanaka (1984), Shiga & Tanaka (1985) and Sznitman (1991)) as $N \to \infty$, the dynamics of the finite-dimensional marginal distribution of limiting representative process is expressed by

$$\mathcal{X}_t = \mathcal{X}_0 - a\int_0^t (\mathcal{X}_s - \mathbb{E}[\mathcal{X}_t])\mathrm{d}s + W_t + \int_0^t \mathbb{E}[\mathcal{X}_{s-}]\mathrm{d}(\mathcal{M}_s - \mathbb{E}[\mathcal{M}_s])\,; \quad t \geq 0\,, \tag{5}$$

where $W.$ is the standard Brownian motion, $\mathcal{M}_t := \sum_{k=1}^{\infty} \mathbf{1}_{\{\tau^k \leq t\}}$, $\tau^k := \inf\{s > \tau^{k-1} : \mathcal{X}_{t-} \leq 0\}$, $k \geq 1$, $\tau^0 = 0$. Then taking expectations of both sides of (5), we obtain $\mathbb{E}[\mathcal{X}_t] = \mathbb{E}[\mathcal{X}_0]$, $t \geq 0$. When $\mathcal{X}_0 = x_0$ a.s. for some $x_0 > 0$, substituting this back into (5), we obtain

$$\mathcal{X}_t = \mathcal{X}_0 - a\int_0^t (\mathcal{X}_s - \mathcal{X}_0)\mathrm{d}s + W_t + \mathcal{X}_0(\mathcal{M}_t - \mathbb{E}[\mathcal{M}_t])\,; \quad t \geq 0\,.$$

Transforming the state space from $[0,\infty)$ to $(-\infty, 1]$ by $\widehat{\mathcal{X}}_t := (x_0 - \mathcal{X}_t)\,/\,x_0$, we see

$$\widehat{\mathcal{X}}_t = -\int_0^t a\widehat{\mathcal{X}}_s\mathrm{d}s + \widehat{W}_t - \widehat{\mathcal{M}}_t + \mathbb{E}[\widehat{\mathcal{M}}_t]\,; \quad t \geq 0\,, \tag{6}$$

where we denote $\widehat{W}. = W.\,/\,x_0$, $\widehat{\mathcal{M}}. = \mathcal{M}.$.

This transformed process $\widehat{\mathcal{X}}.$ is similar to the nonlinear McKean-Vlasov-type stochastic differential equation

$$\widetilde{\mathcal{X}}_t = \widetilde{\mathcal{X}}_0 + \int_0^t \mathrm{b}(\widetilde{\mathcal{X}}_s)\mathrm{d}s + \widetilde{W}_t - \widetilde{\mathcal{M}}_t + \alpha\mathbb{E}[\widetilde{\mathcal{M}}_t]\,; \quad t \geq 0\,, \tag{7}$$

studied by Delarue, Inglis, Rubenthaler & Tanré (2015 a,b). Here $\widetilde{\mathcal{X}}_0 < 1$, $\alpha \in (0,1)$, $\mathrm{b} : (-\infty, 1] \to \mathbb{R}$ is assumed to be Lipschitz continuous with at most linear growth. $\widetilde{W}.$ is the standard Brownian motion, $\widetilde{\mathcal{M}}. = \sum_{k=1}^{\infty} \mathbf{1}_{\{\widetilde{\tau}^k \leq \cdot\}}$ with $\widetilde{\tau}^k := \inf\{s > \tau^{k-1} : \widetilde{\mathcal{X}}_{s-} \geq 1\}$, $k \geq 1$, $\widetilde{\tau}^0 = 0$. When we specify $\widetilde{\mathcal{X}}_0 = 0$, $\mathrm{b}(x) = -ax$, $x \in \mathbb{R}_+$, and $\alpha = 1$, the solution $(\widehat{\mathcal{X}}., \widehat{\mathcal{M}}.)$ to (7) reduces to the solution $(\widetilde{\mathcal{X}}., \widetilde{\mathcal{M}}.)$ to (6), however, the previous study of (7) does not guarantee the uniqueness of solution to (7) in the case $\alpha = 1$.

Proposition 1. *Assume* $x_0 > 1$ *and* $b(x, m) = -a(x - m)$, $x, m \in [0, \infty)$ *for some* $a > 0$. *There exists a unique strong solution to (6) on* $[0, T]$. *Moreover, for every* $T > 0$, *there exists a constant* c_T *such that every solution to (6) satisfies* $(\mathrm{d}/\mathrm{d}t)\mathbb{E}[\widehat{\mathcal{M}}_t] \le c_T$ *for* $0 \le t \le T$.

The proof is based on a fixed point argument. For example, when $a = 0$, we may reformulate the solution $(\widehat{\mathcal{X}}_\cdot, \widehat{\mathcal{M}}_\cdot)$ in (6) as

$$\widehat{\mathcal{Z}}_t = \widehat{\mathcal{X}}_t + \widehat{\mathcal{M}}_t = \widehat{W}_t + \mathbb{E}[\widehat{\mathcal{M}}_t], \quad \widehat{M}_t = \lfloor \sup_{0 \le s \le t} (\widehat{\mathcal{Z}}_s)^+ \rfloor ; \quad t \ge 0, \tag{8}$$

where $\lfloor x \rfloor$ is the integer part. Given a candidate solution e_t for $\mathbb{E}[\widehat{\mathcal{M}}_t]$, $t \ge 0$, we shall consider

$$\widehat{Z}^e_t := \widehat{W}_t + e_t, \quad \widehat{\mathcal{M}}^e_t := \lfloor \sup_{0 \le s \le t} (\widehat{Z}^e_s)^+ \rfloor ; \quad t \ge 0, \tag{9}$$

where the superscripts e of $\widehat{Z}^e_\cdot$ and $\widehat{\mathcal{M}}^e_\cdot$ represent the dependence on $e_\cdot$. Then uniqueness of the solution to (6) is reduced to uniqueness of the fixed point $e^*_\cdot = \mathfrak{M}_\cdot(e^*)$ of the map $\mathfrak{M} : C(\mathbb{R}_+, \mathbb{R}_+) \to C(\mathbb{R}_+, \mathbb{R}_+)$ defined by

$$\mathfrak{M}_t(e) := \mathbb{E}\big[\lfloor \sup_{0 \le s \le t} (\widehat{Z}^e_s)^+ \rfloor \big] = \mathbb{E}[\widehat{\mathcal{M}}^e_t] ; \quad t \ge 0. \tag{10}$$

By utilizing the monotone property of the map $\mathfrak{M}_t$ and the first passage time distribution for diffusions, we show contraction and then find a unique fixed point in the class of continuously differentiable, nonnegative functions bounded by a linear line with slope $1/x_0$.

It follows from Proposition 1 that the propagation-of-chaos result holds for the reformulated solution $(\mathcal{Z}_\cdot, \mathcal{M}_\cdot)$ from the original $X_\cdot$ in (1). Thus we have the following.

Proposition 2. *Under the same assumption as in Proposition 1, for every* $k \ge 1$, $\ell \ge 1$, $t_1, \dots t_\ell$, *as* $N \to \infty$ *the vector* $(X^i_{t_j}, M^i_{t_j})$, $1 \le i \le k$, $1 \le j \le \ell$ *defined from (1) converges towards the finite dimensional marginals at times* $t_1, \dots, t_\ell$ *of* k *independent copies of* $(\mathcal{X}_\cdot, \mathcal{M}_\cdot)$ *in (5).*

We shall also discuss the invariant distribution of $\mathcal{X}_\cdot$ in (5) and relation to the mean field games.

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