An analytic function in 3 complex variables related to the value-distribution of \( \log L \), and the "Plancherel volume"

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The first subject of the talk is of general and elementary nature. For a continuous density measure \( M(x)|dx| \) on \( \mathbb{R}^d \), let \( \mu_M \) denote the variance and \( \nu_M \) the "Plancherel volume"

\[
\nu_M = \int M(x)^2|dx| = \int |\hat{M}(y)|^2|dy|
\]

(\( |dx| \) the self-dual Haar measure of \( \mathbb{R}^d \), \( \hat{\cdot} \) the Fourier transform). We first focus our attention on the natural numerical invariant \( \mu_M^2 \nu_M \); its meaning, basic examples with parameters, the lower bound, etc.

The second (and the main) subject is a complex analytic function in 3 variables defined explicitly as

\[
\tilde{M}(s; z_1, z_2) = \prod_p F(i z_1/2, i z_2/2; 1; p^{-2s}) \quad (\Re(s) > 1/2)
\]

(\( i = \sqrt{-1} \), \( F(a, b; c; t) \) Gauss’ hypergeometric series). When \( \sigma \) is real > 1/2, \( \tilde{M}(\sigma, z_1, z_2) \) can be interpreted as the mean value of \( \{\zeta(s)^{iz_1/2} \zeta(s)^{iz_2/2}\} \) over the vertical line \( \Re(s) = \sigma \). But we consider \( \tilde{M}(s; z_1, z_2) \) as an analytic function also of the complex variable \( s \). We discuss its analytic continuation to the left of \( \Re(s) > 1/2 \), two other infinite product expansions, and the limit behaviours as \( s \to 1/2, +\infty \). This will be applied to the determination of the corresponding limits of the above invariant \( \mu_M^2 \nu_M \) when \( M = M_\sigma \) is the density measure on \( \mathbb{C} \) for the distribution of \( \{\log \zeta(\sigma + ti)\}_{t \in \mathbb{R}} \) or of \( \{\log L(s, \chi)\}_{\chi} \). The connection with value-distributions is joint work with K. Matsumoto.