Short time full asymptotic expansion of hypoelliptic heat kernel at the cut locus

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This is a jointwork with Setsuo Taniguchi (Kyushu University) and can be found at arXiv Preprint Server (arXiv:1603.01386).

We discuss a short time asymptotic expansion of a hypoelliptic heat kernel on a Euclidean space and on a compact manifold. We study the "cut locus" case, namely, the case where energy-minimizing paths which join the two points under consideration form not a finite set, but a compact manifold. Under mild assumptions we obtain an asymptotic expansion of the heat kernel up to any order. Our approach is probabilistic and the heat kernel is regarded as the density of the law of a hypoelliptic diffusion process, which is realized as a unique solution of the corresponding stochastic differential equation (SDE). Our main tools are S. Watanabe's distributional Malliavin calculus and T. Lyons' rough path theory.

Our work has the following three features. To our knowledge, there are no works which satisfy all of these conditions simultaneously:

1. The manifold and the hypoelliptic diffusion process on it are rather general. In other words, this is not a study of special examples.

2. The "cut locus" case is studied. More precisely, we mean by this that the set of energy-minimizing paths (or controls) which connect the two points under consideration becomes a compact manifold of finite dimension.

3. The asymptotic expansion is full, that is, the polynomial part of the asymptotics is up to any order.

On a Euclidean space, however, there are two famous results which satisfy (2), (3) and the latter half of (1). Both of them are probabilistic and use generalized versions of Malliavin calculus. One is Takanobu and Watanabe [3]. They use Watanabe's distributional Malliavin calculus. The other is Kusuoka and Stroock [2]. They use their version of generalized Malliavin calculus. We use the former.

Though we basically follow Takanobu-Watanabe's argument in [3], the main difference is that we use T. Lyons' rough path theory together, which is something like a deterministic version of the SDE theory. The main advantage of using rough path theory is that while the usual Itô map i.e., the solution map of an SDE is discontinuous, the Lyons-Itô map i.e., the solution map of a rough differential equation (RDE) is continuous.

This fact enables us to do "local analysis" of the Lyons-Itô map (for instance, restricting the map on a neighborhood of its critical point and doing a Taylor-like expansion) in a somewhat similar way we do in the Fréchet calculus. Recall that in the standard SDE theory, this type of local operation is very hard and sometimes impossible, due to

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the discontinuity of the Itô map. For this reason, the localization procedure in [3] looks so complicated that it might be difficult to generalize their method if rough path theory did not exist. Of course, there is a possibility that our main result can be proved without rough path theory, but we believe that the theory is quite suitable for this problem and gives us a very clear view (in particular, in the manifold case).

A detailed proof can be found in our preprint [1]. We first reprove and generalize the main result in [3] in the Euclidean setting by using rough path theory. Then, we study the manifold case. Recall that Malliavin calculus for a manifold-valued SDE was studied by Taniguchi [4]. Even in this Euclidean setting, many parts of the proof are technically improved, thanks to rough path theory. We believe that the following are worth mentioning: (i) Large deviation upper bound. (ii) Asymptotic partition of unity. (iii) A Taylor-like expansion of the Lyons-Itô map and the uniform exponential integrability lemma for the ordinary and the remainder terms of the expansion. (iv) Quasi-sure analysis for the solution of the SDE.

References

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