Potential theory of subordinate killed Brownian motion

Panki Kim **Renming Song** and **Zoran Vondraček**

Abstract

Let W^D be a killed Brownian motion in a domain $D \subset \mathbb{R}^d$ and S an independent subordinator with Laplace exponent ϕ. The process Y^D defined by $Y_t^D = W_{S_t}^D$ is called a subordinate killed Brownian motion. It is a Hunt process with infinitesimal generator $-\phi(-\Delta|_D)$, where $\Delta|_D$ is the Dirichlet Laplacian.

In the PDE literature, the operator $-(-\Delta|_D)^{\alpha/2}$, $\alpha \in (0,2)$, which is the generator of the subordinate killed Brownian motion via an $\alpha/2$-stable subordinator, also goes under the name of spectral fractional Laplacian, see [1] and the references therein. This operator has been of interest to quite a few people in the PDE circle. For instance, a version of Harnack inequality was also shown in [6].

In this talk we discuss the potential theory of Y^D under a weak scaling condition on the derivative of ϕ.

For any Borel set $B \subset D$, let $\tau_B = \inf\{t > 0 : Y_t^D \notin B\}$ be the exit time of Y^D from B.

Definition 0.1 A real-valued function f defined on D is said to be *harmonic* in an open set $V \subset D$ with respect to Y^D if for every open set $U \subset \overline{U} \subset V$,

$$\mathbb{E}_x\left[\left|f(Y_{\tau_U}^D)\right|\right] < \infty \quad \text{and} \quad f(x) = \mathbb{E}_x\left[f(Y_{\tau_U}^D)\right] \qquad \text{for all } x \in U. \tag{0.1}$$

Under some mild assumptions on ϕ and D, we show that non-negative harmonic functions of Y^D satisfy the following scale invariant Harnack inequality, which extends [5, 6].

Theorem 0.2 (Harnack inequality) *Let $D \subset \mathbb{R}^d$ be either a bounded Lipschitz domain or an unbounded domain consisting of all the points above the graph of a globally Lipschitz function. There exists a constant $C > 0$ such that for any $r \in (0,1]$ and $B(x_0, r) \subset D$ and any function f which is non-negative in D and harmonic in $B(x_0, r)$ with respect to Y^D, we have*

$$f(x) \le Cf(y), \qquad \textit{for all } x, y \in B(x_0, r/2).$$

The proof of the Harnack inequality is modeled after the powerful method developed in [2].

Subsequently we present two types of scale invariant boundary Harnack principles with explicit decay rates for non-negative harmonic functions of Y^D. The first boundary Harnack principle deals with a $C^{1,1}$ domain D and non-negative functions which are harmonic near the boundary of D.

For any open set $U \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, we use $\delta_U(x)$ to denote the distance between x and the boundary ∂U.

Theorem 0.3 *Let D be a bounded $C^{1,1}$ domain, or a $C^{1,1}$ domain with compact complement or a domain consisting of all the points above the graph of a bounded globally $C^{1,1}$ function. Let (R,Λ) be the $C^{1,1}$ characteristics of D. There exists a constant $C = C(d,\Lambda,R,\phi) > 0$ such that for any $r \in (0,R]$, $Q \in \partial D$, and any non-negative function f in D which is harmonic in $D \cap B(Q,r)$ with respect to Y^D and vanishes continuously on $\partial D \cap B(Q,r)$, we have*

$$\frac{f(x)}{\delta_D(x)} \le C\,\frac{f(y)}{\delta_D(y)} \qquad \text{for all } x,y \in D \cap B(Q,r/2). \tag{0.2}$$

In particular, we see from the theorem above that if a non-negative function which is harmonic with respect to Y^D vanishes near the boundary, then its rate of decay is proportional to the distance to the boundary. This shows that near the boundary of D, Y^D behaves like the killed Brownian motion W^D.

The second one is for a more general domain D and non-negative functions which are harmonic near the boundary of an interior open subset of D.

Theorem 0.4 *Let $D \subset \mathbb{R}^d$ be either a bounded Lipschitz domain or an unbounded domain consisting of all the points above the graph of a globally Lipschitz function. There exists a constant $b = b(\phi,d) > 2$ such that, for every open set $E \subset D$ and every $Q \in \partial E \cap D$ such that E is $C^{1,1}$ near Q with characteristics $(\delta_D(Q) \wedge 1, \Lambda)$, the following holds: There exists a constant $C = C(\delta_D(Q) \wedge 1, \Lambda, \phi, d) > 0$ such that for every $r \le (\delta_D(Q) \wedge 1)/(b+2)$ and every non-negative function f on D which is regular harmonic in $E \cap B(Q,r)$ with respect to Y^D and vanishes on $E^c \cap B(Q,r)$, we have*

$$\frac{f(x)}{\phi(\delta_E(x)^{-2})^{-1/2}} \le C \frac{f(y)}{\phi(\delta_E(y)^{-2})^{-1/2}}\,, \qquad x,y \in E \cap B(Q, 2^{-6}(1+(1+\Lambda)^2)^{-2} r)\,,$$

The obtained decay rates in the above two theorem are not the same, reflecting different boundary and interior behaviors of Y^D. Theorem 0.4 is new even in the case of a stable subordinator. The method of proof of Theorem 0.4 is quite different from that of Theorem 0.3. It relies on a comparison of the Green functions of subprocesses of Y^D and X for small interior subsets of D, and on some already available potential-theoretic results for X obtained in [3].

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References

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