The universality of the Riemann zeta-function

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It is well known that analytic functions can be approximated by polynomials. More precisely, the Mergelian theorem asserts that, for every continuous function $f(s)$, $s = \sigma + it$, on a compact set $K \subset \mathbb{C}$ with connected complement which is analytic in the interior of K, and any $\varepsilon > 0$, there exists a polynomial $p(s)$ in s such that

$$\sup_{s \in K} |f(s) - p(s)| < \varepsilon.$$

Since 1975, it became known that there exist functions $g(s)$ having a property that their shifts $g(s + i\tau)$ approximate uniformly on compact subsets of a certain region every analytic function. The first example of such functions is the Riemann zeta-function $\zeta(s)$, its approximation property is called the universality, and was discovered by Voronin [2]. The last version of the Voronin theorem is the following statement.

Theorem 1 *Let K be a compact subset of the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complement, and let $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\{\tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\} > 0.$$

It is known that certain functions $F(\zeta(s))$ also preserve the universality property. For example, $\log \zeta(s)$, where $\log \zeta(\sigma + it)$, $\frac{1}{2} < \sigma < 1$, is defined from $\log \zeta(2) \in \mathbb{R}$ by continuous variation along the line segments $[2, 2 + it]$ and $[2 + it, \sigma + it]$, provided that the path does not pass a zero or pole of $\zeta(s)$, and if it does, than we take $\log \zeta(\sigma + it) = \lim_{\varepsilon \to +0} \log \zeta(\sigma + i(t + \varepsilon))$, is universal. The derivative $\zeta'(s)$ also has the universality property. In the both examples the approximated function $f(s)$ is not necessarily non-vanishing on K. Therefore a problem arises to describe a set of functions F such that $F(\zeta(s))$ should be universal in the above sense. In the report, we discuss this problem.

Let G be a region on the complex plane. Denote by $H(G)$ the space of analytic on G functions equipped with the topology of uniform convergence on compacta.

Denote by $Lip(c, \alpha)$ a class of functions $F : H(D) \to H(D)$ satisfying the hypotheses:

1° For each polynomial $p = p(s)$ and every compact subset $K \subset D$ with connected complement, the pre-image $F^{-1}p(s) \in H(D)$, and $F^{-1}p(s) \neq 0$ on K.

2° The functions F are of the Lipschitz type, i. e., for all $g_1, g_2 \in H(D)$ and every compact subset $K \subset D$ with connected complement there exist positive constants c and α, and a compact subset $K_1 \subset D$ with connected complement such that,

$$\sup_{s\in K} |F(g_1(s)) - F(g_2(s))| \leq c \sup_{s\in A} |g_1(s) - g_2(s)|^\alpha$$

for some $A \subset K_1$.

Clearly, if $F \in Lip(c, \alpha)$, then the universality of $F(\zeta(s))$ is a simple consequence of the universality of $\zeta(s)$ itself. For example, in view of Cauchy integral formula, the function $F(g(s)) = g'(s)$, $g \in H(D)$, belongs to $Lip(c, 1)$.

Let $S = \{g \in H(D) : \ g^{-1}(s) \in H(D) \text{ or } g(s) \equiv 0\}$. Denote by U the class of continuous functions $F : \ H(D) \to H(D)$ such that, for any open set $G \subset H(D)$,

$$(F^{-1}G) \cap S \neq \emptyset.$$

Theorem 2 *Suppose that $F \in U$. Let K be the same as in Theorem 1 and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\{\tau \in [0, T] : \sup_{s\in K} |F(\zeta(s + i\tau)) - f(s)| < \varepsilon\} > 0.$$

A more convenient is the class U_1 of continuous functions $F : \ H(D) \to H(D)$ such that, for each polynomial $p = p(s)$,

$$(F^{-1}\{p\}) \cap S \neq \emptyset.$$

Theorem 3 *Suppose that $F \in U_1$. Let K and $f(s)$ be the same as in Theorem 2. Then the assertion of Theorem 2 is true.*

Let V be an arbitrary positive number, and $D_V = \{s \in \mathbb{C} : \ \frac{1}{2} < \sigma < 1\}$, $S_V = \{g \in H(D_V) : \ g^{-1}(s) \in H(D_V) \text{ or } g(s) \equiv 0\}$. Denote by $U_{1,V}$ the class of continuous functions $F : \ H(D_V) \to H(D_V)$ such that, for each polynomial $p = p(s)$,

$$(F^{-1}\{p\}) \cap S_V \neq \emptyset.$$

Theorem 4 *Let K and $f(s)$ be the same as in Theorem 2. Suppose that $V > 0$ is such that $K \subset D_V$, and that $F \in U_{1,V}$. Then the assertion of Theorem 2 is true.*

For example, by Theorem 4, the function

$$c_1\zeta'(s) + \cdots + c_r\zeta^{(r)}(s), \quad c_1, \ldots, c_r \in \mathbb{C} \setminus \{0\},$$

is universal.

Define the class $U_{a_1,\ldots,a_r}$, $a_1, \ldots, a_r \in \mathbb{C}$, of continuous functions $F : H(D) \to H(D)$ such that $F(S) = H_{a_1,\ldots,a_r}(D)$,

$$H_{a_1,\ldots,a_r}(D) = \{g \in H(D) : \ g(s) \equiv 0 \text{ or } (g(s) - a_j)^{-1} \in H(D), \ j = 1, \ldots, r\}.$$

Theorem 5 *Suppose that $F \in U_{a_1,\ldots,a_r}$. For $r = 1$, let K be the same as in Theorem 2, and let $f(s)$ be a continuous function on K, $f(s) \neq a_1$ on K, which is analytic in the interior of K. For $r > 1$, let K be any compact subset of D, and $f(s) \in H_{a_1,\ldots,a_r}(D)$. Then the assertion of Theorem 2 is valid.*

Theorem 5 implies the universality of the functions $\zeta^N(s)$, $N \in \mathbb{N}$, and $\mathrm{e}^{\zeta(s)}$.

Also, in the report, the universality of the functions $F(\varphi(s, \widehat{F}))$, where $\varphi(s, \widehat{F})$ is the zeta-function associated to a holomorphic normalized Hecke eigen cusp form $\widehat{F}$, and $F(L(s, \chi_1), \ldots, L(s, \chi_n))$, where $L(s, \chi_1), \ldots, L(s, \chi_n)$ are Dirichlet L-functions with pairwise non-equivalent characters will be discussed.

All proofs of the above universality theorems are based on probabilistic limit theorems in the space of analytic functions. A part of results were obtained jointly with Kohji Matsumoto and Joern Steuding. Universality theorems for the Riemann zeta function are published in [1].

References

[1] A. Laurinčikas, Universality of the Riemann zeta-function, *J. Number Theory*, **130** (2010), 2323–2331.

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