Stochastic complex Ginzburg-Landau equation with space-time white noise *

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In this talk, we prove local well-posedness of the stochastic complex Ginzburg-Landau equation with a complex-valued space-time white noise ξ in the three-dimensional torus $\mathbf{T}^3 = (\mathbf{R}/\mathbf{Z})^3$

$$(\mathrm{P}) \qquad \begin{cases} \partial_t u = (\mathrm{i}+\mu)\triangle u + \nu(1-|u|^2)u + \xi & \text{on } (0,\infty)\times\mathbf{T}^3, \\ u(0,\cdot) = u_0(\cdot). \end{cases}$$

Here, $\mathrm{i} = \sqrt{-1}$, μ is a positive constant and ν is a complex constant.

Before starting our discussion, we introduce notation. We denote by $\mathcal{D}$ the space of all smooth functions on $\mathbf{T}^3$ and by $\mathcal{D}'$ its dual. For every $\alpha \in \mathbf{R}$, $1 \le p, q \le \infty$, we denote by $\mathcal{B}^\alpha_{p,q}$ the Besov space, which is defined by the completion of the space of smooth functions on $\mathbf{T}^3$ under the Besov norm $\|\cdot\|_{\mathcal{B}^\alpha_{p,q}}$. To define the Besov norm, we use the Littlewood-Paley block $\{\triangle_m = \mathcal{F}^{-1}\rho_m\mathcal{F}\}_{m=-1}^\infty$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transformation and its inverse, respectively, and $\{\rho_m\}_{m=-1}^\infty$ is the dyadic partition of unity. For notational simplicity, we set the Hölder-Besov space $\mathcal{C}^\alpha = \mathcal{B}^\alpha_{\infty,\infty}$ and denote by $C_T\mathcal{C}^\alpha$ the space of all $\mathcal{C}^\alpha$-valued continuous functions on $[0,T]$ for every $T > 0$. Next we introduce the notion of paradifferential calculus. For every $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$, we define the resonance $f \odot g$ and the praproduct $f \otimes g$. They give the decomposition $fg = f \otimes g + f \odot g + f \otimes g$. The paraproduct $f \otimes g$ can be defined for any $\alpha, \beta \in \mathbf{R}$, but the resonance $f \odot g$ can be defined for $\alpha + \beta > 0$. Hence, in order define products fg, it is necessary that $\alpha + \beta > 0$ holds. Finally, we set $\mathcal{L}^1 = \partial_t - \{(\mathrm{i}+\mu)\triangle - 1\}$, $P^1_t = e^{t\{(\mathrm{i}+\mu)\triangle-1\}}$ and $I(u)_t = \int_{-\infty}^t P^1_{t-s} u_s\, ds$ for $u : [0,\infty) \to \mathcal{D}'$.

Now we return to well-posedness of the equation (P). For some reason, we write (P) as $\mathcal{L}^1 u = \nu(1-|u|^2)u + u + \xi$ and discuss the problem. To illustrate difficulty of this problem, we consider a stationary solution to the linear equation $\mathcal{L}^1 Z = \xi$ on $(0,\infty)\times\mathbf{T}^3$. The solution is given by $Z_t = I(\xi)_t$ formally and it is not a function but a distribution with respect to the space variable in the dimension three. More precisely, Z_t belongs $\mathcal{C}^{-\frac{1}{2}-\kappa}$ for any $\kappa > 0$. Hence the products Z_t^2, $Z_t\overline{Z_t}$, $Z_t^2\overline{Z_t}$ and so on are not defined a priori. Since the irregularity of the solution to (P) comes from the white noise, it is natural to guess that the space regularity of u_t is not better than that of Z_t and that the product $|u_t|^2 u_t = u_t^2\overline{u_t}$ is not defined a priori.

To overcome this difficulty, we use the theory of paracontrolled distributions developed in [GIP15]. The method consists a deterministic part and a probabilistic part.

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In the deterministic part, we construct the solution map of (P) from the space $\mathcal{X}^{\kappa}_{T_*}$ of driving vectors to the space $\mathcal{D}^{\kappa,\kappa'}_{T_*}$ of solutions, where T_* is a life time of a solution and κ, κ' are positive small parameters, and show that the solution map is continuous. To be precise, for every $0 < \kappa < \kappa' < 1/18$ and $T > 0$, we call a vector of space-time distributions

$$X = (X^{\mathsf{I}}, X^{\mathsf{V}}, X^{\mathsf{V}^{\bullet}}, X^{\mathsf{Y}}, X^{\mathsf{Y}^{\bullet}}, X^{\mathsf{Y}^{\bullet\bullet}}, \dots)$$
$$\in C_T\mathcal{C}^{-\frac{1}{2}-\kappa} \times (C_T\mathcal{C}^{-1-\kappa})^2 \times (C_T\mathcal{C}^{1-\kappa})^2 \times \mathcal{L}_T^{\frac{1}{2}-\kappa,\frac{1}{4}-\frac{1}{2}\kappa} \times (C_T\mathcal{C}^{-\kappa})^6 \times (C_T\mathcal{C}^{-\frac{1}{2}-\kappa})^2$$

which satisfies $\mathcal{L}^1 X^{\mathsf{Y}} = X^{\mathsf{V}}$ and $\mathcal{L}^1 X^{\mathsf{Y}^{\bullet}} = X^{\mathsf{V}^{\bullet}}$ a *driving vector* of (P). We denote by $\mathcal{X}^{\kappa}_T$ the set of all driving vector. The definition of $\mathcal{D}^{\kappa,\kappa'}_T$ is a little complicated. Because we transform (P) to a system of two equations with respect to (v, w) so that $u = X^{\mathsf{I}} - \nu X^{\mathsf{Y}^{\bullet\bullet}} + v + w$ solves (P). The space $\mathcal{D}^{\kappa,\kappa'}_T$ is where (v, w) lives.

We explain the meanings of the graphical symbols I, V, $\mathsf{V}^{\bullet}$, Y, ... They are just coordinates mathematically; however, the dot and the line are icons for the white noise and the operation I, respectively. Hence, I represents $I(\xi) = Z$. Moreover, $\vdots$ and $\mathsf{V}^{\bullet}$ are icons for the complex conjugate of Z and the product $Z\overline{Z}$, respectively. So $\mathsf{Y}^{\bullet\bullet}$ means $I(Z^2\overline{Z})$. Finally, $\circ$ denotes the resonance term; $\mathsf{Y}^{\bullet\bullet}_{\circ}$ represents $I(Z^2\overline{Z}) \odot Z$.

In the probabilistic part, we construct a driving vector X^{ϵ} from a smeared noise ξ^{ϵ} with a parameter $0 < \epsilon < 1$ and show convergence of X^{ϵ} as $\epsilon \downarrow 0$. Of course, we assume that $\xi^{\epsilon} \to \xi$ as $\epsilon \downarrow 0$. More precisely, we set $X^{\epsilon,\mathsf{I}} = Z^{\epsilon} = I(\xi^{\epsilon})_t$, $X^{\epsilon,\vdots} = \overline{Z^{\epsilon}}$ and $X^{\epsilon,\mathsf{V}} = (Z^{\epsilon})^2$; however, since $\mathfrak{c}^{\epsilon}_1 = \boldsymbol{E}[Z^{\epsilon}_t\overline{Z^{\epsilon}_t}]$ diverges as $\epsilon \downarrow 0$, we need to consider renormalization and set $X^{\epsilon,\mathsf{V}^{\bullet}} = Z^{\epsilon}\overline{Z^{\epsilon}} - \mathfrak{c}^{\epsilon}_1$. In order to define $X^{\epsilon,\tau}$ for $\mathsf{Y}^{\bullet\bullet}$, ..., it is necessary to consider renormalization. The other renormalization constants are $\mathfrak{c}^{\epsilon}_{2,1} = \frac{1}{2}\boldsymbol{E}[X^{\epsilon,\mathsf{Y}}_{(t,x)} \odot X^{\epsilon,\mathsf{V}^{\bullet\bullet}}_{(t,x)}]$ and $\mathfrak{c}^{\epsilon}_{2,2} = \boldsymbol{E}[X^{\epsilon,\mathsf{Y}^{\bullet}}_{(t,x)} \odot X^{\epsilon,\mathsf{V}^{\bullet}}_{(t,x)}]$. To show convergence of X^{ϵ}, we express $\triangle_m X^{\tau}$ by the Itô-Wiener integrals and estimate their kernels.

From the discussion above, we obtain our main result:

Theorem 1. *Set* $\mathfrak{c}^{\epsilon} = 2(\mathfrak{c}^{\epsilon}_1 - \overline{\nu\mathfrak{c}^{\epsilon}_{2,1}} - 2\nu\mathfrak{c}^{\epsilon}_{2,2})$. *Let* $u_0 \in \mathcal{C}^{-\frac{2}{3}+\kappa'}$. *Consider the renormalized equation*

$$\text{(P')} \qquad \begin{cases} \partial_t u^{\epsilon} = (\mathrm{i} + \mu)\triangle u^{\epsilon} + \nu(1 - |u^{\epsilon}|^2)u^{\epsilon} + \nu\mathfrak{c}^{\epsilon}u^{\epsilon} + \xi^{\epsilon}, & \text{on } (0,\infty) \times \mathbf{T}^3, \\ u(0,\cdot) = u_0(\cdot). \end{cases}$$

Then $\mathfrak{c}^{\epsilon} \to \infty$ *as* $\epsilon \downarrow 0$ *and there exist a unique process* u^{ϵ} *and a random time* T^{ϵ}_* *such that*

- u^{ϵ} *solves* (P') *on* $[0, T^{\epsilon}_*) \times \mathbf{T}^3$,
- T^{ϵ}_* *converges to some a.s. positive random time* T_* *in probability,*
- u^{ϵ} *converges to some process* u *defined on* $[0, T_*) \times \mathbf{T}^3$ *in the sense that* $\sup_{0 \le s \le T_*/2} \|u^{\epsilon}_s - u_s\|_{\mathcal{C}^{-\frac{2}{3}+\kappa'}} \to 0$ *as* $\epsilon \to 0$ *in probability. Furthermore,* u *is independent of the choice of* ξ^{ϵ}.

References

[GIP15] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math. Pi*, 3:e6, 75, 2015.