# Independence of $L$-functions

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In 1900, Hilbert stated that the Riemann zeta-function $\zeta(s)$ does not satisfy any algebraic differential equation whose coefficients are rational functions of $s$. Much later, in view of the value-distribution of $\zeta(s)$, Voronin obtained another proof and a stronger independence result, which is called the functional independence of $\zeta(s)$ and its derivatives.

Further Voronin established the functional independence of Dirichlet $L$-functions $L(s,\chi_1),\dots,L(s,\chi_r)$ and their derivatives, by proving a result about their *joint* value-distribution, where $L(s,\chi_j)$ are defined for $\operatorname{Re} s > 1$ by

$$L(s,\chi_j) := \sum_{n=1}^{\infty} \frac{\chi_j(n)}{n^s} = \prod_{p\,:\,\text{prime}} \left(1 - \frac{\chi_j(p)}{p^s}\right)^{-1} \qquad (\chi_j : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C})\ .$$

His proof depends deeply on the periodicity of Dirichlet characters $\chi_j$ and their complete orthogonality relation, so his argument cannot be applied to other general number-theoretic $L$-functions.

In this talk, we will report "(hybrid) joint denseness results" for values of general $L$-functions $L_1(s),\dots,L_r(s)$ in a certain class $\mathcal{L}$ and their derivatives, where $L_j(s)$ are functions of the form

$$L_j(s) = \sum_{n=1}^{\infty} \frac{a_{L_j}(n)}{n^s} = \prod_{p\,:\,\text{prime}} \prod_{l=1}^{d_{L_j}} \left(1 - \frac{\alpha_{L_j}(p,l)}{p^s}\right)^{-1} .$$

A typical example of the class $\mathcal{L}$ is the set

$$\bigcup_{n=1}^{\infty} \{\text{cuspidal automorphic } L\text{-functions for } GL(n)/\mathbb{Q} \text{ under the Ramanujan conjecture}\}.$$

In the proofs we use suitably Selberg's orthogonality relation

$$\sum_{p \le x} \frac{a_{L_j}(p)\overline{a_{L_k}(p)}}{p} = \begin{cases} c_{L_j} \log\log x + o(\log\log x) & \text{if } L_j(s) = L_k(s) \\ o(\log\log x) & \text{if } L_j(s) \ne L_k(s) \end{cases}$$

(in place of the periodicity of Dirichlet characters $\chi_j$ and their complete orthogonality relation in Voronin's proof).

Our joint denseness results imply that the functions $L_1(s),\dots,L_r(s)$ and their derivatives are functionally independent and also algebraically independent over a certain class of general Dirichlet series.

If time permits, we would also like to discuss a question about the *probabilistic* independence for $L$-functions $L_1(s),\dots,L_r(s)$, which are random variables on a certain probability space.