On density function concerning maxima of some one-dimensional diffusion processes

Tomonori Nakatsu (Ritsumeikan University)

1 Introduction

This talk is based on [3] and [4].

In this talk, we shall deal with the following one-dimensional stochastic differential equation (SDE),

\[ X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \]

where \( b, \sigma : [0, \infty) \times \mathbb{R} \to \mathbb{R} \) are measurable functions and \( \{W_t, t \in [0, \infty)\} \) denotes a one-dimensional standard Brownian motion defined on a probability space \((\Omega, F, P)\). We will consider discrete time maximum and continuous time maximum which are defined by

\[ M_{nT} := \max\{X_{t_1}, \ldots, X_{t_n}\} \quad \text{and} \quad M_T := \max_{0 \leq t \leq T} X_t, \]

respectively, where the time interval \([0, T]\) and the time partition \(\Delta_n : 0 < t_1 < \cdots < t_n = T, n \geq 2\) are fixed.

The first part of the talk is devoted to prove an integration by parts (IBP) formula of \(M_{nT}\) and \(M_T\). Here, we say that the IBP formula for the random variables \(F\) and \(G\) holds if there exists an integrable random variable \(H(F; G)\) such that

\[ E[\varphi'(M_{nT})G] = E[\varphi(M_{nT})H(F; G)] \]

holds for any \(\varphi \in C^1_b(\mathbb{R}; \mathbb{R})\). Moreover, we will obtain expressions, and upper bounds of the density function of \(M_{nT}\) and \(M_T\) by means of the IBP formula.

In the second part of the talk, we shall obtain some asymptotic behaviors of the density function of \(M_{nT}\). In this part, we will deal with only Gaussian processes: Itô processes with deterministic integrands, the Brownian Bridge and the Ornstein-Uhlenbeck process.

2 Main results

Assumption (A)

(A1) For \(t \in [0, \infty), b(t, \cdot), \sigma(t, \cdot) \in C^2_b(\mathbb{R}; \mathbb{R})\). Furthermore, all constants which bound the derivatives of \(b(t, \cdot)\) and \(\sigma(t, \cdot)\) do not depend on \(t\).

(A2) There exists \(c > 0\) such that

\[ |\sigma(t, x)| \geq c \]

holds, for any \(x \in \mathbb{R}\) and \(t \in [0, \infty)\).

Theorem 1. ([3]) Assume (A). Let \(G \in D^{1,\infty}\). Then there exists a random variable \(H_T^p(G)\) such that \(H_T^p(G)\) belongs to \(L^p(\Omega, F, P)\) for any \(p \geq 1\), and

\[ E^p[\varphi'(M_{nT})G] = E^p[\varphi(M_{nT})H_T^p(G)] \]

holds for any \(\varphi \in C^1_b(\mathbb{R}; \mathbb{R})\).

Assumption (A)'

We assume that the diffusion coefficient of (1) is of the form \(\sigma(t, x) = \sigma_1(t)\sigma_2(x)\) and the following assumption.
function of \( \Psi \) satisfy the following ordinary differential equation (ODE),

\[
\begin{array}{ll}
\frac{d\psi}{dt}(x) = \sigma_2(\psi(x)) \\
\psi(0) = x_0.
\end{array}
\]

Then due to (A3)', \( \Psi^{-1}(x) \) exists for any \( x \in \mathbb{R} \). We define the probability measure \( \tilde{P} \) by

\[
\frac{d\tilde{P}}{dP}\bigg|_{\mathcal{F}_T} := e^{\int_0^T \frac{1}{2} \psi''(\Psi^{-1}(X_s)) \sigma_1^2(s) - b(s, X_s) ds} \quad \tilde{W}_t = W_t - \int_0^t \frac{1}{2} \psi''(\Psi^{-1}(X_s)) \sigma_1^2(s) - b(s, X_s) ds, t \in [0, T],
\]

then \( \{\tilde{W}_t, t \in [0, T]\} \) is a one-dimensional under \( \tilde{P} \). Moreover, it is easy to see that the solution to (1) is expressed as

\[
X_t = \Psi\left(\int_0^t \sigma_1(s) d\tilde{W}_s\right), t \in [0, T].
\]

**Theorem 2.** ([3]) Assume (A)'. Let \( G \in \mathbb{D}^{1,\infty} \) and \( a_0 > x_0 \) be fixed arbitrarily. Then there exists a random variable \( H_T(G, a_0) \) such that \( H_T(G, a_0) \) belongs to \( L^p(\Omega, \mathcal{F}, P) \) for any \( p \geq 1 \), and

\[
E^P[\varphi(M_T)G] = E^P[\varphi(M_T)H_T(G, a_0)]
\]

for any \( \varphi \in C_b^1(\mathbb{R}; \mathbb{R}) \) whose support is included in \( (a_0, \infty) \).

In the talk, formulas (2) and (3) will be used to obtain the expressions and the upper bounds of the density function of \( M_T \) and \( M_T \).

Then, we shall obtain the results on asymptotic behaviors of the density functions which are proved in [4].

**References**


