On density function concerning maxima of some one-dimensional diffusion processes

Tomonori Nakatsu (Ritsumeikan University)

1 Introduction

This talk is based on [3] and [4].

In this talk, we shall deal with the following one-dimensional stochastic differential equation (SDE),

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \tag{1}$$

where $b, \sigma : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ are measurable functions and $\{W_t, t \in [0,\infty)\}$ denotes a one-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. We will consider discrete time maximum and continuous time maximum which are defined by $M_T^n := \max\{X_{t_1}, \cdots, X_{t_n}\}$ and $M_T := \max_{0 \le t \le T} X_t$, respectively, where the time interval $[0, T]$ and the time partition $\Delta_n : 0 < t_1 < \cdots < t_n = T$, $n \ge 2$ are fixed.

The first part of the talk is devoted to prove an integration by parts (IBP) formula of M_T^n and M_T. Here, we say that the IBP formula for the random variables F and G holds if there exists an integrable random variable $H(F; G)$ such that

$$E[\varphi'(F)G] = E[\varphi(F)H(F; G)]$$

holds for any $\varphi \in C_b^1(\mathbb{R}; \mathbb{R})$. Moreover, we will obtain expressions, and upper bounds of the density function of M_T^n and M_T by means of the IBP formula.

In the second part of the talk, we shall obtain some asymptotic behaviors of the density function of M_T^n. In this part, we will deal with only Gaussian processes: Itô processes with deterministic integrands, the Brownian Bridge and the Ornstein-Uhlenbeck process.

2 Main results

Assumption (A)

(A1) For $t \in [0, \infty)$, $b(t, \cdot), \sigma(t, \cdot) \in C_b^2(\mathbb{R}; \mathbb{R})$. Furthermore, all constants which bound the derivatives of $b(t, \cdot)$ and $\sigma(t, \cdot)$ do not depend on t.

(A2) There exists $c > 0$ such that

$$|\sigma(t, x)| \ge c$$

holds, for any $x \in \mathbb{R}$ and $t \in [0, \infty)$.

Theorem 1. *([3]) Assume (A). Let $G \in \mathbb{D}^{1,\infty}$. Then there exists a random variable $H_T^n(G)$ such that $H_T^n(G)$ belongs to $L^p(\Omega, \mathcal{F}, P)$ for any $p \ge 1$, and*

$$E^P\left[\varphi'(M_T^n)G\right] = E^P[\varphi(M_T^n)H_T^n(G)] \tag{2}$$

holds for any $\varphi \in C_b^1(\mathbb{R}; \mathbb{R})$.

Assumption (A)'

We assume that the diffusion coefficient of (1) is of the form $\sigma(t, x) = \sigma_1(t)\sigma_2(x)$ and the following assumption.

(A1)' For $t \in [0,\infty)$, $b(t,\cdot) \in C_b^2(\mathbb{R};\mathbb{R})$. Furthermore, all constants which bound the derivatives of $b(t,\cdot)$ do not depend on t.

(A2)' $\sigma_1(\cdot) \in C_b^0([0,\infty);\mathbb{R})$ and there exists $c_1 > 0$ such that $|\sigma_1(t)| \geq c_1$ for any $t \in [0,\infty)$.

(A3)' $\sigma_2(\cdot) \in C_b^2(\mathbb{R};\mathbb{R})$ and there exists $c_2 > 0$ such that $|\sigma_2(x)| \geq c_2$ for any $x \in \mathbb{R}$.

Let Ψ satisfy the following ordinary differential equation (ODE),

$$\begin{cases} \frac{d\Psi}{dx}(x) = \sigma_2(\Psi(x)) \\ \Psi(0) = x_0. \end{cases}$$

Then due to (A3)', $\Psi^{-1}(x)$ exists for any $x \in \mathbb{R}$. We define the probability measure $\tilde{P}$ by

$$\left.\frac{d\tilde{P}}{dP}\right|_{\mathcal{F}_T} := e^{\int_0^T \frac{\frac{1}{2}\Psi''(\Psi^{-1}(X_s))\sigma_1^2(s) - b(s,X_s)}{\sigma(s,X_s)} dW_s - \frac{1}{2}\int_0^T \left[\frac{\frac{1}{2}\Psi''(\Psi^{-1}(X_s))\sigma_1^2(s) - b(s,X_s)}{\sigma(s,X_s)}\right]^2 ds} \equiv \tilde{K}_T,$$

and

$$\tilde{W}_t := W_t - \int_0^t \frac{\frac{1}{2}\Psi''(\Psi^{-1}(X_s))\sigma_1^2(s) - b(s,X_s)}{\sigma(s,X_s)} ds, t \in [0,T],$$

then $\{\tilde{W}_t, t \in [0,T]\}$ is a one-dimensional under $\tilde{P}$. Moreover, it is easy to see that the solution to (1) is expressed as

$$X_t = \Psi\left(\int_0^t \sigma_1(s) d\tilde{W}_s\right), t \in [0,T].$$

Theorem 2. *([3]) Assume (A)'. Let $G \in \tilde{\mathbb{D}}^{1,\infty}$ and $a_0 > x_0$ be fixed arbitrarily. Then there exists a random variable $H_T(G,a_0)$ such that $H_T(G,a_0)$ belongs to $L^p(\Omega,\mathcal{F},P)$ for any $p \geq 1$, and*

$$E^P[\varphi'(M_T)G] = E^P[\varphi(M_T)H_T(G,a_0)] \tag{3}$$

holds for any $\varphi \in C_b^1(\mathbb{R};\mathbb{R})$ whose support is included in (a_0,∞).

In the talk, formulas (2) and (3) will be used to obtain the expressions and the upper bounds of the density function of M_T^n and M_T.

Then, we shall obtain the results on asymptotic behaviors of the density functions which are proved in [4].

References

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