Central limit theorems for non-symmetric random walks on nilpotent covering graphs

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As a fundamental problem in the theory of random walks (RWs), Donsker's invariance principle or the functional central limit theorem has been studied intensively and extensively. In particular, Ishiwata, Kawabi and Kotani [2] studied this problem for non-symmetric RWs on a crystal lattice from a viewpoint of *discrete geometric analysis* initiated by Sunada [3]. On the other hand, Ishiwata [1] also discussed this problem for symmetric RWs on a *nilpotent covering graph* X, a (locally finite and connected) covering graph of a finite graph X_0 whose covering transformation group Γ is a torsion free and finitely generated nilpotent group. In this talk, we consider a class of non-symmetric RWs on X and discuss Donsker's invariance principle for them as an extension of [1, 2].

Let $X=(V,E)$ be a nilpotent covering graph. Here V is a set of all vertices and E a set of all oriented edges in X. For $e \in E$, we denote the origin, terminus and inverse edge of e by $o(e), t(e)$ and $\overline{e}$, respectively. $E_x := \{e \in E \,|\, o(e)=x\}$ denotes the set of all edges whose origin is $x \in V$. In the following, we introduce basic materials for RWs on X. Now let $p: E \longrightarrow (0,1]$ be a (Γ-invariant) 1-step transition probability and $\{w_n\}_{n=0}^{\infty}$ a RW on X associated with p. We may also consider the RW $\{\pi(w_n)\}_{n=0}^{\infty}$ on the quotient $X_0=(V_0,E_0)$ due to the Γ-invariance of p. Here $\pi : X \longrightarrow X_0$ is a covering map. Let $m: V_0 \longrightarrow (0,1]$ be a normalized invariant measure on X_0 and we also write $m: V \longrightarrow (0,1]$ for the Γ-invariant lift of m to X. Let $\mathrm{H}_1(X_0,\mathbb{R})$ and $\mathrm{H}^1(X_0,\mathbb{R})$ be the first homology group and the first cohomology group of X_0, respectively. We define the *homological direction* of the RW on X_0 by $\gamma_p := \sum_{e\in E_0} p(e)m\big(o(e)\big)e \in \mathrm{H}_1(X_0,\mathbb{R})$. We call the RW on X_0 (m-)*symmetric* if $p(e)m\big(o(e)\big)=p(\overline{e})\big(t(e)\big)$ $(e \in E_0)$. It easily follows that the RW is (m-)symmetric if and only if $\gamma_p=0$.

Thanks to the celebrated theorem of Malćev, we find a connected and simply connected nilpotent Lie group G such that Γ is isomorphic to the lattice in G. In what follows, we always assume that ***G* is free of step 2**. Namely, its Lie algebra $\mathfrak{g}$ has the direct sum decomposition of the form $\mathfrak{g}=\mathfrak{g}^{(1)}\oplus\mathfrak{g}^{(2)}=\mathfrak{g}^{(1)}\oplus[\mathfrak{g}^{(1)},\mathfrak{g}^{(1)}]$. Now we take a canonical surjective linear map $\rho_{\mathbb{R}} : \mathrm{H}_1(X_0,\mathbb{R}) \longrightarrow \mathfrak{g}^{(1)}$ through π. By the discrete Hodge–Kodaira theorem, an inner product

$$\langle\!\langle \omega,\eta\rangle\!\rangle_p := \sum_{e\in E_0} p(e)m\big(o(e)\big)\omega(e)\eta(e) - \langle \omega,\gamma_p\rangle\langle\eta,\gamma_p\rangle \quad \big(\omega,\eta \in \mathrm{H}^1(X_0,\mathbb{R})\big)$$

associated with the transition probability p is induced from the space of (modified) harmonic 1-forms on X_0 to $\mathrm{H}^1(X_0,\mathbb{R})$. Using the map $\rho_{\mathbb{R}}$, we construct a flat metric g_0 on $\mathfrak{g}^{(1)}$ from the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_p$ and this is called the *Albanese metric*. A periodic realization $\Phi_0 : X \longrightarrow G$ is said to be *modified harmonic* if

$$\sum_{e \in E_x} p(e) \log \Big(\Phi_0\big(o(e)\big)^{-1} \cdot \Phi_0\big(t(e)\big) \Big)\Big|_{\mathfrak{g}^{(1)}} = \rho_{\mathbb{R}}(\gamma_p) \quad (x \in V). \tag{$\spadesuit$}$$

The quantity on the right-hand side of ($\spadesuit$) is called the *asymptotic direction*. It should be noted that $\gamma_p = 0$ implies $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}_{\mathfrak{g}}$, however, the converse does not hold in general.

We fix a reference point $x_* \in V$ and take a modified harmonic realization $\Phi_0 : X \longrightarrow G$ such that $\Phi_0(x_*) = \mathbf{1}_G$. Now consider the RW on $\mathfrak{g}$ given by $\Xi_n := \log\big(\Phi_0(w_n)\big)$ $(n = 0, 1, 2, \dots)$ and the sequence of G-valued continuous stochastic processes $\{\mathcal{Y}^{(n)}_\cdot\}_{n=0}^\infty$ given by $\mathcal{Y}^{(n)}_t := \tau_{n^{-1/2}}\big(\exp(\mathfrak{X}^{(n)}_t)\big)$ $(t \in [0,1])$. Here τ_ε $(0 \le \varepsilon \le 1)$ is the dilation operator acting on G and $\mathfrak{X}^{(n)}_t := \Xi_{[nt]} + (nt - [nt])\big(\Xi_{[nt]+1} - \Xi_{[nt]}\big)$. Let $\{V_1, \dots, V_d\}$ be an orthonormal basis of $(\mathfrak{g}^{(1)}, g_0)$. We note that $\{[V_i, V_j] : 1 \le i < j \le d\}$ forms a basis of $\mathfrak{g}^{(2)}$ by the assumption that G is free. Here we put

$$\beta(\Phi_0) := \sum_{e \in E_0} p(e) m\big(o(e)\big) \log \Big(\Phi_0\big(o(e)\big)^{-1} \cdot \Phi_0\big(t(e)\big) \Big)\Big|_{\mathfrak{g}^{(2)}} = \sum_{1 \le i < j \le d} \beta(\Phi_0)^{ij} [V_i, V_j] \in \mathfrak{g}^{(2)}.$$

Note that $\gamma_p = 0 \Longrightarrow \beta(\Phi_0) = \mathbf{0}_{\mathfrak{g}}$. Let $(Y_t)_{t \ge 0}$ be the G-valued diffusion process starting from the unit $\mathbf{1}_G$ which solves a stochastic differential equation

$$dY_t = \sum_{1 \le i \le d} V_i(Y_t) \circ dB^i_t + \beta(\Phi_0)(Y_t)\, dt,$$

where $(B_t)_{t \ge 0} = (B^1_t, \dots, B^d_t)_{t \ge 0}$ is an $\mathbb{R}^d$-valued standard BM. Let $\mathcal{A} := (1/2) \sum_{1 \le i \le d} V_i^2 + \beta(\Phi_0)$ be the infinitesimal generator of $(Y_t)_{t \ge 0}$. Then we obtain

Theorem. *Assume $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}_{\mathfrak{g}}$. For $t \ge 0$ and $f \in C_\infty(G)$, we have*

$$\lim_{n \to \infty} \Big\| L^{[nt]} P_{n^{-1/2}} f - P_{n^{-1/2}} \mathrm{e}^{-t\mathcal{A}} f \Big\|_\infty^X = 0$$

Moreover, we obtain $(\mathcal{Y}^{(n)}_t)_{t \ge 0} \Longrightarrow (Y_t)_{t \ge 0}$ in $C_{\mathbf{1}_G}([0,1]; G)$ as $n \to \infty$.

If time permits, we will discuss a rough path theoretic interpretation of this theorem and give an example of a RW on a nilpotent covering graph with $\Gamma = \mathbb{H}^3(\mathbb{Z})$.

References

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