Long time behavior of the volume of the Wiener sausage on Dirichlet spaces

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In this talk, we consider the long time behavior of the volume of the Wiener sausage on Dirichlet spaces. Here the Wiener sausage $W_{t,\epsilon}$ is the ϵ-neighborhood of the trajectory of a process until time t. We focus on the volume of $W_{t,\epsilon}$, denoted by $V_{t,\epsilon}$, for diffusion process on metric measure space other than the Euclid space. We review known results. Chavel-Feldman [CF86-1, CF86-2, CF86-3] considered $V_{t,\epsilon}$ for Brownian motion on Riemannian manifolds. [CF86-1] shows radial asymptotic results (i.e. $\epsilon \to 0$) on hyperbolic 3-spaces, and a time asymptotic result on Riemannian symmetric spaces of non-positive curvature. [CF86-2] shows radial asymptotic results on complete Riemannian manifolds for the dimension $d \geq 3$. [CF86-3] shows a radial asymptotic result for the Wiener sausage of reflected Brownian motion on a domain in $\mathbb{R}^d, d \geq 2$. Sznitman [Sz89] obtained a time asymptotic result of negative exponentials of Brownian bridge on hyperbolic space, similar to Donsker-Varadhan [DV75]. Chavel-Feldman-Rosen [CFR91] obtained second order radial asymptotic result for 2-dimensional Riemannian manifold, extending Le Gall's expansion [Le88, Theorem 2.1] in $\mathbb{R}^2$. Recently, Gibson-Pivarski [GP15] obtained a time asymptotic result similar to [DV75] for diffusions on local Dirichlet spaces.

Our results are time asymptotics for the volume of the Wiener sausage on *non-symmetric* spaces. First, we will give growth rate of the means on some spaces containing some fractal spaces such as infinite Sierpinski gaskets and carpets. (We state it later.) Second, we will show that the exact growth rate of the means on "finitely modified" Euclidian spaces is identical with the one of the original Euclidian space. Third, we will give an example of a space on which the sequence of the means largely fluctuates. Some analogous results for a discrete framework, specifically, range of random walk on graphs, were obtained by [O14]. Difficulties are that we cannot use symmetries and scalings of spaces and processes. On the Euclid spaces, by Brownian scaling, time asymptotic results can be derived from radial asymptotic results. The time asymptotic results in [Sp64] and [CF86-1] uses such symmetries and scalings of spaces and processes.

Now we state our framework and *one of* our main results. Let (M, d) be a metric space. Assume that any open ball is relatively compact. Let μ be a Borel measure on M such that for any relatively compact open subset U of M, $0 < \mu(U) \leq \mu(\overline{U}) < +\infty$. Here $\overline{U}$ is the closure of U. Let $(\mathcal{E}, \mathcal{F})$ be a strongly-local regular symmetric conservative Dirichlet form on $L^2(M, \mu)$ and (X_t, P^x) be the associated Hunt process. We call a pair $(M, d, \mu, \mathcal{E}, \mathcal{F})$ a metric measure Dirichlet

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space. Let $B(x,r) := \{y \in M : d(x,y) < r\}$. For a Borel measurable set B, let $T_B := \inf\{t > 0 : X_t \in B\}$. Let the volume of the Wiener sausage:

$$V_{t,\epsilon} = V_{t,\epsilon}(X) := \mu\left(\bigcup_{s\in[0,t]} B(X_s,\epsilon)\right).$$

$f \asymp g$ if and only if there are two constants c and C such that $cg(x) \le f(x) \le Cg(x)$ for any x.

Theorem 0.1 (Growth rates)**.** *Fix $\epsilon > 0$. Assume the following:*
(i) There is $C > 0$ such that for any $r \in [0,\epsilon)$,

$$\sup_{x\in M} V(x,r) \le C \inf_{x\in M} V(x,r).$$

(ii) There are an increasing function $f(t)$ and constants $c_1 \in (0,1), c_2 > 1, c_3, c_4 > 0$ such that

$$c_3 f(t) \le \int_0^t p(s,x,y)ds \le c_4 f(t), \quad t \ge 1, c_1\epsilon \le d(x,y) \le c_2\epsilon.$$

Then, there exist constants C_1, C_2 depending on ϵ such that for any $x \in M$,

$$C_1 \le \liminf_{t\to\infty} \frac{E^x[V_{t,\epsilon}]}{t/f(t)} \le \limsup_{t\to\infty} \frac{E^x[V_{t,\epsilon}]}{t/f(t)} \le C_2.$$

The assumptions in the above result are satisfied if the following hold:

(Ahlfors regularity) (V_α): There exist C_1, C_2 such that $C_1 r^\alpha \le V(x,r) \le C_2 r^\alpha$ holds for any x, $r \in (0,\epsilon)$.

fHK(β): There exist c_i, $1 \le i \le 4$, such that for any $t > 0, x, y \in M$,

$$\frac{c_1}{V(x,t^{1/\beta})} \exp\left(-c_2\left(\frac{d(x,y)^\beta}{t}\right)^{1/(\beta-1)}\right) \le p(t,x,y)$$

$$\le \frac{c_3}{V(x,t^{1/\beta})} \exp\left(-c_4\left(\frac{d(x,y)^\beta}{t}\right)^{1/(\beta-1)}\right).$$

Then, for $t \ge 1$,

$$f(t) = 1, \ \alpha > \beta,$$
$$f(t) = \log t, \ \alpha = \beta,$$
$$f(t) = t^{1-\alpha/\beta}, \ \alpha < \beta.$$

Other results will be stated in talk.

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