DISCRETE MOMENTS OF THE RIEMANN ZETA-FUNCTION ON DETERMINISTIC AND RANDOM SEQUENCES

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We survey recent investigations of discrete moments of the Riemann zeta-function $\zeta(s)$ on deterministic sequences inside the critical strip [2, 5] (joint works with Justas Kalpokas from Vilnius University and Elias Wegert from Freiberg University, respectively). Further, we report on recent studies on the distribution of values of $\zeta(s)$ on vertical lines $s = \sigma + i\mathbb{R}$ with respect to the ergodic transformation given by $Tx := \frac{1}{2}(x - \frac{1}{x})$ for $x \neq 0$. Applying Birkhoff's ergodic theorem, we show that whenever $\operatorname{Re} s > -\frac{1}{2}$ the mean-value of $\zeta(s + iT^n x)$ exists for almost all $x \in \mathbb{R}$ as $n \to \infty$ and is independent of x; moreover, we determine its exact value. For instance, for almost all $x \in \mathbb{R}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \zeta(2 + iT^n x) = \zeta(3) = 1.20205\ldots.$$

We discuss our results with respect to the Lindelöf hypothesis on the growth of the zeta-function on the critical line. This is related to a recent article of Lifshits & Weber [3] entitled "*Sampling the Lindelöf Hypothesis with the Cauchy Random Walk*" which describes the content of their interesting paper very well; this probabilistic approach was extended by Shirai [4] to so-called symmetric α-stable processes. These result on the almost sure convergence of the probabilistic moment of the values $\zeta(\frac{1}{2} + i\mathcal{C}_n)$, where $\mathcal{C}_n$ is the Cauchy random walk or any other process considered by Shirai, indicate that the values of the zeta-function on the critical line are small on average. Part of our results may be interpreted in a similar manner.

Moreover, we present an equivalent formulation for the Riemann hypothesis in terms of our ergodic transformation which relies on another equivalent due to Balazard, Saias & Yor [1].

References

[1] M. Balazard, E. Saias, M. Yor, Notes sur la fonction de Riemann, 2. *Advances Math.* **143** (1999), 284-287

[2] J. Kalpokas, J. Steuding, On the value-distribution of the Riemann zeta-function on the critical line, submitted

[3] M. Lifshits, M. Weber, Sampling the Lindelöf hypothesis with the Cauchy random walk, *Proc. London Math. Soc.* **98** (2009), 241-270

[4] T. Shirai, Variance of randomized values of Riemann's zeta function on the critical line, preprint

[5] J. Steuding, E. Wegert, The Riemann Zeta-Function on Arithmetic Progressions, submitted

Date: December 2010.