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Recent progress on conditional randomness

Hayato Takahashi[1]

The set of Hippocratic random sequences w.r.t. P is defined as the compliment of the effective null sets w.r.t. P and denote it by $\mathcal{R}^P$. In particular if P is computable it is called Martin-Löf random sequences.

Lambalgen's theorem (1987) [9] says that a pair of sequences $(x^\infty, y^\infty) \in \Omega^2$ is Martin-Löf (ML) random w.r.t. the product of uniform measures iff x^∞ is ML-random and y^∞ is ML-random relative to x^∞, where Ω is the set of infinite binary sequences. In [10, 5, 6, 7], generalized Lambalgen's theorem is studied for computable correlated probabilities.

Let S be the set of finite binary strings and $\Delta(s) := \{sx^\infty | x^\infty \in \Omega\}$ for $s \in S$, where sx^∞ is the concatenation of s and x^∞. Let $X = Y = \Omega$ and P be a computable probability on $X \times Y$. P_X and P_Y are marginal distribution on X and Y, respectively. In the following we write $P(x, y) := P(\Delta(x) \times \Delta(y))$ and $P(x|y) := P(\Delta(x)|\Delta(y))$ for $x, y \in S$.

Let $\mathcal{R}^P$ be the set of ML-random points and $\mathcal{R}^P_{y^\infty} := \{x^\infty \mid (x^\infty, y^\infty) \in \mathcal{R}^P\}$. In [5, 6], it is shown that conditional probabilities exist for all random parameters, i.e.,

$$\forall x \in S,\ y^\infty \in \mathcal{R}^{P_Y} \quad P(x|y^\infty) := \lim_{y \to y^\infty} P(x|y) \text{ (the right-hand-side exist)}$$

and $P(\cdot|y^\infty)$ is a probability on $(\Omega, \mathcal{B})$ for each $y^\infty \in \mathcal{R}^{P_Y}$.

Let $\mathcal{R}^{P(\cdot|y^\infty), y^\infty}$ be the set of Hippocratic random sequences w.r.t. $P(\cdot|y^\infty)$ with oracle y^∞.

Theorem 1 ([5, 6, 7]) *Let P be a computable probability on $X \times Y$. Then*

$$\mathcal{R}^P_{y^\infty} \supseteq \mathcal{R}^{P(\cdot|y^\infty), y^\infty} \text{ for all } y^\infty \in \mathcal{R}^{P_Y}. \tag{1}$$

Fix $y^\infty \in \mathcal{R}^{P_Y}$ and suppose that $P(\cdot|y^\infty)$ is computable with oracle y^∞. Then

$$\mathcal{R}^P_{y^\infty} = \mathcal{R}^{P(\cdot|y^\infty), y^\infty}. \tag{2}$$

[1] hayato.takahashi@ieee.org
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It is known that there is a non-computable conditional probabilities [4] and in [2] Bauwens showed an example that violates the equality in (2) when the conditional probability is not computable with oracle y^∞. In [8], an example that for all y^∞, the conditional probabilities are not computable with oracle y^∞ and (2) holds. A survey on the randomness for conditional probabilities is shown in [1].

Next we study mutually singular conditional probabilities. In [3], Hanssen showed that for Bernoulli model $P(\cdot|\theta)$,

$$\mathcal{R}^{P(\cdot|\theta)} = \mathcal{R}^{P(\cdot|\theta),\theta} \text{ for all } \theta. \tag{3}$$

We generalize Hanssen's theorem (3) for mutually singular conditional probabilities. In [5, 7], equivalent conditions for mutually singular conditional probabilities are shown.

Theorem 2 ([5, 7]) *Let P be a computable probability on $X \times Y$, where $X = Y = \Omega$. The following six statements are equivalent:*
(1) $P(\cdot \,|\, y) \perp P(\cdot \,|\, z)$ if $\Delta(y) \cap \Delta(z) = \emptyset$ for $y, z \in S$.
(2) $\mathcal{R}^{P(\cdot\,|\,y)} \cap \mathcal{R}^{P(\cdot\,|\,z)} = \emptyset$ if $\Delta(y) \cap \Delta(z) = \emptyset$ for $y, z \in S$.
(3) $P_{Y|X}(\cdot \,|\, x)$ converges weakly to I_{y^∞} as $x \to x^\infty$ for $(x^\infty, y^\infty) \in \mathcal{R}^P$, where I_{y^∞} is the probability that has probability of 1 at y^∞.
(4) $\mathcal{R}^P_{y^\infty} \cap \mathcal{R}^P_{z^\infty} = \emptyset$ if $y^\infty \neq z^\infty$.
(5) There exists $f : X \to Y$ such that $f(x^\infty) = y^\infty$ for $(x^\infty, y^\infty) \in \mathcal{R}^P$.
(6) There exists $f : X \to Y$ and $Y' \subset Y$ such that $P_Y(Y') = 1$ and $f = y^\infty,\ P(\cdot\,; y^\infty) - a.s.$ for $y^\infty \in Y'$.

Generalized form of Hanssen's theorem (3) is as follows.

Theorem 3 *Let P be a computable probability on $X \times Y$, where $X = Y = \Omega$. Under one of the condition of Theorem 2, we have*

$$\mathcal{R}^P_{y^\infty} \supseteq \mathcal{R}^{P(\cdot|y^\infty)} \text{ for all } y^\infty \in \mathcal{R}^{P_Y}.$$

Fix $y^\infty \in \mathcal{R}^{P_Y}$ and suppose that $P(\cdot|y^\infty)$ is computable with oracle y^∞. Then

$$\mathcal{R}^P_{y^\infty} = \mathcal{R}^{P(\cdot|y^\infty)} = \mathcal{R}^{P(\cdot|y^\infty),y^\infty}.$$

References

[1] B. Bauwens, A. Shen, and H. Takahashi. Conditional probabilities and van Lambalgen theorem revisited. arxiv:1607.04240, 2016.

[2] Bruno Bauwens. Conditional measure and the violation of van Lambalgen's theorem for Martin-löf randomness. http://arxiv.org/abs/1103.1529, 2015.

[3] Bjørn Kjos Hanssen. The probability distribution as a computational resource for randomness testing. *Journal of Logic and Analysis*, 2(10):1–13, 2010.

[4] D. M. Roy. *Computability, inference and modeling in probabilistic programming*. PhD thesis, MIT, 2011.

[5] H. Takahashi. Bayesian approach to a definition of random sequences and its applications to statistical inference. In *2006 IEEE International Symposium on Information Theory*, pages 2180–2184, July 2006.

[6] H. Takahashi. On a definition of random sequences with respect to conditional probability. *Inform. and Compt.*, 206:1375–1382, 2008.

[7] H. Takahashi. Algorithmic randomness and monotone complexity on product space. *Inform. and Compt.*, 209:183–197, 2011.

[8] H. Takahashi. Generalization of van lambalgen's theorem and blind randomness for conditional probabilities, 2014. arxiv:1310.0709v3.

[9] M. van Lambalgen. *Random sequences*. PhD thesis, Universiteit van Amsterdam, 1987.

[10] V. G. Vovk and V. V. V'yugin. On the empirical validity of the Bayesian method. *J. R. Stat. Soc. B*, 55(1):253–266, 1993.