

GKZ hypergeometric system

Pfaffian system (Pfaff equation), contiguity relations, cohomology intersection

Version 1.0

December 15, 2022

by S-J. Matsubara-Heo, N.Takayama

1 About this document

This document explains Risa/Asir functions for GKZ hypergeometric system (A-hypergeometric system).

Loading the package:

```
import("mt_gkz.rr");
```

References cited in this document.

- [MT2020] Saiei-Jaeyeong Matsubara-Heo, Nobuki Takayama, Algorithms for Pfaffian Systems and Cohomology Intersection Numbers of Hypergeometric Integrals, Lecture Notes in Computer Science 12097 (2020), 73--84. Errata is posted on <http://arxiv.org/abs/???>. E-attachments can be obtainable at <https://www.math.kobe-u.ac.jp/OpenXM/Math/intersection2>
- [GM2020] Yoshiaki Goto, Saiei-Jaeyeong Matsubara-Heo, Homology and cohomology intersection numbers of GKZ systems, arXiv:2006.07848
- [SST1999] M.Saito, B.Sturmfels, N.Takayama, Hypergeometric polynomials and integer programming, Compositio Mathematica, 155 (1999), 185--204
- [SST2000] M.Saito, B.Sturmfels, N.Takayama, Groebner Deformations of Hypergeometric Differential Equations. Springer, 2000.
- [HNT] T.Hibi, K.Nishiyama, N.Takayama, Pfaffian systems of A-Hypergeometric equations I: bases of twisted cohomology groups. Advances in Mathematics, 306 (2017), 303--327. <https://dx.doi.org/10.1016/j.aim.2016.10.021>

References for maple packages IntegrableConnections and OreMorphisms.

- [BCEW] M.Barkatou, T.Cluzeau, C.El Bacha, J.-A.Weil, IntegrableConnections a maple package for computing closed form solutions of integrable connections (2012). https://www.unilim.fr/pages_perso/thomas.cluzeau/Packages/IntegrableConnections/PDS.html
- [CQ] T.Cluzeau and A.Quadrat, OreMorphisms: A homological algebraic package for factoring, reducing and decomposing linear functional systems (2009). <https://who.rocq.inria.fr/Alban.Quadrat/OreMorphisms/index.html>
- [CQ08] T.Cluzeau, A.Quadrat, Factoring and decomposing a class of linear functional systems, Linear Algebra and its Applications (LAA), 428(1): 324-381, 2008.

2 Pfaff equation

2.1 Pfaff equation for given cocycles

2.1.1 `mt_gkz.pfaff_eq`

`mt_gkz.pfaff_eq(A,Beta,Ap,Rvec,DirX)`
 :: It returns the Pfaff equation for the GKZ system defined by A and $Beta$ with respect to cocycles defined by $Rvec$.

return a list of coefficients of the Pfaff equation with respect to the direction $DirX$

A the matrix A of the GKZ system.

$Beta$ the parameter vector of the GKZ system.

Ap See [MT2020].

$Rvec$ It is used to specify a basis of cocycles as explained below. See also [MT2020].

$DirX$ a list of dx_i 's.

- The independent variables are x_1, x_2, x_3, \dots
- When $Rvec=[v_1, v_2, \dots, v_r]$ where r is the rank of the GKZ system, the set of the cocycles standing for Av_1, Av_2, \dots, Av_r (see [MT2020]) is supposed to be the basis to construct the Pfaffian system. The exponents (q_ℓ, q) of the integral representation $\int \prod h_\ell^{-q_\ell} x^q \frac{dx}{x}$ is shifted by Av_i as $(q_\ell, q) + A_{v_i}$. Let a_1, a_2, \dots, a_n be the column vectors of the matrix A and v be a column vector $(x_1, x_2, \dots, x_n)^T$. Av is defined as $a_1 x_1 + a_2 x_2 + \dots + a_n x_n$.
- When the columns of A are expressed as $e_i \otimes \alpha_{i_j}$, the columns of Ap is $e_i \otimes 0$ where e_i is the i -th unit vector. See [MT2020] on the definition of Ap . Here are some examples. When A is

```
[[1,1,0,0],
 [0,0,1,1],
 [0,1,0,1]]
```

Ap is

```
[[1,1,0,0],
 [0,0,1,1],
 [0,0,0,0]] <-- zero row
```

When A is

```
[[1,1,1,0,0,0],
 [0,0,0,1,1,1],
 [0,1,0,0,1,0],
 [0,0,1,0,0,1]]
```

]

Ap is

```
[[1,1,1,0,0,0],
 [0,0,0,1,1,1],
```

```
[0,0,0,0,0,0], <-- zero row
[0,0,0,0,0,0]  <-- zero row
]
```

See also page 223 of [SST2000].

- Option *xrule*. When the option *xrule* is given, the *x* variables specified by this option are specialized to numbers.
- Option *shift*. When the matrix *A* is not normal (the associated toric ideal is not normal), a proper shift vector must be given to obtain an element of the b-ideal. Or, use the option *b_ideal* below. See [SST1999] on the theory.
- Option *b_ideal*. When the matrix *A* is not normal, the option *b_ideal*=1 obtains b-ideals and the first element of each b-ideal is used as the b-function. The option *shift* is ignored.
- Option *cg*. A constant matrix given by this option is used for the Gauge transformation of the Pfaffian system. In other words, the basis of cocycles specified by *Rvec* is transformed by the constant matrix given by this option.
- By *mt_gkz.use_hilbert_driven*(Rank), the rank of the GKZ system is assumed to be Rank. It makes the computation of Groebner basis by *yang.rr* faster. This option is disabled by *mt_gkz.use_hilbert_driven*(0);

Example: Gauss hypergeometric system, see [GM2020] example ??.

```
[1883] import("mt_gkz.rr");
[2657] PP=mt_gkz.pfaff_eq(A=[[1,1,0,0],[0,0,1,1],[0,1,0,1]],
      Beta=[-g1,-g2,-c],
      Ap = [[1,1,0,0],[0,0,1,1],[0,0,0,0]],
      Rvec = [[1,0,0,0],[0,0,1,0]],
      DirX=[dx4,dx3] | xrule=[[x1,1],[x2,1]],
      cg=matrix_list_to_matrix([[1,0],[-1,1]]))$

Bfunctions=[s_1*s_2-s_1*s_3+s_1^2,s_1*s_3,s_2^2+(-s_3+s_1)*s_2,s_3*s_2]
-- snip --
[2658] PP[0];
[ (g2*x3-g2)/(x4-x3) (g2*x3)/(x4-x3) ]
[ ((-g2*x3-c+g2)*x4+(c-g1)*x3+g1)/(x4^2-x3*x4)
  ((-g2*x3-c)*x4+(c-g1)*x3)/(x4^2-x3*x4) ]
[2659] PP[1];
[ (-g2*x4+g2)/(x4-x3) (-g2*x4)/(x4-x3) ]
[ ((g2*x3+c-g2-1)*x4+(-c+g1+1)*x3-g1)/(x3*x4-x3^2)
  ((g2*x3+c-g2-1)*x4+(-c+g1+g2+1)*x3)/(x3*x4-x3^2) ]
```

Example: The role of shift. When the toric ideal is not normal, a proper shift vector must be given with the option *shift* to find an element of the b-ideal.

```
[1882] load("mt_gkz.rr");
[1883] A=[[1,1,1,1],[0,1,3,4]];
      [[1,1,1,1],[0,1,3,4]]
```

```

[1884] Ap=[[1,1,1,1],[0,0,0,0]];
      [[1,1,1,1],[0,0,0,0]]
[1885] Rvec=[[0,0,0,0],[0,0,1,0],[0,0,0,1],[0,0,0,2]];
      [[0,0,0,0],[0,0,1,0],[0,0,0,1],[0,0,0,2]];
[2674] P=mt_gkz.pfaff_eq(A,[b1,b2],Ap,Rvec,DirX=[dx4]
      | xrule=[[x1,1],[x2,2],[x3,4]])$
dx remains
stopped in step_up at line 342 in file "./mt_gkz/saito-b.rr"
342   if (type(dn(Ans)) > 1) error("dx remains");
(debug) quit
// Since the toric ideal for A is not normal, it stops with the error.
[2675] P=mt_gkz.pfaff_eq(A,[b1,b2],Ap,Rvec,DirX=[dx4]
      | shift=[1,0],xrule=[[x1,1],[x2,2],[x3,4]])$
// It works.

```

Refer to [\[mt_gkz.ff1\]](#), page [\[mt_gkz.ff2\]](#), page [\[un-
defined\]](#), Section 2.1.2 [\[mt_gkz.ff\]](#), page 4, Section 2.1.3 [\[mt_gkz.rvec_to_fvec\]](#),
page 5,

2.1.2 mt_gkz.ff2, mt_gkz.ff1, mt_gkz.ff

```

mt_gkz.ff(Rvec0,A,Ap,Beta)
mt_gkz.ff1(Rvec0,A,Beta,Ap)
mt_gkz.ff2(Rvec0,A,Beta,Ap,BF,C)
      :: ff returns a differential operator whose action to 1 gives the cocycle defined
      by Rvec0

return   ff returns a differential operator whose action to 1 of  $M_A(\beta)$  gives the cocycle
      defined by Rvec0.

return   ff1 returns a composite of step-down operators for the positive part of Rvec0
return   ff2 returns a composite of step-up operators for the positive part of Rvec0
Rvec0    An element of Rvec explained in Section 2.1.1 \[mt\_gkz.pfaff\_eq\], page 2.
BF        the list of b-functions to all directions.
C         the list of the step up operators for all a_1, a_2, ..., a_n.

```

Other arguments are same with those of `pfaff_eq`.

- The function `ff` generates the list of b-functions and the list of step up operators and store them in the cache variable. They can be obtained by calling as `S=mt_gkz.get_bf_step_up()` where `S[0]` is the list of b-functions and `S[1]` is the list of step up operators. Step up operators are obtained by the algorithm given in [\[SST1999\]](#).
- Option `nf`. When `nf=1`, the output operator is reduced to the normal form with respect to the Groebner basis of the GKZ system of the graded reverse lexicographic order.
- Option `shift`. See Section 2.1.1 [\[mt_gkz.pfaff_eq\]](#), page 2.
- Internal info: The function `mt_gkz.bb` gives the constant so that the step up and step down operators (contiguity operators) give contiguity relations for the integral representation in [\[MT2020\]](#). Note that `mt_gkz.ff1` and `mt_gkz.ff2` give contiguity relations which are constant multiple of those for hypergeometric polynomials.

- Internal info: `mt_gkz.step_up` generates step up operators of [SST1999] from b-functions by utilizing `mt_gkz.bf2euler` and `mt_gkz.toric`.

Example: Step up operators compatible with the integral representation in [MT2020]. The function `hgpoly_res` defined in `check-by-hgpoly.rr` returns a multiple of the hypergeometric polynomial which agrees with the residue times a power of $2\pi\sqrt{-1}$ of the integral representation. See [SST1999].

```
[1883] import("mt_gkz.rr")$
[3175] load("mt_gkz/check-by-hgpoly.rr")$
[3176] A=[[1,1,0,0],[0,0,1,1],[0,1,0,1]]$
[3177] B=newvect(3,[5,4,7])$ Ap=[[1,1,0,0],[0,0,1,1],[0,0,0,0]]$
[3179] Beta=[b1,b2,b3]$ R=[0,0,-1,0]$
[3180] F2=hgpoly_res(A,B,2); // HG polynomial. 2 is the number of e_i's
// =the number of non-zero rows of Ap
10*x1^2*x2^3*x4^4+20*x1*x2^4*x3*x4^3+6*x2^5*x3^2*x4^2
[3182] mt_gkz.ff(R,A,Ap,Beta); // the operator standing for R
(x3*x4*dx4+x3^2*dx3+x1*x4*dx2+x1*x3*dx1+x3)/(b1+b2-b3+1)
[3184] S=mt_gkz.get_bf_step_up(A); // b-function and non-reduced step up op's
[[ s_1*s_2-s_1*s_3+s_1^2 s_1*s_3 s_2^2+(-s_3+s_1)*s_2 s_3*s_2 ],
[ x2*x3*dx4+x1*x3*dx3+x1*x2*dx2+x1^2*dx1+x1
x2*x4*dx4+x1*x4*dx3+x2^2*dx2+x1*x2*dx1+x2
x3*x4*dx4+x3^2*dx3+x1*x4*dx2+x1*x3*dx1+x3
x4^2*dx4+x3*x4*dx3+x2*x4*dx2+x2*x3*dx1+x4 ]]
[3185] Fvec=mt_gkz.ff2(R,A,Beta,Ap,S[0],S[1]);
(x3*x4*dx4+x3^2*dx3+x1*x4*dx2+x1*x3*dx1+x3)/(b1+b2-b3+1)
[3188] Fvec = base_replace(Fvec,assoc(Beta,vtol(B)));
1/3*x3*x4*dx4+1/3*x3^2*dx3+1/3*x1*x4*dx2+1/3*x1*x3*dx1+1/3*x3
[3189] R32d = odiff_act(Fvec,F2,[x1,x2,x3,x4]); // Act Fvec to the hg-poly
10*x1^3*x2^2*x4^5+50*x1^2*x2^3*x3*x4^4+50*x1*x2^4*x3^2*x4^3+10*x2^5*x3^3*x4^2
[3190] red(R32d/hgpoly_res(A,B+newvect(3,[0,1,0]),2));
// R32d agrees with the HG polynomial with Beta=[5,4,7]+[0,1,0].
1
```

Refer to Section 2.1.1 [mt_gkz.pfaff_eq], page 2,

2.1.3 mt_gkz.rvec_to_fvec

`mt_gkz.rvec_to_fvec(Rvec,A,Ap,Beta)`

:: It returns a set of differential operators standing for *Rvec*.

return It returns a set of differential operators of which action to $1 \in M_A(\beta)$ give cocycles specified by *Rvec*.

A, Ap, Beta

Same with Section 2.1.1 [mt_gkz.pfaff_eq], page 2,

- Internal info: this function builds the set of operators by calling Section 2.1.2 [mt_gkz.ff], page 4.

Example: The following two expressions are congruent because $2a_1 - a_2 - a_3 + a_4 = a_1$ for this A.

```

[1883] import("mt_gkz.rr");
[3191] mt_gkz.rvec_to_fvec([[2,-1,-1,1],[0,0,1,0]],
    [[1,1,0,0],[0,0,1,1],[0,1,0,1]],
    [[1,1,0,0],[0,0,1,1],[0,0,0,0]],[b1,b2,b3]);
[(x2*x3*x4^2*dx1^2*dx4^3+((x1*x3*x4^2+x2*x3^2*x4)*dx1^2*dx3
+ (x1*x2*x4^2+x2^2*x3*x4)*dx1^2*dx2+(x1^2*x4^2+2*x1*x2*x3*x4+x2^2*x3^2)*dx1^3
+ (x1*x4^2+3*x2*x3*x4)*dx1^2)*dx4^2+(x1*x3^2*x4*dx1^2*dx3^2
+ ((x1^2*x3*x4+x1*x2*x3^2)*dx1^3+(3*x1*x3*x4+x2*x3^2)*dx1^2)*dx3
+ x1*x2^2*x4*dx1^2*dx2^2+((x1^2*x2*x4+x1*x2^2*x3)*dx1^3
+ (3*x1*x2*x4+x2^2*x3)*dx1^2)*dx2+x1^2*x2*x3*dx1^4
+ (x1^2*x4+3*x1*x2*x3)*dx1^3+(x1*x4+x2*x3)*dx1^2)*dx4)
/(b3*b2*b1^3+(b3*b2^2+(-b3^2-2*b3)*b2)*b1^2+(-b3*b2^2+(b3^2+b3)*b2)*b1),
(dx3)/(b2)]
[3192] mt_gkz.rvec_to_fvec([[1,0,0,0],[0,0,1,0]],
    [[1,1,0,0],[0,0,1,1],[0,1,0,1]],
    [[1,1,0,0],[0,0,1,1],[0,0,0,0]],[b1,b2,b3]);
[(dx1)/(b1),(dx3)/(b2)]

```

Refer to Section 2.1.1 [mt_gkz.pfaff.eq], page 2,

2.1.4 mt_gkz.fvec_to_conn_mat

`mt_gkz.fvec_to_conn_mat(Fvec,A,Beta,DirX)`

:: It returns the coefficient matrices of the basis *Fvec* or *DirX*[I]**Fvec* in terms of the set of the standard basis.

return It returns the coefficient matrices of the basis *Fvec* or *DirX*[I]**Fvec* in terms of the set of the standard basis of the Groebner basis explained below.

A Beta Same with Section 2.1.1 [mt_gkz.pfaff.eq], page 2.

DirX When *DirX* is 1, this function returns the matrix which expresses *Fvec* in terms of the set of the standard monomials of the Groebner basis of the GKZ system in the ring of rational function coefficients with respect to the graded reverse lexicographic order. In other cases, it returns the coefficient matrices of *DirX*[I]'s**Fvec* in terms of the set of the standard basis of the Groebner basis.

- It utilizes a Groebner basis computation by the package `yang.rr` and `yang.reduction` to obtain connection matrices.
- This function calls some utility functions `mt_gkz.dmul(Op1,Op2,XvarList)` (multiplication of *Op1* and *Op2* and `mt_gkz.index_vars(x,Start,End | no_=1)` which generates indexed variables without the underbar “_”.
- We note here some other utility functions in this section: `mt_gkz.check_compatibility(P,Q,X,Y)`, which checks if the sytem d/dX -*P*, d/dY -*Q* is compatible.

Example: The following example illustrates how `mt_gkz.pfaff.eq` obtains connection matrices.

```

[1883] import("mt_gkz.rr");
[3201] V=mt_gkz.index_vars(x,1,4 | no_=1);

```



```

[x1,x2,x3,x4]
[3202] mt_gkz.dmul(dx1,x1^2,V);
      x1^2*dx1+2*x1
[3204] A=[[1,1,0,0],[0,0,1,1],[0,1,0,1]]$
      Ap=[[1,1,0,0],[0,0,1,1],[0,0,0,0]]$
      Beta= [b1,b2,b3]$
      Rvec = [[1,0,0,0],[0,0,1,0]]$
      Fvec = mt_gkz.rvec_to_fvec(Rvec,A,Ap,Beta)$
      /* Express cocycles Rvec
         by elements Fvec in the Weyl algebra by contiguity relations. */
      Cg = matrix_list_to_matrix([[1,0],[1,-1]])$
[3208] NN=mt_gkz.fvec_to_conn_mat(Fvec,A,Beta,1);
      // Express Fvec by the standard monomials Std=NN[1].
      1 ooo 2 .ooo
      [[ (x4)/(b1*x1) (b1-b3)/(b1*x1) ]
       [ (-x4)/(b1*x2) (1)/(x3) ],[dx4,1]]
[3209] Std=NN[1];
      [dx4,1]
[3173] NN=NN[0];
      [ (x4)/(b1*x1) (b1-b3)/(b1*x1) ]
      [ (-x4)/(b2*x3) (1)/(x3) ]
[3174] NN1=mt_gkz.fvec_to_conn_mat(Fvec,A,Beta,dx1)[0];
      // Express dx1*Fvec by the standard monomials Std.
      1 ooo 2 .ooo
      [ ((2*b1+b2-b3-1)*x1*x4^2+(-b1+b3+1)*x2*x3*x4)/(b1*x1^3*x4-b1*x1^2*x2*x3)
        ((b1^2+(-2*b3-1)*b1-b3*b2+b3^2+b3)*x1*x4
        +(-b1^2+(2*b3+1)*b1-b3^2-b3)*x2*x3)/(b1*x1^3*x4-b1*x1^2*x2*x3) ]
      [(b1 (-b1*x1*x4^2-b2*x2*x3*x4)/(b2*x1^2*x3*x4-b2*x1*x2*x3^2)
        (b1*x1*x4+(-b1+b3)*x2*x3)/(x1^2*x3*x4-x1*x2*x3^2) ]
[3188] P1=map(red,Cg*NN1*matrix_inverse(NN)*matrix_inverse(Cg));
      [ ((-b2*x3+(b1+b2-b3-1)*x1)*x4+(-b1+b3+1)*x2*x3)/(x1^2*x4-x1*x2*x3)
        (b2*x3*x4)/(x1^2*x4-x1*x2*x3) ]
      [ ((-b2*x3+(b2-b3-1)*x1)*x4+(-b1+b3+1)*x2*x3+b1*x1*x2)/(x1^2*x4-x1*x2*x3)
        ((b2*x3+b1*x1)*x4)/(x1^2*x4-x1*x2*x3) ]

[3191] mt_gkz.pfaff_eq(A,Beta,Ap,Rvec,[dx1]|cg=Cg)[0]-P1;
      [ 0 0 ]
      [ 0 0 ] // P1 agrees with the output of mt_gkz.pfaff_eq.

```

Refer to Section 2.1.1 [mt_gkz.pfaff_eq], page 2,

2.1.5 mt_gkz.contiguity, mt_gkz.contiguity_by_fvec

mt_gkz.contiguity(A,Beta,Ap,Rvec1,Rvec2)

:: It returns the coefficient matrix P that satisfies $Rvec1 = P Rvec2$.

mt_gkz.contiguity_by_fvec(A,Beta,Ap,Fvec1,Fvec2)

:: It returns the coefficient matrix P that satisfies $Fvec1 = P Fvec2$.

return The coefficient matrix P that satisfies $Rvec1 = P \ Rvec2$ or $Fvec1 = P \ Fvec2$

A Beta Ap Rvec1 Rvec2

Same with Section 2.1.1 [mt_gkz.pfaff_eq], page 2.

- It returns the contiguity relation between $Rvec1$ and $Rvec2$

Example:

```
[1883] import("mt_gkz.rr");
[3200] PP=mt_gkz.contiguity(A=[[1,1,0,0],[0,0,1,1],[0,1,0,1]],
    Beta=[-g1,-g2,-c],
    Ap = [[1,1,0,0],[0,0,1,1],[0,0,0,0]],
    Rvec1 = [[1,0,0,0],[0,0,1,0]],
    Rvec2 = [[0,0,1,0],[1,0,0,0]]);

[3366] Fvec411=mt_gkz.rvec_to_fvec(Rvec411=[[1,1,0]],
    A=[[1,1,1],[1,0,1],[0,1,1]],
    Ap=[[1,1,1],[0,0,0],[0,0,0]],
    Beta=[eps,-eps*del,-eps*del])$
Fvec411d=[mt_gkz.dmul(dx1,Fvec411[0],[x1,x2,x3]));
    [(dx1^2*dx2)/(eps^2-eps)]
[3367] mt_gkz.contiguity_by_fvec(A,Beta,Ap,Fvec411d,Fvec411);
    1 .ooo
    [ ((del+1)*eps-1)/(x1) ]
```

Refer to Section 2.1.1 [mt_gkz.pfaff_eq], page 2, Section 2.1.4 [mt_gkz.fvec_to_conn_mat],
page 6,

3 b function

3.1 b function and facet polynomial

3.1.1 `mt_gkz.bf`

`mt_gkz.bf(A, Facet_poly, II0)`
 :: It returns the b-function with respect to the direction *II0*.

return It returns the b-function introduced Saito with respect to the direction *II0* in case of *A* is normal or an element of b-ideal when a proper shift vector is given in case of *A* is not normal.

A the matrix *A* of the GKZ system.

Facet_poly The set of facet polynomials of the convex hull of *A*.

II0 Direction expressed as 0, 1, 2, ... (not 1, 2, 3, ...) to obtain the b function.

- See [SST1999] on the b-function introduced Saito and b-ideal.
- The facet polynomial must be primitive.

Example:

```
[1883] import("mt_gkz.rr");

[3193] A;
      [[1,1,0,0],[0,0,1,1],[0,1,0,1]]
[3194] Fpoly=mt_gkz.facet_poly(A);
      [[s_3,s_1,s_2-s_3+s_1,s_2],[[0,0,1],[1,0,0],[1,1,-1],[0,1,0]]]
[3196] mt_gkz.bf(A,Fpoly,0);
      s_1*s_2-s_1*s_3+s_1^2
[3197] mt_gkz.bf(A,Fpoly,1);
      s_1*s_3
```

Refer to Section 2.1.2 [mt_gkz.ff], page 4, Section 3.1.2 [mt_gkz.facet_poly], page 9,

3.1.2 `mt_gkz.facet_poly`

`mt_gkz.facet_poly(A)`
 :: It returns the set of facet polynomials and their normal vectors of the cone defined by *A*.

return It returns the set of facet polynomials and their normal vectors of the cone generated by the column vectors of the matrix *A*.

A the matrix *A* of the GKZ system.

- The facet polynomial *f* is primitive. In other words, all *f(a_i)* is integer and $\min f(a_i)=1$ for *a_i*'s not being on *f*=0. where *a_i* is the *i*-th column vector of the matrix *A*. It can be checked by `mt_gkz.is_primitive(At, Facets)` where *At* is the transpose of *A* and *Facets* is the second return value of this function.

- This function utilizes the system polymake <https://polymake.org> on our server.

Example:

```
[1883] import("mt_gkz.rr");
[1884] mt_gkz.facet_poly([[1,1,1,1],[0,1,2,3]]);
      oohg_native=0, oohg_curl=1
      [[s_2,-s_2+3*s_1],[[0,1],[3,-1]]]
```

Refer to Section 3.1.1 [mt_gkz.bf], page 9,

3.1.3 mt_gkz.get_bf_step_up

`mt_gkz.get_bf_step_up(A)`

:: It returns the set of b-functions and corresponding step-up operators for A .

return The first entry is a list of b-function for all direction. The second entry is a list of step-up operators (quasi inverse of step-down operators dx_i 's).

A the matrix A of the GKZ system.

- It is an implementation of the algorithm to obtain contiguity relations in [SST1999] M.Saito, B.Sturmfels, N.Takayama, Hypergeometric polynomials and integer programming, Compositio Mathematica, 155 (1999), 185--204.

Example:

```
[1883] import("mt_gkz.rr");;
[3176] A=[[1,1,0,0],[0,0,1,1],[0,1,0,1]];;
[3177] S=mt_gkz.get_bf_step_up(A);
      [[ s_1*s_2-s_1*s_3+s_1^2 s_1*s_3 s_2^2+(-s_3+s_1)*s_2 s_3*s_2 ],
      [ x2*x3*dx4+x1*x3*dx3+x1*x2*dx2+x1^2*dx1+x1
        x2*x4*dx4+x1*x4*dx3+x2^2*dx2+x1*x2*dx1+x2
        x3*x4*dx4+x3^2*dx3+x1*x4*dx2+x1*x3*dx1+x3
        x4^2*dx4+x3*x4*dx3+x2*x4*dx2+x2*x3*dx1+x4 ]]
```

Refer to Section 3.1.1 [mt_gkz.bf], page 9,

4 Utilities

4.1 Some utility functions

4.1.1 `mt_gkz.gkz_b`

`mt_gkz.gkz_b(A,Beta,Simplex)`

:: It returns the so called Horn system (B expression of GKZ) standing for the GKZ system for *A*, *Beta*.

return the Horn system. It is expressed by the Euler operator, e.g., $\langle\langle 3 \rangle\rangle$ stands for $(x_1 dx_1)^3$ in *D*.

A the matrix *A* of the GKZ system.

Beta the vector *beta* of the GKZ system.

Simplex the indices of the variables to put 1. The index starts from 1. It should be a simplex obtained by a regular triangulation.

- When the option `partial=1` is given, the Horn system is expressed by partial differential operators, e.g., $\langle\langle 1,2 \rangle\rangle$ stands for $dx_1 dx_2^2$.
- When *B* is the Gale transform matrix of *A*, in other words, $AB=0$ where the column vectors span $\text{Ker } A$, the associated Horn system is obtained from the series expansion associated to the matrix *B* and the simplex. The 'b' of the function name `gkz_b` comes from this fact.

Example: Pfaffian equation for Appell function *F1*.

```
import("mt_gkz.rr");
import("yang.rr");
A=[[1,1,1,0,0,0],
   [0,0,0,1,1,1],
   [0,1,0,0,1,0],
   [0,0,1,0,0,1]];
Id=mt_gkz.gkz_b(A,[b1,b2,b3,b4],[1,2,3,6] | partial=1);
yang.define_ring(["partial",Xvars=[x4,x5]]);
// Id=map(mt_mm.remove_redundancy,Id,Xvars);
yang.verbose();
G=yang.buchberger(Id);
Std=yang.stdmon(G);
Pf=yang.pfaffian(map(dp_ptod,Std,poly_dvar(Xvars)),G);
// print_xdvi_form(poly_factor(Pf[0]));
end$
```

Outputs are, e.g.,

```
[3104] Id[2];
(x4*x5^2-x4*x5)*<<1,1>>+(x5^3-x5^2)*<<0,2>>+(-b3*x4*x5)*<<1,0>>
+((-b2-b3+1)*x5^2+(b2-b4-1)*x5)*<<0,1>>+(b3*b2*x5)*<<0,0>>
```

```
[3105] G[1];
[(-x5+x4)*<<1,1>>+(b3)*<<1,0>>+(-b1-b2+b3+b4)*<<0,1>>,-x5+x4]
[3106] Std;
[dx4,dx5,1]
```

Pf[0] by `print_tex_form(poly_factor(Pf[0]))` is

$$\left(\begin{array}{ccc} \frac{((b_1+2b_2-2b_3-b_4-1)x_4-b_2+b_3+b_4+1)x_5+(-b_1-2b_2+b_3+b_4+1)x_4^2+(b_2-b_4-1)x_4}{(x_4)(x_4-1)(x_5-x_4)} & \frac{(b_1+b_2-b_3-b_4)(x_5)(x_5-1)}{(x_4)(x_4-1)(x_5-x_4)} & \frac{(-1)(b_2)(b_1+b_2-b_3-b_4)}{(x_4)(x_4-1)} \\ \frac{(b_3)}{(x_5-x_4)} & \frac{(-1)(b_1+b_2-b_3-b_4)}{(x_5-x_4)} & 0 \\ 1 & 0 & 0 \end{array} \right)$$

dx4-Pf[0], dx5-Pf[1] are Pfaffian equations obtained by the Buchberger algorithm.

Refer to `<undefined> [mt_gkz.regular-triangulation]`, page `<undefined>`, Section 5.3.1
`[mt_gkz.top-standard-pairs]`, page 23,

4.1.2 mt_gkz.pf_ratsol_ml

`mt_gkz.pf_ratsol_ml(Pf,V,Param)`
 :: It returns a Maple code to find rational solutions of $d/dV[0]$ -Pf[0],
 $d/dV[1]$ -Pf[1], ...

return The name of the auto generated maple program. The default is
 “auto-generated-pf-ratsol.ml”.

Pf A list of Pfaffian matrices.

V A list of independent variables.

Param A list of parameter variables.

- The option `ntsetting=1` assumes the Maple libraries `OreModules` and `IntegrableConnections` are under `$HOME/Math/Maple_lib`

Example: rational solutions of Pfaffian equation of two variables. The Pfaffian equation is for

```
[2529] F=sm1.mul(x*dx+y*dy+a,x*dx+1,[x,y,a])-x*(x*dx+1);
x^2*dx^2+(y*x*dy-x^2+(a+2)*x)*dx+y*dy-x+a
[2530] G=sm1.mul(x*dx+y*dy+a,y*dy+1,[x,y,a]);
(y*x*dy+x)*dx+y^2*dy^2+(a+2)*y*dy+a
```

It has a rational solution $1/(x*y)$. Input the following codes.

```
import("mt_gkz.rr");
Px=[[ (x-a-1)/(x),0,(x-a+1)/(x^2) ],[ (-1)/(y),(-1)/(x),(-1)/(y*x) ],[ 1,0,0 ]];
Py=[[ (-1)/(y),(-1)/(x),(-1)/(y*x) ],[ 0,(-a-1)/(y),(-a+1)/(y^2) ],[ 0,1,0 ]];
mt_gkz.pf_ratsol_ml([ Px, Py ], [x,y],[a] );
```

Then, outputs are, e.g.,

```
[3033] mt_gkz.pf_ratsol_ml([ Px, Py ], [x,y],[a] | ntsetting=1);
auto-generated-pf-ratsol.ml
```

```
#Load this program by read('auto-generated-pf-ratsol.ml');
libname:="/home/nobuki/Math/Maple_lib", libname;
with(OreModules);
```

```

with(IntegrableConnections);
with(linalg);
C:=Matrix([[ (x-a-1)/(x), 0, (x-a+1)/(x^2) ], [ (-1)/(y), (-1)/(x), (-1)/(y*x) ], [ 1, 0, 0 ]],
  Matrix([[ (-1)/(y), (-1)/(x), (-1)/(y*x) ], [ 0, (-a-1)/(y), (-a+1)/(y^2) ], [ 0, 1, 0 ]]);
RatSols:=RationalSolutions(C, [x,y], ['param', [a]]);

```

Refer to Section 5.1.3 [mt_gkz.generate_maple_file_IC], page 17,

4.1.3 mt_gkz.cbase_by_euler

mt_gkz.cbase_by_euler(A)

:: It returns a standard basis for the GKZ system defined by A.

return The list of a standard basis for the GKZ system defined by A.

A the matrix A of the GKZ system.

option w w is a weight vector (given by a list) for differential operators.

option no_prob

If it is set to 1, a probabilistic algorithm of setting b's to random numbers is not used.

- The returned list of differential operators (monomials) is a standard basis of the GKZ system and it also gives a cohomology basis by applying them to the integration kernel of the GKZ hypergeometric integral.
- It is an implementation of the algorithm in [HNT] <https://dx.doi.org/10.1016/j.aim.2016.10.021> by Hibi-Nishiyama-Takayama.
- The weight vector is used to obtain an initial ideal of the toric ideal for A and is also used to obtain a standard basis of a system of algebraic equations by [HNT].

Example: basis for $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$.

```

import("mt_gkz.rr");
A=[[1,1,1,1],[0,1,2,3]]$
Std1=mt_gkz.cbase_by_euler(A);
Std2=mt_gkz.cbase_by_euler(A | w=[0,0,1,1]);
end$

```

Outputs are, e.g.,

```
[3079] A=[[1,1,1,1],[0,1,2,3]]$
```

```

[3080] Std1=mt_gkz.cbase_by_euler(A);
Gt=[-dx1*dx3+dx2^2, dx1*dx4-dx2*dx3, -dx2*dx4+dx3^2]
T2=[dx2^2, dx2*dx3, dx3^2]
T2(new)=[dx2^2-dx2, dx2*dx3, dx3^2-dx3]
E=[dx4+dx3+dx2+dx1-b_1, 3*dx4+2*dx3+dx2-b_2]
We use a probabilistic method to determine the base.
Note: dx1, dx2, ..., are the Euler operators
[dx3, dx4, 1]

```

```

[3081] Std2=mt_gkz.cbase_by_euler(A | w=[0,0,1,1]);
T2      =[dx2*dx3,dx1*dx3,dx1^2*dx4,dx3^2]
T2(new)=[dx2*dx3,dx1*dx3,(dx1^2-dx1)*dx4,dx3^2-dx3]
E=[dx4+dx3+dx2+dx1-b_1,3*dx4+2*dx3+dx2-b_2]
We use a probabilistic method to determine the base.
Note: dx1, dx2, ..., are the Euler operators
[dx2^2,dx2,1]
[3082]

```

Refer to [\(undefined\)](#) [mt_gkz.mytoric_by_euler], page [\(undefined\)](#),

For other functions, we have not yet written manual entries. We only show examples on these. As for details, please see the source code.

```

[1883] import("mt_gkz.rr");
[2667] mt_gkz.dvar([x1,x2]); // it generates variables starting with d
      [dx1,dx2]
[2669] mt_gkz.p_true_nf_rat((1/3)*x^3-1,[x^2-1],[x],0);
      [x-3,3] // p_true_nf does not accept rational number coefficients
[2670] A=[[1,1,1,1],[0,1,3,4]];
      [[1,1,1,1],[0,1,3,4]]
[2671] mt_gkz.reduce_by_toric(dx3^4,A);
      dx1*dx4^3 // reduction by toric ideal defined by A
[2672] nk_toric.toric_ideal(A);
      [-x1*x4+x2*x3,-x2*x4^2+x3^3,x2^2*x4-x1*x3^2,-x1^2*x3+x2^3]
[2673] mt_gkz.yang_gkz_buch(A,[b1,b2]); // Groebner basis of GKZ system by yang.rr
      1 o 2 ..o 3 ..ooooooooo 4 o 6 ooo 9 o
      [[(x2)*<<0,1,0,0>>+(3*x3)*<<0,0,1,0>>+ ---snip ---*<<0,0,0,0>>,1]],
      [dx1,dx2,dx3,dx4],
      [(1)*<<0,0,0,2>>,(1)*<<0,0,1,0>>,(1)*<<0,0,0,1>>,(1)*<<0,0,0,0>>]]

[2674] mt_gkz.dp_op_to_coef_vec([x1*<<1,0>>+x1*x2*<<0,1>>,x1+1],[<<1,0>>,<<0,1>>]);
      // x1+1 is the denominator
      [ (x1)/(x1+1) (x1*x2)/(x1+1) ]
[2675] mt_gkz.tk_base_is_equal([1,2],[1,2]);
      1
[2676] mt_gkz.tk_base_is_equal([1,2],[1,x,y]);
      0
[2677] mt_gkz.mdifff(sin(x),x,1);
      cos(x)
[2678] mt_gkz.mdifff(sin(x),x,2); //2nd derivative
      -sin(x)
[3164] mt_gkz.ord_xi(V=[x1,x2,x3],II=1);
// matrix to define graded lexicographic order so that V[II] is the smallest.
[ 1 1 1 ]
[ 0 -1 0 ]

```



```

[ -1 0 0 ]
[3166] load("mt_gkz/check-by-hgpoly.rr");
[3187] check_0123(); // check the pfaffian for the A below by hg-polynomial.
A=[[1,1,1,1],[0,1,2,3]]
Ap=[[1,1,1,1],[0,0,0,0]]
--- snip ---
Bfunctions= --- snip ---
0 (vector) is expected:
[[ 0 0 0 ],[ 0 0 0 ]]
[3188] mt_gkz.get_check_fvec();
// get the basis of cocycles used in terms of differential operators.
[1,(dx4)/(b1),(dx4^2)/(b1^2-b1)]
[3189] mt_gkz.clear_bf();
0
[3190] mt_gkz.get_bf_step_up(A=[[1,1,1,1],[0,1,2,3]]);
// b-functions and step-up operators.
// Option b_ideal=1 or shift=... may be used for non-normal case.
[[ -s_2^3+(9*s_1-3)*s_2^2+ ---snip---
   -s_2^3+(3*s_1+1)*s_2^2-3*s_1*s_2 s_2^3-3*s_2^2+2*s_2 ],
 [ x3^3*dx4^2+ ---snip---
   3*x3^2*x4*dx4^2+ --- snip---]]
[3191] mt_gkz.mytoric_ideal(0 | use_4ti2=1);
// 4ti2 is used to obtain a generator set of the toric ideal
// defined by the matrix A
[3192] mt_gkz.mytoric_ideal(0 | use_4ti2=0);
// A slower method is used to obtain a generator set of the toric ideal
// defined by the matrix A. 4ti2 is not needed. Default.
[3193] mt_gkz.cbase_by_euler(A=[[1,1,1,1],[0,1,3,4]]);
// Cohomology basis of the GKZ system defined by A for generic beta.
// Basis is given by a set of Euler operators.
// It is an implementation of the algorithm in https://dx.doi.org/10.1016/j.aim.2016.1
// beta is set by random numbers. Option: no_prob=1

```

5 Cohomology intersection numbers

5.1 Secondary equation

5.1.1 mt_gkz.kronecker_prd

`mt_gkz.kronecker_prd(A,B)`
 :: It returns the Kronecker product of A and B .

return a matrix which is equal to the Kronecker product of A and B (https://en.wikipedia.org/wiki/Kronecker_product).

A,B list

```
[2644] A=[[a,b],[c,d]];
[[a,b],[c,d]]
[2645] B=[[e,f],[g,h]];
[[e,f],[g,h]]
[2646] mt_gkz.kronecker_prd(A,B);
[ e*a f*a e*b f*b ]
[ g*a h*a g*b h*b ]
[ e*c f*c e*d f*d ]
[ g*c h*c g*d h*d ]
```

5.1.2 mt_gkz.secondary_eq

`mt_gkz.secondary_eq(A,Beta,Ap,Rvec,DirX)`
 :: It returns the secondary equation with respect to cocycles defined by $Rvec$.

return a list of coefficients of the Pfaffian system corresponding to the secondary equation (cf. equation (8) of [MT2020]).

$A,Beta,Ap,Rvec,DirX$
 see `pfaff_eq`

- The secondary equation is originally a Pfaffian system for an unknown r by r matrix I with $r = \text{length}(Rvec)$. We set $Y = (I_{11}, I_{12}, \dots, I_{1r}, I_{21}, I_{22}, \dots)^T$. Then, the secondary equation can be seen as a Pfaffian system $\frac{dY}{dx_i} = A_i Y$ with $DirX = \{dx_i\}_i$. The function `mt_gkz.secondary_eq(A,Beta,Ap,Rvec,DirX)` outputs a list obtained by aligning the matrices A_i .
- Let $F := (\omega_i)_i$ be a column vector whose entries are given by the cohomology classes specified by entries of $Rvec$. Then, `pfaff_eq` computes the Pfaffian matrices P_i so that $\frac{dF}{dx_i} = P_i F$. If Q_i denotes the matrix obtained by replacing $Beta$ by $-Beta$, we have $A_i = \text{mt_gkz.kronecker_prd}(E, P_i) + \text{mt_gkz.kronecker_prd}(Q_i, E)$ where E is the identity matrix of size $\text{length}(Rvec)$.
- Options `xrule`, `shift`, `b_ideal`, `cg`. Same as `pfaff_eq`.

Example:

```
[2647] Beta=[b1,b2,b3]$
[2648] DirX=[dx1,dx4]$
```

```

[2649] Rvec=[[1,0,0,0],[0,0,1,0]]$
[2650] A=[[1,1,0,0],[0,0,1,1],[0,1,0,1]]$
[2651] Ap=[[1,1,0,0],[0,0,1,1],[0,0,0,0]]$
[2652] Xrule=[[x2,1],[x3,1]]$
[2653] P=mt_gkz.secondary_eq(A,Beta,Ap,Rvec,DirX|xrule=Xrule)$
--snip--
[2654] length(P);
2
[2655] P[0];
[[(-2*x1^3*x4^2+4*x1^2*x4-2*x1)/(x1^4*x4^2-2*x1^3*x4+x1^2), (b2*x4)/(x1^2*x4-x1),
(-b2*x4)/(x1^2*x4-x1), 0], [(b1)/(x1*x4-1),
((b2-4/3)*x1^2*x4^2+(-b1-b2+8/3)*x1*x4+b1-4/3)/(x1^3*x4^2-2*x1^2*x4+x1), 0,
(-b2*x4)/(x1^2*x4-x1)], [(-b1)/(x1*x4-1), 0,
((-b2-2/3)*x1^2*x4^2+(b1+b2+4/3)*x1*x4-b1-2/3)/(x1^3*x4^2-2*x1^2*x4+x1),
(b2*x4)/(x1^2*x4-x1)], [0, (-b1)/(x1*x4-1), (b1)/(x1*x4-1), 0]]
<--- Paffian matrix in x1 direction.
[2656] P[1];
[[0, (b2)/(x1*x4-1), (-b2)/(x1*x4-1), 0], [(b1*x1)/(x1*x4^2-x4),
((b2-1/3)*x1^2*x4^2+(-b1-b2+2/3)*x1*x4+b1-1/3)/(x1^2*x4^3-2*x1*x4^2+x4), 0,
(-b2)/(x1*x4-1)], [(-b1*x1)/(x1*x4^2-x4), 0,
((-b2+1/3)*x1^2*x4^2+(b1+b2-2/3)*x1*x4-b1+1/3)/(x1^2*x4^3-2*x1*x4^2+x4),
(b2)/(x1*x4-1)], [0, (-b1*x1)/(x1*x4^2-x4), (b1*x1)/(x1*x4^2-x4), 0]]
<--- Paffian matrix in x4 direction.

```

Refer to Section 2.1.1 [mt_gkz.pfaff_eq], page 2,

5.1.3 mt_gkz.generate_maple_file_IC

`mt_gkz.generate_maple_file_IC(A,Beta,Ap,Rvec,DirX)`
 :: It returns the maple input for a solver of a Pfaffian system `IntegrableConnections[RationalSolutions]`.

return a maple input file for the function `IntegrableConnections[RationalSolutions]` (cf. [BCEW]) for the Pfaffian system `mt_gkz.secondary_eq(A,Beta,Ap,Rvec,DirX)`.

A,Beta,Ap,Rvec,DirX
 see `pfaff_eq`.

- A maple package `IntegrableConnections` is available in [BCEW]. In order to use `IntegrableConnections`, you need to add the global path to the file `IntegrableConnections.m` to `libname` on maple. See [BCEW].
- If `Beta` contains unknown variables, they are regarded as generic parameters. For example, if `Beta=[b1,b2,1/5,1/7,b5,...]`, parameters are `[b1,b2,b5,...]`.
- Options `xrule`, `shift`, `b_ideal`, `cg`. Same as `pfaff_eq`.
- Option `filename`. You can specify the file name by specifying the option variable `filename`. If you do not specify it, `generate_maple_file_IC` generates a file "auto-generated-IC.ml".

Example:

```
[2681] Beta=[b1,b2,1/3]$
```

```

[2682] DirX=[dx1,dx4]$
[2683] Rvec=[[1,0,0,0],[0,0,1,0]]$
[2684] A=[[1,1,0,0],[0,0,1,1],[0,1,0,1]]$
[2685] Ap=[[1,1,0,0],[0,0,1,1],[0,0,0,0]]$
[2687] Xrule=[[x2,1],[x3,1]]$
[2688] mt_gkz.generate_maple_file_IC(A,Beta,Ap,Rvec,DirX|xrule=Xrule,filename="Test.ml

```

//A file named Test.ml is automatically generated as follows:

```

with(OreModules);
with(IntegrableConnections);
with(linalg);
C:=Matrix([[(-2*x1^3*x4^2+4*x1^2*x4-2*x1)/(x1^4*x4^2-2*x1^3*x4+x1^2),
(b2*x4)/(x1^2*x4-x1),(-b2*x4)/(x1^2*x4-x1),0],[(b1)/(x1*x4-1),
((b2-4/3)*x1^2*x4^2+(-b1-b2+8/3)*x1*x4+b1-4/3)/(x1^3*x4^2-2*x1^2*x4+x1),0,
(-b2*x4)/(x1^2*x4-x1)], [(-b1)/(x1*x4-1),0,
((-b2-2/3)*x1^2*x4^2+(b1+b2+4/3)*x1*x4-b1-2/3)/(x1^3*x4^2-2*x1^2*x4+x1),
(b2*x4)/(x1^2*x4-x1)], [0,(-b1)/(x1*x4-1),(b1)/(x1*x4-1),0]]),
Matrix([[0,(b2)/(x1*x4-1),(-b2)/(x1*x4-1),0],[(b1*x1)/(x1*x4^2-x4),
((b2-1/3)*x1^2*x4^2+(-b1-b2+2/3)*x1*x4+b1-1/3)/(x1^2*x4^3-2*x1*x4^2+x4),0,
(-b2)/(x1*x4-1)], [(-b1*x1)/(x1*x4^2-x4),0,
((-b2+1/3)*x1^2*x4^2+(b1+b2-2/3)*x1*x4-b1+1/3)/(x1^2*x4^3-2*x1*x4^2+x4),
(b2)/(x1*x4-1)], [0,(-b1*x1)/(x1*x4^2-x4),(b1*x1)/(x1*x4^2-x4),0]]]);
RatSols:=RationalSolutions(C,[x1,x4],[ 'param',[b1,b2]]);

```

```

/*
If you run the output file on maple, you obtain a rational solution of
the secondary equation.
*/

```

```

          [b2*(3*b1-1)/(b1*x1^2)]
RatSols:=[3*b2/x1                ]
          [3*b2/x1                ]
          [3*b2-1                 ]

```

```

/*
Note that the 4 entries of this vector correspond to entries of a 2 by 2 matrix.
They are aligned as (1,1), (1,2), (2,1) (2,2) from the top.
*/

```

Refer to Section 2.1.1 [mt_gkz.pfaff_eq], page 2, Section 4.1.2 [mt_gkz.pf_ratsol_ml], page 12,

5.1.4 mt_gkz.generate_maple_file_MR

`mt_gkz.generate_maple_file_MR(A,Beta,Ap,Rvec,DirX,D1,D2)`

:: It returns the maple input for a solver of a Pfaffian system `MorphismsRat[OreMorphisms]`.

return a maple input file for the function `MorphismsRat[OreMorphisms]` (cf. [CQ]) for the Pfaffian system obtained by `secondary_eq`. If you run the output file on maple, you obtain a rational solution of the secondary equation.

A,Beta,Ap,Rvec,DirX

see `pfaff_eq`.

D1,D2 Positive integers. D1 (resp. D2) is the upper bound of the degree of the numerator (resp. denominator) of the solution.

- We use the same notation as the explanation of `generate_maple_file_IC`. Let D denote the ring of linear differential operators with coefficients in the field of rational functions. We consider D -modules $R := D^{1 \times l} / \sum_{dx_i \in \text{Dir}X} D^{1 \times l} (\partial_i E - P_i)$ and $S := D^{1 \times l} / \sum_{dx_i \in \text{Dir}X} D^{1 \times l} (\partial_i E + Q_i^T)$ where $l = \text{length}(\text{Rvec})$. Then, computing a rational solution of the secondary equation is equivalent to computing a D -morphism from R to S represented by rational function matrix (cf. pp12-13 of [CQ08]).
- A maple package `OreMorphisms` is available in [CQ]. In order to use `OreMorphisms`, you need to add the global path to the file `OreMorphisms.m` to `libname` on maple.
- Options `xrule`, `shift`, `b_ideal`, `cg`. Same as `pfaff_eq`.
- Option `filename`. You can specify the file name as in `generate_maple_file_IC`.
- The difference between `generate_maple_file_IC` and `generate_maple_file_MR` is the appearance of auxiliary variables `D1` and `D2`. If you can guess the degree of the numerator and the denominator of the solution of the secondary equation, `MorphismsRat[OreMorphisms]` can be faster than `RationalSolutions[IntegrableConnections]`.

Example:

```
[2668] Beta=[b1,b2,1/3]$
[2669] DirX=[dx1,dx4]$
[2670] Rvec=[[1,0,0,0],[0,0,1,0]]$
[2671] A=[[1,1,0,0],[0,0,1,1],[0,1,0,1]]$
[2672] Ap=[[1,1,0,0],[0,0,1,1],[0,0,0,0]]$
[2673] Xvar=[x1,x4]$
[2674] Xrule=[[x2,1],[x3,1]]$
[2675] mt_gkz.generate_maple_file_MR(A,Beta,Ap,Rvec,DirX,2,2|xrule=Xrule)$
```

//A file "auto-generated-MR.ml" is automatically generated as follows:

```
with(OreModules);
```

```

with(OreMorphisms);
with(linalg);
Alg:=DefineOreAlgebra(diff=[dx1,x1],diff=[dx4,x4],polynom=[x1,x4],comm=[b1,b2]);
P:=Matrix([[dx1,0],[0,dx1],[dx4,0],[0,dx4]])
-Matrix([[(b1+b2-4/3)*x1*x4-b1+4/3]/(x1^2*x4-x1),(-b2*x4)/(x1^2*x4-x1)],
[(-b1)/(x1*x4-1),(b1*x4)/(x1*x4-1)],[(b2*x1)/(x1*x4-1),(-b2)/(x1*x4-1)],
[(-b1*x1)/(x1*x4^2-x4),(1/3*x1*x4+b1-1/3)/(x1*x4^2-x4)]]);
Q:=Matrix([[dx1,0],[0,dx1],[dx4,0],[0,dx4]])
+Matrix([[(b1+b2-2/3)*x1*x4+b1+2/3]/(x1^2*x4-x1),(b1)/(x1*x4-1)],
[(b2*x4)/(x1^2*x4-x1),(-b1*x4)/(x1*x4-1)],[(b2*x1)/(x1*x4-1),(b1*x1)/(x1*x4^2-x4)],
[(b2)/(x1*x4-1),(-1/3*x1*x4-b1+1/3)/(x1*x4^2-x4)]]);
RatSols:=MorphismsRat(P,Q,Alg,0,2,2);

/*
If you run the output file on maple, you obtain a vector RatSols.
RatSols[1] is the rational solution of the secondary equation:
*/

RatSols[1]:=[(1/3)*n2131*(3*b1-1)/(b1*x1^2*d61)  n2131/(x1*d61)]
              [n2131/(x1*d61)              (1/3)*n2131*(3*b2-1)/(b2*d61)]

/*
Here, n2131 and d61 are arbitrary constants. We can take n2131 = 3*b2 and d61 =
1 to obtain the rational solution of the secondary equation which is iden-
tical to the one obtained from generate_maple_file_IC.
*/

```

Refer to Section 2.1.1 [mt_gkz.pfaff_eq], page 2, Section 5.1.3 [mt_gkz.generate_maple_file_IC], page 17.

5.2 Normalizing the cohomology intersection matrix

5.2.1 mt_gkz.principal_normalizing_constant

mt_gkz.principal_normalizing_constant(A,T,Beta,K)
 :: It returns the normalizing constant of the cohomology intersection matrix in terms of a regular triangulation T.

return a rational function which is the cohomology intersection number $\frac{1}{(2\pi\sqrt{-1})^n} \langle [\frac{dt}{t}], [\frac{dt}{t}] \rangle_{ch}$ in terms of the regular triangulation T. Here, n is the number of integration variables and $\frac{dt}{t}$ is the volume form $\frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}$ of the complex n -torus.

A,Beta see pfaff_eq.

T a regular triangulation of A.

K The number of polynomial factors in the integrand. see [MT2020].

- This function is useful when the basis of the cohomology group $\{\omega_i\}_{i=1}^r$ is given so that $\omega_1 = [\frac{dt}{t}]$.
- One can find a regular triangulation by using a function `mt_gkz.regular_triangulation`.
- `mt_gkz.leading_terms` can be used more generally.

Example:

```
[2676] A=[[1,1,0,0],[0,0,1,1],[0,1,0,1]]$
[2677] Beta=[b1,b2,b3]$
[2678] K=2$
[2679] T=[[1,2,3],[2,3,4]]$
[2680] mt_gkz.principal_normalizing_constant(A,T,Beta,K);
(-b1-b2)/(b3*b1+b3*b2-b3^2)
```

Refer to Section 5.2.2 [`mt_gkz.leading_terms`], page 21.

5.2.2 `mt_gkz.leading_terms`

`mt_gkz.leading_terms(A,Beta,W,Q1,Q2,K,N)`

:: It returns the W-leading terms of a cohomology intersection number specified by Q1 and Q2 up to W-degree=(minimum W-degree)+N.

return a list `[[C1,DEG1],[C2,DEG2],...]`. Each CI is a rational function depending on Beta times a monomial x^m in x-variables. DEGI is the W-degree of x^m . The cohomology intersection number $\frac{1}{(2\pi\sqrt{-1})^n} \langle [h^{-q'_1} t^{q''_1} \frac{dt}{t}], [h^{-q'_2} t^{q''_2} \frac{dt}{t}] \rangle_{ch}$ has a Laurent expansion of the form $C1+C2+....$

A,Beta see `pfaff_eq`.

W a positive and integral weight vector.

Q1,Q2 $Q1 = (q'_1, q''_1)^T$, $Q2 = (q'_2, q''_2)^T$ are integer vectors. The lengths of q'_1 and q'_2 are both equal to K .

K The number of polynomial factors in the integrand. see [MT2020].

N A positive integer.

- For a monomial $x^m = x_1^{m_1} \cdots x_n^{m_n}$ and a weight vector $W = (w_1, \dots, w_n)$, the W-degree of x^m is given by the dot product $m \cdot W = m_1 w_1 + \cdots + m_n w_n$.
- The W-leading terms of the cohomology intersection number $\frac{1}{(2\pi\sqrt{-1})^n} \langle [h^{-q'_1} t^{q''_1} \frac{dt}{t}], [h^{-q'_2} t^{q''_2} \frac{dt}{t}] \rangle_{ch}$ can be computed by means of Theorem 2.6 of [GM2020]. See also Theorem 3.4.2 of [SST2000].
- If the weight vector is not generic, you will receive an error message such as "WARNING(initial_mon): The weight may not be generic". In this case, the output may be wrong and you should retake a suitable W. To be more precise, W should be chosen from an open cone of the Groebner fan.
- Option `xrule`. Same as `pfaff_eq`.

Example:

```
[2922] Beta=[b1,b2,1/3];
[b1,b2,1/3]
```

```

[2923] Q=[[1,0,0],[0,1,0]];
[[1,0,0],[0,1,0]]
[2924] A=[[1,1,0,0],[0,0,1,1],[0,1,0,1]];
[[1,1,0,0],[0,0,1,1],[0,1,0,1]]
[2925] W=[1,0,0,0];
[1,0,0,0]
[2926] K=2;
2
[2927] N=2;
2
[2928] NC=mt_gkz.leading_terms(A,Beta,W,Q[0],Q[1],K,N|xrule=[[x2,1],[x3,1],[x4,1]])$
--snip--
[2929] NC;
[[-3)/(x1),-5],[0,-4],[0,-3]]

/*
This output means that the W-leading term of the (1,2) entry of the cohomology
intersection matrix is  $(-3)/(x1) \times (2\pi\sqrt{-1})$ . In view of examples of generate_
maple_file_IC or generate_maple_file_MR, we can conclude that the cohomology
intersection matrix is given by
*/

[-(3*b1-1)/(b1*x1^2)  -3/x1      ]
[-3/x1                -(3*b2-1)/b2]]

//divided by  $2\pi\sqrt{-1}$ .

Refer to Section 5.2.2 [mt_gkz.leading_terms], page 21, Section 5.1.3
[mt_gkz.generate_maple_file_IC], page 17, Section 5.1.4 [mt_gkz.generate_maple_file_MR],
page 19.

```

5.2.3 mt_gkz.leading_term_rat

`mt_gkz.leading_term_rat(P,W,V)`
 :: It returns the W-leading term of a rational function P depending on variables V.

return It returns the W-leading term of a rational function P.

P a rational function.

W a weight vector.

V a list of variables of P.

- This function is supposed to be combined with `leading_terms` to compute the leading term of a cohomology intersection number.
- If W is chose so that there are several initial terms, you will receive an error message "WARNING(leading_term_rat):The weight vector may not be generic."

Refer to Section 5.2.2 [mt_gkz.leading_terms], page 21.

5.3 Regular triangulations

5.3.1 mt_gkz.toric_gen_initial, mt_gkz.regular_triangulation, mt_gkz.top_standard_pairs

`mt_gkz.toric_gen_initial(A,W)`
`mt_gkz.regular_triangulation(A,W)`
`mt_gkz.top_standard_pairs(A,W)`
 :: utility functions for computing ring theoretic invariants: generic initial ideal for the toric ideal specified by the matrix A and a weight W, its associated regular triangulation, and its associated top-dimensional standard pairs.

return `toric_gen_initial` returns a list [L1,L2] of length 2. L1 is a list of generators of the W-initial ideal of the toric ideal I_A specified by A. L2 is a list of variables of I_A .

return `regular_triangulation` returns a list of simplices of a regular triangulation T_W specified by the weight W.

return `top_standard_pairs` returns a list of the form [[L1,S1],[L2,S2],...]. Each SI is a simplex of T_W . Each LI is a list of exponents.

A a configuration matrix.

W a positive weight vector.

- As for the definition of the standard pair, see Chapter 3 of [SST00].
- We set $n=\text{length}(A)$ and set $BS1:=\{1,2,\dots,n\} \setminus S1$. Then, each $L1[I]$ is an exponent \mathbf{k} of a top-dimensional standard pair $(\partial_{BS1}^{\mathbf{k}}, S1)$. Here, \mathbf{k} is a list of length $n-\text{length}(S1)$ and $\partial_{BS1} = (\partial_J)_{J \in BS1}$.
- If the weight vector is not generic, you will receive an error message such as "WARNING(initial_mon): The weight may not be generic". See also `leading_terms`.
- These functions are utilized in `leading_terms`.

Example: An example of a non-unimodular triangulation and non-trivial standard pairs.

```
[3256] A=[[1,1,1,1,1],[0,1,0,2,0],[0,0,1,0,2]];
[[1,1,1,1,1],[0,1,0,2,0],[0,0,1,0,2]]
[3257] W=[2,0,1,2,2];
[2,0,1,2,2]
[3258] mt_gkz.toric_gen_initial(A,W);
--snip--
[[x1*x5,x1*x4,x3^2*x4],[x1,x2,x3,x4,x5]]
[3259] mt_gkz.regular_triangulation(A,W);
--snip--
[[2,4,5],[2,3,5],[1,2,3]]
[3260] mt_gkz.top_standard_pairs(A,W);
--snip--
[[[0,0],[0,1]],[2,4,5]],[[0,0],[2,3,5]],[[0,0],[1,2,3]]]
```

/*

This means that the regular triangulation of the configuration matrix A is given by $T = \{\{2, 4, 5\}, \{2, 3, 5\}, \{1, 2, 3\}\}$. The normalized volumes of these simplices are 2, 1 and 1. Moreover, the top-dimensional standard pairs are $(1, \{2, 4, 5\}), (\partial_3, \{2, 4, 5\}), (1, \{2, 3, 5\}), (1, \{1, 2, 3\})$.

*/

Refer to Section 5.2.2 [mt_gkz.leading_terms], page 21.

Index

(Index is nonexistent)

(Index is nonexistent)

Short Contents

1	About this document	1
2	Pfaff equation	2
3	b function	9
4	Utilities	11
5	Cohomology intersection numbers	16
	Index	25

Table of Contents

1	About this document	1
2	Pfaff equation	2
2.1	Pfaff equation for given cocycles	2
2.1.1	mt_gkz.pfaff_eq	2
2.1.2	mt_gkz.ff2, mt_gkz.ff1, mt_gkz.ff	4
2.1.3	mt_gkz.rvec_to_fvec	5
2.1.4	mt_gkz.fvec_to_conn_mat	6
2.1.5	mt_gkz.contiguity, mt_gkz.contiguity_by_fvec	7
3	b function	9
3.1	b function and facet polynomial	9
3.1.1	mt_gkz.bf	9
3.1.2	mt_gkz.facet_poly	9
3.1.3	mt_gkz.get_bf_step_up	10
4	Utilities	11
4.1	Some utility functions	11
4.1.1	mt_gkz.gkz_b	11
4.1.2	mt_gkz.pf_ratsol_ml	12
4.1.3	mt_gkz.cbase_by_euler	13
5	Cohomology intersection numbers	16
5.1	Secondary equation	16
5.1.1	mt_gkz.kronecker_prd	16
5.1.2	mt_gkz.secondary_eq	16
5.1.3	mt_gkz.generate_maple_file_IC	17
5.1.4	mt_gkz.generate_maple_file_MR	19
5.2	Normalizing the cohomology intersection matrix	20
5.2.1	mt_gkz.principal_normalizing_constant	20
5.2.2	mt_gkz.leading_terms	21
5.2.3	mt_gkz.leading_term_rat	22
5.3	Regular triangulations	23
5.3.1	mt_gkz.toric_gen_initial, mt_gkz.regular_triangulation, mt_gkz.top_standard_pairs	23
	Index	25

