Rokko Lectures in Mathematics **3**

Lecture Notes on Interacting Particle Systems

By

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PREFACE

In these notes, we study basic contact processes, diffusive θ-contact processes, diffusive contact processes, one-sided contact process and discrete-time growth models. Unfortunately, the rigorous critical values and order parameters for these particle systems are not known. So these notes mainly restrict attention to upper and lower bounds on the critical values and order parameters given by the Harris lemma. In particular, we focus on two typical methods; the Katori-Konno method and the Holley-Liggett one. In general, the Katori-Konno (resp. Holley-Liggett) method gives lower (resp. upper) bounds on critical values and upper (resp. lower) bounds on order parameters. Roughly speaking, the Katori-Konno method corresponds to mean-field type approximation. On the other hand, the Holley-Liggett method corresponds to Gibbsian type one. Most results in these notes already appeared in some journals and books.

The basic contact process is a simple model of a disease with the infection rate λ. This process is a continuous-time Markov process on state space $\eta \in \{0,1\}^{\mathbf{Z}^d}$, where $\mathbf{Z}^d$ is the d-dimensional integer lattice. The dynamics are specified by the following transition rates: at site $x \in \mathbf{Z}^d$,

$$
\begin{aligned}
1 &\to 0 \qquad \text{at rate} \qquad 1, \\
0 &\to 1 \qquad \text{at rate} \qquad \lambda \sum_{y:|y-x|=1} \eta(y),
\end{aligned}
$$

where $|x| = |x_1| + \cdots + |x_d|$. Define ρ_λ as the density of infected individual at a site with respect to a stationary measure. We take ρ_λ as an order parameter of this process. The basic contact process is attractive, so ρ_λ is a nondecreasing function of λ. Then the critical value λ_c is characterized by ρ_λ in the following way: $\lambda_c = \inf\{\lambda \geq 0 : \rho_\lambda > 0\}$. Concerning the processes treated in the rest of these notes, the critical values and the order parameters can be defined in the similar way, since these processes are also attractive. In Chapters 1 and 2, we mainly study the one-dimensional case. Chapter 1 is devoted to lower bounds on the critical value and upper bounds on the order parameter of the basic contact process. In Section 1.3, we give bounds by using the correlation identities and the Harris-FKG inequality. In Section 1.4, we present the Harris lemma. By using it, better bounds are obtained by the Katori-Konno method. In the end of this section, we discuss another derivation of Katori-Konno bounds by assuming correlation inequalities. Chapter 2 deals with upper bounds on the critical value and lower bounds on the order parameter of the basic contact process. In Section 2.2, we obtain bounds by the Holley-Liggett method which uses the Harris lemma. In Section 2.3, we discuss another derivation of Holley-Liggett bounds by assuming correlation inequalities. We should remark that these inequalities are different from ones which appeared in Section 1.4.

The diffusive θ-contact process is a generalization of the one-dimensional basic contact process. Chapter 3 treats bounds on critical values and order parameters for this process. In Section 3.2, we give bounds by using the Katori-Konno method. Section 3.3 provides bounds by the Holley-Liggett method for non-diffusive case. Moreover, in Section 3.4, we discuss another derivation of Holley-Liggett bounds by assuming correlation inequalities as in the case of Section 2.3.

The diffusive contact process is the basic contact process with stirring mechanism. In Chapter 4, we study bounds given by the Katori-Konno method.

Chapter 5 is devoted to basic contact processes on homogeneous trees. In this chapter we obtain both upper and lower bounds on order parameters by the Katori-Konno method. In particular, the result on the lower bound depends on the property of trees.

In Chapter 6, we consider the one-sided contact process which is an asymmetric basic contact process. Section 6.2 gives lower bounds on critical value. In Section 6.3, we study upper bound.

Chapter 7 is devoted to discrete-time growth models on $\mathbf{Z}$. In Section 7.2, we present a discrete-time version of the Harris lemma. Section 6.3 is devoted to bounds by the Katori-Konno method. In Section 6.4, we discuss briefly a derivation of bounds by the Holley-Liggett method.

Finally, Chapter 8 treats 3-state cyclic particle systems. Section 8.2 is devoted to master equations and correlation identities in the d-dimensional case. In Sections 8.3 and 8.4, we study the mean-field and pair approximations respectively. Section 8.5 is devoted to the 3-state cyclic particle system with an external field.

These notes developed as a result of an intensive course given at Kobe University, July 1-5, 1996. They are based on *Lecture Notes on Harris Lemma and Particle Systems* (Universidade de São Paulo, 1996). In these notes, I added two chapters on one-sided contact process (Chapter 6) and 3-state cyclic particle systems (Chapter 8), and 43 exercises for students. Finally I acknowledge gratefully passive and active assistance of Yasunari Higuchi and Katusi Fukuyama during this period.

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CHAPTER 1

BASIC CONTACT PROCESSES I

1.1. Introduction

The basic contact process is a continuous-time Markov process on state space $\eta \in \{0,1\}^{\mathbf{Z}^d}$, where $\mathbf{Z}^d$ is the d-dimensional integer lattice. This process was introduced by Harris[1] in 1974. There are two different types of definition of this process. One is given by the following formal generator

$$\Omega f(\eta) = \sum_{x \in \mathbf{Z}^d} c(x,\eta)[f(\eta^x) - f(\eta)],$$

with flip rates

$$c(x,\eta) = (1-\eta(x)) \times \lambda \sum_{y:|y-x|=1} \eta(y) + \eta(x),$$

where η^x denotes $\eta^x(y) = \eta(y)$ for $y \neq x$ and $\eta^x(x) = 1-\eta(x)$ and $|x| = |x_1|+\cdots+|x_d|$. The details about this definition were discussed in Chapter I of Liggett.[2] In the case of one dimension, the dynamics of the evolution is as follows:

$$\begin{array}{lll} 001 \to 011 & \text{at rate} & \lambda, \\ 100 \to 110 & \text{at rate} & \lambda, \\ 101 \to 111 & \text{at rate} & 2\lambda, \\ \ \ 1 \ \to \ 0 & \text{at rate} & 1. \end{array}$$

The other is given by the *graphical representation.* We consider the state of process $\xi_t \subset \mathbf{Z}^d$, i.e., $\xi_t = \{x \in \mathbf{Z}^d : \eta_t(x) = 1\}$. The points in ξ_t are thought of as being occupied.

(i) If $x \notin \xi_t$, then x becomes occupied at a rate equal to λ times the number of occupied neighbors.

(ii) If $x \in \xi_t$, then x becomes vacant at rate 1.

To construct the basic contact process, we introduce the following two Poisson processes. For each x and y with $|x-y|=1$, let $\{T_n^{(x,y)}, n \geq 1\}$ be a Poisson process with rate λ, and let $\{U_n^x, n \geq 1\}$ be a Poisson process with rate 1. At times $T_n^{(x,y)}$,

we draw an arrow from x to y to indicate that if x is occupied then y will become occupied (if it is not already). At times U_n^x, we put a δ at x. The effect of a δ is to kill the particle at x (if it is present). We call there is a path from $(x,0)$ to (y,t) if there is sequence of times $t_0 = 0 < t_1 < \cdots < t_n < t_{n+1} = t$ and spatial locations $x_0 = x, x_1, \ldots, x_n = y$ so that
(i) for $i = 1, 2, \ldots, n$, there is an arrow from x_{i-1} to x_i at time t_i,
(ii) the vertical segments $\{x_i\} \times (t_i, t_{i-1})$, $i = 0, 1, \ldots, n$ do not contain any δ's.
The basic contact process starting from $A \subset \mathbf{Z}^d$ is defined by

$$\xi_t^A = \{y \in \mathbf{Z}^d : \text{ for some } x \in A \text{ there is a path from } (x,0) \text{ to } (y,t)\}.$$

Concerning the graphical representation, see Durrett,[3,4] for example. We should remark that the first definition is equivalent to the second one.

Let $\wp$ be the collection of all probability measure on $\{0,1\}^{\mathbf{Z}^d}$. For $\mu \in \wp$, $\mu S(t) \in \wp$ denotes the distribution at time t of the basic contact process for initial distribution μ where $S(t)$ is a semigroup corresponding to the formal generator Ω. Let $\Im$ be the set of all stationary measure $\mu \in \wp$:

$$\Im = \{\mu \in \wp : \mu S(t) = \mu \text{ for all } t \geq 0\}.$$

The basic contact process is said to be *ergodic* if

$$(a) \qquad \Im = \{\nu\} \text{ is a singleton,}$$

or

$$(b) \qquad \lim_{t \to \infty} \mu S(t) = \nu \text{ for all } \mu \in \wp.$$

Let δ_0 and δ_1 denote the pointmass on $\eta \equiv 0$ and $\eta \equiv 1$ respectively. By Theorem 2.14 in Chapter III of Liggett,[2] there is a *critical value* $\lambda_c \in [0, \infty]$ so that

$$\text{if } \lambda < \lambda_c, \text{ then the basic contact process is ergodic: } \Im = \{\delta_0\},$$

and

$$\text{if } \lambda > \lambda_c, \text{ then the basic contact process is not ergodic.}$$

So we have

$$\begin{aligned} \lambda_c &= \sup\{\lambda \geq 0 : \text{ the basic contact process is ergodic }\} \\ &= \inf\{\lambda \geq 0 : \text{ the basic contact process is not ergodic }\}. \end{aligned}$$

Bezuidenhout and Grimmett[5] showed

$$\text{even if } \lambda = \lambda_c, \text{ then the basic contact process is ergodic: } \Im = \{\delta_0\}.$$

Exercise 1.1. Show that if we change the infection rate from λ to $\alpha\lambda$ $(\alpha > 0)$ then the critical value λ_c is changed to λ_c/α. For example, if $\lambda \to \lambda/2d$, then $\lambda_c \to 2d\lambda_c$, where d is the dimensionality.

Let
$$\nu_\lambda = \lim_{t\to\infty} \delta_1 S(t),$$
which is called the *upper invariant measure.* The well-definedness of this measure is guaranteed by the attractiveness of this process. In general, the process with rates $c(x,\eta)$ is called to be *attractive* if any η, ζ with $\eta(x) \le \zeta(x)$ for any x,
$$\begin{aligned} c(x,\eta) &\le c(x,\zeta) \quad \text{if} \quad \eta(x) = \zeta(x) = 0, \\ c(x,\eta) &\ge c(x,\zeta) \quad \text{if} \quad \eta(x) = \zeta(x) = 1. \end{aligned}$$
On the other hand, trivial stationary measure in this model,
$$\delta_0 = \lim_{t\to\infty} \delta_0 S(t),$$
is called the *lower invariant measure.* Moreover Bezuidenhout and Grimmett[5] proved that the stationary measure of the basic contact process is a convex combination of δ_0 and ν_λ : for all $\mu \in \wp$,
$$\lim_{t\to\infty} \mu S(t) = \gamma\delta_0 + (1-\gamma)\nu_\lambda$$
where
$$\gamma = \int P_\eta(\tau < \infty)\, \mu(d\eta),$$
and τ is the hitting time of $\eta \equiv 0$. Note that the work of Griffeath[6] pioneered the above complete convergence theorem.

Exercise 1.2. Show that the upper invariant measure ν_λ is translation invariant, that is,
$$\begin{aligned} &\nu_\lambda\{\eta : \eta(x_1) = i_1, \ldots, \eta(x_n) = i_n\} \\ &\quad = \nu_\lambda\{\eta : \eta(x_1+u) = i_1, \ldots, \eta(x_n+u) = i_n\}, \end{aligned}$$
for any $n \ge 1$, $x_1, \ldots, x_n$, $u \in \mathbf{Z}^d$ and $i_1, \ldots, i_n \in \{0,1\}$.

Exercise 1.3. Verify that the basic contact process is attractive.

Exercise 1.4. δ_1 is not the stationary measure of the basic contact process. Explain.

Define ρ_λ as the density of particle at a site x with respect to ν_λ :
$$\rho_\lambda = E_{\nu_\lambda}(\eta(x)) = \nu_\lambda\{\eta : \eta(x) = 1\}.$$

Note that ρ_λ is independent of x, since ν_λ is translation invariant. We take ρ_λ as an *order parameter* of the basic contact process. Then the critical value λ_c can be characterized by ρ_λ :

$$\lambda_c = \sup\{\lambda \geq 0 : \rho_\lambda = 0\} = \inf\{\lambda \geq 0 : \rho_\lambda > 0\}.$$

Unfortunately λ_c and ρ_λ are not known rigorously !

From now on, we will restrict attention to the one-dimensional case. Here we present several rigorous results on the critical values and the order parameters of the basic contact process in one dimension.

Theorem 1.1.1.
(1) $\rho_\lambda = 0$ *for* $\lambda \leq \lambda_c$.
(2) $\rho_\lambda > 0$ *for* $\lambda > \lambda_c$.
(3) ρ_λ *is a nondecreasing function of* λ.
(4) ρ_λ *is continuous for* $\lambda \geq 0$.
(5) $1.539 \leq \lambda_c \leq 1.942$.

Here we would like to give some comments about this theorem. Parts (1) and (2) come from the definition of λ_c immediately. Concerning parts (3) and (4), see Chapter VI of Liggett.[2] The continuity of ρ_λ at λ_c is given by Bezuidenhout and Grimmett.[5] In part (5), the lower bound 1.539 on λ_c is a numerical result by Ziezold and Grillenberger.[7] As for upper bounds, Holley and Liggett[8] proved $\lambda_c \leq 2$. Recently an improved upper bound 1.942 was given by Liggett.[9] Some simulations or (non-rigorous) numerical methods by using computer reported that the critical value is estimated as, $\lambda_c \approx 1.649$. (See, for example, Brower, Furman and Moshe,[10] Konno and Katori.[11]) The above results (1)-(4) hold also for $d \geq 2$.

This chapter is mainly devoted to the lower bounds on critical value and upper bounds on the order parameter of the one-dimensional basic contact process. In general, there are some methods for getting the lower bounds on critical value and/or upper bounds on the order parameter as follows;

(i) Harris-FKG inequality method.
(ii) Katori-Konno method.
(iii) Ziezold-Grillenberger method.
(iv) Griffeath method.

In this chapter we will study (i)-(ii). As for (iii) and (iv), see Konno,[12] for example. In Chapter 4, we will consider the case of higher dimensions as a special case of diffusive contact processes.

1.2. Correlation Identities

In this section we present the correlation identities on the one-dimensional contact process. Let Y be the collection of all finite subsets of $\mathbf{Z}^1$. We define

$$\rho_\lambda(A) = E_{\nu_\lambda}\left(\prod_{x\in A}\eta(x)\right) = \nu_\lambda\{\eta : \eta(x) = 1 \text{ for all } x \in A\},$$

$$\overline{\rho}_\lambda(A) = E_{\nu_\lambda}\left(\prod_{x\in A}(1-\eta(x))\right) = \nu_\lambda\{\eta : \eta(x) = 0 \text{ for all } x \in A\},$$

for any $A \in Y$.

First we consider the correlation identities for $\rho_\lambda(A)$. We begin by rewriting the formal generator of the one-dimensional basic contact process as follows:

$$\Omega f(\eta) = \sum_{x\in\mathbf{Z}}\Big[\lambda\{\eta(x-1)+\eta(x+1)\}(1-\eta(x))+\eta(x)\Big][f(\eta^x)-f(\eta)].$$

The general arguments of the generator of a Markov semigroup give

$$\frac{\partial}{\partial t}E_{\delta_1}(f(\eta_t)) = E_{\delta_1}(\Omega f(\eta_t)). \tag{1.1}$$

Let $A = \{x_1, x_2, \ldots, x_n\}$, where $x_i \in \mathbf{Z}$ for $1 \le i \le n$. Define *n-points correlation functions at time t* as

$$\rho_{t,\lambda}(A) = \rho_{t,\lambda}(x_1x_2\ldots x_n) = E_{\delta_1}(\eta_t(x_1)\eta_t(x_2)\ldots\eta_t(x_n)).$$

Then we have the following system of infinite correlation evolution equations.

Theorem 1.2.1. *For any $A \in Y$,*

$$\begin{aligned}\frac{\partial}{\partial t}\rho_{t,\lambda}(A) = &-\{|A|+2\lambda b(A)\}\rho_{t,\lambda}(A) + \lambda\sum_{x\in A}\sum_{\substack{y\in\triangle A:\\ |y-x|=1}}\rho_{t,\lambda}((A\setminus\{x\})\cup\{y\})\\ &+\lambda\sum_{x\in A}w_A(x)\rho_{t,\lambda}(A\setminus\{x\}) - \lambda\sum_{y\in\triangle A}w_A(y)\rho_{t,\lambda}(A\cup\{y\}),\end{aligned}$$

where $|A|$ denotes the number of elements in A, $b(A)$ denotes the number of bonds inside A, $\triangle A = \{y \in \mathbf{Z}\setminus A : \text{there is an } x \in A \text{ so that } |x-y| = 1\}$ and for $x \in \mathbf{Z}$, $w_A(x) = |\{y \in A : |x-y| = 1\}| \in \{0,1,2\}$, i.e., denotes the number of nearest neighbors of x inside A.

Note that this theorem is satisfied generally in the d-dimensional case by translation: $\mathbf{Z} \to \mathbf{Z}^d$. By applying Theorem 1.2.1 to $A = \{0\}$, $\{0,1\}$, $\{0,2\}$, $\{0,1,2\}$, and the translation-invariance of the initial configuration $\eta \equiv 1$ and evolution of this process, we obtain the following differential equations. Let

$$\rho_{t,\lambda}(\bullet) = \rho_{t,\lambda}(\{0\}), \quad \rho_{t,\lambda}(\bullet\bullet) = \rho_{t,\lambda}(\{0,1\}), \quad \rho_{t,\lambda}(\bullet\times\bullet) = \rho_{t,\lambda}(\{0,2\}), \text{ etc.}$$

Corollary 1.2.2.

(1) $$\frac{\partial}{\partial t}\rho_{t,\lambda}(\bullet) = (2\lambda - 1)\rho_{t,\lambda}(\bullet) - 2\lambda\rho_{t,\lambda}(\bullet\bullet).$$

(2) $$\frac{\partial}{\partial t}\rho_{t,\lambda}(\bullet\bullet) = 2\lambda\rho_{t,\lambda}(\bullet) - 2(\lambda+1)\rho_{t,\lambda}(\bullet\bullet) + 2\lambda\rho_{t,\lambda}(\bullet\times\bullet) - 2\lambda\rho_{t,\lambda}(\bullet\bullet\bullet).$$

(3) $$\begin{aligned}\frac{\partial}{\partial t}\rho_{t,\lambda}(\bullet\times\bullet) = {} & 2\lambda\rho_{t,\lambda}(\bullet\bullet) - 2\rho_{t,\lambda}(\bullet\times\bullet) \\ & - 2\lambda\rho_{t,\lambda}(\bullet\bullet\bullet) + 2\lambda\rho_{t,\lambda}(\bullet\times\times\bullet) - 2\lambda\rho_{t,\lambda}(\bullet\bullet\times\bullet).\end{aligned}$$

(4) $$\begin{aligned}\frac{\partial}{\partial t}\rho_{t,\lambda}(\bullet\bullet\bullet) = {} & 2\lambda\rho_{t,\lambda}(\bullet\bullet) + 2\rho_{t,\lambda}(\bullet\times\bullet) \\ & - (4\lambda+3)\rho_{t,\lambda}(\bullet\bullet\bullet) + 2\lambda\rho_{t,\lambda}(\bullet\bullet\times\bullet) - 2\lambda\rho_{t,\lambda}(\bullet\bullet\bullet\bullet).\end{aligned}$$

Of course, the above corollary is also given by direct computations from the formal generator. For example, in the case of $f(\eta) = \eta(0)$, we see that

$$\begin{aligned}\Omega f(\eta) &= \Big[\lambda\{\eta(-1) + \eta(1)\}(1 - \eta(0)) + \eta(0)\Big][1 - 2\eta(0)] \\ &= \lambda\eta(-1) + \lambda\eta(1) - \eta(0) - \lambda\eta(-1)\eta(0) - \lambda\eta(0)\eta(1),\end{aligned}$$

since $\eta(0)^2 = \eta(0)$. By using this fact and Eq.(1.1), we have

$$\begin{aligned}\frac{\partial}{\partial t}E_{\delta_1}(\eta_t(0)) &= \lambda E_{\delta_1}(\eta_t(-1)) + \lambda E_{\delta_1}(\eta_t(1)) - \lambda E_{\delta_1}(\eta_t(0)) \\ &\quad - \lambda E_{\delta_1}(\eta_t(-1)\eta_t(0)) - \lambda E_{\delta_1}(\eta_t(0)\eta_t(1)) \\ &= (2\lambda - 1)E_{\delta_1}(\eta_t(0)) - 2\lambda E_{\delta_1}(\eta_t(0)\eta_t(1)).\end{aligned}$$

The second equality comes from the translation invariance. Hence we prove part (1). Similarly parts (2)-(4) are obtained.

Exercise 1.5. Prove Corollary 1.2.2 (2) by calculating $\Omega(\eta(0)\eta(1))$.

In the stationary state, that is, in the limit as $t \to \infty$, Theorem 1.2.1 gives

Theorem 1.2.3. *For any $A \in Y$,*

$$\begin{aligned}0 = {} & -\{|A| + 2\lambda b(A)\}\rho_\lambda(A) + \lambda\sum_{x\in A}\sum_{\substack{y\in\triangle A:\\ |y-x|=1}}\rho_\lambda((A\setminus\{x\})\cup\{y\}) \\ & + \lambda\sum_{x\in A} w_A(x)\rho_\lambda(A\setminus\{x\}) - \lambda\sum_{y\in\triangle A} w_A(y)\rho_\lambda(A\cup\{y\}),\end{aligned}$$

where $\rho_\lambda(A) = \nu_\lambda\{\eta : \eta(x) = 1 \ \text{for any} \ x \in A\}$.

Let $\nu_\lambda(\bullet) = \rho_\lambda(\{0\})$, $\nu_\lambda(\bullet\bullet) = \rho_\lambda(\{0,1\})$, $\nu_\lambda(\bullet\times\bullet) = \rho_\lambda(\{0,2\})$, etc. From Theorem 1.2.3, we have

Corollary 1.2.4.

(1) $$0 = (2\lambda - 1)\nu_\lambda(\bullet) - 2\lambda\nu_\lambda(\bullet\bullet).$$

(2) $$0 = \lambda\nu_\lambda(\bullet) - (\lambda+1)\nu_\lambda(\bullet\bullet) + \lambda\nu_\lambda(\bullet\times\bullet) - \lambda\nu_\lambda(\bullet\bullet\bullet).$$

(3) $$0 = \lambda\nu_\lambda(\bullet\bullet) - \nu_\lambda(\bullet\times\bullet) - \lambda\nu_\lambda(\bullet\bullet\bullet) + \lambda\nu_\lambda(\bullet\times\times\bullet) - \lambda\nu_\lambda(\bullet\bullet\times\bullet).$$

(4) $$0 = 2\lambda\nu_\lambda(\bullet\bullet) + 2\nu_\lambda(\bullet\times\bullet) - (4\lambda+3)\nu_\lambda(\bullet\bullet\bullet) + 2\lambda\nu_\lambda(\bullet\bullet\times\bullet) - 2\lambda\nu_\lambda(\bullet\bullet\bullet\bullet).$$

Exercise 1.6. Assume that $\nu_\lambda(\bullet\bullet) = \nu_\lambda(\bullet)^2$ in Corollary 1.2.4 (1), that is, events $\{\eta : \eta(x) = 1\}$ and $\{\eta : \eta(x+1) = 1\}$ are independent with respect to the upper invariant measure ν_λ. Under this assumption, show that if $\nu_\lambda(\bullet) > 0$ then

$$\nu_\lambda(\bullet) = \frac{2\lambda - 1}{2\lambda} = 1 - \frac{1}{2\lambda}.$$

This value is called *mean-field value.* Unfortunately this assumption is not valid. Note that this value can be given as the first bounds both $\rho_\lambda^{(H,1)}$ by the Harris-FKG inequality method and $\rho_\lambda^{(KK,1)}$ by the Katori-Konno method.

Exercise 1.7. Similarly in Exercise 1.6 above, we assume that $\nu_\lambda(\bullet\bullet) = \nu_\lambda(\bullet\times\bullet) = \nu_\lambda(\bullet)^2$ and $\nu_\lambda(\bullet\bullet\bullet) = \nu_\lambda(\bullet)^3$ in Corollary 1.2.4 (2). Show that if $\nu_\lambda(\bullet) > 0$ then

$$\nu_\lambda(\bullet) = \frac{-1+\sqrt{1+4\lambda^2}}{2\lambda} = \sqrt{1+\frac{1}{4\lambda^2}} - \frac{1}{2\lambda}.$$

Moreover, compare this result with that of Exercise 1.6.

Next we consider the correlation identities for $\overline{\rho}_\lambda(A)$. Let

$$H(\eta, A) = \prod_{x\in A}(1-\eta(x)) \quad \text{for any } A \in Y,$$

that is, $H(\eta, A) = 1$ if $\eta(x) = 0$ for any $x \in A$, $= 0$ otherwise. Remark that the product over empty set is 1. We compute Ω applied to $H(\eta, A)$ as a function of η. We begin by computing

$$\begin{aligned} &H(\eta^x, A) - H(\eta, A) \\ &= (1-\eta^x(x)) \prod_{u\in A\setminus\{x\}} (1-\eta^x(u)) - (1-\eta(x)) \prod_{u\in A\setminus\{x\}} (1-\eta(u)) \\ &= \Big[1-(1-\eta(x))\Big] \prod_{u\in A\setminus\{x\}} (1-\eta(u)) - (1-\eta(x)) \prod_{u\in A\setminus\{x\}} (1-\eta(u)) \\ &= (2\eta(x)-1)H(\eta, A\setminus\{x\}), \end{aligned}$$

whenever $x \in A$. By using this, we see that

$$
\begin{aligned}
\Omega H(\eta, A) &= \sum_{x \in \mathbf{Z}} \left[(1-\eta(x)) \times \lambda \sum_{y:|y-x|=1} \eta(y) + \eta(x) \right] [H(\eta^x, A) - H(\eta, A)] \\
&= \sum_{x \in A} (1-\eta(x))(2\eta(x)-1) \times \lambda \sum_{y:|y-x|=1} \eta(y) H(\eta, A \setminus \{x\}) \\
&\qquad + \sum_{x \in A} \eta(x)(2\eta(x)-1) H(\eta, A \setminus \{x\}) \\
&= -\sum_{x \in A} (1-\eta(x)) \times \lambda \sum_{y:|y-x|=1} [1-(1-\eta(y))] \prod_{u \in A \setminus \{x\}} (1-\eta(u)) \\
&\qquad + \sum_{x \in A} [1-(1-\eta(x))] \prod_{u \in A \setminus \{x\}} (1-\eta(u)) \\
&= \sum_{x \in A} \Big[\lambda \sum_{y:|y-x|=1} \big\{ H(\eta, A \cup \{y\}) - H(\eta, A) \big\} + H(\eta, A \setminus \{x\}) - H(\eta, A) \Big].
\end{aligned}
$$

Then we have

Theorem 1.2.5. *For any $A \in Y$,*

$$
\lambda \sum_{x \in A} \sum_{y:|y-x|=1} \Big[\overline{\rho}_\lambda(A \cup \{y\}) - \overline{\rho}_\lambda(A) \Big] + \sum_{x \in A} \Big[\overline{\rho}_\lambda(A \setminus \{x\}) - \overline{\rho}_\lambda(A) \Big] = 0,
$$

where $\overline{\rho}_\lambda(A) = \nu_\lambda\{\eta : \eta(x) = 0 \text{ for any } x \in A\}$.

This theorem shows that the basic contact process is coalescing self-dual. See Section 5 in Chapter III of Liggett[2] for details.

Let $\nu_\lambda(\circ) = \overline{\rho}_\lambda(\{0\})$, $\nu_\lambda(\circ\circ) = \overline{\rho}_\lambda(\{0,1\})$, $\nu_\lambda(\circ \times \circ) = \overline{\rho}_\lambda(\{0,2\})$, etc. From this theorem, we have

Corollary 1.2.6.

(1) $$1 - (2\lambda+1)\nu_\lambda(\circ) + 2\lambda\nu_\lambda(\circ\circ) = 0.$$

(2) $$\nu_\lambda(\circ) - (\lambda+1)\nu_\lambda(\circ\circ) + \lambda\nu_\lambda(\circ\circ\circ) = 0.$$

(3) $$2\nu_\lambda(\circ\circ) - (2\lambda+3)\nu_\lambda(\circ\circ\circ) + \nu_\lambda(\circ\times\circ) + 2\lambda\nu_\lambda(\circ\circ\circ\circ) = 0.$$

(4) $$\nu_\lambda(\circ) + \lambda\nu_\lambda(\circ\circ\circ) - (2\lambda+1)\nu_\lambda(\circ\times\circ) + \lambda\nu_\lambda(\circ\circ\times\circ) = 0.$$

(5) $$\nu_\lambda(\circ\circ\circ) - (\lambda+2)\nu_\lambda(\circ\circ\circ\circ) + \nu_\lambda(\circ\circ\times\circ) + \lambda\nu_\lambda(\circ\circ\circ\circ\circ) = 0.$$

(6) $$\begin{aligned} &\nu_\lambda(\circ\circ) + \nu_\lambda(\circ\times\circ) - (4\lambda+3)\nu_\lambda(\circ\circ\times\circ) + \nu_\lambda(\circ\times\times\circ) \\ &\qquad + 2\lambda\nu_\lambda(\circ\circ\circ\circ) + \lambda\nu_\lambda(\circ\circ\circ\times\circ) + \lambda\nu_\lambda(\circ\circ\times\circ\circ) = 0. \end{aligned}$$

(7) $$\nu_\lambda(\circ) - (2\lambda+1)\nu_\lambda(\circ\times\times\circ) + \lambda\nu_\lambda(\circ\circ\times\circ) + \lambda\nu_\lambda(\circ\circ\times\times\circ) = 0.$$

For example, applying Theorem 1.2.5 to $A = \{0\}$, we get

$$\lambda \sum_{y:|y|=1} \Big[\overline{\rho}_\lambda(\{0\} \cup \{y\}) - \overline{\rho}_\lambda(\{0\})\Big] + \Big[\overline{\rho}_\lambda(\phi) - \overline{\rho}_\lambda(\{0\})\Big] = 0.$$

The translation invariance of $\overline{\rho}_\lambda(A)$ and $\overline{\rho}_\lambda(\phi) = 1$ imply

$$2\lambda\Big[\overline{\rho}_\lambda(\{0,1\}) - \overline{\rho}_\lambda(\{0\})\Big] + \Big[1 - \overline{\rho}_\lambda(\{0\})\Big] = 0.$$

This equation is equivalent to part (1) of Corollary 1.2.6.

Exercise 1.8. Applying Theorem 1.2.5 to $A = \{0,1\}$, show that Corollary 1.2.6 (2).

Exercise 1.9. Assume that $\nu_\lambda(\circ\circ) = \nu_\lambda(\circ)^2$ in Corollary 1.2.6 (1), that is, events $\{\eta : \eta(x) = 0\}$ and $\{\eta : \eta(x+1) = 0\}$ are independent with respect to the upper invariant measure ν_λ. Under this assumption, show that if $\nu_\lambda(\circ) < 1$ then

$$\nu_\lambda(\circ) = \frac{1}{2\lambda},$$

and deduce that

$$\nu_\lambda(\bullet) = \frac{2\lambda - 1}{2\lambda} = 1 - \frac{1}{2\lambda}.$$

Remark that this conclusion is the same as that of Exercise 1.6.

Exercise 1.10. Let

$$\overline{\rho}_{\lambda,t} = E_{\delta_1}(1 - \eta_t(0)).$$

Show that

$$\frac{d}{dt}\overline{\rho}_{\lambda,t}\Big|_{t=0} = 1,$$

$$\frac{d^2}{dt^2}\overline{\rho}_{\lambda,t}\Big|_{t=0} = -(2\lambda+1),$$

$$\frac{d^3}{dt^3}\overline{\rho}_{\lambda,t}\Big|_{t=0} = 4\lambda^2 + 8\lambda + 1,$$

In general, the following result was proved by Belitsky[13]: for $n \geq 1$,

$$\mathrm{sgn}\left[\frac{d^n}{dt^n}\overline{\rho}_{\lambda,t}\Big|_{t=0}\right] = (-1)^{n+1}.$$

1.3. Harris-FKG Inequality Method

This section is devoted to lower bounds on the critical value and upper bounds on the order parameter for the one-dimensional basic contact process given by the *Harris-FKG inequality.* As for the Harris-FKG inequality, for example, see pages 70-83 in Liggett.[2] Results in this section appeared in Konno and Katori.[14] Let M^+ and M^- be the collection of all increasing and decreasing functions on $\{0,1\}^{\mathbf{Z}}$, respectively.

Theorem 1.3.1. *(Harris-FKG inequality) For any $f, g \in M^+$,*

$$E_{\nu_\lambda}(fg) \geq E_{\nu_\lambda}(f)E_{\nu_\lambda}(g).$$

By the definitions of increasing and decreasing functions, we have the following corollary immediately.

Corollary 1.3.2. *(Harris-FKG inequality)*

(1) $$E_{\nu_\lambda}(fg) \geq E_{\nu_\lambda}(f)E_{\nu_\lambda}(g) \quad \text{for any } f, g \in M^-.$$

(2) $$E_{\nu_\lambda}(fg) \leq E_{\nu_\lambda}(f)E_{\nu_\lambda}(g) \quad \text{for any } f \in M^+, g \in M^-.$$

(3) $$E_{\nu_\lambda}(fg) \leq E_{\nu_\lambda}(f)E_{\nu_\lambda}(g) \quad \text{for any } f \in M^-, g \in M^+.$$

Applying the Harris-FKG inequality to $\rho_\lambda(A)$ and $\overline{\rho}_\lambda(A)$, we obtain

Corollary 1.3.3. *(Harris-FKG inequality) For any $A, B \in Y$,*

(1) $$\rho_\lambda(A \cup B) \geq \rho_\lambda(A)\rho_\lambda(B)$$

(2) $$\overline{\rho}_\lambda(A \cup B) \geq \overline{\rho}_\lambda(A)\overline{\rho}_\lambda(B),$$

where $\rho_\lambda(A) = \nu_\lambda\{\eta : \eta(x) = 1 \text{ for any } x \in A\}$ and $\overline{\rho}_\lambda(A) = \nu_\lambda\{\eta : \eta(x) = 0 \text{ for any } x \in A\}$.

Proof. For part (1), define

$$f(\eta) = \prod_{x \in A} \eta(x), \quad g(\eta) = \prod_{x \in B} \eta(x).$$

Since $f, g \in M^+$, the desired result follows from Theorem 1.3.1. For part (2), similarly let

$$f(\eta) = 1 - \prod_{x \in A}(1 - \eta(x)), \quad g(\eta) = 1 - \prod_{x \in B}(1 - \eta(x)).$$

Then $f, g \in M^+$, so the proof is complete.

From Corollary 1.3.3 (1), we have the following result.

Corollary 1.3.4.

(1) $$\nu_\lambda(\bullet\bullet) \geq \nu_\lambda(\bullet)^2.$$

(2) $$\nu_\lambda(\bullet\bullet\bullet) \geq \nu_\lambda(\bullet)\nu_\lambda(\bullet\bullet).$$

(3) $$\nu_\lambda(\bullet\bullet\bullet\bullet) \geq \nu_\lambda(\bullet)\nu_\lambda(\bullet\bullet\bullet).$$

(4) $$\nu_\lambda(\bullet\bullet\times\bullet) \geq \nu_\lambda(\bullet)\nu_\lambda(\bullet\bullet).$$

In the same way, Corollary 1.3.3 (2) gives

Corollary 1.3.5.

(1) $$\nu_\lambda(\circ\circ) \geq \nu_\lambda(\circ)^2.$$

(2) $$\nu_\lambda(\circ\circ\circ) \geq \nu_\lambda(\circ)\nu_\lambda(\circ\circ).$$

(3) $$\nu_\lambda(\circ\circ\circ\circ) \geq \nu_\lambda(\circ)\nu_\lambda(\circ\circ\circ).$$

(4) $$\nu_\lambda(\circ\circ\times\circ) \geq \nu_\lambda(\circ)\nu_\lambda(\circ\circ).$$

(5) $$\nu_\lambda(\circ\circ\circ\circ\circ) \geq \nu_\lambda(\circ)\nu_\lambda(\circ\circ\circ\circ).$$

(6) $$\nu_\lambda(\circ\times\circ\circ\circ) \geq \nu_\lambda(\circ)\nu_\lambda(\circ\circ\circ).$$

(7) $$\nu_\lambda(\circ\circ\times\circ\circ) \geq \nu_\lambda(\circ)\nu_\lambda(\circ\circ\times\circ).$$

(8) $$\nu_\lambda(\circ\circ\times\times\circ) \geq \nu_\lambda(\circ)\nu_\lambda(\circ\circ).$$

We should remark that the definitions of ρ_λ, $\nu_\lambda(\bullet)$ and $\nu_\lambda(\circ)$ give

Lemma 1.3.6.

$$\rho_\lambda = \nu_\lambda(\bullet) = 1 - \nu_\lambda(\circ).$$

1.3.1. *First bound by the Harris-FKG inequality method*

From Corollaries 1.2.6 (1) and 1.3.5 (1), we have

$$1 - (2\lambda + 1)\nu_\lambda(\circ) + 2\lambda\nu_\lambda(\circ)^2 \leq 0.$$

By Lemma 1.3.6, the last inequality can be rewritten as

$$\rho_\lambda\left(\rho_\lambda - \frac{2\lambda - 1}{2\lambda}\right) \leq 0.$$

Combining the last inequality and $\rho_\lambda \geq 0$ gives

$$\rho_\lambda \leq \frac{2\lambda - 1}{2\lambda} \quad \text{for} \quad \frac{2\lambda - 1}{2\lambda} \geq 0,$$

and

$$\rho_\lambda = 0 \quad \text{for} \quad \frac{2\lambda - 1}{2\lambda} \leq 0.$$

So we obtain the following result.

Theorem 1.3.7. *Let* $\lambda_c^{(H,1)} = 1/2$. *Define*

$$\rho_\lambda^{(H,1)} = \begin{cases} (2\lambda - 1)/2\lambda & \text{for } \lambda > \lambda_c^{(H,1)}, \\ 0 & \text{for } \lambda \le \lambda_c^{(H,1)}. \end{cases}$$

Then we have

$$\text{(1)} \qquad \lambda_c^{(H,1)} \le \lambda_c,$$

and

$$\text{(2)} \qquad \rho_\lambda \le \rho_\lambda^{(H,1)} \quad \text{for} \quad \lambda \ge 0.$$

In this way we get a lower bound $\lambda_c^{(H,1)}$ on critical value λ_c and an upper bound $\rho_\lambda^{(H,1)}$ on order parameter ρ_λ simultaneously. Note that Theorem 1.3.7 can be obtained by a different way. That is, Corollaries 1.2.4 (1) and 1.3.4 (1) imply

$$(2\lambda - 1)\nu_\lambda(\bullet) - 2\lambda\nu_\lambda(\bullet)^2 \ge 0,$$

so we have the same conclusion. But, in this case, Corollary 1.2.6 is more suitable than Corollary 1.2.4 as for using the Harris-FKG inequality, since the number of negative terms in Corollary 1.2.6 is always one ! So we use Corollary 1.2.6 to improve bounds instead of Corollary 1.2.4.

1.3.2. *Second bound by the Harris-FKG inequality method*

As the first bound, Corollaries 1.2.6 (1) (2) and 1.3.5 (2) imply

$$\rho_\lambda[\lambda(2\lambda + 1)\rho_\lambda - (2\lambda^2 - 1)] \le 0.$$

Using similar arguments, we obtain

Theorem 1.3.8. *Let* $\lambda_c^{(H,2)} = 1/\sqrt{2} \approx 0.707$. *Define*

$$\rho_\lambda^{(H,2)} = \begin{cases} (2\lambda^2 - 1)/\lambda(2\lambda + 1) & \text{for } \lambda > \lambda_c^{(H,2)}, \\ 0 & \text{for } \lambda \le \lambda_c^{(H,2)}. \end{cases}$$

Then we have

$$\text{(1)} \qquad \lambda_c^{(H,1)} < \lambda_c^{(H,2)} \le \lambda_c,$$

and

$$\text{(2)} \qquad \rho_\lambda \le \rho_\lambda^{(H,2)} \le \rho_\lambda^{(H,1)} \quad \text{for} \quad \lambda \ge 0.$$

1.3.3. *Third bound by the Harris-FKG inequality method*

As the first and second bounds, Corollaries 1.2.6 (1)-(4) and 1.3.5 (3) (4) imply

$$\rho_\lambda[\lambda(2\lambda + 1)(4\lambda + 3\lambda + 2)\rho_\lambda - (8\lambda^4 + 6\lambda^3 - 6\lambda - 3)] \le 0.$$

In a similar fashion as before, the next result is obtained.

Theorem 1.3.9. *Let*

$$\begin{aligned}\lambda_c^{(H,3)} &= \inf\{\lambda \geq \lambda_c^{(H,2)} : 8\lambda^4 + 6\lambda^3 - 6\lambda - 3 \geq 0\} \\ &\approx 0.859.\end{aligned}$$

Define

$$\rho_\lambda^{(H,3)} = \begin{cases} (8\lambda^4 + 6\lambda^3 - 6\lambda - 3)/\lambda(2\lambda+1)(4\lambda^2+3\lambda+2) & \text{for } \lambda > \lambda_c^{(H,3)}, \\ 0 & \text{for } \lambda \leq \lambda_c^{(H,3)}. \end{cases}$$

Then we have

(1) $$\lambda_c^{(H,2)} < \lambda_c^{(H,3)} \leq \lambda_c,$$

and

(2) $$\rho_\lambda \leq \rho_\lambda^{(H,3)} \leq \rho_\lambda^{(H,2)} \quad \text{for} \quad \lambda \geq 0.$$

Note that as in the case of Corollary 1.3.5 (4) we can get

$$\nu_\lambda(\circ \circ \times \circ) \geq \nu_\lambda(\circ)\nu_\lambda(\circ \times \circ). \tag{1.2}$$

From Theorem 1.9 (c) in Chapter VI of Liggett[2] and the definition of $\overline{\rho}_\lambda(A)$, we have

Theorem 1.3.10. *Suppose that, for each* $n \geq 1$, $x_1 < x_2 < \cdots < x_n$ *and* $y_1 < y_2 < \cdots < y_n$ *with* $x_{i+1} - x_i \geq y_{i+1} - y_i$ $(i = 1, 2, \ldots, n-1)$. *Then*

$$\overline{\rho}_\lambda(\{x_1, x_2, \ldots, x_n\}) \leq \overline{\rho}_\lambda(\{y_1, y_2, \ldots, y_n\}).$$

So we get

$$\overline{\rho}_\lambda(\{0,2\}) \leq \overline{\rho}_\lambda(\{0,1\}),$$

that is,

$$\nu_\lambda(\circ \times \circ) \leq \nu_\lambda(\circ \circ).$$

Therefore we observe that estimation by Corollary 1.3.5 (4) is better than that by Eq.(1.2). Similar arguments hold in the next bound.

1.3.4. *Fourth bound by the Harris-FKG inequality method*

As the previous bounds, Corollaries 1.2.6 and 1.3.5 (5)-(8) yield

$$\rho_\lambda[\gamma_1(\lambda)\rho_\lambda^2 + \gamma_2(\lambda)\rho_\lambda + \gamma_3(\lambda)] \leq 0,$$

where

$$\gamma_1(\lambda) = \lambda^2(2\lambda+1)(8\lambda^4+12\lambda^3+9\lambda^2+9\lambda+3),$$
$$\gamma_2(\lambda) = \lambda(32\lambda^6+120\lambda^5+186\lambda^4+182\lambda^3+148\lambda^2+75\lambda+15),$$
$$\gamma_3(\lambda) = -2(2\lambda+1)(24\lambda^6+36\lambda^5+20\lambda^4-2\lambda^3-28\lambda^2-30\lambda-9).$$

Since $\gamma_1(\lambda) > 0$ for $\lambda > 0$, the last inequality can be rewritten as

$$\rho_\lambda \left(\rho_\lambda - \rho_\lambda^{(+)}\right)\left(\rho_\lambda - \rho_\lambda^{(-)}\right) \le 0, \tag{1.3}$$

where

$$\rho_\lambda^{(\pm)} = \frac{-\gamma_2(\lambda) \pm \sqrt{\gamma_2(\lambda)^2 - 4\gamma_1(\lambda)\gamma_3(\lambda)}}{2\gamma_1(\lambda)}.$$

Note that $\gamma_2(\lambda)^2 - 4\gamma_1(\lambda)\gamma_3(\lambda) > 0$ for $\lambda > 0$. Then Eq.(1.3) gives

$$\rho_\lambda \le \rho_\lambda^{(+)} \quad \text{for} \quad \rho_\lambda^{(+)} \ge 0,$$

and

$$\rho_\lambda = 0 \quad \text{for} \quad \rho_\lambda^{(+)} \le 0.$$

So we have the following result.

Theorem 1.3.11. *Let*

$$\lambda_c^{(H,4)} = \inf\{\lambda \ge \lambda_c^{(H,3)} : \gamma_3(\lambda) \le 0\}$$
$$\approx 0.961.$$

Define

$$\rho_\lambda^{(H,4)} = \begin{cases} \rho_\lambda^{(+)} & \text{for } \lambda > \lambda_c^{(H,4)}, \\ 0 & \text{for } \lambda \le \lambda_c^{(H,4)}. \end{cases}$$

Then we have

(1) $$\lambda_c^{(H,3)} < \lambda_c^{(H,4)} \le \lambda_c,$$

and

(2) $$\rho_\lambda \le \rho_\lambda^{(H,4)} \le \rho_\lambda^{(H,3)} \quad \text{for} \quad \lambda \ge 0.$$

As far as the Harris-FKG inequality used for $A \subset \{0,1,2,3\}$ and $B = \{4\}$, a similar discussion in the end of subsection 1.3.3 shows the above bound is the best one.

1.3.5. *Summary*

In this section we obtain the some bounds by using the Harris-FKG inequality:

$$\lambda_c^{(H,1)} = 0.5 < \lambda_c^{(H,2)} \approx 0.707 < \lambda_c^{(H,3)} \approx 0.859 < \lambda_c^{(H,4)} \approx 0.961 \le \lambda_c,$$

and

$$\rho_\lambda \le \rho_\lambda^{(H,4)} \le \rho_\lambda^{(H,3)} \le \rho_\lambda^{(H,2)} \le \rho_\lambda^{(H,1)} \quad \text{for} \quad \lambda \ge 0.$$

We think this method is one of the simplest one. However, as you will see later, when compared with the Katori-Konno method, this is not so good.

1.4. Harris Lemma and Katori-Konno Method

In this section we study lower bounds on critical value and upper bounds on order parameter of the one-dimensional basic contact process by using the Harris lemma. This method was first studied by Katori and Konno[15] in 1991. So we call it the *Katori-Konno method* in these notes. The results in this section appeared in references 14-17. However we will give some new proofs here. For example, Proofs B in Theorems 1.4.4 and 1.4.6. And this method is more powerful than the former Harris-FKG inequality one in the case of the basic contact process.

1.4.1. *Harris lemma*

In this subsection we introduce the Harris lemma which gives lower (resp. upper) bounds on the critical value and upper (resp. lower) bounds on the order parameter. Recall that Y is the collection of all finite subsets in $\mathbf{Z}$. For any $A \in Y$, we let

$$\sigma_\lambda(A) = 1 - E_{\nu_\lambda}\left(\prod_{x\in A}(1-\eta(x))\right) = \nu_\lambda\{\eta : \eta(x) = 1 \text{ for some } x \in A\},$$

$$\overline{\sigma}_\lambda(A) = 1 - E_{\nu_\lambda}\left(\prod_{x\in A}\eta(x)\right) = \nu_\lambda\{\eta : \eta(x) = 0 \text{ for some } x \in A\}.$$

Note that

$$\sigma_\lambda(A) = 1 - \overline{\rho}_\lambda(A),$$
$$\overline{\sigma}_\lambda(A) = 1 - \rho_\lambda(A),$$
$$\rho_\lambda = \rho_\lambda(\{0\}) = 1 - \overline{\rho}_\lambda(\{0\}) = \sigma_\lambda(\{0\}) = 1 - \overline{\sigma}_\lambda(\{0\}),$$

where 0 is the origin. From the definition of $\sigma_\lambda(A)$ and Theorem 1.2.5, we have the same type of the correlation identities immediately:

Theorem 1.4.1. *For any $A \in Y$,*

$$\lambda \sum_{x\in A}\sum_{y:|y-x|=1}\Big[\sigma_\lambda(A\cup\{y\}) - \sigma_\lambda(A)\Big] + \sum_{x\in A}\Big[\sigma_\lambda(A\setminus\{x\}) - \sigma_\lambda(A)\Big] = 0.$$

Similarly the definition of $\overline{\sigma}_\lambda(A)$ and Theorem 1.2.3 give

Theorem 1.4.2. *For any $A \in Y$,*

$$0 = -\{|A| + 2\lambda b(A)\}\overline{\sigma}_\lambda(A) + \lambda \sum_{x\in A} \sum_{\substack{y\in\triangle A:\\ |y-x|=1}} \overline{\sigma}_\lambda((A \setminus \{x\}) \cup \{y\})$$
$$+ \lambda \sum_{x\in A} w_A(x)\overline{\sigma}_\lambda(A \setminus \{x\}) - \lambda \sum_{y\in\triangle A} w_A(y)\overline{\sigma}_\lambda(A \cup \{y\}).$$

For example, in order to obtain lower bounds on λ_c and upper bounds on ρ_λ, we have to look for suitable upper bounds $h_\lambda(A)$ on $\sigma_\lambda(A)$. Because $\rho_\lambda = \sigma_\lambda(\{0\}) \le h_\lambda(\{0\})$ implies that the critical value of $h_\lambda(\{0\})$ gives a lower bound on λ_c. To do this we use the following lemma by Harris[18] in 1976, so we call it the *Harris lemma.*

Let Y^* be the set of all $[0,1]$-valued measurable functions on Y. For any $h \in Y^*$, we let

$$\Omega^* h(A) = \lambda \sum_{x\in A} \sum_{y:|y-x|=1} \Big[h(A \cup \{y\}) - h(A)\Big] + \sum_{x\in A} \Big[h(A \setminus \{x\}) - h(A)\Big].$$

Note that Theorem 1.4.1 implies $\Omega^*\sigma_\lambda(A) = 0$ for any $A \in Y$.

Lemma 1.4.3. *(Harris lemma) Let $h_i \in Y^*$ $(i = 1, 2)$ with*

(1) $$h_i(\phi) = 0,$$

(2) $$0 < h_i(A) \le 1 \quad \text{for any } A \in Y \text{ with } A \ne \phi,$$

(3) $$\lim_{|A|\to\infty} h_i(A) = 1,$$

(4) $$\Omega^* h_1(A) \le 0 \le \Omega^* h_2(A) \quad \text{for any } A \in Y.$$

Then

(5) $$h_2(A) \le \sigma_\lambda(A) \le h_1(A) \quad \text{for any } A \in Y.$$

In particular,

(6) $$h_2(\{0\}) \le \rho_\lambda \le h_1(\{0\}),$$

where 0 is the origin.

Proof. We need the following 4 steps.

Step 1. For any $A \in Y$ and $N \geq 1$, we see that

$$\begin{aligned}\sigma_\lambda(A) &= \lim_{t\to\infty} P(\xi_t^A \neq \phi) \\ &= \lim_{t\to\infty} P(|\xi_t^A| > N) + \lim_{t\to\infty} P(0 < |\xi_t^A| \leq N).\end{aligned} \tag{1.4}$$

The result in pp.168-169 of Liggett[2] implies that for fixed $N \geq 1$,

$$\lim_{t\to\infty} P(0 < |\xi_t^A| \leq N) = 0. \tag{1.5}$$

Combining Eq.(1.4) with Eq.(1.5) gives

$$\sigma_\lambda(A) = \lim_{t\to\infty} P(|\xi_t^A| > N). \tag{1.6}$$

Step 2. From the Markov property and the condition (1),

$$\begin{aligned}E(h(\xi_{t+s}^A)) &= E\Big(E(h(\xi_s^{\xi_t^A}))\Big) \\ &= E\Big(E(h(\xi_s^{\xi_t^A})) : |\xi_t^A| > N\Big) + E\Big(E(h(\xi_s^{\xi_t^A})) : 0 < |\xi_t^A| \leq N\Big).\end{aligned} \tag{1.7}$$

By using condition (2) and Eq.(1.5), we see that for fixed $s > 0$,

$$\lim_{t\to\infty} E\Big(E(h(\xi_s^{\xi_t^A})) : 0 < |\xi_t^A| \leq N\Big) = 0.$$

Using this result and Eq.(1.7), we have

$$\liminf_{t\to\infty} E(h(\xi_{t+s}^A)) = \liminf_{t\to\infty} E\Big(E(h(\xi_s^{\xi_t^A})) : |\xi_t^A| > N\Big). \tag{1.8}$$

On the other hand, condition (3) implies that for any $\varepsilon > 0$ there exist $N \geq 1$ and $s > 0$ such that for any $A \in Y$ with $|A| > N$,

$$E(h(\xi_s^A)) \geq 1 - \varepsilon.$$

Therefore combination of Eqs.(1.6), (1.8) and the above inequality implies that for any $\varepsilon > 0$, there is an $N \geq 1$ such that

$$\begin{aligned}\liminf_{t\to\infty} E(h(\xi_{t+s}^A)) &\geq (1-\varepsilon) \liminf_{t\to\infty} P(|\xi_t^A| > N) \\ &= (1-\varepsilon)\sigma_\lambda(A).\end{aligned} \tag{1.9}$$

Step 3. By using Eq.(1.9), conditions (1) and (2), i.e., $h(\phi) = 0$, $h(A) \leq 1$ for any $A \in Y$ and the definition of $\sigma(A)$, we see that for any $\varepsilon > 0$,

$$\begin{aligned}(1-\varepsilon)\sigma_\lambda(A) &\leq \liminf_{t\to\infty} E(h(\xi_t^A)) \\ &= \liminf_{t\to\infty} E\Big(h(\xi_t^A) : \xi_t^A \neq \phi\Big) \\ &\leq \limsup_{t\to\infty} E\Big(h(\xi_t^A) : \xi_t^A \neq \phi\Big) \\ &\leq \lim_{t\to\infty} P(\xi_t^A \neq \phi) \\ &= \sigma_\lambda(A).\end{aligned}$$

Thus it follows that

$$\sigma_\lambda(A) = \lim_{t\to\infty} E(h(\xi_t^A)). \tag{1.10}$$

Step 4. It is enough to show for the condition (4):

$$\Omega^* h(A) \le 0 \quad \text{for any } A \in Y,$$

since a similar argument holds for

$$\Omega^* h(A) \ge 0 \quad \text{for any } A \in Y.$$

From this condition (4), we obtain

$$\frac{\partial}{\partial t} E(h(\xi_t^A)) = E(\Omega^* h(\xi_t^A)) \le 0.$$

So Eq.(1.10) and this inequality imply

$$\sigma_\lambda(A) = \lim_{t\to\infty} E(h(\xi_t^A)) \le E(h(\xi_0^A)) = h(A),$$

for any $A \in Y$. Thus the proof of the Harris lemma is complete.

Exercise 1.11. Let $h_1 \in Y^*$ with $h_1(\phi) = 0$ and $h_1(A) = 1$ for any non-empty set $A \in Y$. Show that

$$\Omega^* h_1(A) = -1 \qquad \text{if } |A| = 1,$$
$$\Omega^* h_1(A) = \ 0 \qquad \text{otherwise},$$

and deduce that this result gives the trivial upper bound on $\sigma_\lambda(A)$, that is, $\sigma_\lambda(A) \le 1$ for any $A \in Y$.

Exercise 1.12. Show that if $\Omega^* h(A) \le 0$ for any $A \in Y$ then

$$E(h(\xi_t^A)) \le E(h(\xi_s^A)) \qquad \text{for} \qquad 0 \le s \le t.$$

From now on we consider bounds on ρ_λ and λ_c. To get bounds by using the Harris lemma, we need the following 4 steps.

Step 1. First we choose a suitable form of $h_i(A)$.
Step 2. Next we decide $h_i(A)$ explicitly.
Step 3. Third we check conditions (1)–(3) in the Harris lemma.
Step 4. Finally we check condition (4) in the Harris lemma.

In this section we consider upper bounds on $\sigma_\lambda(A)$ by using the Katori-Konno method. On the other hand, the next chapter is devoted to lower bounds on $\sigma_\lambda(A)$ by the Holley-Liggett method.

1.4.2. *First bound by the Katori-Konno method*

Let $|A|$ be the cardinality of A. So we have

Theorem 1.4.4. *Let* $\lambda_c^{(KK,1)} = 1/2$. *Then for* $\lambda \geq \lambda_c^{(KK,1)}$,

$$\sigma_\lambda(A) \leq h_\lambda^{(KK,1)}(A) \quad \text{for all } A \in Y,$$

where

$$h_\lambda^{(KK,1)}(A) = 1 - \alpha_*^{|A|} \quad \text{and} \quad \alpha_* = \frac{1}{2\lambda}.$$

Proof.

Step 1. We let $h(A) = 1 - \alpha^{|A|}$.

Step 2. Next we decide $0 < \alpha_* < 1$ as the unique solution of

$$\Omega^* h(\{0\}) = 0,$$

that is,

$$-2\lambda\alpha^2 + (2\lambda + 1)\alpha - 1 = (2\lambda\alpha - 1)(1 - \alpha) = 0.$$

So we take $\alpha_* = 1/2\lambda$ and let $h(A) = 1 - \alpha_*^{|A|}$ for $\lambda > 1/2$.

Step 3. We check conditions (1)-(3) as follows. For $\lambda > 1/2$, we have $0 < \alpha_* = 1/2\lambda < 1$. So conditions (1) and (3) are trivial. Condition (2) is equivalent to $0 \leq \alpha_*^{|A|} < 1$ for any $A \in Y$ with $A \neq \phi$. This comes from $0 < \alpha_* < 1$.

Step 4. We will give two different proofs; Proof A and Proof B. They are the same in essentials. However Proof B is more refined than Proof A. Therefore we can see the structure of proof of Step 4 explicitly in Proof B.

Proof A. For $k = 0, 1, 2$ and $A \in Y$, let

$$A_k = \big\{x \in A : |\{y \in A : |y - x| = 1\}| = k\big\}.$$

The definitions give $A = A_0 + A_1 + A_2$. Therefore

$$\Omega^* h(A) = \sum_{k=0}^{2} R_k(A),$$

where

$$R_k(A) = \lambda \sum_{x \in A_k} \sum_{y:|y-x|=1} \Big[h(A \cup \{y\}) - h(A)\Big] + \sum_{x \in A_k} \Big[h(A \setminus \{x\}) - h(A)\Big].$$

If $R_k(A) \leq 0$ for any $A \in Y$ and $k = 0, 1, 2$, then $\Omega^* h(A) \leq 0$ for any $A \in Y$, that is, condition (4) is satisfied. From $h(A) = 1 - \alpha_*^{|A|}$, we have

$$\begin{aligned} R_k(A) &= \lambda \sum_{x \in A_k} \sum_{y:|y-x|=1} \Big[\alpha_*^{|A|} - \alpha_*^{|A \cup \{y\}|}\Big] + \sum_{x \in A_k} \Big[\alpha_*^{|A|} - \alpha_*^{|A|-1}\Big] \\ &= \lambda \sum_{x \in A_k} (2 - k)\Big[\alpha_*^{|A|} - \alpha_*^{|A|+1}\Big] + \sum_{x \in A_k} \Big[\alpha_*^{|A|} - \alpha_*^{|A|-1}\Big] \\ &= |A_k| \Big\{(2 - k)\lambda\Big[\alpha_*^{|A|} - \alpha_*^{|A|+1}\Big] + \Big[\alpha_*^{|A|} - \alpha_*^{|A|-1}\Big]\Big\} \\ &= |A_k| \alpha_*^{|A|-1} \Big\{(2 - k)\lambda[\alpha_* - \alpha_*^2] + [\alpha_* - 1]\Big\} \\ &= |A_k| \alpha_*^{|A|-1} (1 - \alpha_*) \Big\{(2 - k)\lambda\alpha_* - 1\Big\}. \end{aligned}$$

So, by using $\lambda\alpha_* = 1/2$, we obtain

$$R_k(A) = |A_k|\alpha_*^{|A|-1}(1-\alpha_*)(-k/2),$$

where $k = 0, 1, 2$. Then $0 \le \alpha_* \le 1$ gives $R_k(A) \le 0$ for any $A \in Y$ and $k = 0, 1, 2$. Therefore the proof for condition (4) is complete.
Proof B. First we let

$$\begin{aligned}
A_{\circ\bullet\circ} &= \{x \in A : x-1, x+1 \notin A\},\\
A_{\circ\bullet\bullet} &= \{x \in A : x-1 \notin A, x+1 \in A\},\\
A_{\bullet\bullet\circ} &= \{x \in A : x-1 \in A, x+1 \notin A\},\\
A_{\bullet\bullet\bullet} &= \{x \in A : x-1, x+1 \in A\}.
\end{aligned}$$

Remark that these definitions give

$$A = A_{\circ\bullet\circ} + A_{\circ\bullet\bullet} + A_{\bullet\bullet\circ} + A_{\bullet\bullet\bullet},$$

and

$$A_0 = A_{\circ\bullet\circ}, \quad A_1 = A_{\circ\bullet\bullet} + A_{\bullet\bullet\circ}, \quad A_2 = A_{\bullet\bullet\bullet}.$$

As in the case of Proof A, by $h(A) = 1 - \alpha_*^{|A|}$, we have

$$\begin{aligned}
\Omega^* h(A) =& \alpha_*^{|A|-1}\Big[|A_0|\Omega^* h(\{0\}) + |A_1|\Big\{\Omega^* h(\{0\}) + \lambda\alpha_*(\alpha_*-1)\Big\} + |A_2|(\alpha_*-1)\Big],\\
=& \alpha_*^{|A|-1}\Big[|A_{\circ\bullet\circ}|\Omega^* h(\{0\})\\
&\qquad + \{|A_{\circ\bullet\bullet}| + |A_{\bullet\bullet\circ}|\}\Big\{\Omega^* h(\{0\}) + \lambda\alpha_*(\alpha_*-1)\Big\} + |A_{\bullet\bullet\bullet}|(\alpha_*-1)\Big],
\end{aligned}$$

for any $A \in Y$. Then $\Omega^* h(\{0\}) = 0$ gives

$$\Omega^* h(A) = \alpha_*^{|A|-1}\Big[\{|A_{\circ\bullet\bullet}| + |A_{\bullet\bullet\circ}|\}\lambda\alpha_*(\alpha_*-1) + |A_{\bullet\bullet\bullet}|(\alpha_*-1)\Big].$$

So $\alpha_* \in (0,1)$ gives $\Omega^* h(A) \le 0$ for any $A \in Y$. Therefore we obtain the desired conclusion.

Applying Theorem 1.4.4 to $A = \{0\}$ gives

Corollary 1.4.5.

$$\begin{aligned}
&\lambda_c \ge \lambda_c^{(KK,1)} = \frac{1}{2},\\
&\rho_\lambda \le \rho_\lambda^{(KK,1)} = 1 - \left(\frac{1}{2\lambda}\right) = \frac{2\lambda-1}{2\lambda} \quad \textit{for} \quad \lambda \ge \frac{1}{2}.
\end{aligned}$$

This corollary is equivalent to Theorem 1.3.7 which is the first bound by the Harris-FKG inequality method.

Exercise 1.13. If we decide $\alpha^* \in (0,1)$ as the unique solution of $\Omega^* h(\{0,1\}) = 0$ instead of $\Omega^* h(\{0\}) = 0$, then we take $\alpha^* = 1/\lambda$. Show that in this choice, the condition (4) in the Harris lemma does not hold.

1.4.3. *Second bound by the Katori-Konno method*

Let $b(A)$ be the number of neighboring pairs of points in A, that is, $b(A) = \left|\left\{x \in \mathbf{Z} : \{x, x+1\} \subset A\right\}\right|$. Then we have

Theorem 1.4.6. *Let* $\lambda_c^{(KK,2)} = 1$. *Then for* $\lambda \geq \lambda_c^{(KK,2)}$,

$$\sigma_\lambda(A) \leq h_\lambda^{(KK,2)}(A) \quad \textit{for all } A \in Y,$$

where

$$h_\lambda^{(KK,2)}(A) = 1 - \alpha_*^{|A|} \beta_*^{b(A)},$$

and

$$\alpha_* = \frac{1}{2\lambda - 1}, \qquad \beta_* = \frac{2\lambda - 1}{\lambda}.$$

Proof.

Step 1. We let $h(A) = 1 - \alpha^{|A|} \beta^{b(A)}$.

Step 2. Next we decide $0 < \alpha_* < 1$ and $\beta_* > 1$ as the unique solutions of

$$\Omega^* h(\{0\}) = 0,$$
$$\Omega^* h(\{0,1\}) = 0,$$

that is,

$$-2\lambda\alpha^2\beta + (2\lambda + 1)\alpha - 1 = 0,$$
$$2\left[-\lambda\alpha^3\beta^2 + (\lambda + 1)\alpha^2\beta - \alpha\right] = 0.$$

Let $w = \alpha\beta$. Then the above equations can be written as

$$(2\lambda w - 1)(w - 1) = 1 - \beta,$$
$$(\lambda w - 1)(w - 1) = 0.$$

Let $w_* = 1/\lambda$ and $\beta_* = (2\lambda - 1)/\lambda$ for $\lambda > 1$. So w_* and β_* satisfy the above equations with $0 < w_* < 1$ and $\beta_* > 1$. By the definition of w, we let $\alpha_* = w_*/\beta_*$. This gives $0 < \alpha_* = 1/(2\lambda - 1) < 1$ for $\lambda > 1$. Remark that a simple computation gives $\alpha_* \beta_*^2 < 1$ for $\lambda > 1$. Therefore we decide

$$h(A) = 1 - \alpha_*^{|A|} \beta_*^{b(A)}$$

with

$$\alpha_* = \frac{1}{2\lambda - 1} \qquad \text{and} \qquad \beta_* = \frac{2\lambda - 1}{\lambda}.$$

Step 3. We check conditions (1)-(3) as follows. Assume that $\lambda > 1$. Condition (1) and $h(A) \le 1$ in condition (2) are trivial. The positivity of $h(A)$ for non-empty set $A \in Y$ is equivalent to $\alpha_*^{|A|}\beta_*^{b(A)} < 1$. On the other hand, we have $b(A) \le 2|A|$ for any $A \in Y$. By $\beta_* > 1$, $\alpha_*\beta_*^2 < 1$ and $b(A) \le 2|A|$, we get

$$\alpha_*^{|A|}\beta_*^{b(A)} \le (\alpha_*\beta_*^2)^{|A|} < 1,$$

for non-empty $A \in Y$. Similarly, concerning condition (3), we obtain

$$h(A) = 1 - \alpha_*^{|A|}\beta_*^{b(A)} \ge 1 - (\alpha_*\beta_*^2)^{|A|}.$$

Then $h(A)$ goes to 1 as $|A|$ goes to infinity, since $\alpha_*\beta_*^2 < 1$ and $h(A) \le 1$.

Step 4. We will also give two proofs; Proof A and Proof B.

Proof A. As in the Proof A of the first bound, for $k = 0,\ 1,\ 2$ and $A \in Y$, we let

$$A_k = \big\{x \in A : |\{y \in A : |y - x| = 1\}| = k\big\}.$$

Remark that $A = A_0 + A_1 + A_2$. Therefore

$$\Omega^* h(A) = \sum_{k=0}^{2} R_k(A),$$

where

$$R_k(A) = \lambda \sum_{x \in A_k} \sum_{y:|y-x|=1} \Big[h(A \cup \{y\}) - h(A)\Big] + \sum_{x \in A_k} \Big[h(A \setminus \{x\}) - h(A)\Big].$$

To prove $\Omega^* h(A) \le 0$ for any $A \in Y$, it suffices to show that $R_k(A) \le 0$ for any $A \in Y$ and $k = 0,\ 1,\ 2$. Recall that $h(A) = 1 - \alpha_*^{|A|}\beta_*^{b(A)}$.

(i) $R_0(A)$: We see that

$$\begin{aligned} R_0(A) &= \lambda \sum_{x \in A_0} \sum_{y:|y-x|=1} \Big[\alpha_*^{|A|}\beta_*^{b(A)} - \alpha_*^{|A|+1}\beta_*^{b(A\cup\{y\})}\Big] \\ &\quad + \sum_{x \in A_0} \Big[\alpha_*^{|A|}\beta_*^{b(A)} - \alpha_*^{|A|-1}\beta_*^{b(A)}\Big] \\ &\le \lambda \sum_{x \in A_0} 2\Big[\alpha_*^{|A|}\beta_*^{b(A)} - \alpha_*^{|A|+1}\beta_*^{b(A)+1}\Big] + \sum_{x \in A_0} \Big[\alpha_*^{|A|}\beta_*^{b(A)} - \alpha_*^{|A|-1}\beta_*^{b(A)}\Big] \\ &= |A_0|\alpha_*^{|A|-1}\beta_*^{b(A)}\Big[2\lambda(\alpha_* - \alpha_*^2\beta_*) + (\alpha_* - 1)\Big] \\ &= |A_0|\alpha_*^{|A|-1}\beta_*^{b(A)}\Omega^* h(\{0\}). \end{aligned}$$

The first equality comes from $|A \cup \{y\}| = |A| + 1$, $|A \setminus \{x\}| = |A| - 1$ and $b(A \setminus \{x\}) = b(A)$ for $x \in A_0$ and $y \in \mathbf{Z}$ with $|y - x| = 1$. The second inequality is given by $\beta_* > 1$ and $b(A \cup \{y\}) \ge b(A) + 1$ in the same condition for x and y. By $\Omega^* h(\{0\}) = 0$, we have that $R_0(A) \le 0$ for any $A \in Y$.

(ii) $R_1(A)$: Similarly we have

$$\begin{aligned}R_1(A) &\le \lambda \sum_{x\in A_1}\Big[\alpha_*^{|A|}\beta_*^{b(A)} - \alpha_*^{|A|+1}\beta_*^{b(A)+1}\Big] + \sum_{x\in A_1}\Big[\alpha_*^{|A|}\beta_*^{b(A)} - \alpha_*^{|A|-1}\beta_*^{b(A)-1}\Big]\\ &= |A_1|\alpha_*^{|A|-2}\beta_*^{b(A)-1}\Big[\lambda(\alpha_*^2\beta_* - \alpha_*^3\beta_*^2) + (\alpha_*^2\beta_* - \alpha_*)\Big]\\ &= |A_1|\alpha_*^{|A|-2}\beta_*^{b(A)-1}\Omega^* h(\{0,1\}).\end{aligned}$$

Therefore $\Omega^* h(\{0,1\}) = 0$ gives $R_1(A) \le 0$ for any $A \in Y$.

(iii) $R_2(A)$: As in the previous cases, we see that

$$\begin{aligned}R_2(A) &= \lambda \sum_{x\in A_2}\sum_{y:|y-x|=1}\Big[\alpha_*^{|A|}\beta_*^{b(A)} - \alpha_*^{|A|}\beta_*^{b(A)}\Big] + \sum_{x\in A_2}\Big[\alpha_*^{|A|}\beta_*^{b(A)} - \alpha_*^{|A|-1}\beta_*^{b(A)-2}\Big]\\ &= |A_2|\alpha_*^{|A|-1}\beta_*^{b(A)-2}\Big(\alpha_*\beta_*^2 - 1\Big).\end{aligned}$$

By using $\alpha_*\beta_*^2 < 1$, we obtain that $R_2(A) \le 0$ for any $A \in Y$.

Therefore we can check condition (4).

Remark. The above Proof A implies that

$$\begin{aligned}\Omega^* h(A) \le & |A_0|\alpha_*^{|A|-1}\beta_*^{b(A)}\Omega^* h(\{0\})\\ &+ |A_1|\alpha_*^{|A|-2}\beta_*^{b(A)-1}\Omega^* h(\{0,1\})\\ &+ |A_2|\alpha_*^{|A|-1}\beta_*^{b(A)-2}\Big(\alpha_*\beta_*^2 - 1\Big).\end{aligned}$$

Proof B. As in the Proof B of the first bound, we divide A into the following 9 disjoint subsets:

$$\begin{aligned}A_{\circ\circ\bullet\circ\circ} &= \{x \in A : x-2, x-1, x+1, x+2 \notin A\},\\ A_{\circ\circ\bullet\circ\bullet} &= \{x \in A : x-2, x-1, x+1 \notin A, x+2 \in A\},\\ A_{\bullet\circ\bullet\circ\circ} &= \{x \in A : x-2 \in A, x-1, x+1, x+2 \notin A\},\\ A_{\bullet\circ\bullet\circ\bullet} &= \{x \in A : x-2 \in A, x-1, x+1 \notin A, x+2 \in A\},\\ A_{\circ\circ\bullet\bullet\times} &= \{x \in A : x-2, x-1 \notin A, x+1 \in A\},\\ A_{\times\bullet\bullet\circ\circ} &= \{x \in A : x-1 \in A, x+1, x+2 \notin A\},\\ A_{\bullet\circ\bullet\bullet\times} &= \{x \in A : x-2 \in A, x-1 \notin A, x+1 \in A\},\\ A_{\times\bullet\bullet\circ\bullet} &= \{x \in A : x-1 \in A, x+1 \notin A, x+2 \in A\},\\ A_{\times\bullet\bullet\bullet\times} &= \{x \in A : x-1 \in A, x+1 \in A\}.\end{aligned}$$

A direct computation gives

$$
\begin{aligned}
\Omega^* h(A) =& |A_{\circ\circ\bullet\circ\circ}|\alpha_*^{|A|-1}\beta_*^{b(A)}\Omega^* h(\{0\}) \\
&+ \Big\{|A_{\circ\circ\bullet\circ\bullet}| + |A_{\bullet\circ\bullet\circ\circ}|\Big\}\alpha_*^{|A|-1}\beta_*^{b(A)}\Big[\Omega^* h(\{0\}) + \lambda\alpha_*^2\beta_*(1-\beta_*)\Big] \\
&+ |A_{\bullet\circ\bullet\circ\bullet}|\alpha_*^{|A|-1}\beta_*^{b(A)}\Big[\Omega^* h(\{0\}) + 2\lambda\alpha_*^2\beta_*(1-\beta_*)\Big] \\
&+ \Big\{|A_{\circ\circ\bullet\bullet\times}| + |A_{\times\bullet\bullet\circ\circ}|\Big\}\alpha_*^{|A|-2}\beta_*^{b(A)-1}\frac{\Omega^* h(\{0,1\})}{2} \\
&+ \Big\{|A_{\bullet\circ\bullet\bullet\times}| + |A_{\times\bullet\bullet\circ\bullet}|\Big\}\alpha_*^{|A|-2}\beta_*^{b(A)-1}\Big[\frac{\Omega^* h(\{0,1\})}{2} + \lambda\alpha_*^3\beta_*^2(1-\beta_*)\Big] \\
&+ |A_{\times\bullet\bullet\bullet\times}|\alpha_*^{|A|-1}\beta_*^{b(A)-2}\Big(\alpha_*\beta_*^2 - 1\Big).
\end{aligned}
$$

Therefore $\Omega^* h(\{0\}) = \Omega^* h(\{0,1\}) = 0$ implies that

$$
\begin{aligned}
\Omega^* h(A) &= \Big\{|A_{\circ\circ\bullet\circ\bullet}| + |A_{\bullet\circ\bullet\circ\circ}|\Big\}\alpha_*^{|A|+1}\beta_*^{b(A)+1}\lambda(1-\beta_*) \\
&\quad + |A_{\bullet\circ\bullet\circ\bullet}|\alpha_*^{|A|+1}\beta_*^{b(A)+1}2\lambda(1-\beta_*) \\
&\quad + \Big\{|A_{\bullet\circ\bullet\bullet\times}| + |A_{\times\bullet\bullet\circ\bullet}|\Big\}\alpha_*^{|A|+1}\beta_*^{b(A)+1}\lambda(1-\beta_*) \\
&\quad + |A_{\times\bullet\bullet\bullet\times}|\alpha_*^{|A|-1}\beta_*^{b(A)-2}\Big(\alpha_*\beta_*^2 - 1\Big) \\
&= \Big\{|A_{\circ\circ\bullet\circ\bullet}| + |A_{\bullet\circ\bullet\circ\circ}| + 2|A_{\bullet\circ\bullet\circ\bullet}| + |A_{\bullet\circ\bullet\bullet\times}| + |A_{\times\bullet\bullet\circ\bullet}|\Big\} \\
&\qquad \times \alpha_*^{|A|+1}\beta_*^{b(A)+1}\lambda(1-\beta_*) \\
&\quad + |A_{\times\bullet\bullet\bullet\times}|\alpha_*^{|A|-1}\beta_*^{b(A)-2}\Big(\alpha_*\beta_*^2 - 1\Big).
\end{aligned}
$$

By $\beta_* > 1$ and $\alpha_*\beta_*^2 < 1$, we have $\Omega^* h(A) \le 0$ for any $A \in Y$, that is, condition (4) is satisfied.

Applying Theorem 1.4.6 to $A = \{0\}$ gives

Corollary 1.4.7.

$$
\begin{aligned}
&\lambda_c \ge \lambda_c^{(KK,2)} = 1, \\
&\rho_\lambda \le \rho_\lambda^{(KK,2)} = 1 - \left(\frac{1}{2\lambda - 1}\right) = \frac{2(\lambda-1)}{2\lambda-1} \quad \textit{for} \quad \lambda \ge 1.
\end{aligned}
$$

Compared with the second bound by the Harris-FKG inequality method (see Theorem 1.3.8), this result is better than that one.

1.4.4. *Third bound by the Katori-Konno method*

Let

$$
\begin{aligned}
c(A) &= \Big|\big\{x \in \mathbf{Z} : \{x, x+1, x+2\} \subset A\big\}\Big|, \\
d(A) &= \Big|\big\{x \in \mathbf{Z} : \{x, x+2\} \subset A\big\}\Big|.
\end{aligned}
$$

Then we have

Theorem 1.4.8. *Let* $\lambda_c^{(KK,3)} = \big(1+\sqrt{37}\big)/6$. *Then for* $\lambda \geq \lambda_c^{(KK,3)}$,

$$\sigma_\lambda(A) \leq h_\lambda^{(KK,3)}(A) \quad \textit{for all } A \in Y,$$

where

$$\begin{aligned}
h_\lambda^{(KK,3)}(A) &= 1 - \alpha_*^{|A|}\beta_*^{b(A)}\gamma_*^{c(A)}\delta_*^{d(A)}, \\
\alpha_* &= \frac{\lambda(2\lambda+3)+\sqrt{D}}{(2\lambda+1)(6\lambda^2-3\lambda-1)}, \\
\beta_* &= \frac{-(24\lambda^4+16\lambda^3+8\lambda^2+\lambda+1)+(6\lambda-1)(\lambda+1)\sqrt{D}}{2\lambda(\lambda-1)^2}, \\
\gamma_* &= \frac{24\lambda^3+16\lambda^2-2\lambda-3)+(4\lambda+3)\sqrt{D}}{8\lambda(2\lambda+1)^2}, \\
\delta_* &= \frac{(\lambda+1)\{12\lambda^3-2\lambda^2-\lambda+1-(3\lambda-1)\sqrt{D}\}}{2\lambda^2(\lambda-1)}, \\
D &= 16\lambda^4+4\lambda^2+4\lambda+1.
\end{aligned}$$

Compared with previous proofs, this proof is more complicated, so we will omit the proof. See Katori and Konno[16] for details. We should remark that α_*, β_*, γ_* and δ_* are unique solutions of

$$\Omega^* h(\{0\}) = \Omega^* h(\{0,1\}) = \Omega^* h(\{0,1,2\}) = \Omega^* h(\{0,2\}) = 0,$$

with $0 < \alpha_* < 1$ and β_*, γ_*, $\delta_* > 1$.

Furthermore, applying Theorem 1.4.8 to $A = \{0\}$ gives

Corollary 1.4.9.

$$\begin{aligned}
\lambda_c &\geq \lambda_c^{(KK,3)} = \frac{1+\sqrt{37}}{6} \approx 1.180, \\
\rho_\lambda &\leq \rho_\lambda^{(KK,3)} = \frac{4\lambda(3\lambda^2-\lambda-3)}{12\lambda^3-2\lambda^2-8\lambda-1+\sqrt{D}} \quad \textit{for} \quad \lambda \geq \frac{1+\sqrt{37}}{6}.
\end{aligned}$$

3.4.5. *Summary and discussions*

In this section we obtained the bounds by using the Harris lemma:

$$\lambda_c^{(KK,1)} = 0.5 < \lambda_c^{(KK,2)} = 1 < \lambda_c^{(KK,3)} \approx 1.180 \leq \lambda_c,$$

and

$$\rho_\lambda \leq \rho_\lambda^{(KK,3)} \leq \rho_\lambda^{(KK,2)} \leq \rho_\lambda^{(KK,1)} \quad \text{for} \quad \lambda \geq 0.$$

We should remark that Corollary 1.4.7 (resp. 1.4.9) can be also obtained by using Corollary 1.2.6 (1), (2) (resp. (1)-(4)) and the following conjecture (1) (resp. (2), (3)). This argument appeared in Chapter 3 of Konno.[12]

Conjecture 1.4.10.

(1) $$\nu_\lambda(\circ)\nu_\lambda(\circ\circ\circ) \geq \nu_\lambda(\circ\circ)^2.$$

(2) $$\nu_\lambda(\circ\circ)\nu_\lambda(\circ\circ\circ\circ) \geq \nu_\lambda(\circ\circ\circ)^2.$$

(3) $$\nu_\lambda(\circ)\nu_\lambda(\circ\circ\times\circ) \geq \nu_\lambda(\circ\circ)\nu_\lambda(\circ\times\circ).$$

That is, we obtain

Theorem 1.4.11. *Assume that*

$$\nu_\lambda(\circ)\nu_\lambda(\circ\circ\circ) \geq \nu_\lambda(\circ\circ)^2.$$

Then we have

$$\begin{aligned}
\lambda_c &\geq \lambda_c^{(KK,2)} = 1,\\
\rho_\lambda &\leq \rho_\lambda^{(KK,2)} = 1 - \left(\frac{1}{2\lambda - 1}\right) = \frac{2(\lambda-1)}{2\lambda-1} \quad \textit{for} \quad \lambda \geq 1.
\end{aligned}$$

Furthermore

Theorem 1.4.12. *Assume that*

$$\begin{aligned}
\nu_\lambda(\circ\circ)\nu_\lambda(\circ\circ\circ\circ) &\geq \nu_\lambda(\circ\circ\circ)^2,\\
\nu_\lambda(\circ)\nu_\lambda(\circ\circ\times\circ) &\geq \nu_\lambda(\circ\circ)\nu_\lambda(\circ\times\circ).
\end{aligned}$$

Then we have

$$\begin{aligned}
\lambda_c &\geq \lambda_c^{(KK,3)} = \frac{1+\sqrt{37}}{6} \approx 1.180,\\
\rho_\lambda &\leq \rho_\lambda^{(KK,3)} = \frac{4\lambda(3\lambda^2-\lambda-3)}{12\lambda^3 - 2\lambda^2 - 8\lambda - 1 + \sqrt{D}} \quad \textit{for} \quad \lambda \geq \frac{1+\sqrt{37}}{6}.
\end{aligned}$$

We should remark that Conjecture 1.4.10 can be rewritten as

Conjecture 1.4.13.

(1) $$\frac{\nu_\lambda(\circ\circ\circ)}{\nu_\lambda(\circ\circ)} \geq \frac{\nu_\lambda(\circ\circ)}{\nu_\lambda(\circ)}.$$

(2) $$\frac{\nu_\lambda(\circ\circ\circ\circ)}{\nu_\lambda(\circ\circ\circ)} \geq \frac{\nu_\lambda(\circ\circ\circ)}{\nu_\lambda(\circ\circ)}.$$

(3) $$\frac{\nu_\lambda(\circ\circ\times\circ)}{\nu_\lambda(\circ\times\circ)} \geq \frac{\nu_\lambda(\circ\circ)}{\nu_\lambda(\circ)}.$$

From Conjecture 1.4.10, we will extend to the following one which was conjectured by Konno[12] (Conjecture 3.4.13).

Conjecture 1.4.14. *For any $A, B \in Y$,*

$$\overline{\rho}_\lambda(A \cap B)\overline{\rho}_\lambda(A \cup B) \geq \overline{\rho}_\lambda(A)\overline{\rho}_\lambda(B),$$

where $\overline{\rho}_\lambda(A) = \nu_\lambda\{\eta : \eta(x) = 0$ for any $x \in A\}$.

For example, Conjecture 1.4.14 for $A = \{0, 1\}$ and $B = \{1, 2\}$ gives Conjecture 1.4.10 (1). If $A \cap B = \phi$, this inequality is equivalent to the Harris-FKG inequality in Corollary 1.3.3 (2):

$$\overline{\rho}_\lambda(A \cup B) \geq \overline{\rho}_\lambda(A)\overline{\rho}_\lambda(B).$$

So it obviously refines the Harris-FKG inequality in this meaning. This approach is called the Markov extension method. (See more details in Katori and Konno.[17]) The reason is as follows. First we choose a subset $Y^{(n)} \subset Y$ and define $h_\lambda^{(n)}$ on $Y^{(n)}$ by a suitable way. Next for the rest of sets in $Y \backslash Y^{(n)}$, the function $h_\lambda^{(n)}$ is constructed from a quotient of simple products of $h_\lambda^{(n)}$'s previously defined on $Y^{(n)}$. The formula used in the latter procedure is similar to the relation found among probability measures in generalized Markov processes, so this extension procedure is called the *Markov extension.* For example, see Schlijper.[19] Recently Conjecture 1.4.14 was proved by Belitsky, Ferrari, Konno and Liggett.[20]

Finally we would like to discuss the relation between the Katori-Konno method and probability measures $\overline{\nu_\lambda}^{(n)}$ $(n = 1, 2, 3)$ on $X = \{0, 1\}^{\mathbf{Z}}$.

We define a probability measure $\overline{\nu_\lambda}^{(1)}$ on X which corresponds to the first bound of the Katori-Konno method by

$$\overline{\nu_\lambda}^{(1)}\{\eta : \eta(x) = 0 \text{ for any } x \in A\} = \alpha^{|A|},$$

where $0 \leq \alpha \leq 1$ which will be determined later. From this definition, the following equality can be easily checked:

$$\overline{\nu_\lambda}^{(1)}(\circ\circ) = \overline{\nu_\lambda}^{(1)}(\circ)^2, \tag{1.11}$$

where $\overline{\nu_\lambda}^{(1)}(\circ\circ) = \overline{\nu_\lambda}^{(1)}\{\eta : \eta(0) = \eta(1) = 0\}$ and $\overline{\nu_\lambda}^{(1)}(\circ) = \overline{\nu_\lambda}^{(1)}\{\eta : \eta(0) = 0\}$. As in the case of Corollary 1.2.6 (1), we assume that the following equation holds:

$$1 - (2\lambda + 1)\overline{\nu_\lambda}^{(1)}(\circ) + 2\lambda\overline{\nu_\lambda}^{(1)}(\circ\circ) = 0. \tag{1.12}$$

Combining Eq.(1.11) with Eq.(1.12) gives

$$\overline{\nu_\lambda}^{(1)}(\circ) = \alpha_* = \left(\frac{1}{2\lambda}\right) \wedge 1,$$

where $a \wedge b$ is the minimum of a and b. This result implies that if $\lambda \leq 1/2$, then $\overline{\nu_\lambda}^{(1)} = \delta_0$.

Similarly we define a probability measure $\overline{\nu_\lambda}^{(2)}$ on X which corresponds to the second bound of the Katori-Konno method by

$$\overline{\nu_\lambda}^{(2)}\{\eta : \eta(x) = 0 \text{ for any } x \in A\} = \alpha^{|A|}\beta^{b(A)}.$$

From this, we have

$$\overline{\nu_\lambda}^{(2)}(\circ\circ\circ) = \frac{\overline{\nu_\lambda}^{(2)}(\circ\circ)^2}{\overline{\nu_\lambda}^{(2)}(\circ)}, \tag{1.13}$$

where $\overline{\nu_\lambda}^{(2)}(\circ\circ\circ) = \overline{\nu_\lambda}^{(2)}\{\eta : \eta(0) = \eta(1) = \eta(2) = 0\}$. As in the case of Corollary 1.2.6 (1) and (2), we assume that the following equations hold:

$$1 - (2\lambda + 1)\overline{\nu_\lambda}^{(2)}(\circ) + 2\lambda\overline{\nu_\lambda}^{(2)}(\circ\circ) = 0. \tag{1.14}$$

$$\overline{\nu_\lambda}^{(2)}(\circ) - (\lambda + 1)\overline{\nu_\lambda}^{(2)}(\circ\circ) + \lambda\overline{\nu_\lambda}^{(2)}(\circ\circ\circ) = 0. \tag{1.15}$$

By Eqs.(1.13-15), we have

$$\overline{\nu_\lambda}^{(2)}(\circ) = \alpha_* = \Big(\frac{1}{2\lambda - 1}\Big) \wedge 1 \quad \text{and} \quad \beta_* = \Big(\frac{2\lambda - 1}{\lambda}\Big) \vee 1,$$

where $a \vee b$ is the maximum of a and b. This result implies that if $\lambda \leq 1$, then $\overline{\nu_\lambda}^{(2)} = \delta_0$. Furthermore we can compute $\overline{\nu_\lambda}^{(2)}(A)$ for any $A \in Y$, for example,

$$\begin{aligned}
\overline{\nu_\lambda}^{(2)}(\circ\circ\times\circ) &= \frac{\overline{\nu_\lambda}^{(2)}(\circ\circ)\overline{\nu_\lambda}^{(2)}(\circ\times)\overline{\nu_\lambda}^{(2)}(\times\circ)}{\overline{\nu_\lambda}^{(2)}(\circ)\overline{\nu_\lambda}^{(2)}(\times)} \\
&= \frac{\overline{\nu_\lambda}^{(2)}(\circ\circ)\overline{\nu_\lambda}^{(2)}(\circ)^2}{\overline{\nu_\lambda}^{(2)}(\circ)} \\
&= \overline{\nu_\lambda}^{(2)}(\circ\circ)\overline{\nu_\lambda}^{(2)}(\circ).
\end{aligned}$$

Moreover we define a probability measure $\overline{\nu_\lambda}^{(3)}$ on X which corresponds to the third bound of the Katori-Konno method by

$$\overline{\nu_\lambda}^{(3)}\{\eta : \eta(x) = 0 \text{ for any } x \in A\} = \alpha^{|A|}\beta^{b(A)}\gamma^{c(A)}\delta^{d(A)}.$$

From this, we have

$$\begin{aligned}
\overline{\nu_\lambda}^{(3)}(\circ\circ\circ\circ) &= \frac{\overline{\nu_\lambda}^{(3)}(\circ\circ\circ)^2}{\overline{\nu_\lambda}^{(3)}(\circ\circ)} \\
\overline{\nu_\lambda}^{(3)}(\circ\circ\times\circ) &= \frac{\overline{\nu_\lambda}^{(3)}(\circ\circ\times)\overline{\nu_\lambda}^{(3)}(\circ\times\circ)}{\overline{\nu_\lambda}^{(3)}(\circ\times)} = \frac{\overline{\nu_\lambda}^{(3)}(\circ\circ)\overline{\nu_\lambda}^{(3)}(\circ\times\circ)}{\overline{\nu_\lambda}^{(3)}(\circ)}.
\end{aligned}$$

where $\overline{\nu_\lambda}^{(3)}(\circ\circ\times\circ) = \overline{\nu_\lambda}^{(3)}\{\eta : \eta(0) = \eta(1) = \eta(3) = 0\}$, etc. As in the case of Corollary 1.2.6 (1), (2), (3) and (4), we assume that the following equations hold:

$$1 - (2\lambda + 1)\overline{\nu_\lambda}^{(3)}(\circ) + 2\lambda\overline{\nu_\lambda}^{(3)}(\circ\circ) = 0.$$

$$\overline{\nu_\lambda}^{(3)}(\circ) - (\lambda + 1)\overline{\nu_\lambda}^{(3)}(\circ\circ) + \lambda\overline{\nu_\lambda}^{(3)}(\circ\circ\circ) = 0.$$

$$2\overline{\nu_\lambda}^{(3)}(\circ\circ) - (2\lambda + 3)\overline{\nu_\lambda}^{(3)}(\circ\circ\circ) + \overline{\nu_\lambda}^{(3)}(\circ\times\circ) + 2\lambda\overline{\nu_\lambda}^{(3)}(\circ\circ\circ\circ) = 0.$$

$$\overline{\nu_\lambda}^{(3)}(\circ) + \lambda\overline{\nu_\lambda}^{(3)}(\circ\circ\circ) - (2\lambda + 1)\overline{\nu_\lambda}^{(3)}(\circ\times\circ) + \lambda\overline{\nu_\lambda}^{(3)}(\circ\circ\times\circ) = 0.$$

By the above 6 equations, we have $\overline{\nu_\lambda}^{(3)}(\circ) = \alpha_* \wedge 1$, β_*, γ_* and δ_* which appeared in Theorem 1.4.8. This implies that if $\lambda \leq \big(1+\sqrt{37}\big)/6$, then $\overline{\nu_\lambda}^{(3)} = \delta_0$. Furthermore we can compute $\overline{\nu_\lambda}^{(3)}(A)$ for any $A \in Y$, for example,

$$\begin{aligned}
\overline{\nu_\lambda}^{(3)}(\circ\circ\circ\times\circ) &= \frac{\overline{\nu_\lambda}^{(3)}(\circ\circ\circ)\overline{\nu_\lambda}^{(3)}(\circ\circ\times)\overline{\nu_\lambda}^{(3)}(\circ\times\circ)}{\overline{\nu_\lambda}^{(3)}(\circ\circ)\overline{\nu_\lambda}^{(3)}(\circ\times)} \\
&= \frac{\overline{\nu_\lambda}^{(3)}(\circ\circ\circ)\overline{\nu_\lambda}^{(3)}(\circ\circ)\overline{\nu_\lambda}^{(3)}(\circ\times\circ)}{\overline{\nu_\lambda}^{(3)}(\circ\circ)\overline{\nu_\lambda}^{(3)}(\circ)} \\
&= \frac{\overline{\nu_\lambda}^{(3)}(\circ\circ\circ)\overline{\nu_\lambda}^{(3)}(\circ\times\circ)}{\overline{\nu_\lambda}^{(3)}(\circ)}.
\end{aligned}$$

In a similar way, we hope that we will extend this argument to the general nth bounds.

Exercise 1.14. Show that for any $n \geq 1$, $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$ with $x_{i+1} - x_i \geq y_{i+1} - y_i$ $(i = 1, 2, \ldots n-1)$,

$$h_\lambda^{(KK,2)}(\{x_1, x_2, \ldots, x_n\}) \geq h_\lambda^{(KK,2)}(\{y_1, y_2, \ldots, y_n\}),$$

for $\lambda \geq \lambda_c^{(KK,2)} = 1$. By using the notation

$$\overline{\nu_\lambda}^{(2)}\{\eta : \eta(x) = 0 \text{ for any } x \in A\} = \alpha^{|A|}\beta^{b(A)},$$

this can be rewritten as

$$\begin{aligned}
&\overline{\nu_\lambda}^{(2)}\{\eta : \eta(u) = 0 \text{ for any } u \in \{x_1, x_2, \ldots, x_n\}\} \\
&\quad \leq \overline{\nu_\lambda}^{(2)}\{\eta : \eta(u) = 0 \text{ for any } u \in \{y_1, y_2, \ldots, y_n\}\}.
\end{aligned}$$

Compare this result with Theorem 1.3.10, that is,

$$\begin{aligned}
&\nu_\lambda\{\eta : \eta(u) = 0 \text{ for any } u \in \{x_1, x_2, \ldots, x_n\}\} \\
&\quad \leq \nu_\lambda\{\eta : \eta(u) = 0 \text{ for any } u \in \{y_1, y_2, \ldots, y_n\}\}.
\end{aligned}$$

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CHAPTER 2

BASIC CONTACT PROCESSES II

2.1. Introduction

In this chapter we consider upper bounds on the critical value and lower bounds on the order parameter of the basic contact process in one dimension. By choosing a suitable renewal measure and using the Harris lemma, Holley and Liggett[1] gave the first upper bound on the critical value:

$$\lambda_c \le \lambda_c^{(HL,1)} = 2,$$

and lower bound on the order parameter:

$$\rho_\lambda \ge \rho_\lambda^{(HL,1)} = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{2\lambda}} \quad \text{for } \lambda \ge 2.$$

By an extension of the Holley-Liggett method, Liggett[2] gave an improved upper bound

$$\lambda_c \le \lambda_c^{(HL,2)} \approx 1.942,$$

where

$$\lambda_c^{(HL,2)} = \sup\{\lambda \ge 0 : 4\lambda^3 - 7\lambda^2 - 2\lambda + 1 \le 0\}.$$

Moreover this argument implies that for $\lambda \ge \lambda_c^{(HL,2)}$,

$$\rho_\lambda \ge \rho_\lambda^{(HL,2)} = \frac{\lambda + \alpha - 1 + \sqrt{(\lambda+\alpha-1)^2 - \alpha\Big[2\lambda + \alpha - 2 + 2\lambda F_1(2)\Big]}}{2\lambda + \alpha - 2 + 2\lambda F_1(2)},$$

where

$$\alpha = \frac{4\lambda - 2}{4\lambda - 1} \quad \text{and} \quad F_1(2) = \frac{4\lambda - 1}{(4\lambda+1)(2\lambda-1)}.$$

The formalism of the general nth approximation is also discussed in Liggett.[2]

This chapter is organized as follows. In Section 2.2, we will consider bounds by the Holley-Liggett method. Section 2.3 is devoted to correlation identities and inequalities which correspond to this method.

2.2. Holley-Liggett Method

2.2.1. *First bound by the Holley-Liggett method*

We introduce the following renewal measure μ on $\{0,1\}^{\mathbf{Z}}$ with density f

$$\mu(\bullet \overbrace{\circ \cdots \circ}^{n_1} \bullet \overbrace{\circ \cdots \circ}^{n_2} \bullet \circ \ldots \circ \bullet \overbrace{\circ \cdots \circ}^{n_k} \bullet) = \frac{f(n_1+1)f(n_2+1)\cdots f(n_k+1)}{\sum\limits_{m=1}^{\infty} mf(m)}.$$

By using the Harris lemma, we have

Theorem 2.2.1. *Let* $\lambda_c^{(HL,1)} = 2$. *Then for* $\lambda \geq \lambda_c^{(HL,1)}$,

$$h_\lambda^{(HL,1)}(A) \leq \sigma_\lambda(A) \qquad \textit{for all } A \in Y,$$

where

$$h_\lambda^{(HL,1)}(A) = \mu\{\eta : \eta(x) = 1 \ \textit{ for some } x \in A\},$$

for a renewal measure μ *on* $\{0,1\}^{\mathbf{Z}}$ *whose density* f *is given by* $\Omega^* h_\lambda^{(HL,1)}(A)= 0$ *for all* A *of the form* $\{1,2,\ldots,n\}$ $(n \geq 1)$.

Applying Theorem 2.2.1 to $A = \{1\}$ gives

Corollary 2.2.2.

$$\lambda_c \leq \lambda_c^{(HL,1)} = 2.$$

$$\rho_\lambda \geq \rho_\lambda^{(HL,1)} = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{2\lambda}} \quad (\lambda \geq 2).$$

Sketch of Proof of Theorem 2.2.1. As in the case of Chapter 1, we need the following 4 steps.

Step 1. First we choose a suitable form of $h_\lambda^{(HL,1)}(A)$.
Step 2. Next we choose $h_\lambda^{(HL,1)}(A)$ explicitly.
Step 3. Thirdly we check conditions (1)-(3) in the Harris lemma.
Step 4. Finally we check condition (4) in the Harris lemma.

In this sketch, we will show only Steps 1 and 2. Concerning Steps 3 and 4, see Chapter VI of Liggett.[3]

Step 1. We choose h of the form

$$\begin{aligned} h(A) &= \mu\{\eta : \eta(x) = 1 \text{ for some } x \in A\} \\ &= 1 - \mu\{\eta : \eta(x) = 0 \text{ for any } x \in A\}, \end{aligned}$$

for a renewal measure μ on $\{0,1\}^{\mathbf{Z}}$.

Step 2. We define the density f so that

$$\Omega^* h(\{1,2,\ldots,n\}) = 0,$$

for any $n \geq 1$, where

$$\Omega^* h(A) = \lambda \sum_{x\in A} \sum_{y:|y-x|=1} \Big[h(A\cup\{y\}) - h(A)\Big] + \sum_{x\in A} \Big[h(A\setminus\{x\}) - h(A)\Big].$$

The definition of Ω^* gives

$$\begin{aligned} \Omega^* h(\{1,2,\ldots,n\}) = \sum_{k=1}^{n} \Big\{ &\lambda\Big[h(\{1,2,\ldots,n\}\cup\{k-1\}) - h(\{1,2,\ldots,n\})\Big] \\ &+ \lambda\Big[h(\{1,2,\ldots,n\}\cup\{k+1\}) - h(\{1,2,\ldots,n\})\Big] \\ &+ \Big[h(\{1,2,\ldots,n\}\setminus\{k\}) - h(\{1,2,\ldots,n\})\Big]\Big\}. \end{aligned}$$

So we have

$$\begin{aligned} \Omega^* h(\{1,2,\ldots,n\}) = \lambda&\Big[h(\{0,1,2,\ldots,n\}) - h(\{1,2,\ldots,n\})\Big] \\ &+ \lambda\Big[h(\{1,2,\ldots,n,n+1\}) - h(\{1,2,\ldots,n\})\Big] \\ &+ \sum_{k=1}^{n}\Big[h(\{1,2,\ldots,n\}\setminus\{k\}) - h(\{1,2,\ldots,n\})\Big]. \end{aligned}$$

We should remark the following relations:

$$\begin{aligned} h(\{1,2,3\}) &= 1 - \mu(\circ\circ\circ), \\ h(\{1,3\}) &= 1 - \mu(\circ\times\circ), \\ h(\{1,2,3\}) - h(\{1,3\}) &= \mu(\circ\times\circ) - \mu(\circ\circ\circ) = \mu(\circ\bullet\circ), \\ &\vdots \end{aligned}$$

For $\lambda \geq 0$ with $\mu(\bullet) > 0$, we let

$$F(n) = \frac{\mu(\bullet\overbrace{\circ\cdots\circ}^{n-1})}{\mu(\bullet)} \quad (n \geq 1).$$

The density f is given by

$$f(n) = \frac{\mu(\bullet\overbrace{\circ\cdots\circ}^{n-1}\bullet)}{\mu(\bullet)} \quad (n \geq 1).$$

So $f(n) = F(n) - F(n+1)$. Then we see that

$$\begin{aligned} h(\{0,1,2,\ldots,n\}) - h(\{1,2,\ldots,n\}) &= \mu(\overbrace{\circ\cdots\circ}^{n}) - \mu(\overbrace{\circ\cdots\circ}^{n+1}) \\ &= \mu(\bullet\,\overbrace{\circ\cdots\circ}^{n}) \\ &= F(n+1)\mu(\bullet). \end{aligned}$$

Similarly we have

$$\begin{aligned} h(\{1,2,\ldots,n,n+1\}) - h(\{1,2,\ldots,n\}) &= \mu(\overbrace{\circ\cdots\circ}^{n}\,\bullet) \\ &= F(n+1)\mu(\bullet). \end{aligned}$$

On the other hand

$$\begin{aligned} &h(\{1,2,\ldots,n\} \setminus \{k\}) - h(\{1,2,\ldots,n\}) \\ &= \mu(\overbrace{\circ\cdots\circ}^{n}) - \mu(\overbrace{\circ\cdots\circ}^{k-1} \times \overbrace{\circ\cdots\circ}^{n-k}) \\ &= -\mu(\overbrace{\circ\cdots\circ}^{k-1}\,\bullet\,\overbrace{\circ\cdots\circ}^{n-k}) \\ &= -\frac{\mu(\overbrace{\circ\cdots\circ}^{k-1}\,\bullet)}{\mu(\bullet)} \times \frac{\mu(\bullet\,\overbrace{\circ\cdots\circ}^{n-k})}{\mu(\bullet)} \times \mu(\bullet) \\ &= -F(k)F(n+1-k)\mu(\bullet). \end{aligned}$$

The third equality comes from the property of the renewal measure μ. Therefore we have

$$\Omega^* h(\{1,2,\ldots,n\}) = \mu(\bullet)\Big[2\lambda F(n+1) - \sum_{k=1}^{n} F(k)F(n+1-k)\Big],$$

for $n \geq 1$. Then $\Omega^* h(\{1,2,\ldots,n\}) = 0$ implies

Lemma 2.2.3. *For $\lambda \geq 0$ with $\mu(\bullet) > 0$,*

$$\begin{aligned} 2\lambda F(n+1) &= \sum_{k=1}^{n} F(k)F(n+1-k) \quad (n \geq 1), \\ F(1) &= 1. \end{aligned}$$

We introduce the following generating function to get $F(n)$ explicitly:

$$\phi(u) = \sum_{n=1}^{\infty} F(n)u^n.$$

By using Lemma 2.2.3, we have the following quadratic equation:

$$\phi^2(u) - 2\lambda\phi(u) + 2\lambda u = 0. \tag{2.1}$$

The nonnegativity of the discriminant of this equation with $u = 1$ is equivalent to $\lambda(\lambda - 2) \geq 0$. So we let $\lambda_c^{(HL,2)} = 2$. By Eq.(2.1) and $\phi'(0) = F(1) = 1$, we get

$$\phi(u) = \lambda - \sqrt{\lambda^2 - 2\lambda u} = \lambda - \lambda\sqrt{1 - \frac{2u}{\lambda}}. \tag{2.2}$$

Here we present a formula to expand Eq.(2.2) in a power series in u:

$$\sqrt{1-s} = 1 - \sum_{n=1}^{\infty} b_n s^n,$$

where

$$b_n = \frac{(2n)!}{4^n (n!)^2 (2n-1)}.$$

Therefore we obtain the explicit form of $F(n)$ as follows.

Lemma 2.2.4. *For* $\lambda \geq \lambda_c^{(HL,2)} = 2$,

$$F(n) = \frac{(2(n-1))!}{(n-1)!\, n!} \left(\frac{1}{2\lambda}\right)^{n-1} \quad (n \geq 1).$$

We should remark that the density f can be given by $f(n) = F(n) - F(n+1)$ for $n \geq 1$. By Lemma 2.2.4 and a direct computation, we can show the positivity of $f(n)$ easily. So Step 2 is complete.

In this way, we have $f(n)$ and $F(n)$ explicitly. By Eq.(2.2), we have

$$\phi(1) = \sum_{n=1}^{\infty} F(n) = \lambda - \sqrt{\lambda(\lambda - 2)}.$$

Note that

$$\rho_\lambda^{(HL,1)} = \mu(\bullet) = \frac{1}{\sum_{n=1}^{\infty} n f(n)} = \frac{1}{\sum_{n=1}^{\infty} F(n)} = \frac{1}{\phi(1)},$$

since the second equality comes from the definition of μ. So we have the following first bounds by using the Holley-Liggett method:

$$\lambda_c \leq \lambda_c^{(HL,1)} = 2.$$

$$\rho_\lambda \geq \rho_\lambda^{(HL,1)} = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{2\lambda}} \quad (\lambda \geq 2).$$

Exercise 2.1. In the first bound, the stationary renewal measure with density f is a reversible measure for the nearest-particle system with birth rates

$$\beta(l, r) = \frac{f(l) f(r)}{f(l + r)}.$$

On the other hand, the one-dimensional contact process is $\beta(1,1) = 2\lambda$, $\beta(l,1) = \beta(1,r) = \lambda$ for l, $r \geq 2$ and $\beta(l,r) = 0$ otherwise. Concerning the nearest-particle system, see Chapter VII of Liggett.[3] We assume that $\lambda \geq 2$.

(a) Show that

$$f(1) = \frac{2\lambda - 1}{2\lambda}, \qquad f(2) = \frac{\lambda - 1}{2\lambda^2}, \qquad f(3) = \frac{4\lambda - 5}{8\lambda^3},$$

$$\beta(1,1) = \frac{(2\lambda - 1)^2}{2(\lambda - 1)}, \qquad \beta(2,1) = \beta(1,2) = \frac{2(\lambda - 1)(2\lambda - 1)}{4\lambda - 5}.$$

(b) Show that

$$\beta(1,1) \geq 2\lambda, \qquad \beta(1,2) = \beta(2,1) \leq \lambda.$$

(c) Verify that the above result (b) implies that the nearest-particle system ξ_t and the basic contact process η_t can not be constructed on the same probability space with

$$\xi_t(x) \leq \eta_t(x) \qquad \text{for any } x \in \mathbf{Z} \text{ and } t \geq 0.$$

However

$$\mu\{\eta : \eta(x) = 1\} \leq \nu_\lambda\{\eta : \eta(x) = 1\},$$

for any $\lambda \geq 2$.

Exercise 2.2. Show that

(1) $$\rho_\lambda \to 1 \qquad \text{as} \quad \lambda \to \infty,$$

moreover,

(2) $$2\lambda(1 - \rho_\lambda) \to 1 \qquad \text{as} \quad \lambda \to \infty.$$

2.2.2. *Second bound by the Holley-Liggett method*

As in the previous subsection, we will consider the second bound which was obtained by Liggett.[2]

Theorem 2.2.5. *Let $\lambda_c^{(HL,2)} \approx 1.942$ be the largest root of the cubic equation of*

$$\lambda^3 - 7\lambda^2 - 2\lambda + 1 = 0.$$

Then for $\lambda \geq \lambda_c^{(HL,2)}$,

$$h_\lambda^{(HL,2)}(A) \leq \sigma_\lambda(A) \qquad \text{for all } A \in Y,$$

where

$$h_\lambda^{(HL,2)}(A) = \mu\{\eta : \eta(x) = 1 \ \text{ for some } x \in A\},$$

for a generalized renewal measure μ on $\{0,1\}^{\mathbf{Z}}$ whose density is given by $\Omega^* h_\lambda^{(HL,2)}(A) = 0$ for all A of the form $\{1, 2, \ldots, n\}$ $(n \geq 1)$ and $\{1, 3\}$.

Concerning the exact definition of μ, see Liggett.[2] Applying Theorem 2.2.5 to $A = \{1\}$ gives

Corollary 2.2.6.

$$\lambda_c \le \lambda_c^{(HL,2)}.$$

$$\rho_\lambda \ge \rho_\lambda^{(HL,2)} = \frac{\lambda+\alpha-1+\sqrt{(\lambda+\alpha-1)^2-\alpha\Big[2\lambda+\alpha-2+2\lambda F_1(2)\Big]}}{2\lambda+\alpha-2+2\lambda F_1(2)}$$

for $\lambda \ge \lambda_c^{(HL,2)}$, *where*

$$\alpha = \frac{4\lambda-2}{4\lambda-1} \quad \textit{and} \quad F_1(2) = \frac{4\lambda-1}{(4\lambda+1)(2\lambda-1)}.$$

As in the case of the first bound, we will discuss only Steps 1 and 2. So Steps 3 and 4 will be omitted.

Step 1. We choose the form of

$$h(A) = \mu\{\eta : \eta(x) = 1 \ \text{ for some } x \in A\},$$

for a generalized renewal measure μ on $\{0,1\}^{\mathbf{Z}}$.

Step 2. We define the density so that

$$\Omega^* h(\{1,2,\ldots,n\}) = 0,$$

for any $n \ge 1$ and

$$\Omega^* h(\{1,3\}) = 0,$$

where

$$\Omega^* h(A) = \lambda \sum_{x\in A}\sum_{y:|y-x|=1}\Big[h(A\cup\{y\}) - h(A)\Big] + \sum_{x\in A}\Big[h(A\setminus\{x\}) - h(A)\Big].$$

We should remark the next relations:

$$\begin{aligned} h(\{1,2,3\}) &= 1-\mu(\circ\circ\circ), \\ h(\{1,3\}) &= 1-\mu(\circ\times\circ), \\ h(\{1,2,3\}) - h(\{1,3\}) &= \mu(\circ\times\circ) - \mu(\circ\circ\circ) = \mu(\circ\bullet\circ), \\ &\vdots \end{aligned}$$

Moreover, following notations of Liggett,[2] we introduce

$$F_1(n) = \frac{\mu(\bullet\bullet\overbrace{\circ\cdots\circ}^{n-1})}{\mu(\bullet\bullet)}, \qquad F_0(n) = \frac{\mu(\circ\bullet\overbrace{\circ\cdots\circ}^{n-1})}{\mu(\circ\bullet)},$$

$$f_1(n) = \frac{\mu(\bullet\bullet\overbrace{\circ\cdots\circ}^{n-1}\bullet)}{\mu(\bullet\bullet)}, \qquad f_0(n) = \frac{\mu(\circ\bullet\overbrace{\circ\cdots\circ}^{n-1}\bullet)}{\mu(\circ\bullet)},$$

for $n \geq 1$. The above definitions give

$$F_1(1) = F_0(1) = 1,$$
$$F_1(n) = \sum_{k=n}^{\infty} f_1(k), \quad F_0(n) = \sum_{k=n}^{\infty} f_0(k).$$

We recall that

$$\begin{aligned}\Omega^* h(\{1,2,\dots,n\}) = \lambda\Big[h(\{0,1,2,\dots,n\}) - h(\{1,2,\dots,n\})\Big] \\ + \lambda\Big[h(\{1,2,\dots,n,n+1\}) - h(\{1,2,\dots,n\})\Big] \\ + \sum_{k=1}^{n}\Big[h(\{1,2,\dots,n\}\setminus\{k\}) - h(\{1,2,\dots,n\})\Big].\end{aligned}$$

From now on we consider two cases; Case A and Case B.

Case A. By using definitions of $F_0(n)$ and $F_1(n)$, we see that

$$\begin{aligned} & h(\{0,1,2,\dots,n\}) - h(\{1,2,\dots,n\}) \\ &= \mu(\overbrace{\circ\cdots\circ}^{n}) - \mu(\overbrace{\circ\cdots\circ}^{n+1}) \\ &= \mu(\bullet\overbrace{\circ\cdots\circ}^{n}) \\ &= \mu(\bullet\bullet\overbrace{\circ\cdots\circ}^{n}) + \mu(\circ\bullet\overbrace{\circ\cdots\circ}^{n}) \\ &= \frac{\mu(\bullet\bullet\overbrace{\circ\cdots\circ}^{n})}{\mu(\bullet\bullet)} \times \mu(\bullet\bullet) + \frac{\mu(\circ\bullet\overbrace{\circ\cdots\circ}^{n})}{\mu(\circ\bullet)} \times \mu(\circ\bullet) \\ &= F_1(n+1)\mu(\bullet\bullet) + F_0(n+1)\mu(\circ\bullet). \end{aligned}$$

On the other hand, the definition of a measure μ implies there is an α such that

$$\alpha = \frac{F_0(n)}{F_1(n)} \quad \text{for } n \geq 1.$$

Note that α is independent of n. $\Omega^* h(\{1\}) = 0$ gives

$$\mu(\bullet\bullet) = (2\lambda - 1)\mu(\circ\bullet).$$

Moreover $\Omega^* h(\{1,2\}) = \Omega^* h(\{1,3\}) = 0$ yields

$$\alpha = \frac{4\lambda - 2}{4\lambda - 1} \qquad \text{and} \qquad F_1(2) = \frac{4\lambda - 1}{(4\lambda + 1)(2\lambda - 1)}.$$

From these, we have

$$\begin{aligned} h(\{0,1,2,\dots,n\}) - h(\{1,2,\dots,n\}) &= (2\lambda - 1)\mu(\circ\bullet)F_1(n+1) + \alpha\mu(\circ\bullet)F_1(n+1) \\ &= (2\lambda - 1 + \alpha)\mu(\circ\bullet)F_1(n+1). \end{aligned}$$

We let

$$\delta = 2\lambda - 1 + \alpha = \frac{(4\lambda+1)(2\lambda-1)}{4\lambda-1}.$$

Remark that

$$\delta = \frac{1}{F_1(2)}.$$

From the above observations, we have

$$h(\{0,1,2,\ldots,n\}) - h(\{1,2,\ldots,n\}) = \mu(\bullet \overbrace{\circ\cdots\circ}^{n}) = \delta\mu(\circ\bullet)F_1(n+1).$$

Similarly

$$h(\{1,2,\ldots,n,n+1\}) - h(\{1,2,\ldots,n\}) = \mu(\bullet \overbrace{\circ\cdots\circ}^{n}) = \delta\mu(\circ\bullet)F_1(n+1).$$

Therefore we obtain

$$\lambda \sum_{x\in A}\sum_{y:|y-x|=1}\Big[h(A\cup\{y\}) - h(A)\Big] = 2\lambda\delta\mu(\circ\bullet)F_1(n+1),$$

where $A = \{1,\ldots,n\}$.

Case B. For $k \in \{2,\ldots,n-1\}$, we see

$$\begin{aligned} h(\{1,2,\ldots,n\}\setminus\{k\}) - h(\{1,2,\ldots,n\}) =& \mu(\overbrace{\circ\cdots\circ}^{n}) - \mu(\overbrace{\circ\cdots\circ}^{k-1} \times \overbrace{\circ\cdots\circ}^{n-k}) \\ =& -\mu(\overbrace{\circ\cdots\circ}^{k-1}\bullet\overbrace{\circ\cdots\circ}^{n-k}) \\ =& -\frac{\mu(\overbrace{\circ\cdots\circ}^{k-1}\bullet\circ)}{\mu(\bullet\circ)} \times \mu(\bullet\overbrace{\circ\cdots\circ}^{n-k}) \\ =& -F_0(k) \times \delta\mu(\circ\bullet)F_1(n+1-k) \\ =& -\alpha F_1(k) \times \delta\mu(\circ\bullet)F_1(n+1-k). \end{aligned}$$

The third equality comes from the definition of a generalized renewal measure μ. The fourth equality is given by the definition of $F_0(k)$ and a similar argument of Case A. The definition of α gives the last equality. So we have

$$h(\{1,2,\ldots,n\}\setminus\{k\}) - h(\{1,2,\ldots,n\}) = -\alpha\delta\mu(\circ\bullet)F_1(k)F_1(n+1-k).$$

For $k = 1$ or $k = n$, a similar argument in Case A implies

$$h(\{1,2,\ldots,n\}\setminus\{k\}) - h(\{1,2,\ldots,n\}) = -\delta\mu(\circ\bullet)F_1(n).$$

Therefore

$$\begin{aligned}\sum_{k=1}^{n} & h(\{1,2,\ldots,n\}\setminus\{k\}) - h(\{1,2,\ldots,n\}) \\ &= -2\delta\mu(\circ\bullet)F_1(n) - \alpha\delta\mu(\circ\bullet)\sum_{k=2}^{n-1}F_1(k)F_1(n+1-k) \\ &= -\delta\mu(\circ\bullet)\Big[2F_1(n) + \alpha\sum_{k=2}^{n-1}F_1(k)F_1(n+1-k)\Big].\end{aligned}$$

From these results, we see that

$$\Omega^* h(\{1,2,\ldots,n\}) = \delta\mu(\circ\bullet)\Big[2\lambda F_1(n+1) - \Big\{2F_1(n) + \alpha\sum_{k=2}^{n-1}F_1(k)F_1(n+1-k)\Big\}\Big],$$

for $n \geq 2$. Then $\Omega^* h(\{1,2,\ldots,n\}) = 0$ $(n \geq 1)$ and $\Omega^* h(\{1,3\}) = 0$ give

Lemma 2.2.7. *Let* $\alpha = (4\lambda - 2)/(4\lambda - 1)$. *Then*

$$\begin{aligned}2\lambda F_1(n+1) &= 2F_1(n) + \alpha\sum_{k=2}^{n-1}F_1(k)F_1(n+1-k) \quad (n \geq 2), \\ F_1(1) &= 1.\end{aligned}$$

We introduce the generating function to get $F_1(n)$ explicitly:

$$\phi(u) = \sum_{n=1}^{\infty} F_1(n)u^n.$$

By using Lemma 2.2.7, we have the following quadratic equation:

$$\alpha\phi^2(u) - 2[\lambda + (\alpha - 1)u]\phi(u) + 2\lambda u + \Big[\alpha - 2 + 2\lambda F_1(2)\Big]u^2 = 0. \tag{2.3}$$

The nonnegativity of the discriminant of this equation with $u = 1$ is equivalent to

$$(\lambda + \alpha - 1)^2 - \alpha\Big[2\lambda + \alpha - 2 + 2\lambda F_1(2)\Big] = \frac{4\lambda^3 - 7\lambda^2 - 2\lambda + 1}{4\lambda + 1} \geq 0.$$

So we let $\lambda_c^{(HL,2)}$ be the largest root of the following cubic equation:

$$4\lambda^3 - 7\lambda^2 - 2\lambda + 1 = 0.$$

From Eq.(2.3) and the definitions of α and $F_1(2)$, we have

$$\alpha\Big[\phi(u) - u\Big] = \lambda - u - \lambda\sqrt{1 - \frac{2u}{\lambda} + \frac{u^2}{\lambda^2(4\lambda + 1)}}. \tag{2.4}$$

Here we present a useful formula to expand Eq.(2.4) in a power series in u:

$$\sqrt{1-s+\left[\frac{1-\beta}{2(1+\beta)}\right]^2 s^2} = 1-\sum_{n=1}^{\infty} c_n s^n, \tag{2.5}$$

with $c_1 = 1/2$ and

$$c_n = \frac{\beta}{2^{n-1}(1+\beta)^n} v(n,\beta) \qquad (n \geq 2),$$

where $v(n,\beta)$ is *Gauss's hypergeometric series of the form*

$$v(n,\beta) = F(-(n-2), -(n-1), 2; \beta) \qquad (n \geq 2).$$

Let

$$s = \frac{2u}{\lambda} \qquad \text{and} \qquad \beta = \frac{2\lambda+1-\sqrt{4\lambda+1}}{2\lambda},$$

for $\lambda > \lambda_c^{(HL,2)}$. It is remarked that $0 < \beta < 1$ for $\lambda > \lambda_c^{(HL,2)}$. By using Eqs.(2.4) and (2.5), we have

Lemma 2.2.8. *For* $\lambda \geq \lambda_c^{(HL,2)}$,

$$F_1(n) = \frac{2\lambda\beta}{\alpha}\frac{v(n,\beta)}{[\lambda(1+\beta)]^n} \qquad \textit{for any}\; n \geq 2,$$
$$F_1(1) = 1,$$

where

$$\alpha = \frac{4\lambda-2}{4\lambda-1}, \qquad \beta = \frac{2\lambda+1-\sqrt{4\lambda+1}}{2\lambda}, \qquad \textit{and}$$
$$v(n,\beta) = F(-(n-2), -(n-1), 2; \beta).$$

Next we will show here the positivity of $f_1(n)(= F_1(n) - F_1(n+1))$ by the explict form of $F_1(n)$ in the above lemma. (A different approach appeared in pp.708-711 in Liggett.[2]) A similar proof is applicable to the case of θ-contact process. See Katori and Konno.[4]

The hypergeometric series $v(n,\beta)$ satisfies the following iterative equation: for $n \geq 1$,

$$(n+2)v(n+2,\beta) - (2n+1)(1+\beta)v(n+1,\beta) + (n-1)(1-\beta)^2 v(n,\beta) = 0, \tag{2.6}$$

with $v(1,\beta) = v(2,\beta) = 1$. Let

$$a_n(\beta) = \frac{v(n+1,\beta)}{v(n,\beta)}.$$

By Eq.(2.6) and the definition of $a_n(\beta)$, we have

$$(n+2)a_{n+1}(\beta) - (2n+1)(1+\beta) + (n-1)(1-\beta)^2 \frac{1}{a_n(\beta)} = 0 \quad (n \geq 1), \tag{2.7}$$

with $a_1(\beta) = 1$. Then we have

Lemma 2.2.9. *For $\lambda \geq \lambda_c^{(HL,2)}$,*

(1) $$a_1(\beta) = 1 \leq a_2(\beta) \leq \cdots \leq a_n(\beta) \leq a_{n+1}(\beta) \leq \cdots,$$

(2) $$\lim_{n\to\infty} a_n(\beta) = a_*(\beta) < \lambda(1+\beta),$$

(3) $$a_n(\beta) < \lambda(1+\beta) \quad \textit{for any}\, n \geq 1.$$

Proof. Using Eq.(2.7), we begin by computing

$$\begin{aligned} a_{n+1}(\beta) - 1 &= \frac{1}{n+2}\Big[(2n+1)(1+\beta) - (n-1)(1-\beta)^2\frac{1}{a_n(\beta)} - (n+2)\Big] \\ &= \frac{1}{(n+2)\lambda a_n(\beta)}\Big[\lambda\Big\{(n-1) + (2n+1)\beta\Big\}\Big(a_n(\beta) - 1\Big) \\ &\qquad + (n-1)\lambda + (2n+1)\beta\Big\{\lambda - \frac{n-1}{2n+1}\Big\}\Big]. \end{aligned}$$

Noting that $\lambda \geq (n-1)/(2n+1)$ for $n \geq 1$ and $\lambda \geq \lambda_c^{(HL,2)}$, we see that if $a_n(\beta) \geq 1$ then $a_{n+1}(\beta) \geq 1$ for $n \geq 1$. Then we have

$$a_n(\beta) \geq 1 \qquad \text{for any}\, n \geq 1, \tag{2.8}$$

since $a_1(\beta) = 1$. Next we use again Eq.(2.7) to get

$$\begin{aligned} a_{n+2}(\beta) - a_{n+1}(\beta) &= \frac{1}{n+3}\Big[(2n+3)(1+\beta) - n(1-\beta)^2\frac{1}{a_{n+1}(\beta)}\Big] \\ &\quad - \frac{1}{n+2}\Big[(2n+1)(1+\beta) - (n-1)(1-\beta)^2\frac{1}{a_n(\beta)}\Big] \\ &= \frac{n(1-\beta)^2}{n+3}\,\frac{\Big[a_{n+1}(\beta) - a_n(\beta)\Big]}{a_n(\beta)a_{n+1}(\beta)} \\ &\quad + \frac{3(1+\beta)}{(n+2)(n+3)a_n(\beta)}\Big[\Big\{a_n(\beta) - 1\Big\} + \Big\{1 - \frac{(1-\beta)^2}{1+\beta}\Big\}\Big]. \end{aligned} \tag{2.9}$$

Noting $a_n(\beta) \geq 1$, i.e., Eq.(2.8), and $1 \geq (1-\beta)^2/(1+\beta)$ for $\lambda \geq \lambda_c^{(HL,2)}$, we prove that if $a_{n+1}(\beta) - a_n(\beta) \geq 0$, then $a_{n+2}(\beta) - a_{n+1}(\beta) \geq 0$ for any $n \geq 1$. Applying Eq.(2.8) to $n = 2$ gives $a_2(\beta) \geq 1$. Combining this result with $a_1(\beta) = 1$ implies $a_2(\beta) - a_1(\beta) \geq 0$. Therefore we have $a_{n+1}(\beta) - a_n(\beta) \geq 0$ for any $n \geq 1$. The proof of part (1) is complete.

Using part (1), we let $a_*(\beta) = \lim_{n\to\infty} a_n(\beta) \in [1, \infty]$. Then Eq.(2.7) implies that $a_*(\beta)$ satisfies the following quadratic equation:

$$f(x) = \lambda x^2 - 2\lambda(1+\beta)x + \beta = 0, \tag{2.10}$$

since $\lambda(1-\beta)^2 = \beta$. We should remark $f(0) = \beta > 0$ and $f(1) = -[\lambda + (2\lambda - 1)\beta] < 0$ for $\lambda \geq \lambda_c^{(HL,2)} (> 1/2)$. So $a_*(\beta)(> 1)$ is larger root of Eq.(2.10). On the other hand, a direct computation shows $\lambda(1+\beta) > 1$ and $f(\lambda(1+\beta)) > 0$. From these facts, we obtain $a_*(\beta) < \lambda(1+\beta)$ for $\lambda \geq \lambda_c^{(HL,2)}$, immediately. So the proof of part (2) is complete. Part (3) comes from parts (1) and (2).

From now on we will turn to the proof of the positivity of $f_1(n)(= F_1(n) - F_1(n+1))$. By using the explicit form of $F_1(n)$ (see Lemma 2.2.8), we see that $f_1(n) > 0$ is equivalent to $a_n(\beta) < \lambda(1+\beta)$ for any $n \geq 2$. So from Lemma 2.2.9 (3), we have $f_1(n) > 0 \; (n \geq 2)$. On the other hand, a direct computation gives

$$f_1(1) = F_1(1) - F_1(2) = \frac{2\lambda(4\lambda - 3)}{(2\lambda - 1)(4\lambda + 1)} > 0 \qquad\qquad (\lambda > \lambda_c^{(HL,2)} > \frac{3}{4}).$$

Therefore we obtain the desired conclusion: $f_1(n) > 0$ for any $n \geq 1$.

Here we present the lower bound $\rho_\lambda^{(HL,2)}$ on ρ_λ in an explicit fashion. By using $\phi(1)$, $\rho_\lambda^{(HL,2)}$ can be given by

$$\rho_\lambda^{(HL,2)} = \frac{1}{\sum\limits_{n=1}^{\infty} F_1(n)} = \frac{1}{\phi(1)},$$

for $\lambda > \lambda_c^{(HL,2)}$. From this and Eq.(2.3), we obtain

$$\rho_\lambda \geq \rho_\lambda^{(HL,2)} = \frac{\lambda + \alpha - 1 + \sqrt{(\lambda + \alpha - 1)^2 - \alpha\Big[2\lambda + \alpha - 2 + 2\lambda F_1(2)\Big]}}{2\lambda + \alpha - 2 + 2\lambda F_1(2)}$$

for $\lambda \geq \lambda_c^{(HL,2)}$.

2.3. Correlation Identities and Inequalities

In Sections 1.2 and 1.4 we gave some correlation identities. (See Theorems 1.2.3, 1.2.5, 1.4.1 and 1.4.2.) This section is devoted to other types of correlation identities. So using them and assuming some correlation inequalities which appear in this section, we obtain easily the first Holley-Liggett bounds. Moreover we consider a similar story concerning the second ones.

2.3.1. *First bound*

For $k \geq 2$ and $n_i \geq 0 \, (i = 1, \ldots, k)$, we let $J(n_1, n_2, \ldots, n_k)$ be the probability of having 1's at $n_1 + 1$, $n_1 + n_2 + 2$, $\ldots$, $n_1 + n_2 + \cdots + n_{k-1} + k - 1$ and 0's at all other sites in $[\,1, n_1 + n_2 + \cdots + n_k + k - 1\,]$ with respect to the upper invariant measure

ν_λ. That is,

$$\begin{aligned}
&J(n_1,n_2,\dots,n_k)\\
&=\nu_\lambda\Big(\overbrace{\circ\cdots\circ}^{n_1}\bullet\overbrace{\circ\cdots\circ}^{n_2}\bullet\circ\dots\circ\bullet\overbrace{\circ\cdots\circ}^{n_k}\Big)\\
&=E_{\nu_\lambda}\left(\prod_{i=1}^{k-1}\Big\{\prod_{q_i=1}^{n_i}[1-\eta(n_1+\cdots+n_{i-1}+q_i+i-1)]\times\eta(n_1+\cdots+n_i+i)\Big\}\right.\\
&\qquad\qquad\left.\times\Big\{\prod_{q_k=1}^{n_k}[1-\eta(n_1+\cdots+n_{k-1}+q_k+k-1)]\Big\}\right).
\end{aligned}$$

For example,

$$\begin{aligned}
J(0,0)&=\nu_\lambda\{\eta:\eta(1)=1\}=\nu_\lambda(\bullet)=\rho_\lambda,\\
J(1,2)&=\nu_\lambda\{\eta:\eta(1)=0,\eta(2)=1,\eta(3)=\eta(4)=0\}=\nu_\lambda(\circ\bullet\circ\circ),\\
J(1,2,3)&=\nu_\lambda\left\{\eta:\begin{array}{l}\eta(1)=0,\eta(2)=1,\eta(3)=\eta(4)=0,\\ \eta(5)=1,\eta(6)=\eta(7)=\eta(8)=0\end{array}\right\}\\
&=\nu_\lambda(\circ\bullet\circ\circ\bullet\circ\circ\circ).
\end{aligned}$$

Using Theorem 1.2.5 for $A=\{1,2,\dots,n\}$, we have

$$\begin{aligned}
&\lambda\Big[\overline{\rho}_\lambda(\{0,1,\dots,n\})-\overline{\rho}_\lambda(\{1,\dots,n\})\Big]+\lambda\Big[\overline{\rho}_\lambda(\{1,\dots,n,n+1\})-\overline{\rho}_\lambda(\{1,\dots,n\})\Big]\\
&\qquad\qquad+\sum_{k=1}^{n}\Big[\overline{\rho}_\lambda(\{1,\dots,n\}\setminus\{k\})-\overline{\rho}_\lambda(\{1,\dots,n\})\Big]=0.
\end{aligned}$$

So the definition of $J(n_1,n_2)$ gives the following correlation identities.

Lemma 2.3.1. *For $n\geq 1$,*

$$2\lambda J(n,0)=\sum_{k=1}^{n}J(k-1,n-k),$$

that is,

$$2\lambda\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}\bullet)=\sum_{k=1}^{n}\nu_\lambda(\overbrace{\circ\cdots\circ}^{k-1}\bullet\overbrace{\circ\cdots\circ}^{n-k}).$$

Exercise 2.3. Show that, when $n=1$, this lemma gives

$$2\lambda\nu_\lambda(\circ\bullet)=\nu_\lambda(\bullet). \tag{2.11}$$

Exercise 2.4. By using

$$\nu_\lambda(\circ\bullet)=\nu_\lambda(\bullet)-\nu_\lambda(\bullet\bullet),$$

and Eq.(2.11), show that

$$(2\lambda - 1)\nu_\lambda(\bullet) - 2\lambda\nu_\lambda(\bullet\bullet) = 0.$$

This equation is equivalent to Corollary 1.2.4 (1) in Chapter 1.

Next we introduce the following conjecture to get the first bound by the Holley-Liggett method.

Conjecture 2.3.2. *For m, $n \geq 1$,*

$$J(0,0)J(m,n) \leq J(m,0)J(n,0),$$

that is,

$$\nu_\lambda(\bullet)\nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet\overbrace{\circ\cdots\circ}^{n}) \leq \nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet)\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}\bullet).$$

Remark. Liggett[5] gave the following argument: if for any m, $n \geq 1$,

$$\nu_\lambda(\bullet)\nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet\overbrace{\circ\cdots\circ}^{n}) \geq \nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet)\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}\bullet),$$

then the Holley-Liggett method does not hold. So he concluded that there are m, $n \geq 1$ such that

$$\nu_\lambda(\bullet)\nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet\overbrace{\circ\cdots\circ}^{n}) \leq \nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet)\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}\bullet).$$

Compared with his conclusion, our conjecture is very strong. Moreover our recent results based on Monte Carlo simulations suggest that the above conjecture is correct for small m and n,[6] for example, $(m,n) = (1,1)$, $(2,1)$, $(1,2)$, $(2,2)$.

Exercise 2.5. Show that

$$J(0,0)J(1,1) \leq J(1,0)^2$$

is equivalent to

$$\rho_\lambda(0)\rho_\lambda(012) \leq \rho_\lambda(01)^2.$$

That is,

$$\nu_\lambda(\bullet)\nu_\lambda(\circ\bullet\circ) \leq \nu_\lambda(\circ\bullet)^2$$

is equivalent to

$$\nu_\lambda(\bullet)\nu_\lambda(\bullet\bullet\bullet) \leq \nu_\lambda(\bullet\bullet)^2.$$

In fact, even when $m = n = 1$, the correlation inequality in Conjecture 2.3.2 is interesting. The reason is as follows. From the Harris-FKG inequality, we have

$$\nu_\lambda(\bullet\bullet) \geq \nu_\lambda(\bullet)^2, \tag{2.12}$$

and

$$\nu_\lambda(\circ\circ) \geq \nu_\lambda(\circ)^2. \tag{2.13}$$

Concerning the Harris-FKG inequality, see Chapter II in Liggett,[3] or page 21 in Konno.[8] We can rewrite Eqs.(2.12) and (2.13) by using the conditional probability:

$$\nu_\lambda(\bullet|\bullet) \geq \nu_\lambda(\bullet), \tag{2.14}$$

and

$$\nu_\lambda(\circ|\circ) \geq \nu_\lambda(\circ), \tag{2.15}$$

where

$$\nu_\lambda(\bullet|\bullet) = \nu_\lambda\{\eta : \eta(1) = 1 \mid \eta(0) = 1\},$$

$$\nu_\lambda(\circ|\circ) = \nu_\lambda\{\eta : \eta(1) = 0 \mid \eta(0) = 0\}.$$

Moreover, in our setting, the following correlation inequalities were proved by Belitsky, Ferrari, Konno and Liggett recently:[7] for any $A, B \subset \mathbf{Z}$,

$$\overline{\rho}_\lambda(A \cap B)\overline{\rho}_\lambda(A \cup B) \geq \overline{\rho}_\lambda(A)\overline{\rho}_\lambda(B), \tag{2.16}$$

where $\overline{\rho}_\lambda(A) = \nu_\lambda\{\eta(x) = 0 \text{ for any } x \in A\}$. In particular, if we take $A = \{-1, 0\}$ and $B = \{0, 1\}$, then we have

$$\nu_\lambda(\circ)\nu_\lambda(\circ\circ\circ) \geq \nu_\lambda(\circ\circ)^2, \tag{2.17}$$

so this becomes

$$\nu_\lambda(\circ|\circ\circ) \geq \nu_\lambda(\circ|\circ), \tag{2.18}$$

where

$$\nu_\lambda(\circ|\circ\circ) = \nu_\lambda\{\eta : \eta(1) = 0 \mid \eta(0) = \eta(-1) = 0\}.$$

On the other hand, when $m = n = 1$ for the correlation inequalities in Conjecture 2.3.2, we have

$$\nu_\lambda(\bullet)\nu_\lambda(\bullet\bullet\bullet) \leq \nu_\lambda(\bullet\bullet)^2, \tag{2.19}$$

that is,

$$\nu_\lambda(\bullet|\bullet\bullet) \leq \nu_\lambda(\bullet|\bullet), \tag{2.20}$$

where

$$\nu_\lambda(\bullet|\bullet\bullet) = \nu_\lambda\{\eta : \eta(1) = 1 \mid \eta(0) = \eta(-1) = 1\}.$$

The interesting thing is that direction of inequality (2.20) is different from those of inequalities (2.14), (2.15) and (2.18). From the attractiveness (see page 72 in Liggett[3]), we can easily expect that inequalities (2.14) and (2.15) hold, moreover, inequality

(2.18) also holds. However, concerning inequality (2.20), we can not easily conclude which sign of inequality is correct. Our estimation by Monte Carlo simulation suggests that inequality (2.20) holds.

For $\lambda > \lambda_c$ and $m, n \geq 0$, define

$$\overline{J}(m,n) = \frac{\nu_\lambda(\overbrace{\circ \cdots \circ}^{m} \bullet \overbrace{\circ \cdots \circ}^{n})}{\nu_\lambda(\bullet)} = \frac{J(m,n)}{J(0,0)}.$$

Note that the definition of λ_c gives $J(0,0) = \nu_\lambda(\bullet) = \rho_\lambda > 0$ for $\lambda > \lambda_c$. By using $\overline{J}(m,n)$, we can rewrite the above conjecture as follows.

Conjecture 2.3.3. *For $\lambda > \lambda_c$ and $m, n \geq 1$,*

$$\overline{J}(m,n) \leq \overline{J}(m,0)\overline{J}(n,0).$$

Let

$$\varphi(u) = \sum_{n=0}^{\infty} \overline{J}(n,0)u^{n+1}.$$

In the previous section we intoduced

$$\phi(u) = \sum_{n=0}^{\infty} F(n+1)u^{n+1}.$$

We should remark that

$$\overline{J}(n,0) = \frac{\nu_\lambda(\overbrace{\circ \cdots \circ}^{n} \bullet)}{\nu_\lambda(\bullet)} \quad \text{and} \quad F(n+1) = \frac{\mu(\overbrace{\circ \cdots \circ}^{n} \bullet)}{\mu(\bullet)}.$$

Lemma 2.3.1 can be rewritten as

$$2\lambda\overline{J}(n,0) = \sum_{k=1}^{n} \overline{J}(k-1, n-k). \tag{2.21}$$

By the definition of $\varphi(u)$, we see that

$$\sum_{n=1}^{\infty} \left[\overline{J}(n,0)u^{n+1}\right] = \varphi(u) - u. \tag{2.22}$$

Assuming Conjecture 2.3.3, we have

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} \overline{J}(k-1, n-k)u^{n+1} \leq \sum_{n=1}^{\infty}\sum_{k=1}^{n} \overline{J}(k-1,0)\overline{J}(n-k,0)u^{n+1}. \tag{2.23}$$

Eq.(2.23) gives

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\overline{J}(k-1,n-k)u^{n+1} \le \varphi(u)^2. \tag{2.24}$$

By Eqs.(2.21), (2.22) and (2.24), we have

$$\varphi^2(u) - 2\lambda\varphi(u) + 2\lambda u \ge 0. \tag{2.25}$$

Note that if $\lambda > \lambda_c$, then

$$\varphi(1) = \sum_{n=0}^{\infty}\overline{J}(n,0) = \frac{1}{J(0,0)} = \frac{1}{\rho_\lambda}. \tag{2.26}$$

Because, for $\lambda > \lambda_c$, the second equality follows from

$$\sum_{n=0}^{\infty}\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}\bullet) = \sum_{n=0}^{\infty}\Big\{\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}) - \nu_\lambda(\overbrace{\circ\cdots\circ}^{n+1})\Big\} = 1 - \lim_{n\to\infty}\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}) = 1.$$

From Eq.(2.25) with $u=1$ and Eq.(2.26), we have

$$2\lambda\rho_\lambda^2 - 2\lambda\rho_\lambda + 1 \ge 0. \tag{2.27}$$

So the continuity and monotonicity of ρ_λ (see Theorem 1.1.1 (3) and (4)) implies

$$\rho_\lambda \ge \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{2\lambda}} \quad \text{for } \lambda \ge 2. \tag{2.28}$$

Note that this lower bound on ρ_λ is nothing but the Holley-Liggett one. So if we assume Conjecture 2.3.3, then Eq.(2.28) gives upper bound on the critical value:

$$\lambda_c \le \lambda_c^{(HL)} = 2.$$

Therefore we have the following theorem which corresponds to Theorems 1.4.11 and 1.4.12. This argument appeared in Chapter 4 of Konno.[8]

Theorem 2.3.4. *Assume that for any $m, n \ge 1$,*

$$\nu_\lambda(\bullet)\nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet\overbrace{\circ\cdots\circ}^{n}) \le \nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet)\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}\bullet).$$

Then we have

$$\lambda_c \le \lambda_c^{(HL,1)} = 2,$$
$$\rho_\lambda \ge \rho_\lambda^{(HL,1)} = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{2\lambda}} \quad (\lambda \ge 2).$$

2.3.2. *Second bound*

First we recall Lemma 2.3.1: For $n \geq 1$,

$$2\lambda\nu_\lambda(\bullet\overbrace{\circ\cdots\circ}^{n}) = \sum_{k=1}^{n}\nu_\lambda(\overbrace{\circ\cdots\circ}^{k-1}\bullet\overbrace{\circ\cdots\circ}^{n-k}). \tag{2.29}$$

Next we let

$$\overline{J}(n,0,0) = \frac{\nu_\lambda(\bullet\bullet\overbrace{\circ\cdots\circ}^{n})}{\nu_\lambda(\bullet\bullet)} \quad \text{and} \quad \gamma(n) = \frac{\nu_\lambda(\circ\bullet\overbrace{\circ\cdots\circ}^{n})}{\nu_\lambda(\bullet\bullet\underbrace{\circ\cdots\circ}_{n})}.$$

Remark the difference between $\overline{J}(n,0,0)$ and

$$\overline{J}(n,0) = \frac{\nu_\lambda(\bullet\overbrace{\circ\cdots\circ}^{n})}{\nu_\lambda(\bullet)}.$$

On the other hand, $\Omega^* h(\{1\}) = 0$ gives $\nu_\lambda(\bullet\bullet) = (2\lambda - 1)\nu_\lambda(\circ\bullet)$. By using these, we have

$$\begin{aligned}\nu_\lambda(\bullet\overbrace{\circ\cdots\circ}^{n}) &= \frac{\nu_\lambda(\bullet\bullet\overbrace{\circ\cdots\circ}^{n})}{\nu_\lambda(\bullet\bullet)} \times \nu_\lambda(\bullet\bullet) + \frac{\nu_\lambda(\circ\bullet\overbrace{\circ\cdots\circ}^{n})}{\nu_\lambda(\circ\bullet)} \times \nu_\lambda(\circ\bullet) \\ &= \overline{J}(n,0,0)(2\lambda-1)\nu_\lambda(\circ\bullet) + \gamma(n)\overline{J}(n,0,0)(2\lambda-1)\nu_\lambda(\circ\bullet).\end{aligned}$$

So we have

$$\nu_\lambda(\bullet\overbrace{\circ\cdots\circ}^{n}) = (2\lambda-1)\nu_\lambda(\circ\bullet)[1+\gamma(n)]\overline{J}(n,0,0). \tag{2.30}$$

By using Eq.(2.30), the left-hand side of Eq.(2.29) is equal to

$$2\lambda(2\lambda-1)\nu_\lambda(\circ\bullet)[1+\gamma(n)]\overline{J}(n,0,0). \tag{2.31}$$

In the above argument, we used the following relation:

$$\frac{\nu_\lambda(\circ\bullet\overbrace{\circ\cdots\circ}^{n})}{\nu_\lambda(\circ\bullet)} = (2\lambda-1)\gamma(n)\overline{J}(n,0,0). \tag{2.32}$$

Next we consider the right-hand side of Eq.(2.29). To do this, we present the following conjecture which is similar to Conjecture 2.3.2:

Conjecture 2.3.5. *For $m, n \geq 1$,*

$$J(1,0)J(m,n) \leq J(m,1)J(n,0),$$

that is,

$$\nu_\lambda(\circ\bullet)\nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet\overbrace{\circ\cdots\circ}^{n}) \leq \nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet\circ)\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}\bullet).$$

From now on we assume this conjecture. For $k \in \{2, \dots, n-1\}$, we see

$$\begin{aligned}
\nu_\lambda(\overbrace{\circ\cdots\circ}^{k-1}\bullet\overbrace{\circ\cdots\circ}^{n-k}) &\le \frac{\nu_\lambda(\overbrace{\circ\cdots\circ}^{k-1}\bullet\circ)}{\nu_\lambda(\bullet\circ)} \times \nu_\lambda(\bullet\overbrace{\circ\cdots\circ}^{n-k}) \\
&= (2\lambda-1)^2\nu(\circ\bullet)\gamma(k-1) \\
&\qquad \times [1+\gamma(n-k)]\overline{J}(k-1,0,0)\overline{J}(n-k,0,0).
\end{aligned}$$

The first inequality comes from Conjecture 2.3.5. The second equality can be obtained by Eqs.(2.30) and (2.32). On the other hand, for $k=1$ or n, Eq.(2.30) gives

$$\nu_\lambda(\bullet\overbrace{\circ\cdots\circ}^{n-1}) = (2\lambda-1)\nu_\lambda(\circ\bullet)[1+\gamma(n-1)]\overline{J}(n-1,0,0).$$

Therefore by using these facts we have

$$\begin{aligned}
2\lambda[1+\gamma(n)]\overline{J}(n,0,0) \le 2[1+\gamma(n-1)]\overline{J}(n-1,0,0) \\
+ (2\lambda-1)\sum_{k=2}^{n-1}\gamma(k-1)[1+\gamma(n-k)] \\
\times \overline{J}(k-1,0,0)\overline{J}(n-k,0,0).
\end{aligned}$$

Then we obtain the following lemma:

Lemma 2.3.6. *Assume Conjecture 2.3.5. Then for $n \ge 3$,*

$$\begin{aligned}
2\lambda\overline{J}(n,0,0) &\le 2\Big[\frac{1+\gamma(n-1)}{1+\gamma(n)}\Big]\overline{J}(n-1,0,0) \\
&\quad + \sum_{k=2}^{n-1}\Big[(2\lambda-1)\frac{\gamma(k-1)[1+\gamma(n-k)]}{1+\gamma(n)}\Big]\overline{J}(k-1,0,0)\overline{J}(n-k,0,0), \\
\lambda\overline{J}(2,0,0) &= \overline{J}(1,0,0), \\
\lambda\overline{J}(1,0,0) &= \overline{J}(0,0,0), \\
\overline{J}(0,0,0) &= 1.
\end{aligned}$$

In the previous section, we obtained

$$2\lambda F_1(n+1) = 2F_1(n) + \alpha\sum_{k=2}^{n-1}F_1(k)F_1(n+1-k).$$

Comparing with this equation, we present the following two assumptions:

$$(A1) \qquad \frac{1+\gamma(n-1)}{1+\gamma(n)} \le 1, \quad \text{i.e.,} \quad \gamma(n-1) \le \gamma(n) \quad \text{for any } n \ge 3.$$

$(A2)$

$$(2\lambda-1)\frac{\gamma(k-1)[1+\gamma(n-k)]}{1+\gamma(n)} \le \alpha = \frac{2(2\lambda-1)}{4\lambda-1} \quad \text{for any } 2 \le k \le n-1 \text{ and } n \ge 3.$$

The assumption (A2) is equivalent to

$$(A2') \qquad \frac{\gamma(k-1)[1+\gamma(n-k)]}{1+\gamma(n)} \le \frac{2}{4\lambda-1} \quad \text{for any } 2 \le k \le n-1 \text{ and } n \ge 3.$$

Under these assumptions (A1) and (A2), Lemma 2.3.6 gives

$$2\lambda\overline{J}(n,0,0) \le 2\overline{J}(n-1,0,0) + \alpha\sum_{k=2}^{n-1}\overline{J}(k-1,0,0)\overline{J}(n-k,0,0). \tag{2.33}$$

Let

$$\varphi(u) = \sum_{n=0}^{\infty}\overline{J}(n,0,0)u^{n+1}. \tag{2.34}$$

In the previous section, we introduced

$$\phi(u) = \sum_{n=0}^{\infty}F_1(n+1)u^{n+1}.$$

By Eqs.(2.33) and (2.34),

$$\alpha\varphi^2(u) - 2\Big[\lambda + (\alpha-1)u\Big]\varphi(u) + 2\lambda u + \Big[\alpha - 2 + 2\lambda\overline{J}(1,0,0)\Big]u^2 \ge 0. \tag{2.35}$$

Here we add the following assumption:

$$(A3) \qquad \overline{J}(1,0,0) \le F_1(2) = \frac{4\lambda-1}{(4\lambda+1)(2\lambda-1)}.$$

Therefore, under this assumption, Eq.(2.35) gives

$$\alpha\varphi^2(u) - 2\Big[\lambda + (\alpha-1)u\Big]\varphi(u) + 2\lambda u + \Big[\alpha - 2 + 2\lambda F_1(2)\Big]u^2 \ge 0. \tag{2.36}$$

On the other hand, we consider the relation between ρ_λ and $\varphi(1)$. For $\lambda > \lambda_c$, we begin by computing

$$\frac{1}{\rho_\lambda} = \frac{1}{\nu_\lambda(\bullet)} = \frac{1}{\nu_\lambda(\bullet)}\sum_{n=0}^{\infty}\nu_\lambda(\bullet\overbrace{\circ\cdots\circ}^{n}),$$

since

$$\sum_{n=0}^{\infty}\nu_\lambda(\bullet\overbrace{\circ\cdots\circ}^{n}) = \sum_{n=0}^{\infty}\Big\{\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}) - \nu_\lambda(\overbrace{\circ\cdots\circ}^{n+1})\Big\} = 1 - \lim_{n\to\infty}\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}) = 1,$$

for $\lambda > \lambda_c$. From this and Eq.(2.30),

$$\begin{aligned}\frac{1}{\rho_\lambda} &= \frac{(2\lambda-1)\nu_\lambda(\circ\bullet)}{\nu_\lambda(\bullet)} \times \sum_{n=0}^{\infty}[1+\gamma(n)]\overline{J}(n,0,0) \\ &= \frac{2\lambda-1}{2\lambda} \times \sum_{n=0}^{\infty}[1+\gamma(n)]\overline{J}(n,0,0),\end{aligned} \tag{2.37}$$

since the second equality comes from $\nu_\lambda(\bullet) = 2\lambda\nu_\lambda(\circ\bullet)$. Finally we present the next assumption:

$$(A4) \qquad \gamma(n) \le \frac{1}{2\lambda-1} \quad \text{for any } n \ge 0.$$

Under assumption (A4), Eq.(2.37) and the definition of $\varphi(1)$ give

$$\frac{1}{\rho_\lambda} \le \frac{2\lambda-1}{2\lambda} \times \frac{2\lambda}{2\lambda-1}\sum_{n=0}^{\infty}\overline{J}(n,0,0) = \varphi(1). \tag{2.38}$$

Taking $u=1$ in Eq.(2.36) implies

$$\Big[\alpha - 2 + 2\lambda F_1(2)\Big]\Big[\frac{1}{\varphi(1)}\Big]^2 - 2[\lambda+\alpha-1]\Big[\frac{1}{\varphi(1)}\Big] + \alpha \ge 0.$$

Then

$$\Big[\frac{1}{\varphi(1)} - \xi_1\Big]\Big[\frac{1}{\varphi(1)} - \xi_2\Big] \ge 0 \quad \text{with} \quad \xi_1 = \rho_\lambda^{(HL,2)} \ge \xi_2.$$

From this and Eq.(2.38), we have

$$\rho_\lambda \ge \frac{1}{\varphi(1)} \ge \xi_1 = \rho_\lambda^{(HL,2)}.$$

Therefore, under Conjecture 2.3.5 and assumptions (A1)-(A4), we have

$$\rho_\lambda \ge \rho_\lambda^{(HL,2)}.$$

Exercise 2.6. Show that

$$\gamma(0) = \frac{1}{2\lambda-1}. \tag{2.39}$$

Exercise 2.7. Verify that the assumption (A3) is equivalent to

$$\frac{2}{4\lambda-1} \le \gamma(1). \tag{2.40}$$

By using Eqs.(2.39) and (2.40) in exercises above, the following result is obtained:

Theorem 2.3.7. *Assume that for* $\lambda \geq \lambda_c$,

$$(1) \qquad \nu_\lambda(\circ\bullet)\nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet\overbrace{\circ\cdots\circ}^{n}) \leq \nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet\circ)\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}\bullet) \quad \text{for any } m, n \geq 1,$$

$$(2) \qquad \gamma(2) \leq \gamma(3) \leq \cdots \leq \gamma(n) \leq \gamma(n+1) \leq \cdots \leq \frac{1}{2\lambda - 1},$$

$$(3) \qquad \frac{\gamma(k-1)[1+\gamma(n-k)]}{1+\gamma(n)} \leq \frac{2}{4\lambda - 1} \quad \text{for any } 2 \leq k \leq n-1 \text{ and } n \geq 3,$$

$$(4) \qquad \frac{2}{4\lambda - 1} \leq \gamma(1) \leq \frac{1}{2\lambda - 1},$$

where

$$\gamma(n) = \frac{\nu_\lambda(\circ\bullet\overbrace{\circ\cdots\circ}^{n})}{\nu_\lambda(\bullet\bullet\underbrace{\circ\cdots\circ}_{n})} \quad (n \geq 0).$$

Then we have

$$\lambda_c \leq \lambda_c^{(HL,2)} \approx 1.942,$$

where

$$\lambda_c^{(HL,2)} = \sup\{\lambda \geq 0 : 4\lambda^3 - 7\lambda^2 - 2\lambda + 1 \leq 0\},$$

and for $\lambda \geq \lambda_c^{(HL,2)}$,

$$\rho_\lambda \geq \rho_\lambda^{(HL,2)} = \frac{\lambda + \alpha - 1 + \sqrt{(\lambda+\alpha-1)^2 - \alpha\Big[2\lambda + \alpha - 2 + 2\lambda F_1(2)\Big]}}{2\lambda + \alpha - 2 + 2\lambda F_1(2)},$$

where

$$\alpha = \frac{4\lambda - 2}{4\lambda - 1} \quad \text{and} \quad F_1(2) = \frac{4\lambda - 1}{(4\lambda+1)(2\lambda-1)}.$$

Exercise 2.7. Show that the condition (1) in this theorem with $m = 1$ and $n = 2$ is equivalent to $\gamma(1) \geq \gamma(2)$.

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CHAPTER 3

DIFFUSIVE θ-CONTACT PROCESSES

3.1. Introduction

The one-dimensional *diffusive θ-contact process* is a continuous-time Markov process on state space $\{0,1\}^{\mathbf{Z}^1}$. The formal generator is given by

$$\begin{aligned}\Omega f(\eta) = \sum_{x\in\mathbf{Z}^1} & \lambda(1-\eta(x)) \\ & \times \Big[\eta(x-1)+\eta(x+1)-(2-\theta)\eta(x-1)\eta(x+1)\Big][f(\eta^x)-f(\eta)] \\ + & \sum_{x\in\mathbf{Z}^1} \eta(x)[f(\eta^x)-f(\eta)] \\ + & \sum_{x\in\mathbf{Z}^1}\sum_{y:|y-x|=1} D\eta(x)(1-\eta(y))[f(\eta^{xy})-f(\eta)],\end{aligned}$$

where $\lambda,\ \theta,\ D \geq 0$, $\eta^x(y)=\eta(y)$ for $y\neq x$, $\eta^x(x)=1-\eta(x)$, and $\eta^{xy}(x)=\eta(y)$, $\eta^{xy}(y)=\eta(x)$, $\eta^{xy}(u)=\eta(u)$ otherwise. That is,

$$\begin{array}{lll} 001 \to 011 & \text{at rate} & \lambda, \\ 100 \to 110 & \text{at rate} & \lambda, \\ 101 \to 111 & \text{at rate} & \theta\lambda, \\ \ \ 1 \ \to \ 0 & \text{at rate} & 1, \\ \ 01 \to 10 & \text{at rate} & D, \\ \ 10 \to 01 & \text{at rate} & D. \end{array}$$

Exercise 3.1. Verify that when $\theta=2$ and $D=0$ this process is equivalent to the basic contact process which was discussed in Chapters 1 and 2.

When $\theta=2$ and $D\geq 0$, this process will be called the diffusive contact process in one dimension and considered in next chapter.

The diffusive θ-contact process with λ, $D\geq 0$ and $\theta\geq 1$ is attractive, so the upper invariant measure is well defined:

$$\nu_{\lambda,\theta,D} = \lim_{t\to\infty}\delta_1 S(t),$$

where δ_1 is the pointmass on $\eta \equiv 1$ and $S(t)$ is the semigroup given by Ω. Then the order parameter $\rho_\lambda(\theta, D)$ and the critical value $\lambda_c(\theta, D)$ are defined as follows:

$$\begin{aligned}\rho_\lambda(\theta, D) &= \nu_{\lambda,\theta,D}\{\eta : \eta(x) = 1\},\\ \lambda_c(\theta, D) &= \inf\{\lambda \geq 0 : \rho_\lambda(\theta, D) > 0\}.\end{aligned}$$

Exercise 3.2. Verify that if λ, $D \geq 0$ and $\theta \geq 1$ then the diffusive θ-contact process is attractive.

Let Y be the collection of all finite subsets of $\mathbf{Z}^1$. We define

$$\sigma_{\lambda,\theta,D}(A) = \nu_{\lambda,\theta,D}\{\eta : \eta(x) = 1 \text{ for some } x \in A\},$$

for any $A \in Y$. Next we consider the coalescing duality of the diffusive θ-contact process. The generator of the coalescing dual process is given by

$$\begin{aligned}\Omega^* h(A) = (2-\theta)\lambda \sum_{x\in A} &\Big[h(A \cup \{x-1, x+1\}) - h(A)\Big]\\ &+ (\theta-1)\lambda \sum_{x\in A}\sum_{y:|y-x|=1} \Big[h(A\cup\{y\}) - h(A)\Big]\\ &+ \sum_{x\in A}\Big[h(A\setminus\{x\}) - h(A)\Big]\\ &+ D\sum_{x\in A}\sum_{y\notin A:|y-x|=1}\Big[h((A\cup\{y\})\setminus\{x\}) - h(A)\Big],\end{aligned}$$

for any $h \in Y^*$, where Y^* is the set of all $[0,1]$-valued measurable functions on Y. Therefore if $1 \leq \theta \leq 2$, then coalescing dualities exist. Remark that $\Omega^*\sigma_{\lambda,\theta,D}(A) = 0$ for any $A \in Y$. From now on we assume that $1 \leq \theta \leq 2$.

Exercise 3.3. Verify that the above Ω^* is the generator of the coalescing dual process.

3.2. Katori-Konno Method

As in the section 1.4, we will give lower bounds on critical values and upper bounds on order parameters for diffusive θ-contact processes in one dimension by the Katori-Konno method. Some parts of results in this section were given by Sato and Konno.[1]

3.2.1. *First bound by the Katori-Konno method*

Let $|A|$ be the cardinality of A. So we have

Theorem 3.2.1. *Assume that $1 \le \theta \le 2$ and $D \ge 0$. Let $\lambda_c^{(KK,1)}(\theta, D) = 1/2$. Then for $\lambda \ge \lambda_c^{(KK,1)}(\theta, D)$,*

$$\sigma_{\lambda,\theta,D}(A) \le h_{\lambda,\theta,D}^{(KK,1)}(A) \quad \text{for all } A \in Y,$$

where

$$h_{\lambda,\theta,D}^{(KK,1)}(A) = 1 - \left(\alpha_*^{(1)}(\theta, D)\right)^{|A|} \quad \text{and} \quad \alpha_*^{(1)}(\theta, D) = \frac{2}{\theta\lambda + \sqrt{\theta^2\lambda^2 + 4(2-\theta)\lambda}}.$$

In particular, for any $\lambda \ge 0$,

$$\rho_{\lambda,\theta,D} \le \rho_{\lambda,\theta,D}^{(KK,1)} = \frac{2(2\lambda - 1)}{(4-\theta)\lambda + \sqrt{\theta^2\lambda^2 + 4(2-\theta)\lambda}} \vee 0,$$

where $x \vee y$ is the maximum of x and y.

Exercise 3.4. Verify that when $\theta = 2$ this result is equivalent to Theorem 1.4.4 in Chapter 1.

Notice that these results are independent of stirring rate D. From now on we will sometimes omit θ and D, and superscript (1) as follows; $\alpha_* = \alpha_*^{(1)}(\theta, D)$.

Proof. First we should remark that the Harris lemma holds for diffusive θ-contact processes when $1 \le \theta \le 2$.

Step 1. We let $h(A) = 1 - \alpha^{|A|}$.

Step 2. Next we define $0 < \alpha_* < 1$ as the unique solution of

$$\Omega^* h(\{0\}) = 0,$$

that is,

$$\Big[(2-\theta)\lambda\alpha^2 + \theta\lambda\alpha - 1\Big](1-\alpha) = 0.$$

Then $f(0) = -1$ and $f(1) = 2\lambda - 1$ implies $0 < \alpha_* < 1$ for $\lambda > 1/2$. In fact we have

$$\alpha_* = \frac{2}{\theta\lambda + \sqrt{\theta^2\lambda^2 + 4(2-\theta)\lambda}}$$

and let $h(A) = 1 - \alpha_*^{|A|}$.

Step 3. We check conditions (1)-(3) as follows. For $\lambda > 1/2$, we have $0 < \alpha_* < 1$. So conditions (1) and (3) are trivial. Condition (2) is equivalent to $0 \le \alpha_*^{|A|} < 1$ for any $A \in Y$ with $A \ne \phi$. This comes also from $0 < \alpha_* < 1$.

Step 4. We will give two different proofs; Proof A and Proof B.

Proof A. For $k = 0, 1, 2$ and $A \in Y$, let

$$A_k = \{x \in A : |\{y \in A : |y - x| = 1\}| = k\}.$$

The definitions give $A = A_0 + A_1 + A_2$. Therefore

$$\Omega^* h(A) = \sum_{k=0}^{2} R_k(A),$$

where

$$\begin{aligned} R_k(A) = (2-\theta)\lambda \sum_{x\in A_k} &\Big[h(A \cup \{x-1, x+1\}) - h(A)\Big] \\ &+ (\theta - 1)\lambda \sum_{x\in A_k} \sum_{y:|y-x|=1} \Big[h(A\cup\{y\}) - h(A)\Big] \\ &+ \sum_{x\in A_k} \Big[h(A\setminus\{x\}) - h(A)\Big] \\ &+ D \sum_{x\in A_k} \sum_{y\notin A:|y-x|=1} \Big[h((A\cup\{y\})\setminus\{x\}) - h(A)\Big]. \end{aligned}$$

If $R_k(A) \le 0$ for any $A \in Y$ and $k = 0, 1, 2$, then $\Omega^* h(A) \le 0$ for any $A \in Y$, that is, condition (4) is satisfied. From $h(A) = 1 - \alpha_*^{|A|}$, we have

$$\begin{aligned} R_k(A) &= (2-\theta)\lambda \sum_{x\in A_k} \Big[\alpha_*^{|A|} - \alpha_*^{|A\cup\{x-1,x+1\}|}\Big] \\ &\quad + (\theta-1)\lambda \sum_{x\in A_k} \sum_{y:|y-x|=1} \Big[\alpha_*^{|A|} - \alpha_*^{|A\cup\{y\}|}\Big] + \sum_{x\in A_k} \Big[\alpha_*^{|A|} - \alpha_*^{|A|-1}\Big] \\ &= (2-\theta)\lambda \sum_{x\in A_k} \Big[\alpha_*^{|A|} - \alpha_*^{|A|+2-k}\Big] \\ &\quad + (\theta-1)\lambda \sum_{x\in A_k} (2-k)\Big[\alpha_*^{|A|} - \alpha_*^{|A|+1}\Big] + \sum_{x\in A_k} \Big[\alpha_*^{|A|} - \alpha_*^{|A|-1}\Big] \\ &= |A_k| \Big\{ (2-\theta)\lambda\Big[\alpha_*^{|A|} - \alpha_*^{|A|+2-k}\Big] \\ &\qquad + (2-k)(\theta-1)\lambda\Big[\alpha_*^{|A|} - \alpha_*^{|A|+1}\Big] + \Big[\alpha_*^{|A|} - \alpha_*^{|A|-1}\Big]\Big\} \\ &= |A_k|\alpha_*^{|A|-1}\Big\{(2-\theta)\lambda[\alpha_* - \alpha_*^{3-k}] + (2-k)(\theta-1)\lambda[\alpha_* - \alpha_*^2] + [\alpha_* - 1]\Big\}. \end{aligned}$$

Therefore we have

$$\begin{aligned} R_0(A) &= |A_0|\alpha_*^{|A|-1}\Big\{(2-\theta)\lambda[\alpha_* - \alpha_*^3] + 2(\theta-1)\lambda[\alpha_* - \alpha_*^2] + [\alpha_* - 1]\Big\} \\ &= |A_0|\alpha_*^{|A|-1}\Omega^* h(\{0\}), \\ R_1(A) &= |A_1|\alpha_*^{|A|-1}\Big\{(2-\theta)\lambda[\alpha_* - \alpha_*^2] + (\theta-1)\lambda[\alpha_* - \alpha_*^2] + [\alpha_* - 1]\Big\} \\ &= |A_1|\alpha_*^{|A|-1}\Big[\Omega^* h(\{0\}) + \lambda\alpha_*(\alpha_* - 1)\{(2-\theta)\alpha_* + \theta - 1\}\Big], \\ R_2(A) &= |A_2|\alpha_*^{|A|-1}[\alpha_* - 1]. \end{aligned}$$

From these, we see that

$$\begin{aligned}\Omega^* h(A) &= \sum_{k=0}^{2} R_k(A)\\ &= |A_0|\alpha_*^{|A|-1}\Omega^* h(\{0\})\\ &\quad + |A_1|\alpha_*^{|A|-1}\Big[\Omega^* h(\{0\}) + \lambda\alpha_*(\alpha_*-1)\{(2-\theta)\alpha_* + \theta - 1\}\Big]\\ &\quad + |A_2|\alpha_*^{|A|-1}[\alpha_* - 1].\end{aligned}$$

By $\Omega^* h(\{0\}) = 0$,

$$\begin{aligned}\Omega^* h(A) &= |A_1|\alpha_*^{|A|-1}\lambda\alpha_*(\alpha_*-1)\Big\{(2-\theta)\alpha_* + \theta - 1\Big\} + |A_2|\alpha_*^{|A|-1}(\alpha_* - 1)\\ &= \alpha_*^{|A|-1}(\alpha_*-1)\Big[|A_1|\lambda\alpha_*\{(2-\theta)\alpha_* + \theta - 1\} + |A_2|\Big].\end{aligned}$$

Therefore $0 < \alpha_* < 1$ and $1 \le \theta \le 2$ imply $\Omega^* h(A) \le 0$ for any $A \in Y$. So the desired result is obtained.

Proof B. This proof is the almost same as Proof A. We let

$$\begin{aligned}A_{\circ\bullet\circ} &= \{x \in A : x-1, x+1 \notin A\},\\ A_{\circ\bullet\bullet} &= \{x \in A : x-1 \notin A, x+1 \in A\},\\ A_{\bullet\bullet\circ} &= \{x \in A : x-1 \in A, x+1 \notin A\},\\ A_{\bullet\bullet\bullet} &= \{x \in A : x-1, x+1 \in A\}.\end{aligned}$$

Remark that

$$A_0 = A_{\circ\bullet\circ}, \quad A_1 = A_{\circ\bullet\bullet} + A_{\bullet\bullet\circ}, \quad A_2 = A_{\bullet\bullet\bullet}.$$

By using these and a similar computation in Proof A, we have

$$\begin{aligned}\Omega^* h(A) &= |A_{\circ\bullet\circ}|\alpha_*^{|A|-1}\Omega^* h(\{0\})\\ &\quad + \Big\{|A_{\circ\bullet\bullet}| + |A_{\circ\bullet\bullet}|\Big\}\alpha_*^{|A|-1}\Big[\Omega^* h(\{0\}) + \lambda\alpha_*(\alpha_*-1)\{(2-\theta)\alpha_* + \theta - 1\}\Big]\\ &\quad + |A_{\bullet\bullet\bullet}|\alpha_*^{|A|-1}[\alpha_* - 1].\end{aligned}$$

The rest of proof is the same as Proof A, so we will omit it.

3.2.2. *Second bound by the Katori-Konno method*

Let $b(A)$ be the number of neighboring pairs of points in A, that is, $b(A) = |\{x \in \mathbf{Z} : \{x, x+1\} \subset A\}|$. Then we have

Theorem 3.2.2. *Assume that* $1 \le \theta \le 2$ *and* $D \ge 0$. *Let*

$$\lambda_c^{(KK,2)}(\theta, D) = \frac{D+1}{2D+1}.$$

Then for $\lambda \geq \lambda_c^{(KK,2)}(\theta, D)$,

$$\sigma_{\lambda,\theta,D}(A) \leq h_{\lambda,\theta,D}^{(KK,2)}(A) \quad \textit{for all } A \in Y,$$

where

$$h_{\lambda,\theta,D}^{(KK,2)}(A) = 1 - \left(\alpha_*^{(2)}(\theta, D)\right)^{|A|} \left(\beta_*^{(2)}(\theta, D)\right)^{b(A)},$$

with $0 < \alpha_*^{(2)}(\theta, D) < 1$, $\beta_*^{(2)}(\theta, D) > 1$, *and* $\alpha_*^{(2)}(\theta, D)$, $\beta_*^{(2)}(\theta, D)$ *are the unique solutions of*

$$\Omega^* h(\{0\}) = -(2-\theta)\lambda\alpha^3\beta^2 - 2(\theta-1)\lambda\alpha^2\beta + (\theta\lambda+1)\alpha - 1 = 0,$$
$$\Omega^* h(\{0,1\}) = \lambda\alpha^3\beta^2 - \Big[\lambda + D + 1\Big]\alpha^2\beta + \alpha\Big[1 + D\alpha\Big] = 0.$$

In particular, for any $\lambda \geq 0$,

$$\rho_\lambda(\theta, D) \leq \rho_\lambda^{(KK,2)}(\theta, D) = (1 - \alpha_*^{(2)}(\theta, D)) \vee 0,$$

where $x \vee y$ *is the maximum of* x *and* y.

Unfortunately, the explicit forms of $\alpha_*^{(2)}(\theta, D)$ and $\beta_*^{(2)}(\theta, D)$ are complicated, so we omit them. As a special case of this theorem, when $D = 0$, we get the following result;

Corollary 3.2.3. *Assume that* $1 \leq \theta \leq 2$ *and* $D = 0$. *Let* $\lambda_c^{(KK,2)}(\theta) = 1$. *Then for* $\lambda \geq \lambda_c^{(KK,2)}(\theta)$,

$$\sigma_{\lambda,\theta}(A) \leq h_{\lambda,\theta}^{(KK,2)}(A) \quad \textit{for all } A \in Y,$$

where

$$h_{\lambda,\theta}^{(KK,2)}(A) = 1 - \left[\frac{\lambda}{\theta\lambda^2 + (3-2\theta)\lambda - (2-\theta)}\right]^{|A|} \left[\frac{\theta\lambda^2 + (3-2\theta)\lambda - (2-\theta)}{\lambda^2}\right]^{b(A)}.$$

In particular, for any $\lambda \geq 0$,

$$\rho_\lambda(\theta) \leq \rho_\lambda^{(KK,2)}(\theta) = \frac{(\lambda-1)(\theta\lambda + 2 - \theta)}{\theta\lambda^2 + (3-2\theta)\lambda - (2-\theta)} \vee 0.$$

Exercise 3.5. Verify that when $\theta = 2$ this result is equivalent to Theorem 1.4.6 in Chapter 1.

In the non-diffusive case ($D = 0$), combining this corollary (upper bound) with Corollary 3.3.2 (lower bound) implies that for any $\lambda \geq \bar{\lambda}_c$,

$$\frac{\lambda(\theta\lambda + 2 - \theta)}{2\theta\lambda^2 + 3(2-\theta)\lambda - (2-\theta)}\left[1 + \frac{1}{\lambda}\sqrt{\frac{\theta\lambda^3 - (3\theta-2)\lambda^2 - 3(2-\theta)\lambda + (2-\theta)}{\theta\lambda + (2-\theta)}}\right]$$
$$\leq \rho_\lambda(\theta) \leq \frac{(\lambda-1)(\theta\lambda + 2 - \theta)}{\theta\lambda^2 + (3-2\theta)\lambda - (2-\theta)},$$

where $\bar{\lambda}_c$ is the largest root of the cubic equation:

$$\theta\lambda^3 - (3\theta - 2)\lambda^2 - 3(2-\theta)\lambda + (2-\theta) = 0.$$

Proof of Theorem 3.2.2.

Step 1. We let $h(A) = 1 - \alpha^{|A|}\beta^{b(A)}$.

Step 2. Next we define $0 < \alpha_* < 1$ and $\beta_* > 1$ as the unique solutions of

$$\Omega^* h(\{0\}) = 0,$$

$$\Omega^* h(\{0,1\}) = 0,$$

that is,

$$(2-\theta)\lambda\alpha^3\beta^2 + 2(\theta-1)\lambda\alpha^2\beta - (\theta\lambda+1)\alpha + 1 = 0, \tag{3.1}$$

$$\lambda\alpha^3\beta^2 - \Big[\lambda + D + 1\Big]\alpha^2\beta + \alpha\Big[1 + D\alpha\Big] = 0, \tag{3.2}$$

for $\lambda \geq \lambda_c^{(2)}(\theta, D)$. First we consider the definitions of α_* and β_*. To do so, we let $w = \alpha\beta$. Eqs.(3.1) and (3.2) can be rewritten as

$$\big[(2-\theta)\lambda w^2 + \theta\lambda w - 1\big]\,(w-1) = 1 - \beta, \tag{3.3}$$

$$(\lambda w - 1)(w-1) = D\alpha(\beta - 1), \tag{3.4}$$

respectively. Combining Eq.(3.3) with Eq.(3.4) implies that w satisfies the following equation of fourth order: $(w-1)f(w) = 0$, where

$$\begin{aligned} f(w) = (2-\theta)\lambda^2 w^3 &+ [2(\theta-1)\lambda + (\theta-2)(1+D)]\lambda w^2 \\ &- [\theta\lambda + (2\theta-1) + \theta D]\lambda w + \theta\lambda + 1 + D. \end{aligned} \tag{3.5}$$

So it is easily checked that

$$f(1) < 0 \quad \Longleftrightarrow \quad \lambda > \lambda_c^{(KK,2)}(\theta, D) = \frac{D+1}{2D+1},$$

where $A \Longleftrightarrow B$ means that A *is equivalent to* B. Note that $f(0) > 0$. So if $\lambda > \lambda_c^{(2)}(\theta, D)$, we see that there must be only one root in the interval (0,1). We will use w_* for this root. The definition of it gives $0 < w_* < 1$ for $\lambda > \lambda_c^{(2)}(\theta, D)$. Using Eq.(3.3) and w_*, we define β_* as follows:

$$\beta_* = [\lambda\{(2-\theta)w_* + \theta\}(1-w_*) + 1]w_*. \tag{3.6}$$

Since $0 < w_* < 1$ and Eq.(3.6), we see $\beta_* > 0$. So from $w = \alpha\beta$, we let

$$\alpha_* = \frac{w_*}{\beta_*}. \tag{3.7}$$

Since β_*, $w_* > 0$ and Eq.(3.7), we have $\alpha_* > 0$. To prove Theorem 3.2.2, we prepare the following results.

Lemma 3.2.4. *We assume* $1 \le \theta \le 2$ *and* $D > 0$. *Let* $\lambda > \lambda_c^{(KK,2)}(\theta, D) = (D+1)/(2D+1)$. *Then*

(1)
$$0 < \alpha_* < 1,$$

(2)
$$\beta_* > 1,$$

(3)
$$0 < w_* \le \left(\frac{1}{\lambda}\right) \wedge 1,$$

where $x \wedge y$ *is the minimum of* x *and* y,

(4)
$$0 < \alpha_* \beta_*^2 < 1.$$

Proof of Lemma 3.2.4. Eq.(3.1) can be rewritten as

$$\lambda w[(2-\theta)w + \theta](w-1) = \beta(\alpha - 1). \tag{3.8}$$

By Eq.(3.8), if w, $\beta > 0$, then

$$\mathrm{sgn}(1-w) = \mathrm{sgn}(1-\alpha), \tag{3.9}$$

where $\mathrm{sgn}(x) = 1$ if $x > 0$, $= 0$ if $x = 0$, $= -1$ if $x < 0$. By Eqs.(3.3) and (3.4),

$$\lambda w[(2-\theta)w + \theta - 1](w-1) = [1 + D\alpha](1-\beta). \tag{3.10}$$

So, by Eq.(3.10), if w, $\alpha > 0$, then

$$\mathrm{sgn}(1-w) = \mathrm{sgn}(\beta - 1). \tag{3.11}$$

Combining Eq.(3.9) with Eq.(3.11) implies that if α, $\beta > 0$, then

$$\mathrm{sgn}(1-w) = \mathrm{sgn}(1-\alpha) = \mathrm{sgn}(\beta - 1). \tag{3.12}$$

So the proofs of parts (1) and (2) follow from α_*, $\beta_* > 0$ and $w_* < 1$. Next, by the assumptions and a direct computation, we get

$$f\left(\frac{1}{\lambda}\right) = -\frac{(2-\theta) + (\theta-1)\lambda}{\lambda} D < 0.$$

From this fact and $w_* < 1$, part (3) is obtained. To show part (4), we observe that Eq.(3.3) gives

$$1 - \alpha\beta^2 = 1 - w\beta = [w-1]\left[(2-\theta)\lambda w^3 + \theta\lambda w^2 - w - 1\right].$$

Let $g(w) = (2-\theta)\lambda w^3 + \theta\lambda w^2 - w - 1$. So it is enough to show $g(w_*) < 0$. Then we have

$$g\left(\frac{1}{\lambda}\right) = \frac{(1-\lambda)(\lambda+2-\theta)}{\lambda^2},$$

and

$$g(1) = 2(\lambda - 1).$$

In the case of $\lambda \geq 1$, from part (3), $g(1/\lambda) \leq 0$, and $g(1) \geq 0$, we get $g(w_*) < 0$. On the other hand, when $\lambda_c^{(2)}(\theta, D) < \lambda < 1$, we see that $g(w_*) < 0$ follows from part (3), $g(1/\lambda) > 0$ and $g(1) < 0$. So the proof of part (4) is complete.

This lemma (1) and (2) complete Step 2.

Step 3. We check conditions (1)-(3) in the Harris lemma by using Lemma 3.2.4. Assume that $\lambda > \lambda^{(KK,2)}(\theta, D)$. Condition (1) and $h(A) \leq 1$ in condition (2) are trivial from Lemma 3.2.4 (1) and (2). The positivity of $h(A)$ for non-empty set $A \in Y$ is equivalent to $\alpha_*^{|A|}\beta_*^{b(A)} < 1$. This also comes from Lemma 3.2.4 (1) and (2). On the other hand, we have $b(A) \leq 2|A|$ for any $A \in Y$. Combining this result with Lemma 3.2.4 (2) and (4) gives

$$\alpha_*^{|A|}\beta_*^{b(A)} \leq (\alpha_*\beta_*^2)^{|A|} < 1,$$

for non-empty set $A \in Y$. Similarly, for condition (3), we get

$$h(A) \geq 1 - \alpha_*^{|A|}\beta_*^{b(A)} \geq 1 - (\alpha_*\beta_*^2)^{|A|}.$$

So $h(A) \to 1$ as $|A| \to \infty$ is due to Lemma 3.2.4 (4) and $h(A) \leq 1$.

Step 4. As usual, we will give two proofs; Proof A and Proof B.

Proof A. For any $A \in Y$, we let

$$A_k = \big\{x \in A : |\{y \in A : |y-x| = 1\}| = k\big\},$$

where $k = 0,\ 1,\ 2$. Then

$$\Omega^* h(A) = \sum_{k=0}^{2} R_k(A),$$

where

$$\begin{aligned} R_k(A) = (2-\theta)\lambda \sum_{x\in A_k} &\Big[h(A \cup \{x-1, x+1\}) - h(A)\Big] \\ &+ (\theta-1)\lambda \sum_{x\in A_k}\sum_{y:|y-x|=1} \Big[h(A\cup\{y\}) - h(A)\Big] \\ &+ \sum_{x\in A_k}\Big[h(A\setminus\{x\}) - h(A)\Big] \\ &+ D\sum_{x\in A_k}\sum_{y\notin A:|y-x|=1}\Big[h((A\cup\{y\})\setminus\{x\}) - h(A)\Big]. \end{aligned}$$

To check condition (4) in the Harris lemma, it is enough to show that $R_k(A) \le 0$ for $k = 0$, 1, 2 and $A \in Y$.
(i) $R_0(A)$: Using Eq.(4.1) and $b((A \cup \{y\}) \setminus \{x\}) \ge b(A) + 1$ for $x \in A_0$ and $y \notin A$ with $|y - x| = 1$, we have

$$R_0(A) \le |A_0| \alpha_*^{|A|} \beta_*^{b(A)} (1 - \beta_*) \big[\lambda \alpha_* \beta_* \big\{ (2 - \theta) \alpha_* \beta_* (2 + \beta_*) + 3(\theta - 1) \big\} + 3D \big]. \quad (3.13)$$

Combining Eq.(3.13) with Lemma 3.2.4 (2) gives $R_0(A) \le 0$.
(ii) $R_1(A)$: Similarly we have

$$R_1(A) \le |A_1| \alpha_*^{|A|} \beta_*^{b(A)-1} (1 - \beta_*)(\lambda \alpha_* \beta_*^2 + D). \quad (3.14)$$

Combining Eq.(3.14) with Lemma 3.2.4 (2) gives $R_1(A) \le 0$.
(iii) $R_2(A)$: As in the previous cases, we have

$$R_2(A) = |A_2| \alpha_*^{|A|-1} \beta_*^{b(A)-2} (\alpha_* \beta_*^2 - 1). \quad (3.15)$$

So $R_2(A) \le 0$ follows from Eq.(3.15) and Lemma 3.2.4 (4).
Proof B. As in the Proof B of the basic contact process, we divide A into the following 9 disjoint subsets:

$$\begin{aligned}
A_{\circ\circ\bullet\circ\circ} &= \{x \in A : x-2, x-1, x+1, x+2 \notin A\}, \\
A_{\circ\circ\bullet\circ\bullet} &= \{x \in A : x-2, x-1, x+1 \notin A, x+2 \in A\}, \\
A_{\bullet\circ\bullet\circ\circ} &= \{x \in A : x-2 \in A, x-1, x+1, x+2 \notin A\}, \\
A_{\bullet\circ\bullet\circ\bullet} &= \{x \in A : x-2 \in A, x-1, x+1 \notin A, x+2 \in A\}, \\
A_{\circ\circ\bullet\bullet\times} &= \{x \in A : x-2, x-1 \notin A, x+1 \in A\}, \\
A_{\times\bullet\bullet\circ\circ} &= \{x \in A : x-1 \in A, x+1, x+2 \notin A\}, \\
A_{\bullet\circ\bullet\bullet\times} &= \{x \in A : x-2 \in A, x-1 \notin A, x+1 \in A\}, \\
A_{\times\bullet\bullet\circ\bullet} &= \{x \in A : x-1 \in A, x+1 \notin A, x+2 \in A\}, \\
A_{\times\bullet\bullet\bullet\times} &= \{x \in A : x-1 \in A, x+1 \in A\}.
\end{aligned}$$

We begin by computing

$$\begin{aligned}
\Omega^* h(A) = {} & |A_{\circ\circ\bullet\circ\circ}| \alpha_*^{|A|-1} \beta_*^{b(A)} \Omega^* h(\{0\}) \\
& + \Big\{ |A_{\circ\circ\bullet\circ\bullet}| + |A_{\bullet\circ\bullet\circ\circ}| \Big\} \alpha_*^{|A|-1} \beta_*^{b(A)} \\
& \quad \times \Big[\Omega^* h(\{0\}) + \lambda \alpha_*^2 \beta_* (1 - \beta_*) \Big\{ (2 - \theta) \alpha_* \beta_* + \theta - 1 \Big\} + D \alpha_* (1 - \beta_*) \Big] \\
& + |A_{\bullet\circ\bullet\circ\bullet}| \alpha_*^{|A|-1} \beta_*^{b(A)} \Big[\Omega^* h(\{0\}) + 2(\theta - 1) \lambda \alpha_*^2 \beta_* (1 - \beta_*) \\
& \qquad + (2 - \theta) \lambda \alpha_*^3 \beta_*^2 (1 - \beta_*^2) + 2D \alpha_* (1 - \beta_*) \Big] \\
& + \Big\{ |A_{\circ\circ\bullet\bullet\times}| + |A_{\times\bullet\bullet\circ\circ}| \Big\} \alpha_*^{|A|-2} \beta_*^{b(A)-1} \frac{1}{2} \Omega^* h(\{0, 1\}) \\
& + \Big\{ |A_{\bullet\circ\bullet\bullet\times}| + |A_{\times\bullet\bullet\circ\bullet}| \Big\} \alpha_*^{|A|-2} \beta_*^{b(A)-1} \\
& \quad \times \Big[\frac{1}{2} \Omega^* h(\{0, 1\}) + \alpha_*^2 (1 - \beta_*)(\lambda \alpha_* \beta_*^2 + D) \Big] \\
& + |A_{\times\bullet\bullet\bullet\times}| \alpha_*^{|A|-1} \beta_*^{b(A)-2} \Big(\alpha_* \beta_*^2 - 1 \Big).
\end{aligned}$$

Therefore $\Omega^* h(\{0\}) = \Omega^* h(\{0,1\}) = 0$ implies that

$$
\begin{aligned}
\Omega^* h(A) = \Big\{ &|A_{\circ\circ\bullet\circ\bullet}| + |A_{\bullet\circ\bullet\circ\circ}| \Big\} \alpha_*^{|A|-1} \beta_*^{b(A)} \\
&\times \Big[\lambda \alpha_*^2 \beta_* (1-\beta_*) \Big\{ (2-\theta)\alpha_* \beta_* + \theta - 1 \Big\} + D\alpha_*(1-\beta_*) \Big] \\
&+ |A_{\bullet\circ\bullet\circ\bullet}| \alpha_*^{|A|-1} \beta_*^{b(A)} \\
&\times \Big[2(\theta-1)\lambda\alpha_*^2 \beta_*(1-\beta_*) + (2-\theta)\lambda\alpha_*^3\beta_*^2(1-\beta_*^2) + 2D\alpha_*(1-\beta_*) \Big] \\
&+ \Big\{ |A_{\bullet\circ\bullet\bullet\times}| + |A_{\times\bullet\bullet\circ\bullet}| \Big\} \alpha_*^{|A|-2} \beta_*^{b(A)-1} \Big[\alpha_*^2 (1-\beta_*)(\lambda\alpha_*\beta_*^2 + D) \Big] \\
&+ |A_{\times\bullet\bullet\bullet\times}| \alpha_*^{|A|-1} \beta_*^{b(A)-2} \Big(\alpha_* \beta_*^2 - 1 \Big).
\end{aligned}
$$

By Lemma 3.2.4 (2) and (4), i.e., $\beta_* > 1$ and $\alpha_* \beta_*^2 < 1$, we have $\Omega^* h(A) \le 0$ for any $A \in Y$. So Proof B is complete.

3.3. Holley-Liggett Method

In this section we consider non-diffusive θ-contact process, i.e., $D = 0$. So in the rest of this section we assume that $D = 0$. By using the Holley-Liggett method, we will get lower bounds on survival probability $\sigma_\lambda(A)$ for the coalescing dual process of the θ-contact process. From now on we will omit θ in some notations, for example, $\lambda_c^{(HL)} = \lambda_c^{(HL)}(\theta)$.

The following result is obtained by Katori and Konno.[2]

Theorem 3.3.1. *Let $\lambda_c^{(HL)}$ be the largest root of the cubic equation:*

$$\theta\lambda^3 - (3\theta - 2)\lambda^2 - 3(2-\theta)\lambda + (2-\theta) = 0.$$

Then for $\lambda \ge \lambda_c^{(HL)}$,

$$h_\lambda^{(HL)}(A) \le \sigma_\lambda(A) \qquad \textit{for all } A \in Y,$$

where

$$h_\lambda^{(HL)}(A) = \mu\{\eta : \eta(x) = 1 \ \textit{ for some } x \in A\},$$

for a renewal measure μ on $\{0,1\}^{\mathbf{Z}}$ whose density f is given by $\Omega^* h_\lambda^{(HL)}(A) = 0$ for all A of the form $\{1, 2, \ldots, n\}$ $(n \ge 1)$.

This result corresponds to the first bound by the Holley-Liggett method in the case of the basic contact process (see Section 2.2). So when $\theta = 2$, Theorem 3.3.1 is equivalent to Theorem 2.2.1.

Sketch of Proof. As in the case of Chapter 2, we need the following 4 steps.

Step 1. First we choose a suitable form of $h_\lambda^{(HL)}(A)$.
Step 2. Next we decide $h_\lambda^{(HL)}(A)$ explicitly.
Step 3. Third we check conditions (1)-(3) in the Harris lemma.
Step 4. Finally we check condition (4) in the Harris lemma.

In this sketch, we will show only Steps 1 and 2. Concerning Steps 3 and 4, see Katori and Konno.[2]

Step 1. We choose h of the form

$$h(A) = \mu\{\eta : \eta(x) = 1 \text{ for some } x \in A\},$$

for a renewal measure μ on $\{0,1\}^{\mathbf{Z}}$.

Step 2. We decide the density f so that

$$\Omega^* h(\{1,2,\ldots,n\}) = 0,$$

for any $n \geq 1$, where

$$\begin{aligned}\Omega^* h(A) = (2-\theta)\lambda \sum_{x\in A} &\Big[h(A \cup \{x-1, x+1\}) - h(A)\Big] \\ &+ (\theta-1)\lambda \sum_{x\in A}\sum_{y:|y-x|=1}\Big[h(A\cup\{y\}) - h(A)\Big] \\ &+ \sum_{x\in A}\Big[h(A\setminus\{x\}) - h(A)\Big].\end{aligned}$$

The definition of Ω^* gives

$$\begin{aligned}&\Omega^* h(\{1,2,\ldots,n\}) \\ &\quad= \sum_{k=1}^{n}(2-\theta)\lambda\Big[h(\{1,2,\ldots,n\}\cup\{k-1,k+1\}) - h(\{1,2,\ldots,n\})\Big] \\ &\qquad+ \sum_{k=1}^{n}(\theta-1)\lambda\Big[h(\{1,2,\ldots,n\}\cup\{k-1\}) - h(\{1,2,\ldots,n\})\Big] \\ &\qquad+ \sum_{k=1}^{n}(\theta-1)\lambda\Big[h(\{1,2,\ldots,n\}\cup\{k+1\}) - h(\{1,2,\ldots,n\})\Big] \\ &\qquad+ \sum_{k=1}^{n}\Big[h(\{1,2,\ldots,n\}\setminus\{k\}) - h(\{1,2,\ldots,n\})\Big].\end{aligned}$$

Then we consider two cases, that is, $A = \{1\}$ and $A = \{1,\ldots,n\}$ for $n \geq 2$.
(i) $A = \{1\}$. We see that

$$\begin{aligned}\Omega^* h(\{1\}) = (2-\theta)\lambda&\Big[h(\{0,1,2\}) - h(\{1\})\Big] \\ &+ 2(\theta-1)\lambda\Big[h(\{1,2\}) - h(\{1\})\Big] \\ &+ \Big[h(\phi) - h(\{1\})\Big].\end{aligned}$$

Remark the following relations as in the case of Chapter 2:

$$\begin{aligned} h(\{1,2,3\}) &= 1 - \mu(\circ\circ\circ), \\ h(\{1,3\}) &= 1 - \mu(\circ\times\circ), \\ h(\{1,2,3\}) - h(\{1,3\}) &= \mu(\circ\times\circ) - \mu(\circ\circ\circ) = \mu(\circ\bullet\circ), \\ &\vdots \end{aligned}$$

Then we have

$$\Omega^* h(\{1\}) = (2-\theta)\lambda\mu(\circ\circ\bullet) + \theta\lambda\mu(\circ\bullet) - \mu(\bullet) = 0. \tag{3.16}$$

Furthermore, for $\lambda \geq 0$ with $\mu(\bullet) > 0$, we let

$$F(n) = \frac{\mu(\overbrace{\circ\cdots\circ}^{n-1}\bullet)}{\mu(\bullet)} \quad (n \geq 1).$$

By using this and Eq.(3.16), we have

$$(2-\theta)\lambda F(3) + \theta\lambda F(2) = F(1). \tag{3.17}$$

(ii) $A = \{1, 2, \ldots, n\}$ for $n \geq 2$. As in the case of the basic contact process in Chapter 2, we have

$$\Omega^* h(\{1,2,\ldots,n\}) = \mu(\bullet)\Big[2\lambda F(n+1) - \sum_{k=1}^{n} F(k)F(n+1-k)\Big],$$

for $n \geq 2$.

Therefore $\Omega^* h(\{1,2,\ldots,n\}) = 0$ for any $n \geq 1$ implies

Lemma 3.3.2.

$$\begin{aligned} (2-\theta)\lambda F(3) + \theta\lambda F(2) &= F(1). \\ 2\lambda F(n+1) &= \sum_{k=1}^{n} F(k)F(n+1-k) \quad (n \geq 2). \\ F(1) &= 1. \end{aligned}$$

Next we introduce the following generating function to get $F(n)$ explicitly;

$$\phi(u) = \sum_{n=1}^{\infty} F(n)u^n.$$

Note that $F(1) = 1$ is equivalent to

$$\left.\frac{d\phi(u)}{du}\right|_{u=0} = 1.$$

Let

$$\begin{aligned} x &= \frac{2-\theta}{\lambda} + (\theta - 1), \\ y &= (2-\theta) + \theta\lambda. \end{aligned} \tag{3.18}$$

So $\phi(u)$ satisfies

$$\phi^2(u) - \frac{2y}{1+x}\phi(u) + \frac{2y}{1+x}u + \frac{1-x}{1+x}u^2 = 0. \tag{3.19}$$

Exercise 3.6. Verify that this equation is equivalent to

$$\phi^2(u) - 2\lambda\phi(u) + 2\lambda u + \Big[2\lambda F(2) - 1\Big]u^2 = 0. \tag{3.20}$$

The unique solution of Eq.(3.19) is given by

$$\phi(u) = \frac{y}{1+x}\left[1 - \sqrt{1 - 2\frac{1+x}{y}u - \frac{1-x}{4(1+x)}\left(2\frac{1+x}{y}u\right)^2}\,\right]. \tag{3.21}$$

From now on we assume that $0 \le x \le 1$ and $y > 0$. Let

$$u_{\pm} = \frac{y}{1 + x \pm \sqrt{2(1+x)}},$$

so $u_- < 0 < u_+$. In fact the function (3.21) is real analytic only when $u_- < u < u_+$. This implies that if $u_+ < 1$ then there is no real solution $F(n)$ which is summable, since $\sum_{n=1}^{\infty} F(n) = \phi(1)$. On the other hand if $u_+ \ge 1$, i.e.,

$$y \ge 1 + x + \sqrt{2(1+x)}, \tag{3.22}$$

then we can obtain a real solution $F(n)$ by expanding Eq.(3.21) in a power series in u, which satisfies

$$\sum_{n=1}^{\infty} F(n)u^n = \frac{y}{1+x}\left[1 - \sqrt{1 - 2\frac{1+x}{y}u - \frac{1-x^2}{y^2}u^2}\,\right]. \tag{3.23}$$

We should remark that $u_+ \ge 1$, i.e., $y \ge 1 + x + \sqrt{2(1+x)}$ is equivalent to

$$\left(\lambda + \frac{2-\theta}{\theta}\right)\left[\theta\lambda^3 - (3\theta-2)\lambda^2 - 3(2-\theta)\lambda + (2-\theta)\right] \ge 0. \tag{3.24}$$

This inequality is equivalent to the nonnegativity of the discriminant of Eq.(3.19) (or Eq.(3.20)). If $\lambda > 0$ and $1 \le \theta \le 2$, then Eq.(3.24) gives the following cubic equation in Theorem 3.3.1:

$$\theta\lambda^3 - (3\theta-2)\lambda^2 - 3(2-\theta)\lambda + (2-\theta) \ge 0.$$

So we take $\lambda_c^{(HL)}$ as the largest root of the above cubic equation.

Exercise 3.7. Verify that when $\theta = 2$ the above inequality becomes $\lambda(\lambda - 2) \geq 0$.

Here we present useful formula to expand Eq.(3.23) in a power series in u. If $0 \leq x, s \leq 1$, then

$$\sqrt{1 - s - \frac{1-x}{4(1+x)} s^2} = 1 - \sum_{n=1}^{\infty} d_n s^n,$$

with

$$d_1 = \frac{1}{2} \qquad \text{and} \qquad d_n = \frac{1}{2^{2n-1}} \left(\frac{2}{1+x} \right)^{n/2} e^{i(n-2)\varphi} v(n, e^{-2i\varphi}) \quad (n \geq 2),$$

where

$$e^{i\varphi} = \sqrt{\frac{1+x}{2}} + i \sqrt{\frac{1-x}{2}},$$

and $v(n, z)$ is *Gauss's hypergeometric series* of the form

$$v(n, z) = F(-(n-2), -(n-1), 2; z) \qquad (n \geq 2).$$

Let $w(1, x) = 1$ and

$$w(n, x) = \left(\frac{1+x}{2} \right)^{n/2-1} e^{i(n-2)\varphi} v(n, e^{-2i\varphi}) \quad (n \geq 2).$$

Applying the above formula to Eq.(3.23), we have

Lemma 3.3.3. *If* $0 \leq x \leq 1$ *and* $y \geq 1 + x + \sqrt{2(1+x)}$, *then*

$$F(n) = \frac{w(n, x)}{y^{n-1}} \qquad (n \geq 1).$$

Therefore we can obtain the density $f(n)(= F(n) - F(n+1))$ by Lemma 3.3.3 immediately. The proof of the positivity of $f(n)$ comes from the explicit form of $F(n)$ in Lemma 3.3.3. See Katori and Konno[2] for details. So Steps 1 and 2 are complete.

To get $\rho_\lambda^{(HL)}$, we return to Eq.(3.20):

$$\phi^2(u) - 2\lambda\phi(u) + 2\lambda u + \Big[2\lambda F(2) - 1 \Big] u^2 = 0.$$

Using Lemma 3.3.3, we get

$$F(2) = \frac{1}{y} = \frac{1}{\theta\lambda + (2-\theta)}. \tag{3.25}$$

Combining Eq.(3.20) with Eq.(3.25) gives

$$\phi^2(u) - 2\lambda\phi(u) + 2\lambda u + \frac{(2-\theta)(\lambda-1)}{\theta\lambda+(2-\theta)}u^2 = 0. \tag{3.26}$$

We should remark that the above equation can be also given by Eq.(3.19). Noting

$$\rho_\lambda^{(HL)} = \mu(\bullet) = \frac{1}{\sum\limits_{n=1}^{\infty} F(n)} = \frac{1}{\phi(1)}, \tag{3.27}$$

we have

$$\left[2\lambda + \frac{(2-\theta)(\lambda-1)}{\theta\lambda+(2-\theta)}\right]\left\{\rho_\lambda^{(HL)}\right\}^2 - 2\lambda\rho_\lambda^{(HL)} + 1 = 0.$$

In this way we obtain the following corollary of Theorem 3.3.1.

Corollary 3.3.4. *Let $\lambda_c^{(HL)}$ be the largest root of the cubic equation:*

$$\theta\lambda^3 - (3\theta-2)\lambda^2 - 3(2-\theta)\lambda + (2-\theta) = 0.$$

Then for $\lambda_c \le \lambda_c^{(HL)}$, we have $\rho_\lambda \ge \rho_\lambda^{(HL)}$, where

$$\rho_\lambda^{(HL)} = \frac{\lambda(\theta\lambda+2-\theta)}{2\theta\lambda^2+3(2-\theta)\lambda-(2-\theta)} \times \left[1 + \frac{1}{\lambda}\sqrt{\frac{\theta\lambda^3-(3\theta-2)\lambda^2-3(2-\theta)\lambda+(2-\theta)}{\theta\lambda+(2-\theta)}}\right].$$

Exercise 3.8. Verify that when $\theta = 2$ this lower bound $\rho_\lambda^{(HL)}$ becomes the first bound for the basic contact process:

$$\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{2\lambda}}.$$

3.4. Correlation Identities and Inequalities

As in Section 2.3, we obtain easily the first Holley-Liggett bound by using correlation identities and assuming correlation inequalities for non-diffusive case ($D = 0$). Recall that for $k \ge 2$ and $n_i \ge 0$ ($i = 1, \ldots, k$), we let $J(n_1, n_2, \ldots, n_k)$ be the probability of having 1's at $n_1 + 1$, $n_1 + n_2 + 2$, $\ldots$, $n_1 + n_2 + \cdots + n_{k-1} + k - 1$ and 0's at all other sites in $[1, n_1 + n_2 + \cdots + n_k + k - 1]$ with respect to the upper invariant measure ν_λ. That is,

$$\begin{aligned} &J(n_1, n_2, \ldots, n_k) \\ &= \nu_\lambda\Big(\overbrace{\circ\cdots\circ}^{n_1}\bullet\overbrace{\circ\cdots\circ}^{n_2}\bullet\circ\ldots\circ\bullet\overbrace{\circ\cdots\circ}^{n_k}\Big) \\ &= E_{\nu_\lambda}\Bigg(\prod_{i=1}^{k-1}\Big\{\prod_{q_i=1}^{n_i}[1-\eta(n_1+\cdots+n_{i-1}+q_i+i-1)]\times\eta(n_1+\cdots+n_i+i)\Big\} \\ &\qquad \times \Big\{\prod_{q_k=1}^{n_k}[1-\eta(n_1+\cdots+n_{k-1}+q_k+k-1)]\Big\}\Bigg). \end{aligned}$$

For example,

$$\begin{aligned}
J(0,0) &= \nu_\lambda\{\eta : \eta(1) = 1\} = \nu_\lambda(\bullet) = \rho_\lambda, \\
J(1,2) &= \nu_\lambda\{\eta : \eta(1) = 0, \eta(2) = 1, \eta(3) = \eta(4) = 0\} = \nu_\lambda(\circ\bullet\circ\circ), \\
J(1,2,3) &= \nu_\lambda\left\{\eta : \begin{matrix} \eta(1) = 0, \eta(2) = 1, \eta(3) = \eta(4) = 0, \\ \eta(5) = 1, \eta(6) = \eta(7) = \eta(8) = 0 \end{matrix}\right\} \\
&= \nu_\lambda(\circ\bullet\circ\circ\bullet\circ\circ\circ).
\end{aligned}$$

So the definition of $J(n_1, n_2)$ gives the following correlation identities.

Lemma 3.4.1. *For* $n \geq 2$,

(1)
$$2\lambda J(n,0) = \sum_{k=1}^{n} J(k-1, n-k),$$

that is,

$$2\lambda\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}\bullet) = \sum_{k=1}^{n} \nu_\lambda(\overbrace{\circ\cdots\circ}^{k-1}\bullet\overbrace{\circ\cdots\circ}^{n-k}).$$

And

(2)
$$(2-\theta)\lambda J(2,0) + \theta\lambda J(1,0) = J(0,0),$$

that is,

$$(2-\theta)\lambda\nu_\lambda(\circ\circ\bullet) + \theta\lambda\nu_\lambda(\circ\bullet) = \nu_\lambda(\bullet).$$

For $\lambda > \lambda_c$ and m, $n \geq 0$, define

$$\begin{aligned}
\overline{J}(m,n) &= \frac{\nu_\lambda\left(\overbrace{\circ\cdots\circ}^{m}\bullet\overbrace{\circ\cdots\circ}^{n}\right)}{\nu_\lambda(\bullet)} \\
&= \frac{J(m,n)}{J(0,0)}.
\end{aligned}$$

Note that the definition of λ_c gives $J(0,0) = \nu_\lambda(\bullet) = \rho_\lambda > 0$ for $\lambda > \lambda_c$. Here we present the following conjecture as in the case of the basic contact process (see Conjecture 2.3.3):

Conjecture 3.4.2. *For* m, $n \geq 0$,

$$\overline{J}(m,n) \leq \overline{J}(m,0)\overline{J}(n,0).$$

Let

$$\varphi(u) = \sum_{n=0}^{\infty} \overline{J}(n,0)u^{n+1}.$$

In the previous section we introduced

$$\phi(u) = \sum_{n=0}^{\infty} F(n+1)u^{n+1}.$$

We should remark that

$$\overline{J}(n,0) = \frac{\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}\bullet)}{\nu_\lambda(\bullet)} \qquad \text{and} \qquad F(n+1) = \frac{\mu(\overbrace{\circ\cdots\circ}^{n}\bullet)}{\mu(\bullet)}.$$

Combining Lemma 3.4.1 with Conjecture 3.4.2 gives

$$\varphi^2(u) - 2\lambda\varphi(u) + 2\lambda u + \Big[2\lambda\overline{J}(1,0) - 1\Big]u^2 \geq 0.$$

On the other hand, if $\lambda > \lambda_c$, then

$$\varphi(1) = \sum_{n=0}^{\infty} \overline{J}(n,0) = \frac{1}{J(0,0)} = \frac{1}{\rho_\lambda}.$$

From these we have

$$\Big[2\lambda + \Big\{2\lambda\overline{J}(1,0) - 1\Big\}\Big]\rho_\lambda^2 - 2\lambda\rho_\lambda + 1 \geq 0. \tag{3.28}$$

Next we obtain $\overline{J}(1,0)$ explicitly. Lemma 3.4.1 (2) is equivalent to

$$(2-\theta)\lambda\overline{J}(2,0) + \theta\lambda\overline{J}(1,0) = 1. \tag{3.29}$$

Applying Lemma 3.4.1 (1) to $n = 2$, we have

$$\lambda\overline{J}(2,0) = \overline{J}(1,0). \tag{3.30}$$

Combining Eq.(3.29) with Eq.(3.30) gives

$$\overline{J}(1,0) = \frac{1}{\theta\lambda + (2-\theta)}\Big(= \frac{1}{y} = F(2)\Big). \tag{3.31}$$

By using Eqs.(3.28) and (3.31), we have

$$\Big[2\lambda + \frac{(2-\theta)(\lambda-1)}{\theta\lambda + (2-\theta)}\Big]\rho_\lambda^2 - 2\lambda\rho_\lambda + 1 \geq 0. \tag{3.32}$$

We should remark that $\rho_\lambda^{(HL)}$ satisfies the equality of Eq.(3.32). By the continuity and monotonicity of ρ_λ in λ, we have

$$\rho_\lambda \geq \frac{\lambda(\theta\lambda + 2 - \theta)}{2\theta\lambda^2 + 3(2-\theta)\lambda - (2-\theta)} \left[1 + \frac{1}{\lambda}\sqrt{\frac{\theta\lambda^3 - (3\theta-2)\lambda^2 - 3(2-\theta)\lambda + (2-\theta)}{\theta\lambda + (2-\theta)}} \right].$$

Note that this lower bound on ρ_λ is nothing but the Holley-Liggett bound which given in the previous section. So if we assume Conjecture 3.4.2, then this lower bound gives upper bound on the critical value:

$$\lambda_c \leq \lambda_c^{(HL)},$$

where $\lambda_c^{(HL)}$ be the largest root of the cubic equation:

$$\theta\lambda^3 - (3\theta-2)\lambda^2 - 3(2-\theta)\lambda + (2-\theta) = 0.$$

Therefore the following result is newly obtained in these notes.

Theorem 3.4.3. *Assume that*

$$\nu_\lambda(\bullet)\nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet\overbrace{\circ\cdots\circ}^{n}) \leq \nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet)\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}\bullet) \quad \textit{for any } m,n \geq 0.$$

Then we have

$$\lambda_c \leq \lambda_c^{(HL)} \qquad \textit{and} \qquad \rho_\lambda \geq \rho_\lambda^{(HL)} \quad (\lambda \geq \lambda_c^{(HL)}),$$

where $\lambda_c^{(HL)}$ be the largest root of the cubic equation:

$$\theta\lambda^3 - (3\theta-2)\lambda^2 - 3(2-\theta)\lambda + (2-\theta) = 0,$$

and

$$\begin{aligned}\rho_\lambda^{(HL)} &= \frac{\lambda(\theta\lambda + 2 - \theta)}{2\theta\lambda^2 + 3(2-\theta)\lambda - (2-\theta)} \\ &\quad \times \left[1 + \frac{1}{\lambda}\sqrt{\frac{\theta\lambda^3 - (3\theta-2)\lambda^2 - 3(2-\theta)\lambda + (2-\theta)}{\theta\lambda + (2-\theta)}} \right].\end{aligned}$$

References

1. K. Sato and N. Konno, Diffusive generalized contact processes and their coalescing dualities, *Proceedings of International Workshop on Soft Computing in Industry '96* (1996) 88-93.

2. M. Katori and N. Konno, Bounds for the critical line of the θ-contact process with $1 \leq \theta \leq 2$. *J. Phys. A* **26** (1993) 1-18.

CHAPTER 4

DIFFUSIVE CONTACT PROCESSES

4.1. Introduction

The *diffusive contact process* is a continuous-time Markov process on $\{0,1\}^{\mathbf{Z}^d}$, where $\mathbf{Z}^d$ is the d-dimensional integer lattice. The formal generator is given by

$$\Omega f(\eta) = \sum_{x\in\mathbf{Z}^d}\left[(1-\eta(x))\lambda \times \sum_{y\in\mathbf{Z}^d:|y-x|=1}\eta(y)+\eta(x)\right][f(\eta^x)-f(\eta)]$$
$$+ D\sum_{x\in\mathbf{Z}^d}\eta(x)\sum_{y\in\mathbf{Z}^d:|y-x|=1}(1-\eta(y))[f(\eta^{xy})-f(\eta)],$$

where $\lambda,\ D\geq 0,\ \eta\in\{0,1\}^{\mathbf{Z}^d},\ \eta^x(y)=\eta(y)$ if $y\neq x$, $\eta^x(x)=1-\eta(x)$, and $\eta^{xy}(x)=\eta(y)$, $\eta^{xy}(y)=\eta(x)$, $\eta^{xy}(u)=\eta(u)$ if $u\neq x,\ y$. Here $|x|=|x_1|+\cdots+|x_d|$ for $x=(x_1,\ldots,x_d)\in\mathbf{Z}^d$. We should remark that the first term of right-hand side is equivalent to the formal generator of the basic contact process and the second one is that of the exclusion process. As for exclusion processes, see Chapter VIII of Liggett,[1] for example.

This process is attractive, so the upper invariant measure is well defined: for λ, $D\geq 0$,

$$\nu_{\lambda,D}=\lim_{t\to\infty}\delta_1 S(t),$$

where δ_1 is the pointmass on $\eta\equiv 1$ and $S(t)$ is the semigroup given by Ω. Then the order parameter $\rho_\lambda(D)$ and the critical value $\lambda_c(D)$ are defined as follows:

$$\rho_\lambda(D)=\nu_{\lambda,D}\{\eta:\eta(x)=1\},$$
$$\lambda_c(D)=\inf\{\lambda\geq 0:\rho_\lambda(D)>0\}.$$

Let Y be the collection of all finite subsets of $\mathbf{Z}^d$. We define

$$\sigma_{\lambda,D}(A)=1-E_{\nu_{\lambda,D}}\left(\prod_{x\in A}(1-\eta(x))\right)=\nu_{\lambda,D}\{\eta:\eta(x)=1 \text{ for some } x\in A\},$$

for any $A \in Y$. We let $|A|$ be the cardinality of A and $b(A) = \big|\big\{\{x, y\} \subset A : |x - y| = 1\big\}\big|$. Note that $b(A)$ is the number of neighboring pairs of sites in A. When $D = 0$, i.e., basic contact process case, Katori and Konno[2] obtained the following second upper bounds on order parameters by using the Harris lemma. When $D = 0$, we sometimes omit D in some notations, for example, $\sigma_\lambda(A)$, ρ_λ, λ_c. Furthermore this chapter is devoted to just the Katori-Konno method, so we will omit superscript KK as follows: $\lambda_c^{(2)} = \lambda_c^{(KK,2)}$.

Theorem 4.1.1. *Assume that $D = 0$. Let $\lambda_c^{(2)} = 1/(2d-1)$. Then for $\lambda \geq \lambda_c^{(2)}$,*

$$\sigma_\lambda(A) \leq h_\lambda^{(2)}(A) \quad \text{for all } A \in Y,$$

where

$$h_\lambda^{(2)}(A) = 1 - \left(\frac{2d-1}{2d(2d-1)\lambda - 1}\right)^{|A|} \left(\frac{2d(2d-1)\lambda - 1}{(2d-1)^2\lambda}\right)^{b(A)}.$$

Remark that $\sigma_\lambda(A) \leq h_\lambda^{(2)}(A) \leq h_\lambda^{(1)}(A)$ for all $A \in Y$ and $\lambda_c^{(1)} \leq \lambda_c^{(2)} \leq \lambda_c$, where the first bound is

$$h_\lambda^{(1)}(A) = 1 - \left(\frac{1}{2d\lambda}\right)^{|A|} \qquad \text{and} \qquad \lambda_c^{(1)} = \frac{1}{2d}.$$

Noting $\rho_\lambda = \sigma_\lambda(\{0\})$, we have

Corollary 4.1.2. *For $\lambda \geq 0$,*

$$\rho_\lambda \leq \rho_\lambda^{(2)} = \left(\frac{2d[(2d-1)\lambda - 1]}{2d(2d-1)\lambda - 1}\right) \vee 0,$$

where $x \vee y$ is the maximum of x and y.

We should remark that

$$\rho_\lambda^{(2)} \leq \rho_\lambda^{(1)} = \left(\frac{2d\lambda - 1}{2d\lambda}\right) \vee 0.$$

The first bound $\rho_\lambda^{(1)}(D)$ of the diffusive contact process is equal to that of the basic contact process, since the stirring mechanisms preserve the cardinality of the set A. So we have $\rho_\lambda(D) \leq \rho_\lambda^{(1)}(D) = \rho_\lambda^{(1)} = [(2d\lambda - 1)/2d\lambda] \vee 0$. In this chapter we extend the above second bounds, $h_\lambda^{(2)}(A)$ and $\rho_\lambda^{(2)}$, to the diffusive case, i.e., $D > 0$. The following result is obtained by Konno and Sato.[3]

Theorem 4.1.3. *Assume that $D \geq 0$. Let*

$$\lambda_c^{(2)}(D) = \frac{(2d-1)D+1}{(2d-1)(2dD+1)}.$$

Then for $\lambda \geq \lambda_c^{(2)}(D)$,

$$\sigma_{\lambda,D}(A) \leq h_{\lambda,D}^{(2)}(A) \quad \text{for all } A \in Y,$$

where

$$h_{\lambda,D}^{(2)}(A) = 1 - \alpha_*^{|A|}\beta_*^{b(A)},$$

and α_* and β_* are the unique solutions of

$$\begin{aligned}\Omega^* h(\{0\}) &= -2d\lambda\alpha^2\beta + (2d\lambda+1)\alpha - 1 = 0,\\ \Omega^* h(\{0,1\}) &= -2\Big[(2d-1)\lambda\alpha^3\beta^2 - \{(2d-1)(\lambda+D)+1\}\alpha^2\beta + \alpha\{1+(2d-1)D\alpha\}\Big]\\ &= 0,\end{aligned}$$

with $0 < \alpha_* \leq 1$, $\beta_* \geq 1$. In particular, for λ, $D \geq 0$, we have

$$\rho_\lambda(D) \leq \rho_\lambda^{(2)}(D) = (1-\alpha_*) \vee 0 \leq \rho_\lambda^{(1)}(D) = \left(\frac{2d\lambda - 1}{2d\lambda}\right) \vee 0.$$

This theorem corresponds to the second bound by the Katori-Konno method. Unfortunately the explicit forms of α_* and β_* are complicated, so we omit them. If we consider non-diffusive case ($D = 0$), then α_* and β_* in Theorem 4.1.3 are equal to those in Theorem 4.1.1: that is, $\alpha_* = (2d-1)/[\,2d(2d-1)\lambda\,]$ and $\beta_* = [\,2d(2d-1)\lambda - 1\,]/[\,(2d-1)^2\lambda\,]$.

On the other hand, De Masi, Ferrari and Lebowitz[4] and Durrett and Neuhauser[5] studied mean field theorem on more general interacting particle systems with the stirring mechanisms. In our setting, as for critical values and order parameters, the following results were obtained by Durrett and Neuhauser:[5] $\lambda_c(D) \to 1/2d$, $\rho_\lambda(D) \to (2d\lambda-1)/2d\lambda$ as $D \to \infty$ for $d \geq 1$. Then, from Theorem 4.1.3, Konno and Sato[3] gave

Theorem 4.1.4. *Let $d \geq 1$. There is a constant $D_0 > 0$ so that if $D \geq D_0$, $C \geq 1/(2d)^2(2d-1)$ and*

$$\lambda - \frac{1}{2d} = \frac{C}{D},$$

then

$$\rho_\lambda(D) \leq \left[\left(1 - \frac{1}{(2d)^2(2d-1)C}\right)\frac{2d\lambda - 1}{2d\lambda}\right] \vee 0.$$

We should remark that Theorem 1.2 of Konno[6] showed that if $d \geq 3$, then $\lambda_c(D) - 1/2d \approx C/D$, where $\approx$ means that if C is small (large) then right-hand side is a lower (upper) bound for large D. Moreover Theorem 1.3 of Konno[6] is as follows: there are positive constants D_1, C_1 and $\theta(d)$ (depending only on the dimensionality d) so that if $D \geq D_1$, $C \geq C_1$ and $\lambda - 1/2d = C/D$, then

$$\left[\left(1 - \frac{1}{\theta(d)C}\right)\frac{2d\lambda - 1}{2d\lambda}\right] \vee 0 \leq \rho_\lambda(D).$$

Therefore combining this with Theorem 4.1.4, we have the following corollary immediately:

Corollary 4.1.5. *Let $d \geq 3$. There are positive constants D_2, C_2 and $\theta(d)$ (depending only on the dimensionality d) so that if $D \geq D_2$, $C \geq C_2$ and*

$$\lambda - \frac{1}{2d} = \frac{C}{D},$$

then

$$\left[\left(1 - \frac{1}{\theta(d)C}\right)\frac{2d\lambda - 1}{2d\lambda}\right] \vee 0 \leq \rho_\lambda(D) \leq \left[\left(1 - \frac{1}{(2d)^2(2d-1)C}\right)\frac{2d\lambda - 1}{2d\lambda}\right] \vee 0,$$

for any $\lambda \geq 0$.

Here we present an interesting open problem:

Open Problem 4.1. If $0 \leq D_1 \leq D_2$ then $\lambda_c(D_1) \geq \lambda_c(D_2)$ in any dimension. (Of course, if D is large then $\lambda_c(D) < \lambda_c(D = 0)$.)

Exercise 4.1. Explain that why we expect the above statement holds.

Finally we would like to discuss about the Holley-Liggett method in the diffusive contact process in one dimension. By a similar computation as in the case of the basic contact process, we have

$$2\lambda F(n+1) + 2DF(n+1) = \sum_{k=1}^{n} F(k)F(n+1-k) + 2DF(2)F(n) \qquad (n \geq 2),$$
$$2\lambda F(2) = F(1).$$

Let $\phi(u) = \sum_{n=1}^{\infty} F(n)u^n$. By using these,

$$\phi(u)^2 - 2\left[(\lambda + D) - DF(2)u\right]\phi(u) + 2(\lambda + D)u = 0.$$

So the nonnegativity of the discriminant of this equation with $u = 1$ is equivalent to

$$\frac{1}{(2\lambda)^2}\left[4\lambda^4 + 8(D-1)\lambda^3 + 4D(D-3)\lambda^2 - 4D^2\lambda + D^2\right] \geq 0.$$

(When $D = 0$, this becomes $\lambda(\lambda - 2) \geq 0$.) From this, we let $\lambda_c^{(HL)}(D)$ is the largest root of

$$4\lambda^4 + 8(D-1)\lambda^3 + 4D(D-3)\lambda^2 - 4D^2\lambda + D^2.$$

Then $\lambda_c^{(HL)}(D)$ converges to $1/2$ as $D \to \infty$. However the condition (4) in the Harris lemma does not hold for large D.

Exercise 4.2. Verify that $F(n)$ satisfy the above two equations.

This chapter is organized as follows. In Section 4.2, we will prove Theorem 4.1.3. Section 4.3 is devoted to proof of Theorem 4.1.4.

4.2. Proof of Theorem 4.1.3.

In this section we will prove Theorem 4.1.3 by using the following Harris lemma. For simplicity, we assume that λ, $D > 0$ in the rest of this chapter. Let Y^* denote the set of all $[0,1]$-valued measurable functions on Y. For any $h \in Y^*$, we let

$$\begin{aligned}\Omega^* h(A) = \lambda \sum_{x\in A} \sum_{y:|y-x|=1} \Big[h(A \cup \{y\}) - h(A)\Big] + \sum_{x \in A} \Big[h(A \setminus \{x\}) - h(A)\Big] \\ + D \sum_{x\in A} \sum_{y \notin A:|y-x|=1} \Big[h((A \cup \{y\}) \setminus \{x\}) - h(A)\Big].\end{aligned}$$

Note that $\Omega^* \sigma_{\lambda,D}(A) = 0$ for any $A \in Y$.

Lemma 4.2.1. *(Harris lemma) Let $h \in Y^*$ with*

(1) $$h(\phi) = 0,$$

(2) $$0 < h(A) \leq 1 \quad \text{for any } A \in Y \text{ with } A \neq \phi,$$

(3) $$\lim_{|A|\to\infty} h(A) = 1,$$

(4) $$\Omega^* h(A) \leq 0 \quad \text{for any } A \in Y.$$

Then

(5) $$\sigma_{\lambda,D}(A) \leq h(A) \quad \text{for any } A \in Y.$$

In particular,

(6) $$\rho_\lambda(D) \leq h(\{0\}).$$

In order to obtain upper bound on $\rho_\lambda(D)$, we need to look for a suitable upper bound $h(A)$ satisfying conditions (1)-(4) in the Harris lemma. To do this, we will need the following 4 steps.

Step 1. First we let $h(A) = 1 - \alpha^{|A|}\beta^{b(A)}$.

Step 2. Next we decide $0 < \alpha_* < 1$ and $\beta_* > 1$ as the unique solutions of

$$\Omega^* h(\{0\}) = \Omega^* h(\{0,1\}) = 0,$$

that is,

$$2d\lambda\alpha^2\beta - (2d\lambda + 1)\alpha + 1 = 0, \tag{4.1}$$

$$(2d-1)\lambda\alpha^3\beta^2 - [(2d-1)(\lambda + D) + 1]\,\alpha^2\beta + \alpha\,[1 + (2d-1)D\alpha] = 0, \tag{4.2}$$

for $\lambda > \lambda_c^{(2)}(D)$. To do so, we let $w = \alpha\beta$. Eqs.(4.1) and (4.2) can be rewritten as

$$(2d\lambda w - 1)(w - 1) = 1 - \beta, \tag{4.3}$$

$$[(2d-1)\lambda w - 1](w - 1) = (2d-1)D\alpha(\beta - 1), \tag{4.4}$$

respectively. Combining Eq.(4.3) with Eq.(4.4) implies that w satisfies the following cubic equation: $(w-1)f(w) = 0$, where

$$f(w) = 2d(2d-1)\lambda^2 w^2 - \lambda[2d(2d-1)(\lambda+D) + 4d - 1]w + 2d\lambda + 1 + (2d-1)D. \tag{4.5}$$

So it is easily checked that

$$f(1) < 0 \quad \Longleftrightarrow \quad \lambda > \lambda_c^{(2)}(D) = \frac{(2d-1)D + 1}{(2d-1)(2dD+1)},$$

where $A \Longleftrightarrow B$ means that A *is equivalent to* B. Note that $f(0) > 0$. So if $\lambda > \lambda_c^{(2)}(D)$, we see that there must be only one root in the interval $(0,1)$. We will use w_* for this root. The definition of it gives $0 < w_* < 1$ for $\lambda > \lambda_c^{(2)}(D)$. Using Eq.(4.3) and w_*, we define β_* as follows:

$$\beta_* = [2d\lambda(1 - w_*) + 1)]w_*. \tag{4.6}$$

Since $0 < w_* < 1$ and Eq.(4.6), we see $\beta_* > 0$. So from $w = \alpha\beta$, we let

$$\alpha_* = \frac{w_*}{\beta_*}. \tag{4.7}$$

Since β_*, $w_* > 0$ and Eq.(4.7), we have $\alpha_* > 0$. To prove Theorem 4.1.3, we prepare the following results.

Lemma 4.2.2. *We assume $D > 0$. Let*

$$\lambda > \lambda_c^{(2)}(D) = \frac{(2d-1)D + 1}{(2d-1)(2dD+1)} > 0.$$

Then

(1) $$0 < \alpha_* < 1,$$

(2) $$\beta_* > 1,$$

(3) $$\frac{1}{2d\lambda} < w_* < \left[\frac{1}{(2d-1)\lambda}\right] \wedge 1,$$

where $x \wedge y$ is the minimum of x and y,

(4) $$\alpha_* \beta_*^{2d} < 1.$$

Proof of Lemma 4.2.2. Eq.(4.1) can be rewritten as

$$2d\lambda w(w-1) = \beta(\alpha - 1). \tag{4.8}$$

By Eq.(4.8), if $w, \beta > 0$, then

$$\mathrm{sgn}(w-1) = \mathrm{sgn}(\alpha - 1), \tag{4.9}$$

where $\mathrm{sgn}(x) = 1$ if $x > 0$, $= 0$ if $x = 0$, $= -1$ if $x < 0$. On the other hand, combining Eq.(4.3) with Eq.(4.4) gives

$$\lambda w(w-1) = [1 + (2d-1)D\alpha](1-\beta). \tag{4.10}$$

So, by Eq.(4.10), if w, $\alpha > 0$, then

$$\mathrm{sgn}(w-1) = \mathrm{sgn}(1-\beta). \tag{4.11}$$

Combining Eq.(4.9) with Eq.(4.11) implies that if $\alpha, \beta > 0$, then

$$\mathrm{sgn}(w-1) = \mathrm{sgn}(\alpha - 1) = \mathrm{sgn}(1-\beta). \tag{4.12}$$

Since α_*, $\beta_* > 0$ and $w_* < 1$, parts (1) and (2) follow from Eq.(4.12). Next by the assumptions and a direct computation, we get

$$f\left(\frac{1}{2d\lambda}\right) = \lambda > \lambda_c^{(2)}(D) > 0, \qquad f\left(\frac{1}{(2d-1)\lambda}\right) = -D < 0.$$

From these facts and $w_* < 1$, part (3) is obtained. To show part (4), we observe that Eq.(4.3) gives

$$1 - \alpha\beta^{2d} = 1 - w\beta^{2d-1} = 1 - w^{2d}[(2d\lambda + 1) - 2d\lambda w]^{2d-1}.$$

Let $g(w) = w^{2d}[(2d\lambda + 1) - 2d\lambda w]^{2d-1}$. So it is enough to show $g(w_*) < 1$. Then we have

$$g'(w) = 2dw^{2d-1}[(2d\lambda + 1) - 2d\lambda w]^{2d-2}[2d\lambda + 1 - (4d-1)\lambda w].$$

So we see that 0, $w_1 = (2d\lambda+1)/[(4d-1)\lambda]$ and $w_2 = (2d\lambda+1)/(2d\lambda)$ are the solutions of $g'(w)=0$. Note that $0 < w_1 < w_2$. Moreover we should remark that

$$w_1 \geq 1 \quad \Longleftrightarrow \quad \lambda \leq \frac{1}{2d-1}.$$

So the above facts, $g(1)=1$ and $w_* < 1$ imply that if $\lambda \leq 1/(2d-1)$, then $g(w_*) < 1$. Next we consider the case of $\lambda > 1/(2d-1)$. In this case, $w_1 < 1$. Noting that $g(1)=1$ and $w_* < 1/[(2d-1)\lambda]$ in part (3), it is enough to show

$$g\left(\frac{1}{(2d-1)\lambda}\right) \leq 1. \tag{4.13}$$

Then Eq.(4.13) is equivalent to

$$[2d(2d-1)\lambda - 1]^{2d-1} \leq (2d-1)^{2d-1}[(2d-1)\lambda]^{2d} \quad \text{for } \lambda > 1/(2d-1). \tag{4.14}$$

Let $z=2d$, $u=(z-1)\lambda$ and

$$G(u) = u^z - \left(\frac{zu-1}{z-1}\right)^{z-1}.$$

So Eq.(4.14) is equivalent to $G(u) \geq 0$ for $u \geq 1$. Note that $G(1)=0$ and

$$G'(u) = z\left[u^{z-1} - \left(\frac{zu-1}{z-1}\right)^{z-2}\right].$$

If $z=2$, then $G'(u) = 2[u-1] \geq 0\,(u \geq 1)$. Therefore, when $z=2$, we have $G(u) \geq 0$ for $u \geq 1$. Next define

$$H(u) = u^{z-1} - \left(\frac{zu-1}{z-1}\right)^{z-2}.$$

Note that $G'(u) = zH(u)$ and $H(1)=0$. Then we have

$$\begin{aligned} H'(u) &= (z-1)u^{z-2} - (z-2)\left(\frac{zu-1}{z-1}\right)^{z-3} \times \frac{z}{z-1} \\ &\geq (z-1)u^{z-2} - (z-2)\left(\frac{zu-1}{z-1}\right)^{z-3} \times \frac{z-1}{z-2} \\ &= (z-1)\left[u^{z-2} - \left(\frac{zu-1}{z-1}\right)^{z-3}\right] \end{aligned}$$

If $z=3$, then $H'(u) \geq 2[u-1] \geq 0$ $(u \geq 1)$, so $H(u) \geq 0$ $(u \geq 1)$. Noting $G'(u) = zH(u)$, when $z=3$, we see that $G(u) \geq 0$ for $u \geq 1$. In a similar way, we can prove that if $z \geq 4$, then $G(u) \geq 0$ for $u \geq 1$. So the proof of part (4) is complete.

Lemma 4.2.2 (1) and (2) complete Step 2.

Step 3. We check conditions (1)-(3) in the Harris lemma by using Lemma 4.2.2. Assume that $\lambda > \lambda_c^{(2)}(D)$. Condition (1) and $h(A) \le 1$ in condition (2) are trivial from Lemma 4.2.2 (1) and (2). The positivity of $h(A)$ for non-empty set $A \in Y$ is equivalent to $\alpha_*^{|A|}\beta_*^{b(A)} < 1$. This also comes from Lemma 4.2.2 (1) and (2). On the other hand, we have $b(A) \le 2d|A|$ for any $A \in Y$. Combining this result with Lemma 4.2.4 (2) and (4) gives

$$\alpha_*^{|A|}\beta_*^{b(A)} \le (\alpha_*\beta_*^{2d})^{|A|} < 1,$$

for non-empty set $A \in Y$. Similarly, for condition (3), we get

$$h(A) \ge 1 - \alpha_*^{|A|}\beta_*^{b(A)} \ge 1 - (\alpha_*\beta_*^{2d})^{|A|}.$$

So $h(A) \to 1$ as $|A| \to \infty$, since Lemma 4.2.2 (4) and $h(A) \le 1$.

Step 4. In this case we will give just Proof A, because Proof B is complicated compared with Proof A. For any $A \in Y$, we let

$$A_k = \big\{x \in A : |\{y \in A : |y - x| = 1\}| = k\big\},$$

where $k \in \{0, 1, \dots, 2d\}$. Then

$$\Omega^* h(A) = \sum_{k=0}^{2d} R_k(A),$$

where

$$\begin{aligned} R_k(A) = \lambda \sum_{x \in A_k} \sum_{y : |y-x|=1} \Big[h(A \cup \{y\}) - h(A)\Big] + \sum_{x \in A_k} \Big[h(A \setminus \{x\}) - h(A)\Big] \\ + D \sum_{x \in A_k} \sum_{y \notin A : |y-x|=1} \Big[h((A \cup \{y\}) \setminus \{x\}) - h(A)\Big]. \end{aligned}$$

To check condition (4) in the Harris lemma, it is enough to show that $R_k(A) \le 0$ for $k \in \{0, 1, \dots, 2d\}$ and $A \in Y$. To do so, first we show

Lemma 4.2.3. *Let $d \ge 1$. Then $R_0(A)$, $R_1(A)$, $R_{2d}(A) \le 0$ for any $A \in Y$.*

Proof of Lemma 4.2.3.
(i) $R_0(A)$: Using Eq.(4.1) and $b((A \cup \{y\}) \setminus \{x\}) \ge b(A) + 1$ for $x \in A_0$ and $y \notin A$ with $|y - x| = 1$, we have

$$R_0(A) \le 2dD|A_0|\alpha_*^{|A|}\beta_*^{b(A)}(1 - \beta_*). \tag{4.15}$$

Combining Eq.(4.15) with Lemma 4.2.2 (2) gives $R_0(A) \le 0$ for any $A \in Y$.

(ii) $R_1(A)$: In a similar fashion we have

$$R_1(A) \le |A_1| \alpha_*^{|A|-1} \beta_*^{b(A)-1} \left[\{(2d-1)\lambda w_* - 1\}(1-w_*) + (2d-1)Dw_*(1-\beta_*) \right]. \tag{4.16}$$

Combining Eq.(4.16) and Lemma 4.2.2 (2), (3) gives $R_1(A) \le 0$ for any $A \in Y$.
(iii) $R_{2d}(A)$: As in the case of R_1, we have

$$R_{2d}(A) = |A_{2d}| \alpha_*^{|A|-1} \beta_*^{b(A)-2d} (\alpha_* \beta_*^{2d} - 1). \tag{4.17}$$

So $R_{2d}(A) \le 0$ follows from Eq.(4.17) and Lemma 4.2.2 (4).

We should remark that by Lemma 4.2.3 we can check the condition (4) of the Harris lemma for $d = 1$, since $R_0(A)$, $R_1(A)$, $R_2(A) \le 0$ for any $A \in Y$. So in the rest of this section, we assume $d \ge 2$.

Lemma 4.2.4. *Let $d \ge 2$. Then $R_k(A) \le 0$ for any $k \in \{2, 3, \ldots, 2d-1\}$ and $A \in Y$.*

Proof of Lemma 4.2.4. As in the proof of Lemma 4.2.3, for $k \in \{2, 3, \ldots, 2d-1\}$, we have

$$R_k(A) \le |A_k| \alpha_*^{|A|} \beta_*^{b(A)} \varphi(k),$$

where $A \in Y$ and

$$\varphi(k) = (2d-k)[\lambda(1-w_*) + D(1-\beta_*)] + 1 - \frac{1}{\alpha_* \beta_*^k}.$$

Then we get

$$\varphi(k+1) - \varphi(k) = -\left[\lambda(1-w_*) + \left(D + \frac{1}{\alpha_* \beta_*^{k+1}} \right)(1-\beta_*) \right]. \tag{4.18}$$

By Eq.(4.18) and Lemma 4.2.2 (2), we see that $\varphi(k+1) - \varphi(k) \le \varphi(1) - \varphi(0)$. So if $\varphi(1) - \varphi(0) \le 0$, then $\varphi(2d-1) \le \varphi(2d-2) \le \cdots \le \varphi(1) \le 0$. The inequality: $\varphi(1) \le 0$ comes from the proof of $R_1(A)$ in Lemma 4.2.3. Therefore, to prove $R_k(A) \le 0$ ($k \in \{2, \ldots, 2d-1\}$), it is enough to show $\varphi(1) - \varphi(0) \le 0$. By using Eq.(4.3), we observe that $\varphi(1) - \varphi(0) \le 0$ is equivalent to

$$(Dw_* + 1)(2d\lambda w_* - 1) - \lambda w_* \le 0. \tag{4.19}$$

To prove Eq.(4.19), we let $I(w) = (Dw+1)(2d\lambda w - 1) - \lambda w$. That is, our objective is to show $I(w_*) \le 0$. In the rest of this proof, define $z = 2d$, for simplicity. Let w_3 and w_4 with $w_3 < 0 < w_4$ be the two roots of $I(w) = 0$. On the other hand, Eq.(4.5) can be rewritten as

$$f(w) = z(z-1)\lambda^2 w^2 - \lambda[z(z-1)(\lambda + D) + 2z - 1]w + z\lambda + 1 + (z-1)D.$$

Recall that $f(w_*) = 0$ and $0 < w_* < 1$. Therefore, noting $I(0) = -1$ and $I(w_4) = 0$, we know that if $w_* \le w_4$, then the proof is complete. To check $w_* \le w_4$, it is enough to show $f(w_4) \le 0$, since $f(0) > 0$, $f(w_*) = 0$, $f(1) < 0$ and $0 < w_* < 1$. Let $J(w) = Df(w) - (z-1)\lambda I(w)$. Then

$$J(w) = -\big[(z-1)(zD+z-1)\lambda + zD + z(z-1)D^2\big]\lambda w + (zD+z-1)\lambda + D + (z-1)D^2.$$

By $D > 0$, $I(w_4) = 0$ and the definitions of $J(w)$ and w_4, we obtain

$$\begin{aligned} f(w_4) \le 0 &\Longleftrightarrow J(w_4) \le 0 \\ &\Longleftrightarrow \begin{array}{l} 2zDK + [(z-1)\lambda - D][zK - (z-1+zD)\lambda] \\ \quad \le [zK - (z-1+zD)\lambda]\sqrt{L} \end{array} \end{aligned}$$

where $K = (z-1+zD)\lambda + D + (z-1)D^2$ and $L = (z-1)^2\lambda^2 + 2(z+1)D\lambda + D^2$. By a direct computation, we see that the last inequality is equivalent to

$$(z^2 - 3z + 1)(z - 1 + zD)\lambda + [1 + (z-1)D][1 + z(z-2)D] \ge 0.$$

Note that if $z = 2d \ge 4$, then $z^2 - 3z + 1 \ge 0$. So when $d \ge 2$, we have $f(w_4) \le 0$. The proof of Lemma 4.2.4 is complete.

By using Lemma 4.2.4, we can check condition (4) in the case of $d \ge 2$. So the proof of Theorem 4.1.3 is complete.

4.3. Proof of Theorem 4.1.4.

In this section we will give the proof of Theorem 4.1.4. By $w_* = \alpha_*\beta_*$ and Eq.(4.1), we have

$$1 - \alpha_* = \frac{\beta_* - w_*}{\beta_*} = \frac{2d\lambda(1 - w_*)}{2d\lambda(1 - w_*) + 1}.$$

Using the explicit form of w_*, a direct computation gives

$$1 - \alpha_* = \frac{2(2d)^2(2d-1)\lambda D + 2d(2d-1)(\lambda - D) - 1 - \sqrt{M}}{2(2d)^2(2d-1)\lambda D}, \tag{4.20}$$

where $M = [2d(2d-1)(\lambda - D) - 1]^2 + 4(2d)^2(2d-1)^2\lambda D$. Then

$$\sqrt{M} = 2d(2d-1)(\lambda + D) + 1 + o\left(\frac{1}{D}\right), \tag{4.21}$$

where $f(D)$ is $o(1/D)$ as $D \to \infty$ means $Df(D) \to 0$ as $D \to \infty$. Combining Eq.(4.20) with Eq.(4.21) gives

$$1 - \alpha_* = \frac{2d\lambda - 1}{2d\lambda}\left[1 - \frac{1}{(2d\lambda - 1)2d(2d-1)}\frac{1}{D} + o\left(\frac{1}{D}\right)\right].$$

If $\lambda - 1/2d = C/D$, then

$$1 - \alpha_* = \frac{2d\lambda - 1}{2d\lambda}\left[1 - \frac{1}{(2d)^2(2d-1)}\frac{1}{C} + o\left(\frac{1}{D}\right)\right].$$

So the proof of Theorem 4.1.4 is complete.

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CHAPTER 5

BASIC CONTACT PROCESSES ON TREES

5.1. Introduction

We consider the contact process on the homogeneous tree, $\mathbf{T}$, which is also called the Bethe lattice. This is an infinite graph without cycles, in which each vertex has the same number of nearest neighbors, which we will denote by $\kappa \geq 3$. Let o be a distinguished vertex of $\mathbf{T}$, which we call the origin. For x, $y \in \mathbf{T}$, the natural distance between x and y, $|x - y|$, is defined by the number of edges in the unique path of $\mathbf{T}$ joining x to y.

We introduce the contact process, ξ_t, on $\mathbf{T}$, which is a continuous-time Markov process on $\mathbf{T}$. The dynamics of this process are given by the following transition rates: for $x \in \xi$ with $\xi \subset \mathbf{T}$,

$$\begin{aligned} \xi &\to \xi \cup \{y\} && \text{at rate } \lambda \text{ for } y \text{ with } |y - x| = 1, \\ \xi &\to \xi \setminus \{x\} && \text{at rate } 1. \end{aligned}$$

Let ξ_t^o denote the contact process starting from the origin. Define the global survival probability $\rho^g(\lambda)$ by

$$P(\xi_t^o \neq \phi \ \text{ for all } t \geq 0).$$

The critical value of the global survival probability is defined by

$$\lambda_c^g = \inf\{\lambda \geq 0 : \rho^g(\lambda) > 0\}.$$

For contact processes on $\mathbf{T}$, the following local survival probability $\rho^l(\lambda)$ is also introduced:

$$\rho^l(\lambda) = P(\limsup_{t\to\infty} \xi_t^o(o) = 1) = P(o \in \xi_t^o \ \text{ infinitely often}).$$

Using this, we define the critical value of the local survival probability in the following way:

$$\lambda_c^l = \inf\{\lambda \geq 0 : \rho^l(\lambda) > 0\}.$$

We should remark that the above definitions give $\lambda_c^g \le \lambda_c^l$. In the case of the d-dimensional integer lattice, $\mathbf{Z}^d$, Bezuidenhout and Grimmett[1] showed that $\lambda_c^g = \lambda_c^l$. On the other hand, in the case of $\mathbf{T}$, Pemantle[2] ($\kappa \ge 4$), Liggett[3] and Stacey[4] ($\kappa = 3$) proved that $\lambda_c^g < \lambda_c^l$. Unfortunately the rigorous values of both critical values are not known.

From now on we consider only λ_c^g and $\rho^g(\lambda)$. As regards bounds on λ_c^g, Pemantle[2] proved that for $\kappa \ge 3$,

$$\frac{1}{\kappa - 1} \le \lambda_c^g \le \frac{\sqrt{9 + \frac{16}{\kappa - 2}} - 1}{2\kappa}. \tag{5.1}$$

When $\kappa = 3$ (binary tree case), this gives $1/2 = 0.5 \le \lambda_c^g \le 2/3 \approx 0.667$. In this case, better bounds were obtained by Griffeath[5] (lower bound) and Liggett[3] (upper bound) as follows:

$$\frac{\sqrt{109} - 1}{18} \approx 0.524 \le \lambda_c^g \le 0.605.$$

The lower bound comes from the submodularity of the survival probability. On the other hand, the upper bound is derived by the Holley-Liggett method. Recently the following estimate of this critical value was given by the time dependent simulations, see Tretyakov and Konno:[6]

$$\lambda_c^g = 0.5420 \pm 0.0005.$$

Exercise 5.1. Estimate λ_c^l by using simulations when $\kappa = 3$. Rigorous lower bound is 0.609 given by Liggett.[3]

Next we consider the critical exponent for the global survival probability. Results on the continuity of $\rho^g(\lambda)$ at λ_c^g were given by Pemantle[2] for $\kappa \ge 4$ and by Morrow, Schinazi and Zhang[7] for $\kappa = 3$. That is, $\rho^g(\lambda_c^g) = 0$ ($\kappa \ge 3$). As regards the critical exponent, the following is conjectured that for $\kappa \ge 3$,

$$\beta = \lim_{\lambda \downarrow \lambda_c^g} \frac{\log \rho^g(\lambda)}{\log(\lambda - \lambda_c^g)}$$

exists and $\beta = 1$. Note that $\beta = 1$ is the mean field value. Concerning this conjecture, recently Wu[8] proved that for $\kappa \ge 6$,

$$\liminf_{\lambda \downarrow \lambda_c^g} \frac{\rho^g(\lambda)}{\lambda - \lambda_c^g} > 0 \qquad \text{and} \qquad \limsup_{\lambda \downarrow \lambda_c^g} \frac{\rho^g(\lambda)}{\lambda - \lambda_c^g} < \infty.$$

This result proves that $\beta = 1$ for $\kappa \ge 6$. Tretyakov and Konno[6] reported that the time dependent simulations gave $\beta = 0.95 \pm 0.15$ for $\kappa = 3$. That is, this estimation indicates that β takes the mean field value of 1 for a tree with $\kappa = 3$. Assuming that increase of the coordination number increases the tendency of a system to exhibit mean field behavior, we would expect mean field values for the exponents for any $k \geq 3$.

From now on we write just $\rho_\lambda = \rho^g(\lambda)$ and $\lambda_c = \lambda_c^g$ for simplicity. In this chapter we discuss bounds on ρ_λ and λ_c by using the Harris lemma. Let $Y = \{A \subset \mathbf{T} : |A| < \infty\}$ and

$$\sigma_\lambda(A) = P(\xi_t^A \neq \phi \ \text{ for all } t \geq 0) = \nu_\lambda\{\eta : \eta(x) = 1 \ \text{ for some } x \in A\},$$

for any $A \in Y$, where ν_λ is the upper invariant measure of the basic contact process on trees. The first result is as follows:

Theorem 5.1.1. *Assume $\kappa \geq 3$. For any $\lambda \geq 0$ and $A \in Y$, we have*

$$\left[1 - \left(\frac{1}{(\kappa-2)\lambda}\right)^{|A|}\right] \vee 0 \leq \sigma_\lambda(A) \leq \left[1 - \left(\frac{1}{\kappa\lambda}\right)^{|A|}\right] \vee 0,$$

where $a \vee b$ is the maximum of a and b.

Applying Theorem 5.1.1 to $A = \{0\}$ gives

Corollary 5.1.2. *Assume $\kappa \geq 3$. For any $\lambda \geq 0$,*

$$\frac{(\kappa-2)\lambda - 1}{(\kappa-2)\lambda} \vee 0 \leq \rho_\lambda \leq \frac{\kappa\lambda - 1}{\kappa\lambda} \vee 0,$$

and

$$\frac{1}{\kappa} \leq \lambda_c \leq \frac{1}{\kappa - 2}.$$

The both bounds on $\sigma_\lambda(A)$ in Theorem 5.1.1 correspond to the first bounds by the Katori-Konno method. However the proof of the lower bound on $\sigma_\lambda(A)$ depends on the property of trees. So we do not have similar results in the case of $\mathbf{Z}^d$. Parts of these results are already known, see Liggett,[3] for example. When $\kappa = 3$, this corollary gives

$$\frac{1}{3} \leq \lambda_c \leq 1.$$

Compared with Eq.(5.1), the above first bounds are not so good.

Exercise 5.2. Try to obtain an improved lower bound on $\sigma_\lambda(A)$ by using the form $h(A) = 1 - \alpha^{|A|}\beta^{b(A)}$ where $b(A)$ is the number of neighboring pairs of points in A. Concerning upper bound for the same form, the story is almost the same as in the case of the basic contact process, so it is not interesting.

Next by choosing another type of $h(A)$ in the Harris lemma, the following result is obtained in Konno.[9]

Theorem 5.1.3. *Assume $\kappa \geq 3$. For any $A \in Y$ and*

$$\lambda \geq \Big[\frac{\sqrt{\kappa}+1}{\kappa+\sqrt{\kappa}-2}\Big]^2,$$

we have

$$1 - \frac{\kappa-2}{|A|+\kappa-2} \leq \sigma_\lambda(A).$$

From this theorem, we obtain

Corollary 5.1.4. *Assume $\kappa \geq 3$.*

(1) $$\rho_\lambda \geq \frac{1}{\kappa-1} \qquad \text{for any } \lambda \geq \Big[\frac{\sqrt{\kappa}+1}{\kappa+\sqrt{\kappa}-2}\Big]^2,$$

and

(2) $$\lambda_c \leq \Big[\frac{\sqrt{\kappa}+1}{\kappa+\sqrt{\kappa}-2}\Big]^2.$$

The above results also depend on the property of trees, so we do not get similar results in the case of $\mathbf{Z}^d$.

Here we consider bounds on ρ_λ in the case of $\kappa = 3$. Corollary 5.1.2 gives

$$\frac{\lambda-1}{\lambda} \vee 0 \leq \rho_\lambda \leq \frac{3\lambda-1}{3\lambda} \vee 0 \qquad \text{for any } \lambda \geq 0.$$

Moreover Corollary 5.1.4 implies

$$\rho_\lambda \geq \frac{1}{2} \qquad \text{for any } \lambda \geq 1.$$

Therefore we see that when $1 \leq \lambda \leq 2$, the result of Corollary 5.1.4 is better than that of Corollary 5.1.2 concerning ρ_λ. In particular, if we take $\lambda = 1$, then

$$\frac{1}{2} \leq \rho_1 \leq \frac{2}{3}.$$

This chapter is organized as follows. In Section 5.2, we will prove Theorem 5.1.1. Section 5.3 gives the proof of Theorem 5.1.3.

5.2. Proof of Theorem 5.1.1.

In this section we will prove Theorem 5.1.1 by using the Harris lemma. Let $Y = \{A \subset \mathbf{T} : |A| < \infty\}$ and Y^* be the set of all $[0,1]$-valued measurable functions on Y. For any $h \in Y^*$, we let

$$\Omega^* h(A) = \lambda \sum_{x\in A} \sum_{y:|y-x|=1} \Big[h(A \cup \{y\}) - h(A)\Big] + \sum_{x \in A} \Big[h(A \setminus \{x\}) - h(A)\Big].$$

As in the case of $\mathbf{Z}^d$, we consider the following 4 steps.

Step 1. First we let $h(A) = 1 - \alpha^{|A|}$ with $0 \leq \alpha \leq 1$.

Here we need some observations. Let

$$N(A) = \big|\{(x,y) : |x-y| = 1,\, x \in A,\, y \notin A\}\big|.$$

By using this, we have

$$\begin{aligned} \Omega^* h(A) &= \lambda \sum_{x\in A} \sum_{y \notin A : |y-x|=1} \Big[\alpha^{|A|} - \alpha^{|A|+1}\Big] + \sum_{x \in A} \Big[\alpha^{|A|} - \alpha^{|A|-1}\Big] \\ &= \lambda N(A) \Big[\alpha^{|A|} - \alpha^{|A|+1}\Big] + |A| \Big[\alpha^{|A|} - \alpha^{|A|-1}\Big] \\ &= (1-\alpha)\alpha^{|A|-1} \Big[\alpha N(A) \lambda - |A|\Big]. \end{aligned} \tag{5.2}$$

On the other hand, the definition of $N(A)$ gives the following result: for any $A \in Y$ and $\kappa \geq 3$,

$$(\kappa - 2)|A| + 2 \leq N(A) \leq \kappa |A|. \tag{5.3}$$

Remark that this lower bound on $N(A)$ depends on the property of trees. Combining Eq.(5.2) with Eq.(5.3), we have

$$(1-\alpha)\alpha^{|A|-1}|A| \Big[(\kappa-2)\alpha\lambda - 1\Big] \leq \Omega^* h(A) \leq (1-\alpha)\alpha^{|A|-1}|A| \Big[\kappa\alpha\lambda - 1\Big], \tag{5.4}$$

for any $A \in Y$ and $\kappa \geq 3$.

First we consider the lower bound on $\sigma_\lambda(A)$.

Step 2. We take $\alpha_* = 1/(\kappa-2)\lambda$ for $\lambda > 1/(\kappa - 2)$. So we have $0 < \alpha_* < 1$ for $\lambda > 1/(\kappa-2)$.

Step 3. The proof of Step 3 is trivial.

Step 4. Using Eq.(5.4) and $(\kappa-2)\lambda\alpha_* = 1$, we have

$$\Omega^* h(A) \geq 0 \quad \text{for any } A \in Y.$$

Therefore we finish the proof of the lower bound on $\sigma_\lambda(A)$. Concerning the upper bound, we take $\alpha_* = 1/\kappa\lambda$ for $\lambda > 1/\kappa$. The rest is the almost same as the previous proof. So we will omit it.

Exercise 5.3. Verify that $\alpha_* = 1/\kappa\lambda$ is the solution of $\Omega^* h(\{0\}) = 0$ as in the case of the basic contact process on $\mathbf{Z}^d$ with $\kappa = 2d$.

Next we will give another proof of Step 4 concerning the lower bound on $\sigma_\lambda(A)$ which corresponds to Proof B in the basic contact process on $\mathbf{Z}^d$. As for Proofs A and B, see Chapter 1 in these notes. On the other hand, the above proofs in Step 4 correspond to Proof A. Moreover Proof B for the upper bound on $\sigma_\lambda(A)$ is the almost same as Proof B of Theorem 1.4.4, so we do not present here. From now on we will discuss about Proof B in the case of the lower bound on $\sigma_\lambda(A)$.

Let $A_n = \{x_1 = o, x_2, \ldots, x_n\}$ with $|x_{i+1} - x_i| = 1$ $(1 \le i \le n-1)$ and $x_{i+1} \ne x_{i-1}$ $(2 \le i \le n-1)$. Then we see that for $n \ge 1$

$$\begin{aligned}
\Omega^* h(A_n) &= \lambda \sum_{x \in A_n} \sum_{y \in A_n : |y-x|=1} \Big[\alpha^{|A_n|} - \alpha^{|A_n \cup \{y\}|}\Big] + \sum_{x \in A_n} \Big[\alpha^{|A_n|} - \alpha^{|A_n|-1}\Big] \\
&= \lambda\Big[(\kappa-2)|A_n| + 2\Big]\Big[\alpha^{|A_n|} - \alpha^{|A_n|+1}\Big] + |A_n|\Big[\alpha^{|A_n|} - \alpha^{|A_n|-1}\Big] \\
&= (1-\alpha)\alpha^{|A_n|-1}\Big[\Big\{(\kappa-2)|A_n| + 2\Big\}\alpha\lambda - |A_n|\Big] \\
&= (1-\alpha)\alpha^{|A_n|-1}\Big\{(\kappa-2)|A_n| + 2\Big\}\lambda \\
&\qquad \times \Big[\Big\{\alpha - \frac{1}{(\kappa-2)\lambda}\Big\} + \frac{2}{(\kappa-2)[(\kappa-2)|A_n| + 2]\lambda}\Big].
\end{aligned}$$

If we take $\alpha_* = 1/(\kappa-2)$, then $\Omega^* h(A_n) \ge 0$ for any $n \ge 1$. Next we consider the general case of A. For any $A \in Y$, we let $A = \sum_{k=1}^{N} B_k$ where B_k is a connected component of A. Define $b_n = |\{B_k : |B_k| = n\}|$. Then we have

$$\begin{aligned}
\Omega^* h(A) &= \sum_{k=1}^{N}\Big[\lambda \sum_{x \in B_k} \sum_{y : |y-x|=1} \Big[h(A \cup \{y\}) - h(A)\Big] + \sum_{x \in B_k} \Big[h(A \setminus \{x\}) - h(A)\Big]\Big] \\
&\ge \sum_{n=1}^{|A|} b_n \Big[\lambda \sum_{x \in A_n} \sum_{y : |y-x|=1} \Big[h(A \cup \{y\}) - h(A)\Big] + \sum_{x \in A_n} \Big[h(A \setminus \{x\}) - h(A)\Big]\Big] \\
&= (1-\alpha)\alpha^{|A|-1} \sum_{n=1}^{|A|} b_n \Big[\Big\{(\kappa-2)|A_n| + 2\Big\}\alpha\lambda - |A_n|\Big] \\
&= \alpha^{|A|-1} \sum_{n=1}^{|A|} \frac{b_n}{\alpha^{n-1}} \Omega^* h(A_n),
\end{aligned}$$

since the second inequality comes from the property of trees. Therefore we have

$$\Omega^* h(A) \ge \sum_{n=1}^{|A|} \alpha^{|A|-n} b_n \Omega^* h(A_n).$$

This inequality implies that $\Omega^* h(A_n) \ge 0$ for any $n \ge 1$ gives $\Omega^* h(A) \ge 0$ for any $A \in Y$. So the Proof B is complete.

5.3. Proof of Theorem 5.1.3.

In this section we will prove Theorem 5.1.3 by using the Harris lemma. We assume that $\kappa \ge 3$.

Steps 1 and 2. We take

$$h(A) = 1 - \frac{\kappa-2}{|A| + \kappa - 2}.$$

This choice appears for the first time in these notes.

Step 3. By the definition of $h(A)$, we can check conditions (1)-(3) immediately.

Step 4. We begin by computing

$$\begin{aligned}\Omega^* h(A) &= \lambda \sum_{x\in A} \sum_{y\notin A:|y-x|=1} \Big[h(A\cup\{y\}) - h(A)\Big] + \sum_{x\in A}\Big[h(A\setminus\{x\}) - h(A)\Big] \\ &= \lambda N(A)\Big[\frac{\kappa-2}{|A|+\kappa-2} - \frac{\kappa-2}{|A|+\kappa-1}\Big] + |A|\Big[\frac{\kappa-2}{|A|+\kappa-2} - \frac{\kappa-2}{|A|+\kappa-3}\Big] \\ &= (\kappa-2)\Big[\frac{\lambda N(A)}{(|A|+\kappa-2)(|A|+\kappa-1)} - \frac{|A|}{(|A|+\kappa-2)(|A|+\kappa-3)}\Big].\end{aligned}$$

By using $N(A) \geq (\kappa-2)|A| + 2$,

$$\Omega^* h(A) \geq (\kappa-2)\Big[\frac{\lambda\{(\kappa-2)|A|+2\}}{(|A|+\kappa-2)(|A|+\kappa-1)} - \frac{|A|}{(|A|+\kappa-2)(|A|+\kappa-3)}\Big]. \quad (5.5)$$

Let $x = |A|$. So Eq.(5.5) can be written as

$$\Omega^* h(A) \geq \frac{(\kappa-2)\Big[\{(\kappa-2)x+2\}(x+\kappa-3)\lambda - x(x+\kappa-1)\Big]}{(x+\kappa-3)(x+\kappa-2)(x+\kappa-1)} \quad (5.6)$$

We define

$$\lambda_*(\kappa) = \sup_{x\geq 1} \frac{x(x+\kappa-1)}{[(\kappa-2)x+2](x+\kappa-3)}.$$

Eq.(5.6) implies that if $\lambda \geq \lambda_*(\kappa)$, then $\Omega_* h(A) \geq 0$ for any $A \in Y$. By a direct computation, we have

$$\lambda_*(\kappa) = \Big[\frac{\sqrt{\kappa}+1}{\kappa+\sqrt{\kappa}-2}\Big]^2.$$

Therefore we can check condition 4 and have the desired conclusion.

We should remark that this proof does not hold in the case of $\mathbf{Z}^d$, since we use the following inequality to get Eq.(5.5): $N(A) \geq (\kappa-2)|A| + 2$, which holds only in the case of trees.

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CHAPTER 6

ONE-SIDED CONTACT PROCESS

6.1. Introduction

We consider the one-sided contact process ξ_t which is a continuous-time Markov process on $\mathbf{Z}^1$. The dynamics of this process are given by the following transition rates: for $x \in \xi$ with $\xi \subset \mathbf{Z}^1$,

$$\begin{aligned} &\xi \to \xi \setminus \{x\} && \text{at rate } 1, \\ &\xi \to \xi \cup \{x-1\} && \text{at rate } \lambda. \end{aligned}$$

Concerning the one-sided contact process for details, see Griffeath[1] and Schonmann.[2]

Let ξ_t^0 denote the one-sided contact process starting from the origin. Define the survival probability ρ_λ by

$$P(\xi_t^0 \neq \phi \ \text{ for all } t \geq 0).$$

The critical value of the survival probability is defined by

$$\lambda_c = \inf\{\lambda \geq 0 : \rho_\lambda > 0\}.$$

As for bounds on λ_c, Griffeath[1] showed

$$\sqrt{6} = 2.449... \leq \lambda_c \leq 4.$$

This lower bound is obtained by submodulality for the survival probability. On the other hand, this upper bound can be given by the Holley-Liggett method.

Exercise 6.1. Verify that 4 is the upper bound on λ_c as in the sketch of proof of Theorem 2.2.1 in Chapter 2.

Improved bounds were given by Konno[3] as follows:

$$\underline{\lambda}_c = 2.577... \leq \lambda_c \leq \overline{\lambda}_c = 3.882...,$$

where $\underline{\lambda}_c = \sup\{\lambda \geq 0 : 8\lambda^7 + 126\lambda^5 + 55\lambda^6 - 53\lambda^4 - 902\lambda^3 - 1945\lambda^2 - 1958\lambda - 756 < 0\}$ and $\overline{\lambda}_c = \sup\{\lambda \geq 0 : 2\lambda^3 - 7\lambda^2 - 4\lambda + 4 < 0\}$. As for the lower bound, we get it by using the Ziezold and Grillenberger method. The upper bound can be obtained by a generalization of the Holley-Liggett method. In the case of basic contact process, Liggett[4] used this method and gave an improved upper bound. See subsection 2.2.2 in these notes.

Concerning the estimated value on λ_c, Tretyakov, Inui and Konno[5] obtained the following numerical result by using Monte Carlo simulation and power series expansion techniques:

$$\lambda_c = 3.306 \pm 0.002.$$

The interesting thing is that this estimated value is slightly different from the best estimation of the critical value λ_c^{bcp} for the basic contact process with the same total infection rate λ given by Jensen and Dickman[6]:

$$\lambda_c^{bcp} = 3.297824.$$

Unfortunately we can not use a usual coupling technique when we try to compare these critical values.

Open Problem 6.1.1. Which is correct in the following three possibilities ?

$$\text{a)} \quad \lambda_c > \lambda_c^{bcp}, \qquad \text{b)} \quad \lambda_c = \lambda_c^{bcp}, \qquad \text{c)} \quad \lambda_c < \lambda_c^{bcp}.$$

As for critical exponent of the survival probability, numerical results by Tretyakov, Inui and Konno[5] suggest that both exponents are expected as same. That is, the one-sided contact process and basic contact process belong to the same universality class. This conclusion is not so surprising from the physical point of view.

This chapter is organized as follows. In Section 6.2, we give lower bounds on critical value of the one-sided contact process. Section 6.3 is devoted to upper bounds.

6.2. Lower Bound

In this section, we will give a lower bound on the critical value of the one-sided contact process by the Ziezold-Grillenberger method. Let ξ_t^- and ξ_t^+ denote the one-sided contact process starting from $\{\ldots, -1, 0\}$ and $\{0, 1, \ldots\}$, respectively. Here we introduce edge processes of the one-sided contact process as follows:

$$r_t = \max \xi_t^-, \qquad l_t = \min \xi_t^+.$$

Let $\alpha_t^1(\lambda) = E(r_t)$ and $\alpha_t^2(\lambda) = -E(l_t)$. Shonmann[2] showed

$$\lim_{t \to \infty} \frac{r_t}{t} = \alpha_1(\lambda) \qquad \text{a.s.}$$
$$\lim_{t \to \infty} \frac{l_t}{t} = -\alpha_2(\lambda) \qquad \text{a.s.}$$

where

$$\alpha_1(\lambda) = \lim_{t\to\infty} \frac{\alpha_t^1(\lambda)}{t} = \inf_{t>0} \frac{\alpha_t^1(\lambda)}{t} \in [-\infty, \infty),$$
$$\alpha_2(\lambda) = \lim_{t\to\infty} \frac{\alpha_t^2(\lambda)}{t} = \inf_{t>0} \frac{\alpha_t^2(\lambda)}{t} \in [-\infty, \infty).$$

Now we can define the *edge speed* $\alpha(\lambda)$ by

$$\alpha(\lambda) = \frac{\alpha_1(\lambda) + \alpha_2(\lambda)}{2}.$$

The critical value of the one-sided contact process can be characterized by the edge speed: $\lambda_c = \inf\{\lambda \geq 0 : \alpha(\lambda) > 0\}$.

Next we modify ξ_t^- by keeping all coordinates which lie strictly to left of $r_t - n$ identically equal to one, where n is a fixed nonnegative integer. Similarly, we modify ξ_t^+ by keeping all coordinates which lie strictly to left of $l_t + n$ identically equal to one. Let $r_t^{(n)}$ (resp. $l_t^{(n)}$) be the position of rightmost (resp. leftmost) one in the modified process. Define $\alpha_t^{1,(n)}(\lambda) = E(r_t^{(n)})$ and $\alpha_t^{2,(n)}(\lambda) = -E(l_t^{(n)})$. In a similar fashion, we have

$$\lim_{t\to\infty} \frac{r_t^{(n)}}{t} = \alpha_1^{(n)}(\lambda) \qquad \text{a.s.}$$
$$\lim_{t\to\infty} \frac{l_t^{(n)}}{t} = -\alpha_2^{(n)}(\lambda) \qquad \text{a.s.}$$

where

$$\alpha_1^{(n)}(\lambda) = \lim_{t\to\infty} \frac{\alpha_t^{1,(n)}(\lambda)}{t} = \inf_{t>0} \frac{\alpha_t^{1,(n)}(\lambda)}{t} \in [-\infty, \infty),$$
$$\alpha_2^{(n)}(\lambda) = \lim_{t\to\infty} \frac{\alpha_t^{2,(n)}(\lambda)}{t} = \inf_{t>0} \frac{\alpha_t^{2,(n)}(\lambda)}{t} \in [-\infty, \infty).$$

So we can define the edge speed $\alpha^{(n)}(\lambda)$ by

$$\alpha^{(n)}(\lambda) = \frac{\alpha_1^{(n)}(\lambda) + \alpha_2^{(n)}(\lambda)}{2}.$$

From $r_t^{(n)} \geq r_t$ and $l_t^{(n)} \leq l_t$, we have $\alpha^{(n)} \geq \alpha(\lambda)$, for $n = 0, 1, 2, \ldots$ Moreover, the definition of the modified one-sided contact process gives $\alpha^{(n)}(\lambda) \geq \alpha^{(n+1)}(\lambda)$. Then the critical value $\lambda_c^{(n)}$ of the modified one-sided contact process is introduced by the edge speed as follows: $\lambda_c^{(n)} = \inf\{\lambda \geq 0 : \alpha^{(n)}(\lambda) > 0\}$. Following the argument in Ziezold and Grillenberger, we see that $\lambda_c^{(n)} \nearrow \lambda_c$ as $n \nearrow \infty$. Concerning the next computation, see Chapter 3 of Konno[7] for details.

6.2.1. $n = 0$

In this case, $\alpha_1^{(0)}(\lambda) = -1$ and $\alpha_2^{(0)}(\lambda) = \lambda - 1$. So we have

$$\alpha^{(0)}(\lambda) = \frac{\alpha_1^{(0)}(\lambda) + \alpha_2^{(0)}(\lambda)}{2} = \frac{\lambda - 2}{2}.$$

Then $\lambda_c^{(0)} = 2$. This gives $2 \le \lambda_c$.

6.2.2. $n = 1$

In this case,

$$\alpha_1^{(1)}(\lambda) = -\frac{\lambda+3}{\lambda+2}, \qquad \alpha_2^{(1)}(\lambda) = \frac{2\lambda^2-3}{2(\lambda+1)}.$$

So we have

$$\alpha^{(1)}(\lambda) = \frac{\alpha_1^{(1)}(\lambda)+\alpha_2^{(1)}(\lambda)}{2} = \frac{2\lambda^3+2\lambda^2-11\lambda-12}{4(\lambda+1)(\lambda+2)}.$$

Then $\lambda_c^{(1)} = 2.376\ldots$ This gives $2.376 \le \lambda_c$.

6.2.3. $n = 2$

Similarly,

$$\alpha_1^{(2)}(\lambda) = -\frac{\lambda^3+7\lambda^2+21\lambda+27}{\lambda^3+6\lambda^2+14\lambda+14}, \qquad \alpha_2^{(2)}(\lambda) = \frac{8\lambda^4+15\lambda^3+3\lambda^2-34\lambda-27}{8\lambda^3+23\lambda^2+30\lambda+14}.$$

So we have

$$\begin{aligned}\alpha^{(2)}(\lambda) &= \frac{\alpha_1^{(2)}(\lambda)+\alpha_2^{(2)}(\lambda)}{2} \\ &= \frac{8\lambda^7+55\lambda^6+126\lambda^5-53\lambda^4-902\lambda^3-1945\lambda^2-1958\lambda-756}{8\lambda^6+71\lambda^5+280\lambda^4+628\lambda^3+826\lambda^2+616\lambda+196}.\end{aligned}$$

Then $\lambda_c^{(2)} = 2.577\ldots$ This gives $2.577 \le \lambda_c$.

6.3. Upper Bound

In this section, we consider upper bounds on the critical value of the one-sided contact process. Here we review the case of the basic contact process in one dimension. By the Holley-Liggett method, the first upper bound 4 on the critical value of the basic contact process was given, as you saw in Chapter 2 in these notes. This argument holds in the case of the one-sided contact process and Griffeath[1] showed that 4 is also the first upper bound on λ_c for the one-sided contact process. For the basic contact process, using the Holley-Liggett method, Liggett[4] gave an improved upper bound which is the largest root of the cubic equation of $2\lambda^3 - 7\lambda^2 - 4\lambda + 4 = 0$. In this section, we will consider the second bound for the one-sided contact process.

Theorem 6.3.1. *Let $\lambda_c^{(HL,2)} \approx 3.882$ be the largest root of the cubic equation of*

$$2\lambda^3 - 7\lambda^2 - 4\lambda + 4 = 0.$$

Then for $\lambda \geq \lambda_c^{(HL,2)}$,

$$h_\lambda^{(HL,2)}(A) \leq \sigma_\lambda(A) \qquad \textit{for all } A \in Y,$$

where

$$h_\lambda^{(HL,2)}(A) = \mu\{\eta : \eta(x) = 1 \ \textit{ for some } x \in A\},$$

for a generalized renewal measure μ *on* $\{0,1\}^{\mathbf{Z}}$ *whose density is given by* $\Omega^* h_\lambda^{(HL,2)}(A) = 0$ *for all* A *of the form* $\{1,2,\dots,n\}$ $(n \geq 1)$ *and* $\{1,3\}$.

First we choose the form of

$$h(A) = \mu\{\eta : \eta(x) = 1 \ \text{ for some } x \in A\},$$

for a generalized renewal measure μ on $\{0,1\}^{\mathbf{Z}}$ whose density is given by

$$\Omega^* h(\{1,2,\dots,n\}) = 0,$$

for any $n \geq 1$ and

$$\Omega^* h(\{1,3\}) = 0,$$

where

$$\Omega^* h(A) = \lambda \sum_{x \in A} \sum_{y:|y-x|=1} \Big[h(A \cup \{y\}) - h(A)\Big] + \sum_{x \in A} \Big[h(A \setminus \{x\}) - h(A)\Big].$$

We should remark the next relations:

$$\begin{aligned} h(\{1,2,3\}) &= 1 - \mu(\circ\circ\circ), \\ h(\{1,3\}) &= 1 - \mu(\circ\times\circ), \\ h(\{1,2,3\}) - h(\{1,3\}) &= \mu(\circ\times\circ) - \mu(\circ\circ\circ) = \mu(\circ\bullet\circ), \\ &\vdots \end{aligned}$$

Moreover, following notations of Liggett,[4] we introduce

$$F_1(n) = \frac{\mu(\bullet\bullet\overbrace{\circ\cdots\circ}^{n-1})}{\mu(\bullet\bullet)}, \qquad F_0(n) = \frac{\mu(\circ\bullet\overbrace{\circ\cdots\circ}^{n-1})}{\mu(\circ\bullet)},$$

$$f_1(n) = \frac{\mu(\bullet\bullet\overbrace{\circ\cdots\circ}^{n-1}\bullet)}{\mu(\bullet\bullet)}, \qquad f_0(n) = \frac{\mu(\circ\bullet\overbrace{\circ\cdots\circ}^{n-1}\bullet)}{\mu(\circ\bullet)},$$

for $n \geq 1$. The above definitions give

$$\begin{aligned} &F_1(1) = F_0(1) = 1, \\ &F_1(n) = \sum_{k=n}^{\infty} f_1(k), \quad F_0(n) = \sum_{k=n}^{\infty} f_0(k). \end{aligned}$$

We recall that

$$\begin{aligned}\Omega^* h(\{1,2,\ldots,n\}) = \lambda\Big[&h(\{0,1,2,\ldots,n\}) - h(\{1,2,\ldots,n\})\Big] \\ &+ \lambda\Big[h(\{1,2,\ldots,n,n+1\}) - h(\{1,2,\ldots,n\})\Big] \\ &+ \sum_{k=1}^{n}\Big[h(\{1,2,\ldots,n\}\setminus\{k\}) - h(\{1,2,\ldots,n\})\Big].\end{aligned}$$

From now on we consider two cases; Case A and Case B.

Case A. By using definitions of $F_0(n)$ and $F_1(n)$, we see that

$$\begin{aligned}h(\{0,1,2,\ldots,n\}) - h(\{1,2,\ldots,n\}) &= \mu(\overbrace{\circ\cdots\circ}^{n}) - \mu(\overbrace{\circ\cdots\circ}^{n+1}) \\ &= \mu(\bullet\overbrace{\circ\cdots\circ}^{n}) \\ &= \mu(\bullet\bullet\overbrace{\circ\cdots\circ}^{n}) + \mu(\circ\bullet\overbrace{\circ\cdots\circ}^{n}) \\ &= \frac{\mu(\bullet\bullet\overbrace{\circ\cdots\circ}^{n})}{\mu(\bullet\bullet)} \times \mu(\bullet\bullet) + \frac{\mu(\circ\bullet\overbrace{\circ\cdots\circ}^{n})}{\mu(\circ\bullet)} \times \mu(\circ\bullet) \\ &= F_1(n+1)\mu(\bullet\bullet) + F_0(n+1)\mu(\circ\bullet).\end{aligned}$$

On the other hand, the definition of a measure μ implies there is an α such that

$$\alpha = \frac{F_0(n)}{F_1(n)} \quad \text{for} \ \ n \geq 1.$$

Note that α is independent of n. $\Omega^* h(\{1\}) = 0$ gives

$$\mu(\bullet\bullet) = (\lambda-1)\mu(\circ\bullet).$$

Moreover $\Omega^* h(\{1,2\}) = \Omega^* h(\{1,3\}) = 0$ yields

$$\alpha = \frac{2(\lambda-1)}{2\lambda-1} \qquad \text{and} \qquad F_1(2) = \frac{2\lambda-1}{(\lambda-1)(2\lambda+1)}.$$

From these, we have

$$\begin{aligned}h(\{0,1,2,\ldots,n\}) - h(\{1,2,\ldots,n\}) &= (\lambda-1)\mu(\circ\bullet)F_1(n+1) + \alpha\mu(\circ\bullet)F_1(n+1) \\ &= (\lambda-1+\alpha)\mu(\circ\bullet)F_1(n+1).\end{aligned}$$

We let

$$\delta = \lambda - 1 + \alpha = \frac{(\lambda-1)(2\lambda+1)}{2\lambda-1}.$$

Remark that $\delta = 1/F_1(2)$. From the above observations, we have

$$h(\{0,1,2,\ldots,n\}) - h(\{1,2,\ldots,n\}) = \mu(\bullet\overbrace{\circ\cdots\circ}^{n}) = \delta\mu(\circ\bullet)F_1(n+1).$$

Similarly

$$h(\{1,2,\dots,n,n+1\}) - h(\{1,2,\dots,n\}) = \mu(\bullet\overbrace{\circ\cdots\circ}^{n}) = \delta\mu(\circ\bullet)F_1(n+1).$$

Therefore we obtain

$$\lambda\sum_{x\in A}\sum_{y:|y-x|=1}\Big[h(A\cup\{y\}) - h(A)\Big] = 2\lambda\delta\mu(\circ\bullet)F_1(n+1),$$

where $A=\{1,\dots,n\}$.

Case B. For $k\in\{2,\dots,n-1\}$, we see

$$\begin{aligned}
h(\{1,2,\dots,n\}\setminus\{k\}) - h(\{1,2,\dots,n\}) =& \mu(\overbrace{\circ\cdots\circ}^{n}) - \mu(\overbrace{\circ\cdots\circ}^{k-1}\times\overbrace{\circ\cdots\circ}^{n-k}) \\
=& -\mu(\overbrace{\circ\cdots\circ}^{k-1}\bullet\overbrace{\circ\cdots\circ}^{n-k}) \\
=& -\frac{\mu(\overbrace{\circ\cdots\circ}^{k-1}\bullet\circ)}{\mu(\bullet\circ)}\times\mu(\bullet\overbrace{\circ\cdots\circ}^{n-k}) \\
=& -F_0(k)\times\delta\mu(\circ\bullet)F_1(n+1-k) \\
=& -\alpha F_1(k)\times\delta\mu(\circ\bullet)F_1(n+1-k).
\end{aligned}$$

The third equality comes from the definition of a generalized renewal measure μ. The fourth equality is given by the definition of $F_0(k)$ and a similar argument of Case A. The definition of α gives the last equality. So we have

$$h(\{1,2,\dots,n\}\setminus\{k\}) - h(\{1,2,\dots,n\}) = -\alpha\delta\mu(\circ\bullet)F_1(k)F_1(n+1-k).$$

For $k=1$ or $k=n$, a similar argument in Case A implies

$$h(\{1,2,\dots,n\}\setminus\{k\}) - h(\{1,2,\dots,n\}) = -\delta\mu(\circ\bullet)F_1(n).$$

Therefore

$$\begin{aligned}
&\sum_{k=1}^{n} h(\{1,2,\dots,n\}\setminus\{k\}) - h(\{1,2,\dots,n\}) \\
&\qquad = -2\delta\mu(\circ\bullet)F_1(n) - \alpha\delta\mu(\circ\bullet)\sum_{k=2}^{n-1}F_1(k)F_1(n+1-k) \\
&\qquad = -\delta\mu(\circ\bullet)\Big[2F_1(n) + \alpha\sum_{k=2}^{n-1}F_1(k)F_1(n+1-k)\Big].
\end{aligned}$$

From these results, we see that

$$\Omega^* h(\{1,2,\dots,n\}) = \delta\mu(\circ\bullet)\Big[2\lambda F_1(n+1) - \Big\{2F_1(n) + \alpha\sum_{k=2}^{n-1}F_1(k)F_1(n+1-k)\Big\}\Big],$$

for $n\ge 2$. Then $\Omega^* h(\{1,2,\dots,n\}) = 0$ $(n\ge 1)$ and $\Omega^* h(\{1,3\}) = 0$ give

Lemma 6.3.2. *Let* $\alpha = 2(\lambda - 1)/(2\lambda - 1)$. *Then*

$$\lambda F_1(n+1) = 2F_1(n) + \alpha \sum_{k=2}^{n-1} F_1(k) F_1(n+1-k) \quad (n \geq 2),$$
$$F_1(1) = 1.$$

We introduce the following generating function to get $F_1(n)$ explicitly:

$$\phi(u) = \sum_{n=1}^{\infty} F_1(n) u^n.$$

By using Lemma 6.3.2, we have the following quadratic equation:

$$\alpha\phi^2(u) - [\lambda + 2(\alpha - 1)u]\phi(u) + \lambda u + \Big[\alpha - 2 + \lambda F_1(2)\Big] u^2 = 0.$$

The nonnegativity of the discriminant of this equation with $u = 1$ is equivalent to

$$[\lambda + 2(\alpha - 1)]^2 - 4\alpha\Big[\lambda + \alpha - 2 + \lambda F_1(2)\Big] \geq 0.$$

This becomes $2\lambda^3 - 7\lambda^2 - 4\lambda + 4 \geq 0$. So we let $\lambda_c^{(2)}$ be the largest root of the following cubic equation which is the same as the case of basic contact process:

$$2\lambda^3 - 7\lambda^2 - 4\lambda + 4 = 0.$$

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CHAPTER 7

DISCRETE-TIME GROWTH MODELS

7.1. Introduction

We consider the discrete-time growth model ξ_n^A at time n starting from $A \in Y$ which is a Markov chain on $\{0,1\}^{\mathbf{Z}^1}$, where Y is the collection of all finite subsets of $\mathbf{Z}^1$. The dynamics of this model is as follows. We write ξ_n^A as a union of maximal subintervals

$$\xi_n^A = \bigcup_{i=1}^{k} I_i,$$

where $I_i = \{m_i + 1, m_i + 2, \ldots, n_i\}$ and $m_i < n_i < m_{i+1}$. Then ξ_{n+1}^A is obtained by choosing points in $\{m_i + 1, m_i + 2, \ldots, n_i - 1\}$ each with probability q, and points m_i and n_i each with probability p. The choices are made independently. Throughout this chapter, we assume that

$$0 \le p \le q \le 1,$$

so this process is attractive; that is, if $\xi_n^A \subset \xi_n^B$, then we can guarantee that $\xi_{n+1}^A \subset \xi_{n+1}^B$ by using appropriate coupling.

Exercise 7.1. Show that if $q = p$ (resp. $q = p(2-p)$) then this process is equivalent to the oriented site (resp. bond) percolation in two dimension.

We define the survival probability by

$$\sigma(A) = P(\xi_n^A \neq \phi \quad \text{for all } n \ge 0),$$

for any $A \in Y$. Then the order parameter is defined as $\rho(p,q) = \sigma(\{0\})$, where 0 is the origin. For given q, define the critical value $p_c(q)$ by

$$p_c(q) = \inf\{p \ge 0 : \rho(p,q) > 0\}.$$

The above-mentioned models were recently studied by Liggett[1] who gave the following bounds on critical values.

Theorem 7.1.1. *Let* $0 \le q \le 1$. *Then*

$$\frac{1-q}{2} \le p_c(q) - \frac{1}{2} \le \frac{\sqrt{1-q}}{2}.$$

In particular, the upper bound in this theorem is obtained by the Holley-Liggett method. As for it, we will discuss in Section 7.4. Concerning the above theorem, we present the following open problem which appeared in page 98 of Durrett.[2]

Open Problem 7.1.2. Find $\theta \in [0,1]$ satisfying that there exist C_1, $C_2 > 0$ such that

$$C_1(1-q)^{\theta} \le p_c(q) - \frac{1}{2} \le C_2(1-q)^{\theta}$$

as q goes to 1.

Theorem 7.1.1 implies $1/2 \le \theta \le 1$.

This chapter is organized as follows. In Section 7.2, we will present the discrete-time version of the Harris lemma. Section 7.3 gives results by the Katori-Konno method. Section 7.4 treats the Holley-Liggett method.

7.2. Harris Lemma

Here we present discrete-time version of the Harris lemma which appeared in Konno.[3] Let Y^* denote the set of all $[0,1]$-valued measurable functions on Y.

Lemma 7.2.1. *(Harris lemma) Let* $h_i \in Y^*$ $(i = 1, 2)$ *with*

$$h_i(\phi) = 0, \tag{1}$$

$$0 < h_i(A) \le 1 \quad \text{for any } A \in Y \text{ with } A \neq \phi. \tag{2}$$

For any $\varepsilon > 0$, *there is an* $N \ge 1$ *such that if* $|A| \ge N$, *then*

$$E\left(h_i(\xi_1^A)\right) \ge 1 - \varepsilon, \tag{3}$$

$$E\left(h_1(\xi_1^A) - h_1(A)\right) \le 0 \le E\left(h_2(\xi_1^A) - h_2(A)\right) \quad \text{for any } A \in Y. \tag{4}$$

Then

$$h_2(A) \le \sigma(A) \le h_1(A) \quad \text{for any } A \in Y. \tag{5}$$

In particular,

$$h_2(\{0\}) \le \rho(p,q) \le h_1(\{0\}), \tag{6}$$

where 0 *is the origin.*

In the rest of this section, we assume that $p \le q < 1$. When $q = 1$, the proof is almost trivial, so we will omit them. To prove Lemma 7.2.1, we require some preliminary lemmas. Write A as a union of maximal subintervals

$$A = \bigcup_{i=1}^{k} I_i, \tag{7.1}$$

where $I_i = \{m_i + 1, m_i + 2, \ldots, n_i\}$ and $m_i < n_i < m_{i+1}$. Define

$$L = |\{m_1 + 1, \ldots, n_1 - 1, m_2 + 1, \ldots, n_2 - 1, \ldots, m_k + 1, \ldots, n_k - 1\}|,$$
$$M = |\{m_1, n_1, m_2, n_2, \ldots, m_k, n_k\}|.$$

The definitions of L and M give

$$L + M = |A| + k, \tag{7.2}$$

$$M = 2k. \tag{7.3}$$

Then the following is easily shown by the property of binomial distribution.

Lemma 7.2.2. *For any* $A \in Y$ *and* $n \in \{0, 1, \ldots, |A| + k\}$,

$$P(|\xi_1^A| = n) = \sum_{l=0}^{L} \sum_{m=0}^{M} 1_n(l+m) \binom{L}{l} q^l (1-q)^{L-l} \binom{M}{m} p^m (1-p)^{M-m},$$

where $1_x(y) = 1$ *if* $y = x$, *and* $= 0$ *otherwise and*

$$\binom{i}{j} = \frac{i!}{j!\,(i-j)!} \qquad \textit{for} \quad 0 \le j \le i.$$

Exercise 7.2. Prove Lemma 7.2.2.

Furthermore we require the next lemma.

Lemma 7.2.3. *For any* $A \in Y$ *and* $N \ge 1$,

$$\lim_{n \to \infty} P(0 < |\xi_n^A| \le N) = 0.$$

Proof. It is enough to show that for any $A \in Y$ and $r \ge 1$,

$$\lim_{n \to \infty} P(|\xi_n^A| = r) = 0.$$

By the Markov property,

$$\begin{aligned} P(|\xi_{n+1}^A| = r-1) &= \sum_{m=0}^{\infty} E\Big(P(|\xi_1^{\xi_n^A}| = r-1) : |\xi_n^A| = m\Big) \\ &\geq E\Big(P(|\xi_1^{\xi_n^A}| = r-1) : |\xi_n^A| = r\Big) \\ &\geq c(r)P(|\xi_n^A| = r), \end{aligned}$$

where

$$c(r) = \inf_{B:|B|=r} P(|\xi_1^B| = r-1) > 0.$$

Note that the positivity of $c(r)$ follows from Lemma 7.2.2. Therefore it suffices to prove that for any $A \in Y$,

$$\lim_{n\to\infty} P(|\xi_n^A| = 1) = 0.$$

To do so, we will show that

$$\sum_{n=1}^{\infty} P(|\xi_n^A| = 1) < \infty.$$

By the Markov property,

$$\begin{aligned} P(|\xi_{n+1}^A| = 0) - P(|\xi_n^A| = 0) &= P(|\xi_{n+1}^A| = 0) - P(|\xi_n^A| = 0, |\xi_{n+1}^A| = 0) \\ &= \sum_{k=1}^{\infty} P(|\xi_n^A| = k, |\xi_{n+1}^A| = 0) \\ &\geq P(|\xi_n^A| = 1, |\xi_{n+1}^A| = 0) \\ &= E\Big(P(|\xi_1^{\xi_n^A}| = 0) : |\xi_n^A| = 1\Big) \\ &= (1-p)^2 P(|\xi_n^A| = 1). \end{aligned}$$

Then we see that $p < 1$ gives

$$\sum_{n=1}^{\infty} P(|\xi_n^A| = 1) \leq \frac{1}{(1-p)^2} < \infty.$$

Thus the proof is complete.

Exercise 7.3. Verify that for $r \geq 1$,

$$c(r) = \inf_{B:|B|=r} P(|\xi_1^B| = r-1) > 0.$$

From now on we will consider the proof of Lemma 7.2.1. For any $A \in Y$ and $N \geq 1$, Lemma 7.2.3 gives

$$\begin{aligned} \sigma(A) &= \lim_{n\to\infty} P(\xi_n^A \neq \phi) \\ &= \lim_{n\to\infty} P(|\xi_n^A| > N) + \lim_{n\to\infty} P(0 < |\xi_n^A| \leq N) \\ &= \lim_{n\to\infty} P(|\xi_n^A| > N). \end{aligned} \tag{7.4}$$

From the Markov property and condition (1),

$$\begin{aligned} E(h(\xi_{n+1}^A)) &= E\Big(E(h(\xi_1^{\xi_n^A}))\Big) \\ &= E\Big(E(h(\xi_1^{\xi_n^A})) : |\xi_n^A| > N\Big) + E\Big(E(h(\xi_1^{\xi_n^A})) : 0 < |\xi_n^A| \le N\Big). \end{aligned} \tag{7.5}$$

By using $h(A) \le 1$ for any $A \in Y$ and Lemma 7.2.3, we have

$$\lim_{n\to\infty} E\Big(E(h(\xi_1^{\xi_n^A})) : 0 < |\xi_n^A| \le N\Big) = 0.$$

Using this result and Eq.(7.5), we have

$$\liminf_{n\to\infty} E(h(\xi_{n+1}^A)) = \liminf_{n\to\infty} E\Big(E(h(\xi_1^{\xi_n^A})) : |\xi_n^A| > N\Big). \tag{7.6}$$

Therefore combination of Eqs.(7.4), (7.6) and condition (3) implies that for any $\varepsilon > 0$, there is an $N \ge 1$ such that

$$\begin{aligned} \liminf_{n\to\infty} E(h(\xi_{n+1}^A)) &\ge (1-\varepsilon) \liminf_{n\to\infty} P(|\xi_n^A| > N) \\ &= (1-\varepsilon)\sigma(A). \end{aligned} \tag{7.7}$$

By using Eq.(7.7), $h(\phi) = 0$, $h(A) \le 1$ for any $A \in Y$ and the definition of $\sigma(A)$, we see that for any $\varepsilon > 0$,

$$\begin{aligned} (1-\varepsilon)\sigma(A) &\le \liminf_{n\to\infty} E(h(\xi_n^A)) \\ &= \liminf_{n\to\infty} E\Big(h(\xi_n^A) : \xi_n^A \ne \phi\Big) \\ &\le \limsup_{n\to\infty} E\Big(h(\xi_n^A) : \xi_n^A \ne \phi\Big) \\ &\le \lim_{n\to\infty} P(\xi_n^A \ne \phi) \\ &= \sigma(A). \end{aligned}$$

Thus it follows that

$$\sigma(A) = \lim_{n\to\infty} E(h(\xi_n^A)). \tag{7.8}$$

From the Markov property and condition (4), we obtain

$$E(h(\xi_2^A)) = E\Big(E(h(\xi_1^{\xi_1^A}))\Big) \le E(h(\xi_1^A)) \le h(A).$$

Using a similar argument repeatedly, we see that for any $n \ge 1$,

$$E(h(\xi_n^A)) \le h(A). \tag{7.9}$$

Combining Eqs.(7.8) and (7.9) gives

$$\sigma(A) \le h(A),$$

for any $A \in Y$. Thus the proof of part (5) in Lemma 7.2.1 is complete. Part (6) follows from taking $A = \{0\}$ in part (5).

Exercise 7.4. Verify that for $A \in Y$,

$$E(h(\xi_{n+1}^A)) \le E(h(\xi_n^A)) \qquad (n \ge 0).$$

Exercise 7.5. Let $\xi_n^0 = \xi_n^{\{0\}}$. Show that for $1/2 \le p \le 1$,

$$E(h(\xi_0^0)) = E(h(\xi_1^0)) = -\frac{1}{p^2} + \frac{2}{p}, \tag{1}$$

$$E(h(\xi_2^0)) = -\frac{4}{p} + 14 - 16p + 9p^2 - 2p^3. \tag{2}$$

Exercise 7.6. Check the following fact:

$$E(h(\xi_2^0)) \le E(h(\xi_1^0)).$$

7.3. Katori-Konno Method

In this section, we will give the upper bound on $\sigma(A)$ by the Katori-Konno method. Let $|A|$ be the cardinality of $A \in Y$. Then the following result was obtained by Konno.[3]

Theorem 7.3.1. *Let $1/2 \le p \le q \le 1$.*

$$\sigma(A) \le \sigma^{(KK)}(A) = 1 - \left(\frac{1-p}{p}\right)^{2|A|} \qquad \text{for all } A \in Y. \tag{1}$$

In particular,

$$\rho(p,q) \le \rho^{(KK)}(p,q) = 1 - \left(\frac{1-p}{p}\right)^2. \tag{2}$$

This result corresponds to the first bound by the Katori-Konno method. Part (2) can be also obtained from the result on page 97 of Durrett[2] in the following way. Observing the behavior of edge processes of this model gives

$$\rho(p,1) = 1 - \left(\frac{1-p}{p}\right)^2.$$

Therefore part (2) follows from $\rho(p,q) \le \rho(p,1)$ for $q \le 1$.

In the rest of this section, we assume $q < 1$. When $q = 1$, the proof is almost trivial, so we will omit them.

To prove this theorem, we need the following 4 steps.

Step 1. First we let $h(A) = 1 - \alpha^{|A|}$ with $0 \le \alpha \le 1$.

Step 2. Next we decide $0 < \alpha_* < 1$ as the unique solution of

$$E(h(\xi_1^{\{0\}})) = h(\{0\}),$$

that is,

$$\left[p^2\alpha - (1-p)^2\right][\alpha - 1] = 0. \tag{7.10}$$

Here we assume that

$$\frac{1}{2} < p \le q < 1. \tag{7.11}$$

From Eq.(7.10), we take

$$\alpha_* = \left(\frac{1-p}{p}\right)^2, \tag{7.12}$$

and let $h(A) = 1 - \alpha_*^{|A|}$. Remark that Eq.(7.11) gives

$$0 < \alpha_* < 1. \tag{7.13}$$

Step 3. Now we will check conditions (1)-(3) in Lemma 7.2.1 (Harris lemma). Condition (1) and $h(A) \le 1$ in condition (2) are trivial. The positivity of $h(A)$ for a non-empty set $A \in Y$ is equivalent to $\alpha_*^{|A|} < 1$. The last inequality comes from Eq.(7.13). As for (3), it is sufficient to show that

$$\lim_{|A| \to \infty} E(\alpha_*^{|\xi_1^A|}) = 0, \tag{7.14}$$

when $q = p$. By Lemma 7.2.2, in the case of general p and q, we obtain

$$E(\alpha_*^{|\xi_1^A|}) = [\alpha_* q + 1 - q]^L [\alpha_* p + 1 - p]^M. \tag{7.15}$$

By using Eq.(7.15), $q = p$, and $k \ge 1$, we see that

$$E(\alpha_*^{|\xi_1^A|}) = [\alpha_* p + 1 - p]^{|A|+k} \le [\alpha_* p + 1 - p]^{|A|+1}.$$

Therefore Eq.(7.14) follows from the last result, since $0 < \alpha_* p + 1 - p < 1$.

Step 4. Finally we will check condition (4). Note that Eq.(7.10) gives

$$\alpha_* = [\alpha_* p + 1 - p]^2. \tag{7.16}$$

By Eq.(7.16), condition (4) is equivalent to

$$[\alpha_* p + 1 - p]^{2|A|} \le E(\alpha_*^{|\xi_1^A|}). \tag{7.17}$$

From Eq.(7.15), we see that Eq.(7.17) can be rewritten as

$$[\alpha_* p + 1 - p]^{2|A|} \le [\alpha_* q + 1 - q]^L \, [\alpha_* p + 1 - p]^M \,. \tag{7.18}$$

By Eqs.(7.2) and (7.3), we see that Eq.(7.18) is equivalent to

$$[\alpha_* p + 1 - p]^{2L} \le [\alpha_* q + 1 - q]^L \,. \tag{7.19}$$

Then Eq.(7.16) implies that Eq.(7.19) is equivalent to

$$\alpha_*^L \le [\alpha_* q + 1 - q]^L \,.$$

The last inequality holds, since $(1-q)\alpha_* \le 1-q$. Therefore we can check condition (4), and thus the proof of Theorem 7.3.1 (1) is complete. Part (2) follows from taking $A = \{0\}$ in part (1).

7.4. Holley-Liggett Method

In this section we will discuss the Holley-Liggett method briefly. The argument of Liggett[1] implies

Theorem 7.4.1. *Let*

$$p_c^{(HL)}(q) = \frac{1+\sqrt{1-q}}{2} \quad \textit{and} \quad \rho^{(HL)}(p,q) = \frac{2p - q + \sqrt{q[q-4p(1-p)]}}{2p^2}.$$

Then we have

$$p_c(q) \le p_c^{(HL)}(q) \qquad \textit{for } 0 \le q \le 1,$$

and

$$\rho(p,q) \ge \rho^{(HL)}(p,q) \qquad \textit{for } \frac{1}{2} \le p \le 1 \textit{ and } q \ge 4p(1-p).$$

In this section, we will show just how to get these bounds as in the case of the basic contact process in one dimension. Let μ be a renewal measure on $\{0,1\}^{\mathbf{Z}}$ whose density $f(n)(= F(n) - F(n+1))$ is given by

$$E(h(\xi_1^A)) = h(A),$$

for all A of the form $\{1, 2, \ldots, n\}$ $(n \ge 1)$, where

$$h(A) = \mu\{\eta : \eta(x) = 1 \text{ for some } x \in A\}.$$

Remark that

$$f(n) = \frac{\mu(\bullet \overbrace{\circ \cdots \circ}^{n-1} \bullet)}{\mu(\bullet)} \quad \text{and} \quad F(n) = \frac{\mu(\bullet \overbrace{\circ \cdots \circ}^{n-1})}{\mu(\bullet)} \quad \text{for } n \ge 1.$$

The definition of $F(n)$ gives $F(1) = 1$. A direct computation implies that

$$E(h(\xi_1^{\{1\}})) = h(\{1\})$$

gives

$$F(2) = \Big(\frac{1-p}{p}\Big)^2.$$

Similarly, for $n \geq 2$,

$$E(h(\xi_1^{\{1,2,\ldots,n\}})) = h(\{1, 2, \ldots, n\})$$

gives

$$p^2 F(n+1) = (1-q)\sum_{k=1}^{n-1} F(k)F(n+1-k) + (1-p)^2 F(n). \qquad (7.20)$$

We introduce the generating function

$$\phi(u) = \sum_{n=1}^{\infty} F(u)u^n.$$

Multiplying Eq.(7.20) by u^{n+1}, summing by $n \geq 2$ and using the value of $F(2)$ leads to the following quadratic equation of ϕ:

$$(1-q)\phi^2(u) - \Big[p^2 + \Big\{1 - q - (1-p)^2\Big\}u\Big]\phi(u) + p^2 u = 0.$$

The nonnegativity of the discriminant of this equation with $u = 1$ is equivalent to

$$q\Big[q - 4p(1-p)\Big] \geq 0.$$

The last inequality gives the upper bound $p_c^{(HL)}(q)$ on $p_c(q)$. Let $M = 1/\{\phi(1)\}$. So M satisfies the following equation:

$$(1-q)M^2 + (q-2p)M + p^2 = 0. \qquad (7.21)$$

Note that

$$\rho^{(HL)}(p,q) = \mu(\bullet) = \frac{1}{\phi(1)} = M. \qquad (7.22)$$

Combining Eq.(7.21) with Eq.(7.22) gives the lower bound on $\rho(p,q)$:

$$\rho^{(HL)}(p,q) = \frac{2p - q + \sqrt{q[q-4p(1-p)]}}{2p^2}.$$

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CHAPTER 8

3-STATE CYCLIC PARTICLE SYSTEMS

8.1. Introduction

In this chapter we consider 3-state interacting particle systems, in particular, cyclic type models. Until the previous chapter we consider just 2-state models, i.e., 0 and 1. Here we mainly study the next cyclic model. The dynamics is as follows: for $x \in \mathbf{Z}^d$ and $\eta \in \{0,1,2\}^{\mathbf{Z}^d}$,

$$\begin{aligned} 0 &\to 1 && \text{at rate} \quad n(1;x,\eta), \\ 1 &\to 2 && \text{at rate} \quad n(2;x,\eta), \\ 2 &\to 0 && \text{at rate} \quad n(0;x,\eta), \end{aligned}$$

where $n(i;x,\eta) = |\{y : |x-y| = 1 \text{ and } \eta(x) = i-1, \ \eta(y) = i \mod 3\}|$. We call this model *cyclic particle system* for short in these notes. In general, the N-state cyclic particle system can be considered in the following way; for $x \in \mathbf{Z}^d$ and $\eta \in \{0,1,\ldots,N-1\}^{\mathbf{Z}^d}$,

$$i \to i+1 \qquad \text{mod } N \qquad \text{at rate} \quad n(i+1;x,\eta).$$

where $n(i;x,\eta) = |\{y : |x-y| = 1 \text{ and } \eta(x) = i-1, \ \eta(y) = i \mod N\}|$. When $N = 2$, the cyclic model is equivalent to the standard two-state voter model, see Chapter V of Liggett,[1] for example. Our process is $N = 3$. This model can be considered as the biological model for three competing species. In other words, it is a lattice version of the Lotka-Volterra model. It is easy to see that there are three trivial invariant measures; δ_0, δ_1, and δ_2. In the case of one dimension, considering the walls made by two different species, we can easily show that the above three measures are only extreme measures, that is, any invariant measure can be written as a convex combination of δ_0, δ_1 and δ_2. Since the number of walls just decrease. So the interesting phenomena would appear in higher dimensions. The case of most biological interest is $d = 2$. On the other hand, concerning many-state cyclic particle system ($N \geq 3$) in one dimension, Bramson and Griffeath[2] studied the results on fixation. The many-state cyclic particle system ($N \geq 3$) is not reversible, additive or even attractive, so they restricted attention to the one dimensional case. Recently Tainaka and Yamasaki[3] considered the case $N = 4$ to introduce vortices in two dimensions and strings in three dimensions by using Monte Carlo simulations.

Cyclic particle systems (i.e. $N = 3$) in one and two dimensions were first studied by Tainaka[4] by Monte Carlo simulations. He reported that the two-dimensional cyclic particle system approaches the nontrivial stable state regardless of initial conditions. On the other hand, by the mean-field theory, the d-dimensional cyclic particle system reveals a neutrally stable center; the density of each species oscillates around the fixed points where three species coexists with equal densities, for example, see Itoh.[5–7] Moreover Tainaka[8] showed fixed point for the dynamics of pair-approximation becomes unstable. So, following Tainaka's statements[8], in the case of two dimensions (probably also higher dimensions,) one of the interesting things is as follows: for nontrivial fixed point,

1. Monte Carlo simulations show it becomes stable focus.
2. Mean-field approximation (the first approximation) shows it becomes neutrally stable center.
3. Pair-approximation (the second approximation) shows it becomes unstable point.

That is, the second approximation is not better than the first one for the stability of the cyclic particle system in two dimensions.

This chapter is organized as follows. Section 8.2 is devoted to master equations and correlation identities for the d-dimensional cyclic particle systems. In Sections 8.3 and 8.4, we consider the mean-field and pair approximations for this system respectively. Finally Section 8.5 is devoted to the cyclic particle system with an external field.

8.2. Correlation Identities

In this section, we consider the correlation identities for the d-dimensional cyclic particle system. To do so, first we introduce the following formal generator of this process as in the case of the basic contact process. Let $y \sim x$ denote that y is a nearest neighbor of x.

$$\begin{aligned}\Omega f(\eta) = &\sum_x I_0(\eta(x)) \sum_{y\sim x} I_1(\eta(y))[f(\eta^{x,0\to 1}) - f(\eta)] \\ &+ \sum_x I_1(\eta(x)) \sum_{y\sim x} I_2(\eta(y))[f(\eta^{x,1\to 2}) - f(\eta)] \\ &+ \sum_x I_2(\eta(x)) \sum_{y\sim x} I_0(\eta(y))[f(\eta^{x,2\to 0}) - f(\eta)],\end{aligned}$$

where $I_i(j) = 1$ if $j = i$, $= 0$ if $j \neq i$, and $\eta^{x,i\to j}(k) = \eta(k)$ if $k \neq x$, $= \eta(x)$ if $\eta(x) \neq i$, $= j$ if $\eta(x) = i$.

From now on we assume the initial measures we consider here are translation, rotation and reflection invariant. Let $\rho_t(i)$ be the density of i species for a site at time

t, that is, $\rho_t(i) = E[I_i(\eta_t(0))]$, where η_t is the cyclic particle system at time t and 0 is the origin. If we take $f(\eta) = I_0(\eta(0))$, then

$$\Omega f(\eta) = -\sum_{y\sim 0} I_0(\eta(0))I_1(\eta(y)) + \sum_{y\sim 0} I_2(\eta(0))I_0(\eta(y)).$$

Hence translation, rotation and reflection invariance for initial measure give

$$\frac{d\rho_t(0)}{dt} = z\left[\rho_t(20) - \rho_t(01)\right].$$

where $z = 2d$ (d is the dimensionality) and $\rho_t(ij) = E\left[I_i(\eta_t(0))I_j(\eta_t(e_1))\right]$ where $e_1 = (1, 0, \ldots, 0)$ be a unit vector.

Exercise 8.1. Verify the following differential equations;

$$\frac{d\rho_t(1)}{dt} = z\left[\rho_t(01) - \rho_t(12)\right],$$
$$\frac{d\rho_t(2)}{dt} = z\left[\rho_t(12) - \rho_t(20)\right].$$

Next we will compute the differential equations for two-point correlation functions $\rho_t(ij)$. In a similar way, if $f(\eta) = I_0(\eta(0))I_1(\eta(e_1))$, then

$$\begin{aligned}
\Omega f(\eta) = I_0(\eta(0)) \sum_{y\sim 0} I_1(\eta(y))[I_0(1)I_1(\eta(e_1)) - I_0(\eta(0))I_1(\eta(e_1))] \\
+ I_0(\eta(e_1)) \sum_{y\sim e_1} I_1(\eta(y))\left[I_0(\eta(0))I_1(1) - I_0(\eta(0))I_1(\eta(e_1))\right] \\
+ I_1(\eta(0)) \sum_{y\sim 0} I_2(\eta(y))\left[I_0(2)I_1(\eta(e_1)) - I_0(\eta(0))I_1(\eta(e_1))\right] \\
+ I_1(\eta(e_1)) \sum_{y\sim e_1} I_2(\eta(y))\left[I_0(\eta(0))I_1(2) - I_0(\eta(0))I_1(\eta(e_1))\right] \\
+ I_2(\eta(0)) \sum_{y\sim 0} I_0(\eta(y))\left[I_0(0)I_1(\eta(e_1)) - I_0(\eta(0))I_1(\eta(e_1))\right] \\
+ I_2(\eta(e_1)) \sum_{y\sim e_1} I_0(\eta(y))\left[I_0(\eta(0))I_1(0) - I_0(\eta(0))I_1(\eta(e_1))\right] \\
= -I_0(\eta(0))I_1(\eta(e_1)) - \sum_{y\sim 0, y\neq e_1} I_0(\eta(0))I_1(\eta(e_1))I_1(\eta(y)) \\
+ \sum_{y\sim e_1, y\neq 0} I_0(\eta(0))I_0(\eta(e_1))I_1(\eta(y)) \\
- \sum_{y\sim e_1, y\neq 0} I_0(\eta(0))I_1(\eta(e_1))I_2(\eta(y)) \\
+ \sum_{y\sim 0, y\neq e_1} I_2(\eta(0))I_1(\eta(e_1))I_0(\eta(y)).
\end{aligned}$$

Therefore we have

$$\frac{d\rho_t(01)}{dt} = -\rho_t(01) + \sum_{y\sim e_1, y\neq 0} [\rho_t(00,1) + \rho_t(12,0) - \rho_t(10,1) - \rho_t(01,2)],$$

where $\rho_t(ij,k) = E\left[I_i(\eta_t(0))I_j(\eta_t(e_1))I_k(\eta_t(y))\right]$ for y with $y \sim e_1$ and $y \neq 0$. Note that we omit y in $\rho_t(ij,k)$ for simplicity. Similarly we have differential equations for other $\rho_t(ij)$. Hence the following result can be obtained.

Theorem 8.2.1. *Let $z = 2d$. For any initial measure with translation, rotation and reflection invariances, we have*

(1) $$\frac{d\rho_t(0)}{dt} = z\left[\rho_t(20) - \rho_t(01)\right].$$

(2) $$\frac{d\rho_t(1)}{dt} = z\left[\rho_t(01) - \rho_t(12)\right].$$

(3) $$\frac{d\rho_t(2)}{dt} = z\left[\rho_t(12) - \rho_t(20)\right].$$

(4) $$\frac{d\rho_t(00)}{dt} = 2\rho_t(20) + 2\sum_{y\sim e_1, y\neq 0} [\rho_t(02,0) - \rho_t(00,1)].$$

(5) $$\frac{d\rho_t(11)}{dt} = 2\rho_t(01) + 2\sum_{y\sim e_1, y\neq 0} [\rho_t(10,1) - \rho_t(11,2)].$$

(6) $$\frac{d\rho_t(22)}{dt} = 2\rho_t(12) + 2\sum_{y\sim e_1, y\neq 0} [\rho_t(21,2) - \rho_t(22,0)].$$

(7) $$\frac{d\rho_t(01)}{dt} = -\rho_t(01) + \sum_{y\sim e_1, y\neq 0} [\rho_t(00,1) + \rho_t(12,0) - \rho_t(10,1) - \rho_t(01,2)].$$

(8) $$\frac{d\rho_t(12)}{dt} = -\rho_t(12) + \sum_{y\sim e_1, y\neq 0} [\rho_t(11,2) + \rho_t(20,1) - \rho_t(21,2) - \rho_t(12,0)].$$

(9) $$\frac{d\rho_t(20)}{dt} = -\rho_t(20) + \sum_{y\sim e_1, y\neq 0} [\rho_t(22,0) + \rho_t(01,2) - \rho_t(02,0) - \rho_t(20,1)].$$

Exercise 8.2. Verify Theorem 8.2.1 (4)-(6), (8) and (9).

In the stationary state, we have

Corollary 8.2.2.

(1) $$\rho(01) = \rho(12) = \rho(20).$$

$$(2) \qquad -\rho(01) = \sum_{y\sim e_1, y\neq 0} [\rho(02,0) - \rho(00,1)] = \sum_{y\sim e_1, y\neq 0} [\rho(10,1) - \rho(11,2)]$$
$$= \sum_{y\sim e_1, y\neq 0} [\rho(21,2) - \rho(22,0)].$$

$$(3) \qquad \rho(01) = \sum_{y\sim e_1, y\neq 0} [\rho(00,1) + \rho(12,0) - \rho(10,1) - \rho(01,2)]$$
$$= \sum_{y\sim e_1, y\neq 0} [\rho(11,2) + \rho(20,1) - \rho(21,2) - \rho(12,0)]$$
$$= \sum_{y\sim e_1, y\neq 0} [\rho(22,0) + \rho(01,2) - \rho(02,0) - \rho(20,1)]$$

8.3. Mean-Field Approximation

In this section we consider the mean-field approximation for the cyclic particle system. Let $I_t = \rho_t(0) + \rho_t(1) + \rho_t(2)$ and $J_t = \rho_t(0)\rho_t(1)\rho_t(2)$. Then $I_t = 1$ for any t, so it is trivial conservative quantity. In the mean-field theory, J_t also becomes conservative quantity. That is, if we assume $\rho_t(ij) = \rho_t(i)\rho_t(j)$ for any t and i, j, (mean-field approximation,) then Theorem 8.2.1 (1)-(3) give

$$\frac{dJ_t}{dt} = 0.$$

Concerning the stationary density, if we assume $\rho(ij) = \rho(i)\rho(j)$ for any i, j, (mean-field approximation), then Corollary 8.2.2 (1) implies

$$(\rho(0), \rho(1), \rho(2)) = (1,0,0),\ (0,1,0),\ (0,0,1),\ (1/3,1/3,1/3).$$

Next we discuss the stability for the cyclic particle system by mean-field approximation. For simplicity, we assume $z = 1$. Let

$$\rho_t(0) = \rho_t^*(0) + x_t,$$
$$\rho_t(1) = \rho_t^*(1) + y_t,$$
$$\rho_t(2) = \rho_t^*(2) + z_t,$$

where $(\rho_t^*(0), \rho_t^*(1), \rho_t^*(2))$ is a stationary density and (x_t, y_t, z_t) is a fluctuation with $x_t + y_t + z_t = 0$.
(i) $(\rho_t^*(0), \rho_t^*(1), \rho_t^*(2)) = (1,0,0)$. In this case, we begin with

$$\frac{dx_t}{dt} = \frac{d\rho_t(0)}{dt} = \rho_t(0)\rho_t(2) - \rho_t(0)\rho_t(1)$$
$$= (\rho_t^*(0) + x_t)(\rho_t^*(2) + z_t) - (\rho_t^*(0) + x_t)(\rho_t^*(1) + y_t)$$
$$= (1 + x_t)z_t - (1 + x_t)y_t$$

Here we neglect $x_t z_t$ and $x_t y_t$ and use $z_t = -x_t - y_t$, so we have the following linearized equation:

$$\frac{dx_t}{dt} = -x_t - 2y_t.$$

Similarly we get

$$\frac{dy_t}{dt} = y_t.$$

Therefore

$$\frac{d}{dt}\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_t \\ y_t \end{pmatrix}.$$

The eigenvalues of this matrix are -1 and 1. So we conclude that $(1, 0, 0)$ is unstable. In a similar way, we see that $(0, 1, 0)$ and $(0, 0, 1)$ are also unstable.
(ii) $(\rho_t^*(0), \rho_t^*(1), \rho_t^*(2)) = (1/3, 1/3, 1/3)$. In this case, we obtain

$$\frac{d}{dt}\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_t \\ y_t \end{pmatrix}.$$

The eigenvalues of this matrix are $\pm\sqrt{3}i$. So we conclude that $(1/3, 1/3, 1/3)$ is neutrally stable.

8.4. Pair-Approximation

This section is devoted to pair-approximation. In the stationary states, if we assume $\rho(j)\rho(ij, k) = \rho(ij)\rho(jk)$ for any i, j, k, (pair-approximation), then Corollary 8.2.2 (1)-(3) give a nontrivial stationary density;

$$\rho(i) = \frac{1}{3}, \quad \rho(ii) = \frac{z+1}{9(z-1)}, \quad \rho(ij) = \frac{z-2}{9(z-1)} \quad (i \neq j), \tag{8.1}$$

where $z = 2d$. When $d = 1$, Eq.(8.1) becomes

$$\rho(i) = \rho(ii) = \frac{1}{3}, \quad \rho(ij) = 0 \quad (i \neq j). \tag{8.2}$$

So it suggests any invariant measure in one dimension is a convex combination of δ_0, δ_1 and δ_2. In any dimension, we see that $\rho(ii) > 1/9$ and $\rho(ij) < 1/9$ $(i \neq j)$. Furthermore, if d goes to infinity, both $\rho(ii)$ and $\rho(ij)$ with $i \neq j$ approach to 1/9, that is, mean-field limit. However, as Tainaka[8] pointed out, pair-approximation does not explain the stability for the cyclic particle system.

8.5. Cyclic Particle System with External Field

In this section we consider the cyclic particle system with an external field from $1 \to 0$ as follows; for $x \in \mathbf{Z}^d$ and $\eta \in \{0, 1, 2\}^{\mathbf{Z}^d}$,

$$\begin{aligned}
0 &\to 1 && \text{at rate} \quad n(1; x, \eta), \\
1 &\to 2 && \text{at rate} \quad n(2; x, \eta), \\
2 &\to 0 && \text{at rate} \quad n(0; x, \eta), \\
1 &\to 0 && \text{at rate} \quad \delta,
\end{aligned}$$

where $n(i,x;\eta) = |\{y : |x-y| = 1 \text{ and } \eta(y) = i\}|$. This system was first studied by Tainaka.[9] The formal generator is given by

$$\begin{aligned}\Omega f(\eta) = &\sum_x I_0(\eta(x)) \sum_{y\sim x} I_1(\eta(y))[f(\eta^{x,0\to 1}) - f(\eta)]\\ &+ \sum_x I_1(\eta(x)) \sum_{y\sim x} I_2(\eta(y))[f(\eta^{x,1\to 2}) - f(\eta)]\\ &+ \sum_x I_2(\eta(x)) \sum_{y\sim x} I_0(\eta(y))[f(\eta^{x,2\to 0}) - f(\eta)]\\ &+ \sum_x I_1(\eta(x))[f(\eta^{x,1\to 0}) - f(\eta)],\end{aligned}$$

where $I_i(j) = 1$ if $j = i$, $= 0$ if $j \neq i$, and $\eta^{x,i\to j}(k) = \eta(k)$ if $k \neq x$, $= \eta(x)$ if $\eta(x) \neq i$, $= j$ if $\eta(x) = i$. As in the cyclic particle system, we obtain

Theorem 8.5.1. *Let $z = 2d$. For any initial measure with translation, rotation and reflection invariances, we have*

(1) $$\frac{d\rho_t(0)}{dt} = z\left[\rho_t(20) - \rho_t(01)\right] + \delta\rho_t(1).$$

(2) $$\frac{d\rho_t(1)}{dt} = z\left[\rho_t(01) - \rho_t(12)\right] - \delta\rho_t(1).$$

(3) $$\frac{d\rho_t(2)}{dt} = z\left[\rho_t(12) - \rho_t(20)\right].$$

(4) $$\frac{d\rho_t(00)}{dt} = 2\rho_t(20) + 2\sum_{y\sim e_1, y\neq 0}[\rho_t(02,0) - \rho_t(00,1)] + 2\delta\rho_t(01).$$

(5) $$\frac{d\rho_t(11)}{dt} = 2\rho_t(01) + 2\sum_{y\sim e_1, y\neq 0}[\rho_t(10,1) - \rho_t(11,2)] - 2\delta\rho_t(11).$$

(6) $$\frac{d\rho_t(22)}{dt} = 2\rho_t(12) + 2\sum_{y\sim e_1, y\neq 0}[\rho_t(21,2) - \rho_t(22,0)].$$

(7) $$\begin{aligned}\frac{d\rho_t(01)}{dt} = &-\rho_t(01) + \sum_{y\sim e_1, y\neq 0}[\rho_t(00,1) + \rho_t(12,0) - \rho_t(10,1) - \rho_t(01,2)]\\ &+ \delta\rho_t(11) - \delta\rho_t(01).\end{aligned}$$

(8) $$\begin{aligned}\frac{d\rho_t(12)}{dt} = &-\rho_t(12) + \sum_{y\sim e_1, y\neq 0}[\rho_t(11,2) + \rho_t(20,1) - \rho_t(21,2) - \rho_t(12,0)]\\ &- \delta\rho_t(02).\end{aligned}$$

(9) $$\begin{aligned}\frac{d\rho_t(20)}{dt} = &-\rho_t(20) + \sum_{y\sim e_1, y\neq 0}[\rho_t(22,0) + \rho_t(01,2) - \rho_t(02,0) - \rho_t(20,1)]\\ &+ \delta\rho_t(12).\end{aligned}$$

In a stationary state, if we assume $\rho(ij) = \rho(i)\rho(j)$ for any i, j (mean-field approximation), then Theorem 8.5.1 gives

$$\rho(0) = \rho(1) = \frac{z+\delta}{3z}, \quad \rho(2) = \frac{z-2\delta}{3z}.$$

Then, one of the interesting open problems presented by Tainaka[9] is as follows:

Open Problem 8.5.2. Let $d = 2$. Suppose that $\eta_0(x)$, $x \in \mathbf{Z}^2$ are independent and $P(\eta_0(x) = i) = 1/3$ for any x and $i = 0, 1, 2$. Then there exists $\delta_c > 0$ such that for any $0 < \delta < \delta_c$,

$$\rho(1) > \frac{1}{3}.$$

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