CONTENTS

0. Introduction

The aim of these notes is to consider the following problems in various cases.

Problem 1. *Given two manifolds of the same dimension in a projective space, how can we decide that they are projectively equivalent to each other?*

Let $f\colon M \to \mathbf{P}^n$ and $g\colon N \to \mathbf{P}^n$ be immersions. The problem is to look for conditions that enable us to find a diffeomorphism $\varphi\colon M \to N$ and a projective transformation p of $\mathbf{P}^n$ such that $p \circ f = g \circ \varphi$. This is in general of global nature and seems more difficult than the next

Problem 2. *Let f_1 and f_2 be two immersions of a manifold M into $\mathbf{P}^n$. Find conditions so that $f_2 = p \circ f_1$ for some projective transformation p.*

This is in principle a local problem and the main concern of these notes. The relation with a system of differential equations is seen by the following rough arguments. Let $(x^1, \ldots, x^m)$ be local coordinates of M and make a set of vectors $\{\, f, \partial f/\partial x^i, \partial^2 f/\partial x^i \partial x^j, \ldots \}$. Since the maximum possible number of independent vectors are $n+1$, there will be a *linear* relation among each $n+2$ vectors of this set. These relations make a system of linear homogeneous differential equations satisfied by f. By the linearity, each set of independent solutions, the number of which is assumed to be $n+1$, define an immersion projectively equivalent to f. Conversely, given such a system whose rank, the dimension of solution space, is $n+1$, the fundamental set of solutions define an immersion.

The method we now take for the above problems is to draw some geometrical information out of this system, that is sufficient to characterize immersion. This method originates in Halphen's study of ordinary linear differential equations and in Wilczynski's work for curves and surfaces among others.

In Chapter 1 we deal with a mapping from M^n to a same-dimensional $\mathbf{P}^n$. Such a mapping when $n = 1$ is called a projective motion by E. Cartan. The aim is to review Schwarzian derivatives from a geometric point of view. Main references are [CAR] and [Y]. Chapter 2 treats curves in a projective plane and recalls the theory by Laguerre-Forsyth. We give some applications to linear ordinary differential equations. In Chapter 3 we recall the theory of ruled surfaces and give a generalization of treatments of plane curves and ruled surfaces. References for these two chapters are [W1], [LAN] and [BOL] among other many volumes.

Chapter 4 reformulates the theory of hypersurfaces in a projective space. The main emphasis is laid on the definition of several invariants and on the formulation of a fundamental theorem. Chapter 5 is an application in the study of a system of linear differential equations with n variables of rank $n+2$. The principal role is played by the conformal geometry. The case $n = 2$ is separately treated. In Chapter 6 we discuss the projective minimality. Transforms of surfaces due to Demoulin will be formulated in our point of view. References for these four chapters will be given in the context.

Notations: Throughout these notes, $\mathbf{P}^n$ denotes an n-dimensional projective space over $\mathbf{R}$. The coefficient field is $\mathbf{R}$. Functions are assumed to be C^∞. However, the most of arguments holds also for the complex case: $\mathbf{P}^n$ over $\mathbf{C}$, the coefficient field $\mathbf{C}$ and the holomorphic functions. Some parts are valid only for the complex case, which will be stated explicitly.

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1. Motion of Points in $\mathbf{P}^n$

We treat a mapping from an n-dimensional manifold into $\mathbf{P}^n$. When $n = 1$, it is called a projective motion. The projective equivalence class of such a mapping is described by Schwarzian derivatives. Sections 1–4 treat the case $n = 1$ and Section 5 the case $n \geq 2$.

§1. Schwarzian derivative

M denotes a 1-dimensional manifold with a coordinate t. A mapping from M to $\mathbf{P}^1$: $t \to p(t)$ is called a *motion* of points in $\mathbf{P}^1$. We forget sometimes to mention M saying "a motion $p(t)$". General problem concerning the projective equivalence is now stated for motions as follows.

Problem. *Given two motions $p(t)$ and $q(t)$, decide whether they are projectively equivalent or not. In other words when is the one transformed to the other by a projective transformation?*

Recall first the euclidean case: let $x(t)$ and $y(t)$ be motions in $\mathbf{R}^1$. Then they are equivalent under a rigid motion, i.e. $x(t) = y(t) + b$ for a constant b, if and only if $x'(t) = y'(t)$. So in the projective case we should ask for the condition replacing the derivative x'. To make the discussion easier we fix an affine coordinate of $\mathbf{P}^1$ as

$$p(t) = [\,1, f(t)\,] \quad \text{and} \quad q(t) = [\,1, q(t)\,].$$

Then the above problem is reduced to the problem to find a condition that assures

$$g = \frac{af + b}{cf + d}. \tag{1.1}$$

Assume first this identity and take derivatives successively to get

$$g' = \frac{(ad - ec)f'}{(cf + d)^2},$$

$$\frac{g''}{g'} = \frac{f''}{f'} - \frac{2f'}{cf+d},$$

and

$$\frac{g'''}{g'} - \left(\frac{g''}{g'}\right)^2 = \frac{f'''}{f'} - \left(\frac{f''}{f'}\right)^2 - \frac{2f''}{cf+d} + \frac{2(f')^2}{(cf+d)^2}.$$

The last two terms are equal to

$$\frac{1}{2}\left(\frac{f''}{f'} - \frac{2f'}{cf+d}\right)^2 - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{1}{2}\left(\left(\frac{g''}{g'}\right)^2 - \left(\frac{f''}{f'}\right)^2\right).$$

Hence, if we put

$$\{f;t\} = \frac{1}{2}\frac{f'''}{f'} - \frac{3}{4}\left(\frac{f''}{f'}\right)^2 = \frac{1}{2}\left(\frac{f''}{f'}\right)' - \frac{1}{4}\left(\frac{f''}{f'}\right)^2 \tag{1.2}$$

and similarly for g, then we have an identity

$$\{f;t\} = \{g;t\}.$$

Here we have assumed f' and g' do not vanish; such a motion is called a *regular* motion.

Proposition 1.1. *If two regular motions p and q are projectively equivalent, then $\{f;t\} = \{g;t\}$ and conversely.*

To see the converse statement we need

Lemma 1.2. 1° $\{(at+b)/(ct+d);t\} = 0$.
2° *If t is a non-constant function of s, then $\{f;t\}\,dt^2 = \{f;s\}\,ds^2 + \{s;t\}\,dt^2$.*
3° $\{t;s\}\,ds^2 = -\{s;t\}\,dt^2$.
4° *If $\{f;t\} = 0$, then $f = (at+b)/(ct+d)$.*

Assuming Lemma 1.2, we prove the converse. Replacing s in 2° by g, we get $\{f;g\}\,dg^2 = [\,\{f;t\} - \{g;t\}\,]\,dt^2 = 0$. Hence, by 4°, $f = (ag+b)/(cg+d)$.

Proof of Lemma 1.2. The assertion 1° follows from the first part of Proposition 1.1, since $\{t; t\} = 0$. 2° is shown by computation. 3° is a special case of 2° when $f = t$. The assertion 4° is seen by a simple integration.

Definition. We call $\{f; t\}$ the *Schwarzian derivative* of f with respect to t.

We notice here that Proposition 1.1 says

$$S(t) = \{f; t\}$$

is defined independently of a choice of coordinates of $\mathbf{P}^1$. This is an (differential) invariant of a projective equivalence class of motions. Furthermore the quadratic differential $S(t)\,dt^2$ is invariant under any projective coordinate change of t.

§2. Ordinary linear differential equation of order 2

We next associate with every motion a linear differential equation and understand the invariant $S(t)$ in terms of this equation. We write the motion $p(t)$ in homogeneous coordinates as $p(t) = (x_1(t), x_2(t))$. Assume $p(t)$ is regular, *i.e.*,

$$x_1' x_2 - x_1 x_2' \neq 0. \tag{2.1}$$

Functions $x_i(t)$ satisfy the differential equation with unknown $x(t)$:

$$\begin{vmatrix} x'' & x' & x \\ x_1'' & x_1' & x_1 \\ x_2'' & x_2' & x_2 \end{vmatrix} = 0.$$

This can be written, by the assumption (2.1), as

$$x'' + p_1 x' + p_2 x = 0, \tag{2.2}$$

which is the equation associated with the motion p. Let y_1 and y_2 be a set of independent solutions. Then $y_i = \sum_j a_i^j x_j$ for a constant matrix (a_{ij}) with $\det a_{ij} \neq 0$. Hence a motion (y_1, y_2) is equivalent to the original p. Namely the differential equation of type (2.2) gives a projective equivalence class of motions. Then, by referring the result in §1, the invariant $S(t)$ may have some relation with p_1 and p_2. In fact the explicit form is given in the following way.

First note that the equation (2.2) is not the only one associated with the motion, since the choice of x_i is not unique. Put $y_i = \lambda(t)x_i$ for a non-vanishing function λ. Then y_i also represent the same motion and satisfy the differential equation

$$x'' + \left(-\frac{2\lambda'}{\lambda} + p_1\right)x' + \left(-\frac{\lambda''}{\lambda} + 2\frac{\lambda'^2}{\lambda^2} - p_1\frac{\lambda'}{\lambda} + p_2\right)x = 0. \tag{2.3}$$

Hence for any λ this equation may be considered as the same equation in our viewpoint. To kill the freedom of choice of λ, we impose that (2.3) should have a special form. This is done by choosing λ so that

$$p_1 - \frac{2\lambda'}{\lambda} = 0$$

and (2.3) turns out to be

$$x'' + Q(t)x = 0, \tag{2.4}$$

where

$$Q(t) = p_2 - \frac{1}{4}p_1^2 - \frac{1}{2}p_1'. \tag{2.5}$$

Proposition 1.3. *$Q(t) = S(t)$.*

Proof. Let $p(t)$ be a motion. It is represented by functions $x_i(t)$ of the form

$$x_1(t) = \lambda(t)f(t), \quad x_2(t) = \lambda(t).$$

Let us find λ so that x_i satisfies (2.4). From equations $(\lambda f)'' + Q(\lambda f) = 0$ and $\lambda'' + Q\lambda = 0$ follows $2\lambda' f' + \lambda f'' = 0$. Hence

$$\lambda = (f')^{-1/2}. \tag{2.6}$$

Then

$$Q(t) = -\frac{\lambda''}{\lambda} = \{f;t\} = S(t),$$

which is the desired identity.

The equality (2.6) implies that, for a given f,

$$x_1 = f(f')^{-1/2} \quad \text{and} \quad x_2 = (f')^{-1/2}$$

satisfy (2.4) for $Q = \{f;t\}$.

§3. Normal form of a motion

We now try to find a normal form of the function $f(t)$ relating with value of Q. Assume $Q(t)$ is defined around $t = 0$ and compute approximate solutions of (2.4). Let

$$Q(t) = Q(0) + Q'(0)t + \cdots$$

and put

$$x = 1 + \frac{a_2}{2}t^2 + \frac{a_3}{6}t^3 + \cdots .$$

Inserting these expressions into (2.4), we get

$$\begin{aligned} 0 &= (a_2 + a_3 t + \cdots) + (Q(0) + Q'(0)t + \cdots)\Big(1 + \frac{a_2}{2}t^2 + \cdots\Big) \\ &= (a_2 + Q(0)) + (a_3 + Q'(0))t + \cdots \end{aligned}$$

Hence one approximate solution is

$$x_1(t) = 1 - \frac{Q(0)}{2}t^2 + \cdots$$

Another is obtained similarly by putting

$$x = t + \frac{b_3}{6}t^3 + \cdots$$

The result is

$$x_2(t) = t - \frac{Q(0)}{6}t^3 + \cdots .$$

Since $f = \dfrac{x_2}{x_1}$ is an affine coordinate function, we have

Proposition 1.4. *For any regular motion around $t = 0$, there is an affine coordinate such that the motion is written as $(1, f(t))$ where*

$$f(t) = t + \frac{Q(0)}{3}t^3 + \cdots$$

§4. Example by H. A. Schwarz

We will in this section explain how the invariant was used by H. A. Schwarz himself. For details and further results refer the books [F-K], [Y].

The coefficient field is now $\mathbf{C}$ and the category is that of holomorphic functions. Consider the Gauss hypergeometric differential equation

$$y'' + \left(\frac{c}{x} + \frac{a+b+1-c}{x-1}\right)y' + \frac{ab}{x(x-1)}y = 0$$

where a, b, c are real parameters. This equation is defined on $\mathbf{P}^1(x)$ and has regular singularities at $x = 0,\ 1,\ \infty$. Let y_1 and y_2 be independent solutions and put $z = y_1/y_2$. We are interested in the multivalued mapping from $\mathbf{P}^1(x)$ to $\mathbf{P}^1(z)$. The behavior of this mapping near singularities are seen by the next

Lemma 1.5. *Let z be a non-constant function around $x = 0$ and of the form $z = x^e h(x)$ or $z = (\log x)h(x)$, where $h(x)$ is a non-vanishing holomorphic function. Then $\lim_{t\to 0} 4x^2\{z; x\} = 1 - e^2$ or 1 respectively.*

To find the value e, let us compute the invariant $Q(x)$. Since

$$\begin{aligned} \frac{1}{2}p_1' &= -\frac{c}{2x^2} - \frac{c'}{2(x-1)^2} \\ \frac{1}{4}p^2 &= \frac{c^2}{4x^2} + \frac{cc'}{2x(x-1)} + \frac{c'^2}{4(x-1)^2}, \end{aligned}$$

where $c' = a + b + 1 - c$, we get

$$Q(x) = \frac{2c - c^2}{4x^2} + \frac{2c' - c'^2}{4(x-1)^2} + \frac{cc' + 2ab}{2x(x-1)}.$$

Then, by Lemma 1.5, the values e at $x = 0,\ 1,\ \infty$ are $\pm(1-c)$, $\pm(c-a-b)$, $\pm(a-b)$ respectively (for $x=\infty$, rewrite the equation by introducing a new coordinate $t = 1/x$ and compute Q at $t=0$). For simplicity assume these numbers are not integers. Then Lemma 1.5 shows that around each singularity the mapping z behaves like $x^{|1-c|}$, $(1-x)^{|c-a-b|}$ and $(1/x)^{|a-b|}$ up to a projective transformation. From this fact we can prove that the mapping z restricted to the upper half-plane has its image in the inside of the triangle whose sides are circular arcs and that the angles at the edge of the triangle are $\pi|1-c|$, $\pi|c-a-b|$ and $\pi|a-b|$.

This mapping is continued analytically to the lower half plane through the intervals $(0,1)$, $(1,\infty)$, and $(\infty,0)$ and we obtain the multi-valued mapping z. Put

$$\lambda = \frac{1}{|1-c|}, \quad \mu = \frac{1}{|c-a-b|}, \quad \nu = \frac{1}{|a-b|}$$

and assume λ, μ and ν are integers. Then Schwarz found the following fact.

(H. A. Schwarz, 1872). The image of $\mathbf{P}^1(z)$ is $\mathbf{P}^1$, $\mathbf{C}^1$ or the unit disc according as

$$\frac{1}{\lambda}+\frac{1}{\mu}+\frac{1}{\nu} > 1, \quad =1 \quad \text{or} \quad <1 \quad \text{respectively.}$$

§5. Schwarzian derivatives of several variables

Let M be an n-dimensional manifold with local coordinates (x^i). A motion of M on a projective space $\mathbf{P}^n$ is now understood to be a non-degenerate mapping

$$M \ni (x) \mapsto z(x) \in \mathbf{P}^n.$$

Problem in §1 can be read similarly. We need Schwarzian derivatives of several variables defined below.

Let $(z^1, \ldots, z^n)$ be affine coordinates of z and $j(z,x)$ be the jacobian:

$$j(z,x) = (j_i^k)\,;\; j_i^k = \frac{\partial z^k}{\partial x^i}.$$

Its inverse is written

$$J_i^k(z,x) = \frac{\partial x^k}{\partial z^i}.$$

Put

$$\sigma(z,x) = \frac{1}{n+1} \log \det j(z,x),$$

$$\sigma_i(z,x) = \frac{\partial \sigma}{\partial x^i},$$

$$\gamma_{ij}^k(z,x) = \sum_\ell \frac{\partial^2 z^\ell}{\partial x^i\, \partial x^j} J_\ell^k(z,x).$$

Then *Schwarzian derivatives* S_{ij}^k are defined by

$$S_{ij}^k(z;x) = \gamma_{ij}^k(z,x) - \delta_i^k \sigma_j(z,x) - \delta_j^k \sigma_i(z,x). \tag{5.1}$$

Remark that these expressions are given by derivatives up to order 2. When $n = 1$, this is trivial. So assume $n \geq 2$. The next is an analogue of Lemma 1.2.

Lemma 1.6. 1° *$S_{ij}^k(z;x) = S_{ji}^k(z;x)$.*
2° *$\sum_k S_{ik}^k = 0$.*
3° *$S_{ij}^k(Az;x) = S_{ij}^k(z;x)$ for any projective transformation $A \in PGL_{n+1}$. In particular, $S_{ij}^k(Ax;x) = 0$.*
4° *Let x be a non-degenerate map of y. Then*

$$S_{ij}^k(z;y) - S_{pq}^r(z;x) j_i^p(x;y) j_j^q(x;y) J_r^k(x,y) = S_{ij}^k(x;y)$$

5° *If $S_{ij}^k(z;x) = 0$ for all i, j and k, then $z = Ax$ for a projective transformation $A \in PGL_{n+1}$.*

Proof. 1° is trivial. 3° and 5° will be shown later. 4° is verified by a direct calculation. 2° is seen as follows:

$$\sum_k S^k_{ik}(z;x) = \sum_{k,\ell} \frac{\partial^2 z\ell}{\partial x^i\,\partial x^k}\frac{\partial x^k}{\partial z^\ell} - (n+1)\sigma_i = 0$$

by definition of σ.

From this lemma follows an analogue of Proposition 1.1:

Proposition 1.7. *Two non-degenerate mappings z_1 and z_2 are projectively equivalent if and only if*

$$S^k_{ij}(z_1;x) = S^k_{ij}(z_2;x) \quad \textit{for all} \quad i,\ j \ \textit{and}\ \ k.$$

The differential equations satisfied by the mapping z will next be derived. Let y be a vector in $\mathbf{A}^{n+1}$ representing z. Consider $n+1$ vectors $y, y_1, y_2, \dots, y_n$ ($y_i = \partial y/\partial x^i$). They are linearly independent by non-degeneracy. Hence the second-order derivatives $z_{ij} = \partial^2 z/\partial x^i \partial x^j$ are linear combinations of these vectors; we have

$$z_{ij} = \sum_k A^k_{ij} z_k + A^0_{ij} z, \tag{5.2}$$

for some functions A^k_{ij} and A^0_{ij}. These are equations defining a non-degenerate mapping into $\mathbf{P}^n$. If we take $w = \lambda^{-1} y$ instead of y for a scalar function λ, then new equations are

$$w_{ij} = \sum_k \left(A^k_{ij} - \delta^k_j \frac{\lambda_i}{\lambda} - \delta^k_i \frac{\lambda_j}{\lambda} \right) w_k + \left(A^0_{ij} + \sum A^k_{ij} \frac{\lambda_k}{\lambda} - \frac{\lambda_{ij}}{\lambda} \right) w.$$

This verifies that we can choose y so that

$$\sum_k A^k_{ik} = 0. \tag{5.3}$$

When this condition is satisfied we call (5.2) a *normalized* system.

Proposition 1.8. *Assume* (5.2) *is normalized. Then*

$$1^o \quad A^k_{ij} = S^k_{ij}(z;x) \qquad 2^o \quad A^0_{ij} = \frac{1}{n-1}\Big(\sum_{\ell,k} S^\ell_{ik}S^k_{\ell j} - \sum_k \frac{\partial}{\partial x^k}S^k_{ij}\Big).$$

Proof. The components of y are λ, λz^1, ... , λz^n for some λ; they satisfy (5.2). So

$$\begin{aligned}\lambda_{ij} &= \sum A^k_{ij}\lambda_k + A^0_{ij}\lambda \\ (\lambda z^\ell)_{ij} &= \sum A^k_{ij}(\lambda z^\ell)_k + A^0_{ij}(\lambda z^\ell),\end{aligned}$$

whence we have

$$A^k_{ij} = \delta^k_j \frac{\lambda_i}{\lambda} + \delta^k_i \frac{\lambda_j}{\lambda} + \sum z^\ell_{ij} J^k_\ell .$$

Then the normalization condition implies

$$0 = \sum A^k_{ik} = (n+1)\frac{\lambda_i}{\lambda} + (n+1)\sigma_i .$$

Hence we get 1°. To see 2°, differentiate (5.2) once getting

$$\begin{aligned} z_{ij\ell} &= \sum A^k_{ij} z_{k\ell} + \sum \frac{\partial A^k_{ij}}{\partial x^\ell} z_k + A^0_{ij} z_\ell + \frac{\partial A^0_{ij}}{\partial x^\ell} z \\ &= \sum \Big(\frac{\partial A^k_{ij}}{\partial x^\ell} + \sum A^m_{ij}A^k_{m\ell} + A^0_{ij}\delta^k_\ell\Big) z_k + \Big(\frac{\partial A^0_{ij}}{\partial x^\ell} + \sum A^k_{ij}A^0_{k\ell}\Big) z .\end{aligned}$$

Since this expression does not change by interchange of j and ℓ, we have the identity

$$\frac{\partial A^k_{ij}}{\partial x^\ell} + \sum A^m_{ij}A^k_{m\ell} + A^0_{ij}\delta^k_\ell = \frac{\partial A^k_{i\ell}}{\partial x^j} + \sum A^m_{i\ell}A^k_{mj} + A^0_{i\ell}\delta^k_j ,$$

whence follows easily the identity 2^o by use of the condition (5.3).

We now give

Proof of 3^o *and* 5^o *of Lemma 1.6.* We have seen that, for some λ, $y = \lambda(1, z^1, \ldots, z^n)$ satisfies the normalized equation (5.2). Let $A \in PGL_{n+1}$. Then $y_1 = \mu(1, (Az)^1, \ldots, (Az)^n)$ is a *linear* transformation of y for some μ. Hence y_1 also satisfies the same equations. This implies

$$S^k_{ij}(Az; x) = A^k_{ij} = S^k_{ij}(z; x); \quad \text{i.e.} \quad 3^o.$$

Next assume $S^k_{ij}(z; x) = 0$. Then $y = \lambda(1, z^1, \ldots, z^n)$ satisfies a system of equations $y_{ij} = 0$ for some λ. Then every component of y must be a linear combination of 1, x^1, ... , x^n. Hence (z^i) is a projective transformation of (x^i). This shows 5^o.

As we have defined Schwarzian derivatives, it will be better to mention projective structure and projective connection.

Definition. An n-dimensional manifold is said to admit a *projective structure* if it is covered by a coordinate system $\{U_\alpha, z_\alpha\}$ such that z_α maps U_α diffeomorphically into an open set of $\mathbf{P}^n$ and the coordinate change $z_\beta \circ z_\alpha^{-1}$ is a projective transformation so far as it is defined.

Definition. A *normal projective connection* on an n-dimensional manifold is a pair of a local coordinate system $\{U_\alpha, z_\alpha\}$ and a system of functions $\{P^k_{\alpha ij}\}$ attached to every U_α so that they satisfy

1) $P^k_{\alpha ij} = P^k_{\alpha ji}$,

2) $P^k_{\alpha ij} - P^r_{\beta pq} j^p_i(z_\beta, z_\alpha) j^q_j(z_\beta, z_\alpha) J^k_r(z_\beta, z_\alpha) = S^k_{ij}(z_\beta; z_\alpha)$.

By these definitions, a projective structure is seen to be a normal projective connection where $P^k_{ij} = 0$. We call a projective connection is *flat* if it arises from a projective structure, in other words, if there exists a coordinate system such that the projective connection is given by $P^k_{ij} = 0$ (more precisely, is compatible with the connection defined by $P^k_{ij} = 0$). We cite a following characterization of flatness. Proof can be given by use of Lemma 1.6, 4^o.

Proposition 1.9. *A projective connection* $\{z_\alpha, P^k_{\alpha ij}\}$ *is flat if and only if the system of equations*

$$z_{ij} = \sum P^k_{\alpha ij} z_k + P^0_{\alpha ij} z,$$

has $n+1$ independent solutions. Here $P^0_{\alpha ij}$ is given by

$$(n-1)P^0_{\alpha ij} = \sum\Big(P^k_{\alpha ik,j} - P^k_{\alpha ij,k} + \sum P^m_{\alpha ik}P^k_{\alpha mj} - \sum P^m_{\alpha ij}P^k_{\alpha mk}\Big).$$

Remark. As for the role of holomorphic projective structures, refer [GUN] and [K-O]. A system of differential equations called Appell-Lauricella's system is one of important examples of type (5.2). See [Y] and references therein.

2. Plane curves

A plane curve is an immersion p of a 1-dimensional manifold into $\mathbf{P}^2$. This chapter treats the equivalence problem of plane curves. The fundamental invariant of a plane curve is Laguerre-Forsyth cubic differential invariant defined in §1. If this invariant does not vanish, then we can define the projective curvature, by use of which the projective Frenet formula of a plane curve is given in §3. Since a plane curve is described by an ordinary linear homogeneous differential equation of degree 3, the theory of plane curves can be seen as a projective treatment of such equations. Notions such as symmetric product and exterior product of linear differential equations will be introduced.

§1. Plane curves

Let $p(t)$ be a plane curve, i.e. an immersion of a 1-dimensional manifold with a coordinate t into $\mathbf{P}^2$. We denote by (x_1, x_2, x_3) a system of homogeneous coordinates of $\mathbf{P}^2$ and express the immersion p as

$$p(t) = (x_1(t), x_2(t), x_3(t)).$$

Each coordinate function $x_i(t)$ satisfies a differential equation

$$\begin{vmatrix} x''' & x'' & x' & x \\ x_1''' & x_1'' & x_1' & x_1 \\ x_2''' & x_2'' & x_2' & x_2 \\ x_3''' & x_3'' & x_3' & x_3 \end{vmatrix} = 0.$$

We assume the coefficient of x''' does not vanish:

$$\begin{vmatrix} x_1'' & x_1' & x_1 \\ x_2'' & x_2' & x_2 \\ x_3'' & x_3' & x_3 \end{vmatrix} \neq 0.$$

The point where this determinant vanishes is called an *inflection* point. When this determinant vanishes everywhere, then, as is easily seen, the curve is contained in a projective line. Under the assumption that the curve has no inflection points, the equation reduces to

$$x''' + p_1 x'' + p_2 x' + p_3 x = 0. \tag{1.1}$$

Conversely, similarly as in the case of motions in $\mathbf{P}^1$, each set of independent solutions defines a curve that is equivalent to the original curve p. Hence (1.1) represents one and only one class of plane curves. The equation (1.1) is not, however, the unique equation associated to the curve p because, if we replace x with $y = \lambda^{-1}x$, we arrive at another equation also corresponding to this curve. By calculation we see that y satisfies

$$\begin{aligned}\lambda y''' + (3\lambda' + p_1\lambda)y'' + (3\lambda'' + 2p_1\lambda' + p_2\lambda)y''& \\ +(\lambda''' + p_1\lambda'' + p_2\lambda' + p_3\lambda)y &= 0.\end{aligned}$$

Now we choose λ by the condition

$$3\lambda' + p_1\lambda = 0 \tag{1.2}$$

so that the equation becomes (rewrite x for y)

$$x''' + P_2 x' + P_3 x = 0, \tag{1.3}$$

where

$$\begin{cases} P_2 = p_2 - p_1' - \dfrac{1}{3}p_1^2 \\ P_3 = p_3 - \dfrac{1}{3}p_1'' + \dfrac{2}{27}p_1^3 - \dfrac{1}{3}p_1 p_2. \end{cases} \tag{1.4}$$

We next want to see how the form (1.3) changes under a transformation of variables

$$(t, x) \mapsto (s = f(t), y = g(t)^{-1}x), \tag{1.5}$$

(see Remark below). Denoting by "$\cdot$" derivations with respect to s, we get

$$\begin{aligned} x' &= g'y + gf'\dot{y} \\ x'' &= g''y + (2g'f' + gf'')\dot{y} + g(f')^2\ddot{y} \\ x''' &= g'''y + (3g''f' + 3g'f'' + gf''')\dot{y} + 3(g'(f')^2 + gf'f'')\ddot{y} + g(f')^3\dddot{y}. \end{aligned}$$

Hence, to keep the vanishing of the coefficient of $\ddot{y}$ as (1.3), it is necessary to assume

$$g'(f')^2 + gf'f'' = 0, \quad \text{i.e.} \quad g = c/f'.$$

In this case y satisfies

$$\begin{aligned} &(f')^2\dddot{y} + (P_2 - 4\{f;t\})\dot{y} \\ &\qquad + \left[P_3/f' - f''P_2/(f')^2 - f'''/(f')^2 + 2(f''^2/(f')^3)'\right]y = 0. \end{aligned}$$

So, letting f be a solution of

$$\{f;t\} = \frac{1}{4}P_2, \tag{1.6}$$

we have, again replacing y with x and s with t,

$$x''' + Rx = 0, \tag{1.7}$$

where

$$R = \left(P_3 - \frac{1}{2}P_2'\right)\Big/(f')^3. \tag{1.8}$$

We define for later use

$$P = P_3 - \frac{1}{2}P_2', \tag{1.9}$$

which is called *Laguerre-Forsyth invariant* of the original equation (1.1). And the equation of form (1.7) is called a Laguerre-Forsyth canonical form of (1.1). Now arises a question: which transformation (1.5) keeps this form? Repeating the above process, we know such a transformation is given by

$$g = c/f' \quad \text{and} \quad \{f;t\} = 0.$$

By Lemma 1.2, f is a linear fractional transformation. So we have proved

Proposition 2.1. 1^o *The differential equation* (1.1) *is transformed by a change of variables to the equation of Laguerre-Forsyth canonical form.*
2^o *Any change of variables* $(t,x) \mapsto (s,y)$ *preserving this form is given by*

$$s = \frac{at+b}{ct+d} \quad \textit{and} \quad y = C(ct+d)^{-2}x.$$

These transformations form a group isomorphic to $\mathbf{R}^* \times SL_2$.
3^o *Under this group the differential form* $P\,dt^3$ *is invariant.*

Definition. We call $P\,dt^3$ the Laguerre-Forsyth *cubic differential invariant* and $ds = P^{1/3}\,dt$ the *projective arc length element* of the curve.

Proposition 2.2. *Assume the invariant* P *vanishes everywhere. Then the curve is a conic.*

Proof. The associated equation of such a curve is normalized as $x''' = 0$ for an appropriate choice of a coordinate t. The independent solutions are 1, t and t^2. Hence the given curve is projectively equivalent to the curve $(1,t,t^2)$ which is a conic.

Example. Let $Y = f(X)$ be a plane curve in inhomogeneous coordinate (X,Y). The associated equation, in case $f'' \neq 0$, is

$$x''' - \frac{f'''}{f''}x'' = 0.$$

Then the invariant P is computed to yield

$$P = -\frac{1}{6}\left(\frac{f'''}{f''}\right)'' - \frac{2}{27}\left(\frac{f'''}{f''}\right)^3 + \frac{1}{3}\left(\frac{f'''}{f''}\right)\left(\frac{f'''}{f''}\right)'.$$

Or, if we put $\xi = (f'')^{-2/3}$, then

$$P = \frac{1}{4}\frac{\xi'''}{\xi}.$$

Hence we have the curve $Y = f(X)$ is a conic if and only if

$$9(f'')^2 f^{(5)} - 45 f'' f''' f^{(4)} + 40(f''')^3 = 0.$$

Remark. Consider a transformation φ in (t,x)-space given by $(t,x) \mapsto (\tau,\xi) = (\tau(t,x), \xi(t,x))$ where $\partial(\tau,\xi)/\partial(t,x) \neq 0$. We can prove that the equation (1.1) is transformed into a linear homogeneous differential equation with unknown ξ and the variable τ if and only if φ has a form $\xi = g(t)x$ and $\tau = f(t)$ as (1.5).

§2. Sextactic points and normal form of curves

So far we have normalized equations to define the invariant P. We next see the role of P in a local representation of a curve.

Let $p(t)$ be a plane curve defined around $t = 0$. Under the generality assumption in §1 three vectors $p(0)$, $p'(0)$ and $p''(0)$ are linearly independent. Hence any point $p(t)$ is written as

$$p(t) = z\, p(0) + x\, p'(0) + y\, p''(0)$$

for scalar functions x, y and z. We want to express these functions in terms of the invariant P around $t = 0$. Expand $p(t)$ at $t = 0$, then

$$p(t) = p(0) + t\, p'(0) + \frac{1}{2} t^2 p''(0) + \frac{1}{6} t^3 p'''(0) + \cdots .$$

Assume the parameter t is already chosen so as

$$p''' + P\, p = 0.$$

Then, by a simple computation, we see that

$$\begin{aligned} p(t) = \Big(1 - \frac{1}{6} a\, t^3 + \cdots\Big) p(0) + \Big(t - \frac{1}{24} a\, t^4 + \cdots\Big) p'(0) \\ + \Big(\frac{1}{2} t^2 - \frac{1}{120} a\, t^5 + \cdots\Big) p''(0), \end{aligned}$$

therefore

$$(2.1)\qquad \begin{cases} z = 1 - \dfrac{1}{6}a\,t^3 + \cdots \\ x = t - \dfrac{1}{24}a\,t^4 + \cdots \\ y = \dfrac{1}{2}t^2 - \dfrac{1}{120}a\,t^5 + \cdots\,, \end{cases}$$

where $a = P(0)$. Let $X = x/z$ and $Y = y/z$ be inhomogeneous coordinates. Then

$$X = t + \frac{1}{8}a\,t^4 + \cdots$$
$$Y = \frac{1}{2}t^2 + \frac{1}{24}a\,t^5 + \cdots\,.$$

We have shown

Proposition 2.3. *Any plane curve has a local expression*

$$(2.2)\qquad Y = \frac{1}{2}X^2 - \frac{1}{12}aX^5 + \cdots$$

at a non-inflectional point for an appropriate choice of inhomogeneous coordinates (X, Y).

The value a that was the value $P(0)$ for variable t has no absolute meaning by the ambiguity explained in Proposition 2.1, 2^o. Whether a is 0 or not is however projectively invariant and has a geometrical meaning: consider the conic defined by $Y = \frac{1}{2}X^2$ (in homogeneous coordinates $p(0) + t\,p'(0) + \frac{1}{2}t^2 p''(0)$). Proposition 2.3 says that this conic tangents to the curve at $t = 0$ to the highest order of contact unless $a = 0$ and to the order higher by at least one if $a = 0$. This conic is called an *osculating conic* of the curve. Be careful that conics $Y = \frac{1}{2}bX^2$ $(b \neq 0)$ are also osculating conics. We have no invariant way about how to choose b. The quantity which measures the difference of a curve with its associated osculating conics will be given in the next section.

Definition. We call a point where P vanishes a *sextactic point*.

This notion resembles that of inflection points in the euclidean theory of plane curves. In fact we have an analogue of the four-vertex theorem.

Theorem 2.4. (G. Herglotz-J. Radon) *The number of sextactic points of a strictly convex simply closed smooth curve is at least six.*

We first prepare a lemma:

Lemma 2.5. *Let $p(t)$ be a closed smooth curve in $\mathbf{P}^2$ without inflection points. Let $(x_1(t), x_2(t), x_3(t))$ be one of closed lifts. The coordinate functions x_i are periodic with period, say, 1 and satisfy $x''' + p_1 x'' + p_2 x' + p_3 x = 0$. Let P be the Laguerre-Forsyth invariant of this equation. Then*

$$\int_0^1 P x_i x_j \exp\left(\frac{2}{3}\int^t p_1(u)\,du\right) dt = 0 \quad \textit{for} \quad 1 \le i, j \le 3. \tag{2.3}$$

Proof. Define a new parameter $s = f(t)$ by $f'(t) = \exp\bigl(-\frac{1}{3}\int^t p_1(u)\,du\bigr)$. Then the new differential equation is $\dddot{x} + q_2\dot{x} + q_3 x = 0$; $(\,\cdot\, = d/ds)$. Let Q be the invariant of this equation: $Q = q_3 - \frac{1}{2}\dot{q}_2$. We have seen the identity $Q\,ds^3 = P\,dt^3$. Hence

$$\int P x_i x_j \exp\left(\frac{2}{3}\int^t p_1\,du\right) dt = \int Q x_i x_j\,ds.$$

Now Stokes's theorem proves Lemma as follows.

$$\begin{aligned}
2\int Q x_i x_j\,ds &= \int (2q_3 - \dot{q}_2) x_i x_j\,ds \\
&= -\int \{(\dddot{x}_i + q_2\dot{x}_i)x_j + (\dddot{x}_j + q_2\dot{x}_j)x_i + \dot{q}_2 x_i x_j\}\,ds \\
&= -\int \{(\dddot{x}_i x_j + x_i\dddot{x}_j) + (q_2 x_i x_j)^{\cdot}\}\,ds \\
&= \int (\ddot{x}_i\dot{x}_j + \dot{x}_i\ddot{x}_j)\,ds \\
&= 0.
\end{aligned}$$

Remark. Any closed curve in $\mathbf{P}^2$ without inflection points is homotopically trivial ([SS]).

Proof of Theorem 2.4. Since the curve is assumed to be simply closed and convex, it is contained in one affine plane. So we can take $x_3(t) = 1$ throughout. The strong convexity implies that the curve has no inflection points. Assume first P has no zeros or has only one zero; P must be of constant sign. Then taking a line outside the curve and denoting this line by a linear equation $\ell(x_i) = 0$, we see

$$\int P\ell \exp\left(\frac{2}{3}\int p_1\,du\right)dt \neq 0.$$

This contradicts to (2.3). We next assume that P has only two or three zeros and changes sign. Then take a line through two zeros where P changes sign. This also leads a contradiction as above. Hence P has at least four zeros. But, if only four, then it is seen that the sign of P on each arc changes alternately. Then we can find two lines, given by linear functions ℓ_1 and ℓ_2, through zeros so that

$$\int P\ell_1\ell_2 \exp\left(\frac{2}{3}\int p_1\,dt\right)du \neq 0,$$

which contradicts to (2.3). The five-zero case is also cleared by this argument. Hence we have the theorem.

The first part of this proof shows also

Proposition 2.6. *The number of sextactic points of a locally strongly convex smooth closed curve that is contained in one affine plane is at least two.*

The following two examples show that the numbers six and two above are best possible.

Example. Define a curve (x, y) in $\mathbf{A}^2$ by

$$\begin{cases} x = -\alpha \sin t + \dfrac{\sin 2t}{4} + \dfrac{\sin 4t}{8} \\[2ex] y = \alpha \cos t + \dfrac{\cos 2t}{4} - \dfrac{\cos 4t}{8} \end{cases} \quad ;\ 0 \le t \le 2\pi.$$

This is simply closed and strongly convex when $\alpha > 1$. Its sextactic points are given by

$$\sin 3t(7\alpha \cos 3t + 8 - \alpha^2) = 0.$$

So, when $\alpha \geq 8$, the number of sextactic points is six (see Figure 1; sectactic points are marked by quadrangles).

Example. Consider a curve (x, y) in $\mathbf{A}^2$ given by

$$\begin{cases} x = \cos t \cdot \cos(t/3) \\ y = \sin t \cdot \cos(t/3) \end{cases} \quad ; \; 0 \leq t \leq 4\pi.$$

This is closed and locally strongly convex. Its sextactic points are $(1, 0)$ and $(0, 0)$ (see Figure 2).

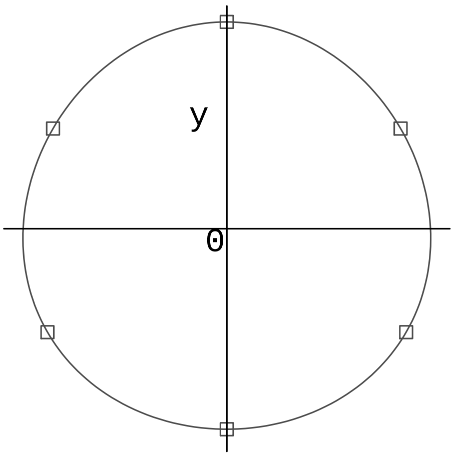

Figure 1

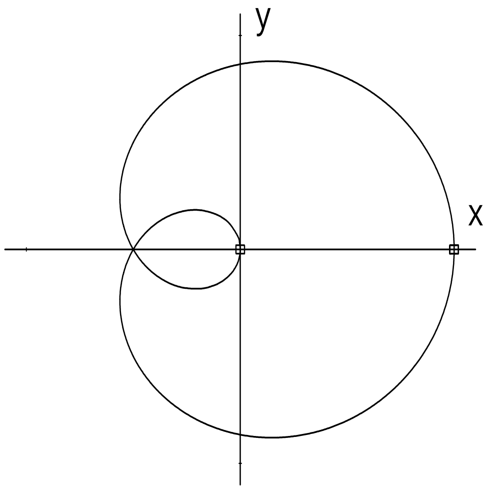

Figure 2

§3. Projective curvature

We have seen that P itself is not a scalar invariant. So, assuming $P \neq 0$, we restart the normalization process with the projective length s as a parameter. Then the equation (1.3) in §1 is written as

$$\frac{d^3}{ds^3}x + 2k\frac{d}{ds}x + hx = 0.$$

Since P is now equal to 1, $h = 1 + dk/ds$. Namely

$$x''' + 2k\,x' + (1 + k')x = 0 \tag{3.1}$$

is the equation with respect to the projective length parameter.

Definition. We call the coefficient k the *projective curvature* of a plane curve. It is defined outside the set of sextactic points.

Since k is uniquely determined, we have

Theorem 2.7. *Let p_1 and p_2 be two connected plane curves with parameters t_1 and t_2 respectively and without sextactic points. Let ds_i and k_i denote the projective length element and the projective curvature of p_i for $i = 1, 2$.*
1^o. Assume p_1 and p_2 are projectively equivalent. Then there exists a mapping φ between parameters, $t_2 = \varphi(t_1)$, such that

$$ds_1 = \varphi^* \, ds_2 \quad \textit{and} \quad k_1 = \varphi^* k_2.$$

2^o. Conversely, if there exists a mapping φ satisfying these conditions, then p_1 and p_2 are projectively equivalent.

We here remark that the equation (3.1) provides us a formula called the projective Frenet formula. Define vector valued functions $e_0(s)$, $e_1(s)$ and $e_2(s)$ by

$$\begin{aligned} e_0 &= x \\ e_1 &= x' \\ e_2 &= x'' + k\,x. \end{aligned} \tag{3.2}$$

Then (3.1) yields

$$d \begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix} = \omega \begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix}, \tag{3.3}$$

where

$$\omega = \begin{pmatrix} 0 & 1 & 0 \\ -k & 0 & 1 \\ -1 & -k & 0 \end{pmatrix} ds. \tag{3.4}$$

This set of vectors (e_0, e_1, e_2) is called a projective frame along a curve. Theorem 2.7 says that this matrix-valued 1-form ω describes a projective equivalence class of plane curves uniquely.

Proposition 2.8. *The projective curvature k of the curve* (1.1) *is given by*

$$k = P^{-2/3}\left(\frac{1}{2}P_2 - \frac{1}{3}\frac{P''}{P} + \frac{7}{18}\left(\frac{P'}{P}\right)^2\right). \tag{3.5}$$

Proof. We denote by $'$ the derivation with respect to the original parameter t. The parameter used for the normalization (1.7) is denoted by u. It is determined by $\{u;s\} = \frac{1}{2}k$. Since $\{u;s\}\,ds^2 = [\,\{u;t\} - \{s;t\}\,]\,dt^2$ by Lemma 1.2 , we have

$$k = 2\frac{\{u;t\} - \{s;t\}}{(ds/dt)^2}.$$

By the identity $ds = P^{1/3}\,dt$,

$$\{s;t\} = \frac{1}{6}\frac{P''}{P} - \frac{7}{36}\left(\frac{P'}{P}\right)^2.$$

Hence, combined with (1.6) in §1, the formula follows.

Remark. Wilczynski has used other expressions Θ_3 and Θ_8 to denote the invariants. They are defined by

$$\begin{aligned}\Theta_3 &= P\\ \Theta_8 &= 6PP'' - 7(P')^2 - 9P_2(P)^2.\end{aligned}$$

Subindices means the weight showing the order of relative invariance: by a transformation $(t,x) \mapsto (a\,t, x)$, Θ_i changes into $a^{-i}\Theta_i$. And it is shown that, in case $P = \Theta_3 \neq 0$, Θ_8^3/Θ_3^8 is an absolute invariant. The formula (3.5) says that this ratio is equal to k^3 up to a constant multiple.

Example. Let us find curves whose projective curvatures are constant. Let λ be one of roots of the characteristic equation $\lambda^3 + 2k\lambda + 1 = 0$. Then $e^{\lambda s}$ is a solution of (3.1). Following three cases occur.

1. Different three real roots: the curve is $(e^{\lambda_1 s}, e^{\lambda_2 s}, e^{\lambda_3 s})$ equivalent to $Y = X^m$. Here m takes values other than ± 1, ± 2 and $\pm 1/2$.

2. Different roots but two complex conjugates: the curve is $(e^{\lambda s}, e^{\mu s} + e^{\overline{\mu} s}, i(e^{\mu s} - e^{\overline{\mu} s}))$. In inhomogeneous coordinates, $X = e^{\nu s}\cos\nu' s$ and $Y = e^{\nu s}\sin\nu' s$; ν, ν' are constants. This curve is called a logarithmic spiral.

3. Double roots: the curve is $Y = Xe^X$.

These are all projectively homogeneous: the action of $\mathbf{R} \ni t$ is given by a projective transformation

$$1. \begin{pmatrix} t & 0 & 0 \\ 0 & t^m & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad 2. \begin{pmatrix} e^{-\nu t}\cos\nu' t & e^{-\nu t}\sin\nu' t & 0 \\ -e^{-\nu t}\sin\nu' t & e^{-\nu t}\cos\nu' t & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad 3. \begin{pmatrix} 1 & 0 & t \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If a curve is projectively homogeneous and if it is not a conic nor a line, then it is equivalent to one of above curves; because the projective curvature of such a curve is constant.

Example. Consider an equation with a regular singularity at the origin:

$$y''' + \frac{a}{x^2}y' + \frac{b}{x^3}y = 0.$$

By a computation, $P = (a+b)/x^3$. Hence, if $a + b = 0$, this generates a conic. Assume $a + b \neq 0$. Then the projective curvature of the corresponding curve is seen to be $k = \frac{1}{2}(a-1)(a+b)^{-2/3}$. So, if $a = 1$, this belongs to the case 2 of the above Example.

§4. Symmetric product of differential equations

We will consider the condition $P = 0$ in the different view point introducing a new notion.

Let us treat first a second-order equation.

$$x'' + q\,x = 0.$$

We choose two independent solutions x_1 and x_2. Their symmetric products $y_1 = (x_1)^2$, $y_2 = (x_1 x_2)$ and $y_3 = (x_2)^2$ satisfy

$$y''' + 4q\,y' + 2q'y = 0.$$

Then, by the definition (1.9), P vanishes. Since y_i satisfies a quadratic relation $y_1 y_3 = (y_2)^2$, this is obvious by the meaning of P. But in some cases, this process gives non-trivial examples. See an example in the end of this section.

We next treat a third-order equation

$$x''' + p_2 x' + p_3 x = 0. \tag{4.1}$$

We try to find the equation satisfied by $y = \frac{1}{2}x^2$. Successive derivations yield

$$\begin{aligned} y' &= xx' \\ y'' &= (x')^2 + xx'' \\ y''' &= 3xx'' + xx''' = 3x'x'' - p_2 y' - 2p_3 y. \end{aligned}$$

Define

$$Y = y''' + p_2 y' + 2p_3 y = 3x'x''.$$

Taking derivations further we get

$$Y' = 3x'x''' + 3(x'')^2 = -3p_3 y' - 3P_2(x')^2 + 3(x'')^2$$

$$\begin{aligned} (Y' + 3p_3 y') &= 6x''x''' - 6p_2 x'x'' - 3p_2'(x')^2 \\ &= -4p_2 Y - 6p_3(y'' - (x')^2) - 3p_2'(x')^2. \end{aligned}$$

Hence

$$(Y' + 3p_3 y')' + 4p_2 Y + 6p_3 y'' = 6\Big(p_3 - \frac{1}{2}p_2'\Big)(x')^2. \tag{4.2}$$

The left side contains only derivatives of y. Taking one more derivation and cancelling derivatives of x, we will obtain a six-order equation with respect to y. This is the equation satisfied by any product of solutions of (4.1). We call this equation the *symmetric product*. Note that this definition can be generalized for higher order case or for several variables case (see [HSY] and cf. Chapter 8).

Our concern here is, however, the fifth-order equation (4.2). If the coefficient of the right side, that is the invariant P of the equation (4.1), vanishes, then y satisfies a homogeneous fifth-order equation. This phenomenon is observed also in view of the meaning of P. If P vanishes, then independent solutions satisfy a quadratic relation as we have seen in §1. This, in turn, imples that there holds a *linear* relation between products. So the number of independent products reduces at least by one and, hence, they satisfy a fifth-order equation given by the left hand side of (4.2). Conversely, if there is a linear relation between solutions of the symmetric product of (4.1), that is, if solutions satisfy a fifth-order equation, then there holds a relation such as $\sum a_{ij}(x_i x_j) = 0$ for a non-trivial matrix a_{ij}, x_i being solutions. This means that the mapping (x_i) has image in a conic, and $P = 0$. So we have proved

Proposition 2.9. *The symmetric product of* $x''' + p_2 x' + p_3 x = 0$ *reduces to a fifth-order equation if and only if* $2p_3 - p_2' = 0$.

This seems very simple but has important applications. One application is an explanation of the following fact due to L. Fuchs. He has considered the problem how to integrate the equation (4.1). In other words, he treated the structure of the automorphism group of the equation now called Picard-Vessiot group. Fuchs reduced this problem to the problem to consider algebraic relations between solutions. Let $f \in \mathbf{C}[X_1, X_2, X_3]$ be a polynomial satisfying $f(x_1, x_2, x_3) = 0$ for a set of independent solutions x_i. Define $d =$ the minimum of degree (f) for such polynomials. Then

(L. Fuchs, 1882) If $d \geq 3$, then the automorphism group is finite and the equation is integrated by algebraic operations and by quadrature. If $d = 2$, then the integration reduces to solving an equation of second order.

The latter half follows from Proposition 2.9. G. Fano has solved this problem for fourth- and fifth-order equations (cf. [Fan]).

Example. Let us define a function ${}_qF_p$ by a series

$$ {}_qF_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_p; x) = \sum_{n=0}^{\infty} \frac{(\alpha_1, n) \cdots (\alpha_q, n)}{(\beta_1, n) \cdots (\beta_p, n)} \frac{x^n}{n!}. $$

Here $(a, n) = a(a+1) \cdots (a+n-1)$. Parameters α_i and β_j are complex numbers. Assume β_j are not negative integers. We define a differential operator $\theta = x\, d/dx$. Then $z = {}_qF_p$ satisfies

$$ ({}_qE_p) : \theta(\theta + \beta_1 - 1) \cdots (\theta + \beta_p - 1)z - x(\theta + \alpha_1) \cdots (\theta + \alpha_q)z = 0. $$

This equation is called the generalized hypergeometric equation and the function ${}_qF_p$ is called the generalized hypergeometric functions. When $(p, q) = (1, 2)$, they are Gauss hypergeometric equation and Gauss hypergeometric functions. (see [E]). We here quote an identity found by Clausen, 1828:

$$ {}_2F_1\Big(\alpha, \beta; \alpha + \beta + \frac{1}{2}; x\Big)^2 = {}_3F_2\Big(2\alpha, 2\beta, \alpha + \beta; 2(\alpha + \beta), \alpha + \beta + \frac{1}{2}; x\Big). $$

This can be proved by making the symmetric product of Gauss equation and by connecting it with $({}_3E_2)$.

§5. Dual curve and exterior product of differential equations

The dual vector space of $\mathbf{P}^2$, denoted by $\mathbf{P}^{2*}$, is defined as the set of all lines in $\mathbf{P}^2$. Since each line is determined by two points (x_i) and (y_i) on it, we associate a vector (ξ_i) by $\xi_1 = x_2y_3 - x_3y_2$, $\xi_2 = x_3y_1 - x_1y_3$ and $\xi_3 = x_1y_2 - x_2y_1$ called the Pluücker coordinates of a line. We express these coordinates simply by

$$ \xi = x \wedge y. $$

Let now $p(t)$ be a plane curve. Then two vectors $p(t) = (x_i(t))$ and $p'(t) = (x_i(t))$ span a line $\pi(t) = (\xi_i)$ (recall the generality assumption in §1). π is a curve in $\mathbf{P}^{2*}$ and is called the *dual curve* of P. We are interested in relations between p and π. By the above convention

$$\xi = x \wedge x'. \tag{5.1}$$

Assume p is defined by the equation (1.1):

$$x''' + p_1 x'' + p_2 x' + p_3 x = 0.$$

Then taking derivations of (5.1) and making use of this equation, we get

$$\begin{aligned} \xi' &= x \wedge x'' \\ \xi'' &= x' \wedge x'' + x \wedge x'' = x' \wedge x'' - p_1 \xi' - p_2 \xi \\ \xi''' &= x' \wedge x''' - (p_1 \xi' + p_2 \xi)'. \end{aligned}$$

Hence ξ satisfies

$$\xi''' + 2p_1 \xi'' + (p_1' + p_1^2 + p_2)\xi' + (p_2' + p_1 p_2 - p_3)\xi = 0. \tag{5.2}$$

Recall here the definition of the adjoint equation of (1.1). It is by definition

$$\xi''' - (p_1 \xi)'' + (p_2 \xi)' - p_3 \xi = 0. \tag{5.3}$$

Although these two equations seem different, it is easy to see that both are projectively equivalent to

(5.4)

$$\xi''' + \Big(p_2 - p_1' - \frac{1}{3} p_1^2\Big)\xi' + \Big(p_2' - \frac{2}{3} p_1'' - p_3 - \frac{2}{27} p_1^3 + \frac{1}{3} p_1 p_2 - \frac{2}{3} p_1 p_1'\Big)\xi = 0.$$

This fact clarifies the meaning of the adjointness in our viewpoint. We call the (projective equivalent class of) equation (5.2) the *exterior product* of the given equation (1.1).

We have defined P_2, P_3 and P in §1 to denote coefficients of normalized equations. Let P_2^*, P_3^* and P^* denote those for π. Then (5.4) shows

$$P_2^* = P_2, \qquad P_3^* = -P_3 + P_2'$$

and hence

$$P^* = -P. \tag{5.5}$$

We say a curve p is *self-dual* if the immersion p and its dual immersion π is projectively equivalent. Here we identify $\mathbf{P}^{2*}$ and $\mathbf{P}^2$ by identifying respective homogeneous coordinates. Though this identification has ambiguity up to a projective automorphism, this does not affect the definition of self-duality. Then from (5.5) and Proposition 2.2, we have

Proposition 2.10. *A curve is self-dual if and only if it is a conic curve.*

Assume next $P \neq 0$ and that the equation is given by

$$x''' + 2kx' + (1 + k')x = 0 \tag{5.6}$$

Then the equation of the dual curve is

$$\xi''' + 2k\xi' - (1 - k')\xi = 0 \tag{5.7}$$

with respect to the same parameter. Hence

Proposition 2.11. *The projective length element of the dual curve is the minus of that of the original curve. The projective curvature is the same at the corresponding points.*

Example. The exterior product of the generalized hypergeometric equation ${}_3E_2(\alpha, \beta, \gamma; \delta, \varepsilon; z)$ (see Example in §4) is equivalent to the equation ${}_3E_2(1-\alpha, 1-\beta, 1-\gamma; 2-\delta, 2-\varepsilon; z)$. This was first shown by Darling (1932). From this identity follows a remarkable formula:

$$\begin{aligned}
&{}_3F_2(\alpha,\beta,\gamma;\delta,\varepsilon;z)\,{}_3F_2(1-\alpha,1-\beta,1-\gamma;2-\delta,2-\varepsilon;z)\\
&\quad= \frac{\varepsilon-1}{\varepsilon-\gamma}\,{}_3F_2(1+\alpha-\delta,1+\beta-\delta,1+\gamma-\delta;2-\delta,1+\varepsilon-\delta;z)\\
&\qquad\qquad\cdot\,{}_3F_2(\delta-\alpha,\delta-\beta,\delta-\gamma;\delta,1+\delta-\varepsilon;z)\\
&\qquad+\frac{\delta-1}{\delta-\varepsilon}\,{}_3F_2(1+\alpha-\varepsilon,1+\beta-\varepsilon,1+\gamma-\varepsilon;1+\delta-\varepsilon,2-\varepsilon;z)\\
&\qquad\qquad\cdot\,{}_3F_2(\varepsilon-\alpha,\varepsilon-\beta,\varepsilon-\gamma;1+\varepsilon-\delta,\varepsilon;z).
\end{aligned}$$

3. Ruled surfaces

We say a surface in $\mathbf{P}^3$ is *ruled* if through each point of the surface passes one straight line lying entirely on the surface or, equivalently, if the surface is paved with one parameter family of lines. In other words, a ruled surface is a curve in a Grassman manifold $G_{2,4}$ of lines in $\mathbf{P}^3$. In this chapter we present some of fundamentals on ruled surfaces.

§1. System of differential equations associated with a ruled surface

To write differential equations for ruled surfaces it will be convenient to understand that a ruled surface is given by a pair of curves in $\mathbf{P}^3$ with a common parameter u: Let $x_1(u)$ and $x_2(u)$ be such curves. The ruling is given by lines connecting two points $x_1(u)$ and $x_2(u)$ and the surface is a mapping $(u, v) \mapsto x(u, v) = x_1(u) + v\, x_2(u)$ or $x(u, v) = x_2(u) + v\, x_1(u)$. By abuse of notation we denote this surface sometimes by (x_1, x_2).

A typical example of ruled surfaces is a quadratic surface. It is the surface where $x_1 = (1, u, 0, 0)$ and $x_2 = (0, 0, 1, u)$. Then $x = (1, u, v, u\,v)$. This is doubly ruled. As an another example, $x(u, v) = (1 + a\,v)x_1(u)$ for $x_2 = a\,x_1$. This looks like a cone and is a developable surface. In the following consideration we avoid the latter case: Assume

$$\det |x_1, x_2, x_1', x_2'| \neq 0. \tag{1.1}$$

namely assume that four vectors x_1, x_2, x_1' and x_2' are linearly independent. Under this assumption, the second derivatives of x_i are linearly dependent on x_i and x_i'. Hence we have

$$x_i''(u) = \sum p_i^j x_j' + \sum q_i^j x_j. \tag{1.2}$$

for some functions p_i^j and q_i^j. Conversely, if we are given a system of differential equations of type (1.2), then the number of independent solutions are four and they define a ruled surface. Hence we can regard

(1.2) as a system of differential equations for a ruled surface (x_1, x_2) with condition (1.1).

§2. Normalization of a system

As we have seen for curves, the system (1.2) is not unique for a given ruled surface. Even if we change variables $(u, x) \mapsto (w, y)$ by

$$
\begin{aligned}
& w = f(u) \\
(2.1) \qquad & y_i = \sum a_i^j(u) x_j\,, \qquad \det(a_i^j) \neq 0,
\end{aligned}
$$

the surface is unchanged. Let us first see that p_i^j may be assumed to vanish. For this purpose we here employ a geometrical reasoning.

Definition. Let $x(u, v)$ be a surface. A curve on this surface defined by $u = u(t)$ and $v = v(t)$ is said an *asymptotic* curve if four vectors x, x_u, x_v, and x_{tt} are linearly dependent.

Since first three vectors generate the tangent plane of the surface, this definition is equivalent to say the second osculating vector x_{tt} of the curve is included in this tangent plane along the curve. For the sake of simplicity we write this condition as

$$
(2.2) \qquad x \wedge x_u \wedge x_v \wedge x_{tt} = 0.
$$

($\wedge$ means the wedge product in $\mathbf{R}^4$.) Apply this definition to a ruled surface (x_1, x_2). By differentiation, we have

$$
\begin{aligned}
x \wedge x_u \wedge x_v &= (x_1 + v\,x_2) \wedge (x_1' + v\,x_2') \wedge x_2 \\
&= x_1 \wedge x_1' \wedge x_2 + v\,x_1 \wedge x_2' \wedge x_2.
\end{aligned}
$$

Denote by " · " the derivation with respect to t. Then

$$
\begin{aligned}
x_t &= (x_1' + v\,x_2')\dot{u} + x_2 \dot{v} \\
x_{tt} &= (x_1' + v\,x_2')\ddot{u} + (x_1'' + v\,x_1'')(\dot{u})^2 + 2x_2' \dot{u}\dot{v} + x_2 \ddot{v},
\end{aligned}
$$

hence

$$
x \wedge x_u \wedge x_v \wedge x_{tt} = 2\dot{u}\,\dot{v}\,x_1 \wedge x_1' \wedge x_2 \wedge x_2' - \dot{u}^2 A,
$$

where

$$A = x_1 \wedge x_2 \wedge x_1' \wedge x_1'' + v(x_1 \wedge x_2 \wedge x_2' \wedge x_1'' + x_1 \wedge x_2 \wedge x_1' \wedge x_2'') \\ + v^2 x_1 \wedge x_2 \wedge x_2' \wedge x_2''.$$

Therefore the condition (2.2) is

$$\dot{u}\{2\dot{v}\, x_1 \wedge x_1' \wedge x_2 \wedge x_2' - \dot{u}\, A\} = 0. \tag{2.3}$$

Since we have assumed $x_1 \wedge x_1' \wedge x_2 \wedge x_2' \neq 0$, the equation (2.3) has always two different solutions; through each point pass two asymptotic curves. One is a ruling line and the other is given by a differential equation of Riccati type

$$2x_1 \wedge x_1' \wedge x_2 \wedge x_2' dv - A\, du = 0. \tag{2.4}$$

Now we reparametrize the surface assuming both x_1 and x_2 are asymptotic curves. Then from (2.4) we see

$$x_1 \wedge x_2 \wedge x_1' \wedge x_1'' = x_1 \wedge x_2 \wedge x_2' \wedge x_2'' = 0. \tag{2.5}$$

This says $p_1^2 = p_2^1 = 0$ as asserted. We next replace x_1 and x_2 by λx_1 and μx_2 for scalars λ, μ. Then the coefficients p_1^1 and p_2^2 vary by λ'/λ and μ'/μ respectively. So we can always find λ and μ so that $p_1^1 = p_2^2 = 0$. Hence we have proved

Proposition 3.1. *A ruled surface with condition (1.1) is given by a system of differential equations*

$$\begin{cases} x_1'' = p\, x_1 + q\, x_2 \\ x_2'' = r\, x_1 + s\, x_2. \end{cases} \tag{2.6}$$

This has the following

Corollary 3.2. *If two curves defining a ruled surface satisfy* (2.6)*, then asymptotic curves through each point are a ruling line (v-curve) and a u-curve through this point.*

Proof. Because the equation (2.3) becomes $du\,dv = 0$.

Consider next a surface x ruled by two ways. Let (x_1, x_2) give one ruling such that curves x_i are asymptotic. Since only v-curves and u-curves are asymptotic, x_i must be lines of another ruling. Hence x_i are assumed to be linear in u. Then we see $x = a + b\,u + c\,v + d\,u\,v$ for some constant vectors a, b, c and d; hence, x is a quadratic surface:

Proposition 3.3. *If a surface is ruled in two ways both with condition* (1.1)*, then the surface is a quadratic surface.*

Examples. (1) The system for a quadratic surface is $x_1'' = x_2'' = 0$.
(2) The surface defined by $(x^2)^3 + x^1(x^1x^4 + x^2x^3) = 0$ in $\mathbf{P}^3$ in homogeneous coordinates (x^i) is called a *Cayley's cubic scroll.* This is a ruled surface with generating curves $x_1 = (1, -u, -u^2, 0)$ and $x_2 = (0, 0, 1, u)$. Hence the equations are

$$\begin{cases} x_1'' = -2x_2 + 2u\,x_2' \\ x_2'' = 0. \end{cases}$$

The asymptotic curves other than ruling lines are given by $2x_1 \wedge x_1' \wedge x_2 \wedge x_2'\,dv = x_1 \wedge x_2 \wedge x_1' \wedge x_1''\,du$, i.e. $dv = -u\,du$. Hence they are twisted cubics defined by $x_3 = x_1 - \frac{1}{2}u^2x_2 + a\,x_2 = \left(1, -u, -\frac{3}{2}u^2 + a, -\frac{1}{2}u^3 + a\,u\right)$ for a constant a. Then, the system of equations with respect to (x_2, x_3) is

$$\begin{cases} x_3'' = -3x_2 \\ x_2'' = 0. \end{cases} \tag{2.7}$$

§3. Fundamental invariant

We will continue to normalize systems further.

Lemma 3.4. *Every change of variables* (2.1) *that preserves the form* (2.6) *is given by*

$$w = f(u)\,, \quad y_1 = (f')^{1/2}(a\,x_1 + b\,x_2)\,, \quad \textit{and} \quad y_2 = (f')^{1/2}(c\,x_1 + d\,x_2)$$

where a, b, c, d are constants.

Proof. Put $X = {}^t(x_1, x_2)$. The transformation (2.1) is written as

$$w = f(u); \quad Y = AX, \quad A = (a_i^j)$$

in matrix notation, A being non-singular. Similarly (2.5) is written as

$$X'' = QX, \quad Q = \begin{pmatrix} p & q \\ r & s \end{pmatrix}. \tag{3.1}$$

Then, denoting by "$\cdot$" the w-derivation, we have

$$f'\dot{Y} = A'X + AX'$$

$$(f')^2\ddot{Y} + f''Y = A''X + 2A'X' + AX''.$$

These yield

$$\begin{aligned}(f')^2\ddot{Y} &= (2f'A'A^{-1} - f'')\dot{Y} \\ &\quad + (A''A^{-1} + AQA^{-1} - 2A'A^{-1}A'A^{-1})Y.\end{aligned} \tag{3.2}$$

To preserve the form (3.1), $2f'A'A^{-1} - f'' = 0$ is necessary. Putting $B = (f')^{-1/2}A$, we see $B' = 0$. This proves the lemma.

From (3.2), more can be said. If we put $A = (f')^{1/2}B$, where B is constant, then the coefficient of Y is $h''/h - 2(h'/h)^2 + BQB^{-1}$ where $h = (f')^{1/2}$. Now note that $h''/h - 2(h'/h)^2 = \{f; u\}$. Then the trace of this coefficient is equal to $\operatorname{tr} Q + 2\{f; u\}$. So, we can assume $\operatorname{tr} Q = 0$ by an appropriate choice of f. We have

Proposition 3.5. 1^o. *Each system* (1.2) *representing a ruled surface can be normalized to have a form*

$$X'' = Q\,X; \quad X = {}^t(x_1, x_2)\,, \quad Q = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

with the condition

$$\operatorname{tr} Q = 0. \tag{3.3}$$

2°. *The transformation* (2.1) *preserving this normalization is given by*

$$w = \frac{\alpha u + \beta}{\gamma u + \delta}, \quad Y = (\gamma u + \delta)^{-1} B X, \tag{3.4}$$

where α, β, γ, δ and B are constant.
3^o. *The matrix-valued quadratic form $Q\,du^2$ changes, under a transformation* (3.4), *into $B(Q\,du^2)B^{-1}$. In particular the quartic form* $(\det Q)\,du^4$ *is a differential invariant of the surface.*

Proof. 1^o is shown already. 2^o follows from the condition $\{f; u\} = 0$. Then 3^o is seen by (3.2).

Definition. We call $Q\,du^2$ the *fundamental invariant* of a ruled surface.

Theorem 3.6. *Let x and y be two ruled surfaces with condition* (1.1) *given respectively by curves $(x_1(u), x_2(u))$ and by curves $(y_1(w), y_2(w))$. Let $Q\,du^2$ and $R\,dw^2$ denote fundamental invariants of x and y, respectively.*
1^o. *Assume x and y are projectively equivalent. Then there exists a diffeomorphism between parameters, $w = f(u)$, and a non-singular constant matrix B such that*

$$f^*(R\,dw^2) = B(Q\,du^2)B^{-1}. \tag{3.5}$$

2^o. *Conversely, if there exists a mapping f and a matrix B satisfying this identity, then ruled surfaces x and y are projectively equivalent.*

Proof. If one of surfaces is a quadratic surface, then the other is also a quadratic surface and (3.5) holds trivially. So we assume both x and

y are not quadratic surfces. Then, letting φ be a projective transformation mapping x to y, we have a mapping f such that the line with parameter u is mapped by φ to the line with parameter $w = f(u)$ (see Proposition 3.3). Then Proposition 3.5 implies 1^o. Conversely, if (3.5) holds, then we can find new curves defining the same surfaces so that these curves satisfy the same system of differential equations under the identification of parameters by f. This proves the theorem.

We put, after Wilczynski,

$$\theta_4 = -4 \det Q. \tag{3.6}$$

This relative invariant for the system (3.1), is given by

$$\theta_4 = (p - s)^2 + 4rq. \tag{3.7}$$

This invariant has the following geometrical meaning. Let $x(u, v) = x_1(u) + v\, x_2(u)$ be a ruled surface. Fixing v to a certain value, consider the curve $c(u) = x(u, v)$ of u. Then the point $x(u, v)$ is called a *flecnodal* point of the surface if it is a flecnodal point of the space curve c, i.e.

$$c \wedge c_u \wedge c_{uu} = 0$$

Since $c_{uu} = (p + v\, r)x_1 + (q + v\, s)x_2$, this condition is equivalent to

$$r\, v^2 + (p - s)v - q = 0.$$

Hence, on each line, there are generally two flecnodal points. The invariant θ_4 is the discriminant of this quadratic equation of v. Therefore $\theta_4 = 0$ means that two flecnodal points coincide. For later use we introduce one more terminology. The flecnodal points on each line draw curves on the surface. We call these curves *flecnodal curves.*

§4. Scalar differential invariants

Wilczynski has shown that, other than θ_4, there exist three basic scalar invariants denoted by $\theta_{4\cdot 1}$, θ_9 and θ_{10}. We will reproduce them in our notation.

We have seen that the fundamental invariant changes under the rule

$$(f')^2 Q_w = B Q_u B^{-1} \tag{4.1}$$

by the transformation (3.4), the suffices being used to denote the parameter of curve. Let I_u be a differential polynomial of components of Q_u and I_w the corresponding value for Q_w. We say I (or I_u by abuse of language) is a (relative) scalar invariant of weight a if

$$I_w = I_u (f')^{-a}. \tag{4.2}$$

The invariant θ_4 is of weight 4.

Lemma 3.7. *If I is an invariant of weight a, then its derivatives satisfy*

$$\dot{I}_w = I'_u (f')^{-a-1} - a\, I_u (f')^{-a-2} f'' \tag{4.3}$$

and

(4.4)

$$\ddot{I}_w = I''_u (f')^{-a-2} - (2a+1) I'_u (f')^{-a-3} f'' + a(a+\frac{1}{2}) I_u (f')^{-a-4} (f'')^2.$$

($\prime$ *and* $\cdot$ *denote derivations with respect to* u *and* w *respectively.*)

Proof. Differentiate (4.2) and use the identity $f' f'' = \frac{3}{2}(f'')^2$.

From this lemma follows

Lemma 3.8. 1^o. *If I is an invariant of weight a, then $2aI''I - (2a+1)(I')^2$ is an invariant of weight $2a+2$.*
2^o. *If I and J are invariants of weight a and b respectively, then $aIJ' - bI'J$ is an invariant of weight $a+b+1$.*

By this lemma

$$\theta_{4\cdot 1} = 32(\det\, Q)'' \det\, Q - 36((\det\, Q)')^2 \tag{4.5}$$

is an invariant of weight 10. Lemma 3.7 applied to det Q yields

$$\begin{aligned}(\det Q_w)^{\cdot} &= (\det Q_u)'(f')^{-5} - 4(\det Q_u)(f')^{-6}f''\\ (\det Q_w)^{\cdot\cdot} &= (\det Q_u)''(f')^{-6} - 9(\det Q_u)'(f')^{-7}f''\\ &\quad + 18(\det Q_u)(f')^{-8}(f'')^2.\end{aligned}$$

On the other hand

$$B^{-1}Q^{\cdot}_w B = Q'_u(f')^{-3} - 2Q_u(f')^{-4}f'',$$

whence we have

$$\begin{aligned}\det(Q^{\cdot}_w) &= (f')^{-6}\det\left(Q' - 2\frac{f''}{f'}Q\right)\\ &= (f')^{-6}\left(\det(Q') - 2\frac{f''}{f'}(\det Q)' + 4\left(\frac{f''}{f'}\right)^2 \det Q\right).\end{aligned}$$

These identities show easily

$$9\det(Q^{\cdot}_w) - 2(\det Q_w)^{\cdot\cdot} = (f')^{-6}\left\{9\det(Q'_u) - 2(\det Q_u)''\right\}. \tag{4.6}$$

Hence,

$$\theta_6 := 9\det(Q') - 2(\det Q)'' \tag{4.7}$$

is an invariant of weight six. He has also introduced

$$\theta_{10} = -4\det Q\det(Q') + ((\det Q)')^2. \tag{4.8}$$

These are not independent as he claimed. They satisfy

$$\theta_{4\cdot 1} + 36\theta_{10} - 4\theta_4\theta_6 = 0.$$

Let $\begin{pmatrix} p & q \\ r & -p\end{pmatrix}$ be components of Q as before and define

$$\theta_9 = \det\begin{pmatrix} p & q & r\\ p' & q' & r'\\ p'' & q'' & r''\end{pmatrix}. \tag{4.9}$$

This is an invariant of weight 9; due to three identities (4.1), (4.6) and

$$B^{-1}Q^{\cdot\cdot}_w B = Q''(f')^{-4} - 5Q'(f')^{-5}f'' + 5Q(f')^{-6}(f'')^2. \tag{4.10}$$

The geometrical meaning of θ_9 is given in

Proposition 3.9. ([W1, Chapter VIII]). $\theta_9 = 0$ *if and only if the associated curve in* $G_{2,4}$*, which is considered as a curve in* $\mathbf{P}^5$ *by the Plücker embedding of* $G_{2,4}$ *into* $\mathbf{P}^5$*, is included in a hyperplane of* $\mathbf{P}^5$*.*

Proof. Let us denote by y the associated curve in $\mathbf{P}^5$. It is given by $y = x_1 \wedge x_2$. Differentiation gives

$$\begin{aligned} y' &= x_1' \wedge x_2 + x_1 \wedge x_2' \\ y'' &= x_1'' \wedge x_2 + 2x_1' \wedge x_2' + x_1 \wedge x_2'' \\ &= 2x_1' \wedge x_2'. \quad \text{(by (2.6) and } p+s=0) \end{aligned}$$

Then successively we have

$$\frac{1}{2}y'' + py' = 2p\,x_1 \wedge x_2' + q\,x_2 \wedge x_2' - r\,x_1 \wedge x_1'. \tag{4.11}$$

(4.12)
$$\left(\frac{1}{2}y''' + py'\right)' = 2p' x_1 \wedge x_2' + q' x_2 \wedge x_2' - r' x_1 \wedge x_1' + py'' - 2(p^2 + q\,r)y.$$

$$\begin{aligned} &\left(\left(\frac{1}{2}y''' + py'\right)' - py'' + 2(p^2 + q\,r)y\right)' \\ &= 2p'' x_1 \wedge x_2' + q'' x_2 \wedge x_2' - r'' x_1 \wedge x_1' + p'' y'' - 2((p')^2 + q' r')y. \end{aligned} \tag{4.13}$$

These equalities (4.11)–(4.l3) show that if $\theta_9 = 0$, then y satisfies an equation of order 5, which means that there exists a linear relation among coordinates of y and that the curve y is included in a hyperplane. Conversely, if y is included in a hyperplane, then vectors y, $y', \ldots, y^{(5)}$ are linearly dependent and we obtain a linear relation among six vectors given by the right hand sides of above equalities. But, if $\theta_9 \neq 0$, they span a six-dimensional vector space by the generality assumption (1.1). This proves Lemma.

Remark. If $\theta_4 \neq 0$, then we have absolute invariants, say $(\theta_{4\cdot 1})^4/(\theta_4)^{10}$, $(\theta_9)^4/(\theta_4)^9$, ... Moreover, if $\theta_4 \neq 0$, we can normalize the system further so that $|\theta_4| = 4$, i.e. $\det Q = \pm 1$, and reduce the transformation

(3.4) to that composed of a translation of u and a multiplication by B. Hence the procedure to get invariants is completely reduced to the PGL_2-invariant theory. Then, generally speaking, two invariants suffice to determine Q because $\operatorname{tr} Q = 0$ and $\det Q = \pm 1$. Wilcyznski showed that we can take θ_9 and θ_{10} for these invariants assuming $\theta_{10} \neq 0$. (see [W1, p. 120-121]).

§5. A generalization of plane curves and ruled surfaces

Recall that a ruled surface is defined as a pair of two curves in $\mathbf{P}^3$. This situation may be generalized to that for a certain number of curves in $\mathbf{P}^N$. We here treat the case of r curves in $\mathbf{P}^{nr-1}$, because similar arguments are possible. Namely, we consider a one-parameter family of $(r-1)$-plane in $\mathbf{P}^{nr-1}$. When $r = 1$, this is the case of a single curve; Chapter 1 for $n = 2$ and Chapter 2 for $n = 3$. When $r = n = 2$, the case of ruled surfaces.

Let $x_1(t), \dots, x_r(t)$ be curves in $\mathbf{P}^{nr-1}$. They define an $(r-1)$-plane $x(t, u_2, \dots, u_r) = x_1(t) + \sum u_j x_j(t)$. We assume that the vectors $x_1, \dots, x_r, x_1', \dots, x_r', \dots, x_1^{(n-1)}, \dots, x_r^{(n-1)}$ are linearly independent. Then we have a system of differential equations

$$x_i^{(n)} = \sum_{k=0}^{n-1} P_{ik}^i x_j^{(k)} \tag{5.1}$$

We define matrices P_k by $P_k = (p_{ik}^j)$. By putting $X = {}^t(x_1, \dots, x_r)$ this system is written

$$X^{(n)} = \sum P_k X^{(k)}. \tag{5.2}$$

The ambiguity to determine the planes $x(t, u_2, \dots, u_r)$ lies in the choice of parameter t and the choice of generating curves $x_1, \dots, x_r$. So transformations that we should consider have the form

$$\begin{cases} s = f(t) \\ y_i = \sum a_i^j(t) x_j(t) \quad ; \quad Y = AX. \end{cases} \tag{5.3}$$

By a transformation $Y = AX$ and $f = t$, the equation (5.2) is transformed into

$$\begin{aligned} Y^{(n)} &= AX^{(n)} + nA'X^{(n-1)} + \cdots \\ &= (AP_1 + nA')X^{(n-1)} + \cdots \\ &= (AP_1 + nA')A^{-1}Y^{(n-1)} + \cdots \end{aligned}$$

Then a choice of A by $AP_1 + nA' = 0$ leads us to the situation

$$P_1 = 0. \tag{5.4}$$

Under a general transformation (5.3), the equation changes into

$$\begin{aligned} &(f')^n Y^{(n)} + a_n (f')^{n-2} f'' Y^{(n-1)} \\ &\qquad + \big(b_n (f')^{n-3} f''' + c_n (f')^{n-4} (f'')^2\big) Y^{(n-2)} + \cdots \\ &\quad = AX^{(n)} + nA'X^{(n-1)} + \frac{n(n-1)}{2} A'' X^{(n-2)} + \cdots, \end{aligned}$$

where

$$a_n = \frac{n(n-1)}{2},\ b_n = \frac{n(n-1)(n-2)}{6},\ c_n = \frac{n(n-1)(n-2)(n-3)}{8}.$$

From this we get

$$\begin{aligned} (f')^n Y^{(n)} &= (n(f')^{n-1} A' A^{-1} \\ &\qquad - a_n (f')^{n-2} f'') Y^{(n-1)} + (f')^{n-2} \widetilde{P}_2 Y^{(n-2)} + \cdots, \end{aligned}$$

where

$$\begin{aligned} \widetilde{P}_2 &= AP_2A^{-1} - b_n \frac{f''}{f'} - c_n \left(\frac{f''}{f'}\right)^2 + n\, a_{n-1} \frac{f''}{f'} A' A^{-1} \\ &\qquad - n(n-1) A' A^{-1} A' A^{-1} + \frac{n(n-1)}{2} A'' A^{-1}. \end{aligned} \tag{5.5}$$

Hence, to preserve the condition (5.4),

$$A = (f')^{(n-1)/2} B \quad \text{for a constant matrix } B$$

is necessary.

On the other hand, from (5.5) we see

$$\frac{\operatorname{tr}\widetilde{P}_2}{r} = \frac{n(n-1)(n+1)}{6}\{f;t\} + \frac{\operatorname{tr}P_2}{r}.$$

Then, solving $\operatorname{tr}\widetilde{P}_2 = 0$, we get the next proposition which is similar to Proposition 3.4.

Proposition 3.10. 1^o *The equation* (5.4) *can be normalized so as*

$$P_1 = 0 \quad \textit{and} \quad \operatorname{tr}P_2 = 0.$$

2^o *A transformation* (5.3) *preserving this condition has the form*

$$\begin{aligned} s &= \frac{\alpha t+\beta}{\gamma t+\delta} \\ Y &= (\gamma t+\delta)^{1-n}BX\,;\ B \quad \textit{is constant.} \end{aligned}$$

3^o *Under this transformation, the matrix-valued quadratic form* $P_2\,dt^2$ *changes only by a conjugate action of* B*.*

Remark. We can moreover see the k-differential form

$$R_k = \sum_{j=0}^{k-2} a_{k,j}\left(\frac{d}{dt}\right)^j P_{k-j}(dt)^k\,;\ k \geq 2,$$

$$a_{k,j} = (-1)^j\frac{(2k-j-2)!\,(n-k+j)!}{j!\,(k-j-1)!}$$

are also invariants up to conjugation of B. ([W1, II,§4]; [MOR],[SEA1]). When $r=1$, P_2 does not appear. When $r \geq 2$, $R_2 = P_2\,dt^2$. Wilczynski's computation for $r=1$ holds also for $r \geq 2$. Note that $R_2, \ldots, R_n$ are fundamental invariants in the sense that they determine the projective equivalence class of motions of $(r-1)$-planes in general. See [SEA1].

Example. When $n = 2$ and $r = 3$, we have a three-dimensional submanifold in $\mathbf{P}^5$, which is a one-parameter family of 2-planes. When the invariants vanish, i.e. $P_2 = P_3 = 0$, the equation has six solutions $(1, 0, 0, 0, 0, 0)$, $(0, t, 0, 0, 0, 0)$, $(0, 0, 1, 0, 0, 0)$, $(0, 0, 0, t, 0, 0)$, $(0, 0, 0, 0, 1, 0)$ and $(0, 0, 0, 0, 0, t)$. The submanifold is given by a map $(t, s, u) \mapsto (1, t, s, st, u, ut)$ and written in homogeneous coordinates by equations $x_1x_4 - x_2x_3 = 0$ and $x_1x_6 - x_2x_5 = 0$. The 2-plane through $(1, t, 0, 0, 0, 0)$ is given by $x_1 = t\,x_2$, $x_3 = t\,x_4$, $x_5 = t\,x_6$.

4. Projective theory of hypersurfaces

In this chapter we will treat hypersurfaces in $\mathbf{P}^{n+1}$. The aim is to formulate the projective fundamental theorem of hypersurfaces. For its better understanding, we recall first the outline of the euclidean theory of hypersurfaces in $\mathbf{R}^{n+1}$.

Let M be an immersed hypersurface in $\mathbf{R}^{n+1}$. The euclidean metric induces a Riemannian metric on M and the second fundamental form is defined on M. These two are related in such ways that the Riemannian curvature tensor is expressible in terms of the second fundamental form (Gauss equation) and that the covariant derivatives of the second fundamental form are written by this form and by the metric (Codazzi-Minardi equation). Then the fundamental theorem of hypersurfaces says that these two equations characterize the immersion: given a Riemannian metric and a quadratic form on an n-manifold M which satisfy these equations, we can find an immersion of M into $\mathbf{R}^{n+1}$ up to a rigid motion so that the given metric and the form are the induced metric and the second fundamental form respectively.

Now consider an immersed hypersurface M in $\mathbf{P}^{n+1}$. Then it turns out that the conformal class, denoted by h, of the second fundamental form of M is a projective invariant. Assume this class is non-degenerate and $n \geqq 3$. Then we can define a matrix-valued one-form τ, which is essentially the same as the so-called cubic form. These invariants h and τ play a similar role as the induced metric and the second fundamental form in the euclidean case: the conformal curvature tensor is expressed by τ and the covariant derivative of τ have a certain relation with τ and h. We call these relations also Gauss equation and Codazzi-Minardi equation and we can find an immersion up to a projective motion so that the given class and the given form are the induced conformal class and the induced one-form respectively.

In §1, we recall some terminologies about conformal connections. In §2, some fundamental invariants of a hypersurface in $\mathbf{P}^{n+1}$ will be defined and explicit expressions of these invariants will be given in §3. The §4 explains a relation with the unimodular affine treatment of hypersurfaces. In §5, we prove a theorem by Pick and Berwald about the characterization of quadratic hypersurfaces. Fundamental theory of hypersurfaces will be formulated and proved in §6, when the dimension $n \geqq 3$; the case when $n = 2$ is treated in §7. In the last section 8 we give some formulae about the projective metric.

§1. Quadratic hypersurfaces

Let h be a non-singular $n \times n$ symmetric matrix of signature (p, q). The orthogonal group with respect to h is

$$O(h) = \{\, g \in GL_n \,;\, g\, h\, {}^t g = h \,\}.$$

The conformal orthogonal group $CO(h)$ is defined by

$$CO(h) = \{\, \lambda g \,;\, g \in O(h),\ \lambda \in \mathbf{R}^* \,\}.$$

Let M be an n-dimensional manifold. The bundle of linear frames is denoted by $L(M)$. This is a principal GL_n-bundle. A subbundle of $L(M)$ with $CO(h)$ as the structure group is called a $CO(h)$-*structure* of the manifold M. This has the unique correspondence with the conformal class of a pseudo-riemannian metric of type (p, q).

Let Q be a matrix in GL_{n+2}:

$$Q = \begin{pmatrix} 0 & 0 & -1 \\ 0 & h & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Define the orthogonal group with respect to Q by

$$O(Q) = \{\, g \in GL_{n+2} \,;\, gQ\,{}^t g = Q \,\}.$$

Its Lie algebra $\mathfrak{o}(Q)$ is generated by elements

$$\begin{pmatrix} \Lambda & D & 0 \\ B & A & E \\ 0 & C & -\Lambda \end{pmatrix}, \tag{1.1}$$

where B and E are vertical and C and D are horizontal vectors and A is an $n \times n$-matrix. Λ is a scalar. They satisfy relations

$$B = h\,{}^t C, \quad E = h\,{}^t D, \quad \text{and} \quad Ah + h\,{}^t A = 0. \tag{1.2}$$

A quadratic hypersurface Q^n is defined by

$$Q^n = \{\, [\,x\,] \in \mathbf{P}^{n+1}\,;\, xQ\,{}^t x = 0\,\}.$$

The group $O(Q)$ acts on Q^n transitively on the right: $(x^i) \mapsto \left(\sum g^i_j x^j\right)$. The isotropy subgroup at $(1, 0, \ldots, 0)$ is

$$H = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ b & a & 0 \\ \mu & c & \nu \end{pmatrix}; \begin{array}{l} \lambda\nu = 1,\ a \in O(h) \\ b = \lambda a\, h\,{}^t c,\ \mu = \dfrac{1}{2}\lambda c\, h\,{}^t c \end{array} \right\}. \tag{1.3}$$

Hence $O(Q)$ is a principal H-bundle over Q^n. Let $\mathfrak{h}$ denote the Lie algebra of H; whose element is written as

$$\begin{pmatrix} \Lambda & 0 & 0 \\ B & A & 0 \\ 0 & C & -\Lambda \end{pmatrix}; \quad B = h\,{}^t C, \quad Ah + h\,{}^t A = 0.$$

The linear representation of H at $(1, 0, \ldots, 0)$ is not faithful and has a kernel consisting of elements

$$\begin{pmatrix} 0 & 0 & 0 \\ B & 0 & 0 \\ 0 & C & 0 \end{pmatrix}; \quad B = h\,{}^t C.$$

Let N be the corresponding normal subgroup of H. Then we see $H/N \cong CO(h)$. Hence the bundle $O(Q)/N$ is a principal $CO(h)$-bundle over Q^n. This defines the canonical conformal structure on Q^n. Put $\varphi = -2dx^o dx^{n+1} + \sum h_{ij} dx^i dx^j$, a non-degenerate quadratic form on $\mathbf{R}^{n+2}$. The restriction to Q^n of the pull-back of φ by a section of $\mathbf{R}^{n+2} - \{0\} \to \mathbf{P}^{n+1}$ gives the conformal class associated to the bundle

$O(Q)/N$. It is independent of the choice of a section. Moreover, the bundle $O(Q)$ itself is the first prolongation of $O(Q)/N$ and the Maurer-Cartan form of $O(Q)$ defines a normal conformal connection.

We here briefly recall the definition of a normal conformal connection (see Chapter 4 of [KOB]). Let M be an n-manifold, G a Lie group, G_0 a closed subgroup with $\dim G/G_0 = n$ and P a principal G_0-bundle over M. The group G_0 acts on P on the right and the right translation by $g \in G_0$ is denoted by R_g. Through this action every element A of the Lie algebra $\mathfrak{g}_0$ of G_0 defines a vertical vector field on P denoted by A^*. Then a pair (P, ω) of the bundle P and a 1-form ω with values in the Lie algebra of G is called a Cartan connection if the following conditions are satisfied:

$$
\begin{aligned}
&a) \quad \omega(A^*) = A \quad \text{for every} \quad A \in \mathfrak{g}_0 \\
(1.4) \qquad &b) \quad (R_g)^*\omega = \mathrm{ad}\,(g^{-1})\omega \quad \text{for every} \quad g \in G_0, \\
&c) \quad \omega(X) \neq 0 \quad \text{for every non-zero vector} \quad X \quad \text{of} \quad P.
\end{aligned}
$$

When $G = O(Q)$ and $G_0 = H$, this connection is called a conformal connection of type Q or of type (p, q). Let (P, ω) be one conformal connection. The components of ω are written as ω_α^β, $0 \leq \alpha, \beta \leq n+1$. Corresponding to the decomposition (1.1), $\mathfrak{o}(Q)$ has a grading $\mathfrak{o}_{-1} + \mathfrak{o}_0 + \mathfrak{o}_1$, where $\mathfrak{o}_{-1} = \{B, C\}$, $\mathfrak{o} = \{\Lambda, A\}$ and $\mathfrak{o}_1 = \{D, E\}$. The decomposition of ω according to this grading is denoted by $\omega_{-1} = (\omega_i^0)$, $\omega_0 = (\omega_0^0, \omega_i^j)$ and $\omega_1 = (\omega_0^i)$. By (1.2) we can forget the C and E parts. The condition (1.4) shows that the component ω_1 is basic: the tangent vector X of P is vertical if $\omega_0^i(X) = 0$. Let $\Omega = d\omega - \omega \wedge \omega$ be the curvature form and Ω_α^β be components of Ω. Put

$$\Omega_i^j - \delta_i^j \Omega_0^0 = \frac{1}{2} \sum K_{ik\ell}^j \omega^k \wedge \omega^\ell.$$

Then the connection ω is called *normal* if

$$(1.5) \qquad \sum K_{ij\ell}^j = 0.$$

We will use the following fact: Let P be a principal H-bundle over a manifold of dimension $\geqq 3$. Given 1-forms $\omega_1 = (\omega^i)$ and $\omega_0 = (\omega_0^0, \omega_i^j)$

there exists a unique normal conformal connection ω extending ω_1 and ω_0 provided that they satisfy $d\omega^i = \omega_0^0 \wedge \omega^i + \sum \omega^j \wedge \omega_j^i$ and (a′) $\omega_i(A^*) = 0$, $\omega_0(A^*) =$ the $\mathfrak{o}_0$-component of A for $A \in \mathfrak{o}_0 + \mathfrak{o}_1$; (b′) $R_g^*(\omega_1 + \omega_0) = \mathrm{ad}\,(g^{-1})(\omega_1 + \omega_0)$ for every $g \in H$ and (c′) a tangent vector X of P is vertical if $\omega_1(X) = 0$. See Theorem 4.2 of [KOB].

§2. Projective invariants of a hypersurface

2.1. projective frames

Let $\overline{\mathcal{F}}$ be the set of linear bases $e = (e_0, e_1, \ldots, e_{n+1})$ of $\mathbf{R}^{n+2}$. The group GL^{n+2} acts on $\overline{\mathcal{F}}$ simply transitively by $g(e_\alpha) = (g_\alpha^\beta e_\beta)$. Between two bases e and $\overline{e}$ define a relation $\sim$ by $e_\alpha = \lambda \overline{e}_\alpha$ for some $\lambda \in \mathbf{R}^*$. Then the quotient space

$$\mathcal{F} := \overline{\mathcal{F}}/\sim$$

is defined, whose element is called a *projective frame.* This space is identified with the projective linear group $G = SL_{n+2}/\text{centre}$. Define a mapping $\pi\colon \mathcal{F} \to \mathbf{P}^{n+1}$ by

$$\pi(e) = [\,e_0\,].$$

Then $\mathcal{F}$ is a principal bundle over $\mathbf{P}^n$ with π as its projection. The fibre group is isomorphic to

$$\begin{aligned} G_0 &= \{\, g \in G\,;\, e_0(g\,e) = e_0(e) \text{ for any } e \in \mathcal{F}\,\} \\ &\cong \left\{ \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ * & & * & \end{pmatrix} \in G \right\}. \end{aligned}$$

A local section is called a *projective frame field* (or simply a frame). We will denote it by the same letter $e = (e_0, e_1, \ldots, e_{n+1})$. Fix an element $e^0 \in \mathcal{F}$. Then any projective frame e is written as $e = g\,e^0$. So $de = dg\,e^0 = dg \cdot g^{-1}e$. We write $\omega = dg \cdot g^{-1}$, the Maurer-Cartan form of G. Let ω_α^β, $0 \le \alpha,\ \beta \le n+1$, be components of ω. Then

$$d\,e_\alpha = \omega_\alpha^\beta e_\beta, \tag{2.1}$$

and

$$d\omega_\alpha^\beta = \omega_\alpha^\gamma \wedge \omega_\gamma^\beta\,, \quad \omega_\alpha^\alpha = 0. \tag{2.2}$$

Here and in the following we use the summation convention: repeated indices, once in upper case and once in lower case, are summed on its range. The range will be from 0 to $n+1$ for $\alpha, \beta, \dots$ and from 1 to n for i, j, … Note that the 1-forms $\omega_0^1, \dots, \omega_0^{n+1}$ are basic forms with respect to the projection π.

Let e be a projective frame field. The induced form $e^*\omega$ is denoted by $\omega(e)$, or simply by ω if there is no confusion. Let $\widetilde{e}$ be another frame field with a relation

$$\widetilde{e} = g\, e \tag{2.3}$$

for a G_0-valued function g. By definition

$$\omega(\widetilde{e}) = dg\, g^{-1} + g\omega(e)g^{-1}. \tag{2.4}$$

Let now $f\colon M^n \to \mathbf{P}^{n+1}$ be an immersion of an n-manifold M. The pull-back of the bundle $\mathcal{F}$ onto M is denoted by $f^*\mathcal{F}$. Frame fields with the property $\omega_0^{n+1} = 0$ generate a subbundle $\mathcal{F}_1$. We write this fact as

$$\mathcal{F}_1 = \{e \in f^*\mathcal{F}\, ;\, \omega_0^{n+1} = 0\}. \tag{2.5}$$

Each fibre is isomorphic to

$$G_1 = \left\{ \begin{pmatrix} \lambda & \overbrace{0 \cdots 0}^{n} & 0 \\ & & 0 \\ * & * & \vdots \\ & & 0 \\ * & * & \nu \end{pmatrix} \in G_0 \right\}. \tag{2.6}$$

The first component e_0 of $e \in \mathcal{F}_1$ represents a point of M in $\mathbf{R}^{n+2}$ and the next n components together with e_0 span the tangent space of the cone over M at e_0. When $\omega_0^0 = 0$, these n components are tangent vectors at $e_0(M)$.

2.2 The fundamental form

Let e be a frame field in $\mathcal{F}_1$. The exterior derivation of the condition $\omega_0^{n+1} = 0$ gives

$$\omega^i \wedge \omega_i^{n+1} = 0.$$

Hence

$$\omega_i^{n+1} = h_{ij}\omega^j\,, \quad h_{ij} = h_{ji} \tag{2.7}$$

for some functions h_{ij}. Define

$$\varphi_2 = h_{ij}\omega^i\omega^j\,, \quad h = (h_{ij}) \quad \text{and} \quad H = \det h. \tag{2.8}$$

We call φ_2 the *fundamental form* of the immersed hypersurface.

To see the dependence on frames we rewrite (2.4) componentwise for $g \in G_1$:

$$\begin{aligned}
\widetilde{\omega}_0^0 &= \omega_0^0 + d\,\log\lambda - b_i A_j^i \omega^j \\
\widetilde{\omega}^i &= \lambda A_j^i \omega^j \\
\widetilde{\omega}_i^{n+1} &= \nu^{-1} a_i^k \omega_k^{n+1} \\
\widetilde{\omega}_i^k &= da_i^j A_j^k + a_i^\ell \omega_\ell^j A_j^k + b_i \omega^j A_j^k - \nu^{-1} a_i^\ell \omega_\ell^{n+1} c^j A_j^k \\
\widetilde{\omega}_{n+1}^{n+1} &= \omega_{n+1}^{n+1} + d\,\log\nu + \nu^{-1} c^i \omega_i^{n+1} \\
\widetilde{\omega}_{n+1}^i + c^j A_j^i \widetilde{\omega}_{n+1}^{n+1} &= (dc^j + c^k\omega_k^j + \mu\omega^j + \nu\omega_{n+1}^j) A_j^i \\
\lambda\widetilde{\omega}_i^0 + \widetilde{\omega}_i^j b_j + \mu\widetilde{\omega}_i^{n+1} &= a_i^j \omega_j^0 + db_i + b_i\omega_0^0 \\
\lambda\widetilde{\omega}_{n+1}^0 + b_i\widetilde{\omega}_{n+1}^i + \mu\widetilde{\omega}_{n+1}^{n+1} &= \nu\omega_{n+1}^0 + d\mu + \mu\omega_0^0 + c^i\omega_i^0,
\end{aligned} \tag{2.9}$$

where

$$g = \begin{pmatrix} \lambda & 0 & 0 \\ b & a & 0 \\ \mu & c & \nu \end{pmatrix}, \quad A = a^{-1} \quad \text{and} \quad \widetilde{\omega} = \omega(\widetilde{e}).$$

From the second and the third equations we have

$$\widetilde{h} = (\lambda\nu)^{-1} a\, h\, {}^t a, \quad \text{i.e.} \quad \widetilde{h}_{ij} = (\lambda\nu)^{-1} a_i^k h_{k\ell} a_j^\ell, \tag{2.10}$$

$$(2.11) \qquad \widetilde{H} = (\det a)^{n+2} H.$$

($\sim$ denotes that the reference frame is $\widetilde{e}$). The formula (2.10) shows that rank h and $|\,\mathrm{index}\, h\,|$ are independent of choice of frames. *We assume from now on that* $\mathrm{rank}\, h = n$ (see §2.5). In this case we say that the hypersurface is *non-degenerate*. The identity (2.11) then shows that we may assume

$$(2.12) \qquad |H| = 1,$$

and consequently $|\det a| = |\lambda\nu| = 1$. With this assumption the first and the fifth equations of (2.9) give the identity

$$\widetilde{\omega}_0^0 + \widetilde{\omega}_{n+1}^{n+1} = \omega_0^0 + \omega_{n+1}^{n+1} + \nu^{-1} c^i \omega_i^{n+1} - b_i A_j^i \omega^j.$$

So we can find a frame with the property

$$(2.13) \qquad \omega_0^0 + \omega_{n+1}^{n+1} = 0.$$

Define a bundle $\mathcal{F}_2$ by

$$(2.14) \qquad \mathcal{F}_2 = \{\, e \in \mathcal{F}_1 \,;\, |H| = 1\,,\, \omega_0^0 + \omega_{n+1}^{n+1} = 0 \,\}$$

The fibre group is

$$(2.15) \qquad G_2 = \{\, g \in G \,;\, |\det a| = 1\,,\, |\lambda\nu| = 1\,,\, b = \nu^{-1} a\, h\, {}^t c \,\}.$$

2.3. The cubic form

We next take the exterior derivation of (2.7):

$$\begin{aligned}
0 &= -d\omega_i^{n+1} + dh_{ij} \wedge \omega^j + h_{ij} d\omega^j \\
&= -\omega_i^j \wedge \omega_j^{n+1} - \omega_i^{n+1} \wedge \omega_{n+1}^{n+1} + dh_{ij} \wedge \omega^j \\
&\qquad + h_{ij}(\omega_0^0 \wedge \omega^j + \omega^k \wedge \omega_k^j) \qquad \text{(by (2.2))} \\
&= \{dh_{ij} - h_{ij}\omega_j^k - h_{kj}\omega_i^k + h_{ij}(\omega^0 + \omega_{n+1}^{n+1})\} \wedge \omega^j \\
&= (dh_{ij} - h_{ik}\omega_j^k - h_{kj}\omega_i^k) \wedge \omega^j \qquad \text{(by (2.13))}.
\end{aligned}$$

Hence we can define a symmetric tensor h_{ijk} by

$$h_{ijk}\omega^k = d\,h_{ij} - h_{kj}\omega_j^k - h_{ik}\omega_j^k. \tag{2.16}$$

Put

$$\begin{aligned} \varphi_3 &= h_{ijk}\omega^i\omega^j\omega^k, \\ F &= h_{ijk}h_{pqr}h^{ip}h^{jq}h^{kr}, \end{aligned} \tag{2.17}$$

which we call the *cubic form* and the *Fubini-Pick invariant.* Here $(h^{ij}) = (h_{ij})^{-1}$. The cubic form satisfies an identity called the *apolarity condition*:

$$h^{ij}h_{ijk} = 0. \tag{2.18}$$

This is seen as follows:

$$\begin{aligned} 0 = d\log|H| &= h^{ij}dh_{ij} = h^{ij}(h_{ijk}\omega^k + h_{ik}\omega_j^k + h_{kj}\omega_i^k) \\ &= h^{ij}h_{ijk}\omega^k \qquad \text{by (2.2).} \end{aligned}$$

The next formula can be seen by a straightforward calculation

$$\lambda^2\nu\widetilde{h}_{ijk} = h_{pqr}a_i^p a_j^q a_k^r. \tag{2.19}$$

We have seen

Proposition 4.1. *Assume the hypersurface is non-degenerate. Then the fundamental form φ_2 and the Fubini-Pick invariant transform as*

$$\widetilde{\varphi}_2 = \lambda\nu^{-1}\varphi_2, \quad \widetilde{\varphi}_3 = \lambda\nu^{-1}\varphi_3, \quad \widetilde{F} = \lambda^{-1}\nu F \tag{2.20}$$

when frames change as $\widetilde{e} = g\,e$ for $g \in G_2$. The quadratic form $F\varphi_2$ is independent of choice of frames.

Definition. We call $F\varphi_2$ *the projective metric.*

2.4. The structure bundle

From (2.13) we have the following identity by taking exterior derivation:

$$(h_{ij}\omega^j_{n+1} - \omega^0_i) \wedge \omega^i = 0.$$

This enables us to put

$$h_{ij}\omega^j_{n+1} - \omega^0_i = L_{ij}\omega^j\,, \quad L_{ij} = L_{ji} \tag{2.21}$$

and to define

$$L = \frac{1}{n} h^{ij} L_{ij}. \tag{2.22}$$

The sixth and seventh formulae of (2.9) then yield

$$\lambda\widetilde{L}_{ij} = \lambda^{-1} L_{pq} a^p_i a^q_j + (2\mu - \nu^{-1} h_{k\ell} c^k c^\ell)\widetilde{h}_{ij} - (\lambda\nu)^{-1} h_{pqr} c^p a^q_i a^r_j$$

and

$$\lambda\widetilde{L} = \lambda^{-1} L + (2\mu - \nu^{-1} h_{k\ell} c^k c^\ell). \tag{2.23}$$

Consequently, we can find a frame with the property

$$L = 0. \tag{2.24}$$

Define a bundle $\mathcal{F}_3$ by

$$\mathcal{F}_3 = \{\, e \in \mathcal{F}_2\, ;\, L = 0 \,\}. \tag{2.25}$$

The fibre group is

$$G_3 = \left\{ g \in G_2\, ;\, \mu = \frac{1}{2}\nu^{-1} c\, h\, {}^t c \right\}. \tag{2.26}$$

With respect to frames in $\mathcal{F}_3$, L_{ij} transforms as

$$\lambda^2 \widetilde{L}_{ij} = (L_{k\ell} - \lambda h_{k\ell m} c^m) a^k_i a^\ell_j. \tag{2.27}$$

In summary, we have

Proposition 4.2. *For a non-degenerate hypersurface there exists a local projective frame field satisfying*

$$\omega_0^{n+1} = \omega_0^0 + \omega_{n+1}^{n+1} = 0, \quad |H| = 1, \quad L = 0.$$

Such frames generate a bundle $\mathcal{F}_3$ with G_3 as its fibre group:

$$G_3 = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ b & a & 0 \\ \mu & c & \nu \end{pmatrix} \in G_1\,; \quad \begin{aligned} |\lambda\nu| = 1\,,\ b = \nu^{-1} a\, h^t c\,, \\ \mu = \frac{1}{2}\nu^{-1} c\, h^t c \end{aligned} \right\}.$$

The formulae in (2.9) for g in G_3 become simpler. The last three are written as

$$\begin{aligned} \lambda\widetilde{\omega}_i^0 &= a_i^k \omega_k^0 + \{d(b_j A_k^j) - b_j A_\ell^j \omega_k^\ell\} a_i^k + b_i \omega_0^0 \\ &\qquad - \mu\nu^{-1} a_i^k \omega_k^{n+1} - b_i b_j A_k^j \omega^k + \nu^{-1} a_i^\ell \omega_\ell^{n+1} \left(c^j A_j^k b_k\right), \\ \widetilde{\omega}_{n+1}^i a_i^j &= \nu\omega_{n+1}^j + d\,c^j + c^k \omega_k^j + c^j(\omega_0^0 + d\log\lambda) + \mu\omega^j - c^j c^k \omega_k^{n+1}, \\ \widetilde{\omega}_{n+1}^0 &= \lambda^{-1}\nu\omega_{n+1}^0 - \left(\lambda^{-1} c^i L_{ik} - \frac{1}{2}(\lambda\nu)^{-1} c^i c^j h_{ijk}\right)\omega^k. \end{aligned} \tag{2.28}$$

Put

$$\omega_{n+1}^0 = -\gamma_j \omega^j. \tag{2.29}$$

Then we see

$$\lambda\widetilde{\gamma}_j = \left(\lambda^{-1}\nu\gamma_k + \lambda^{-1} c^i L_{ik} - \frac{1}{2}(\lambda\nu)^{-1} c^i c^\ell h_{i\ell k}\right) a_j^k. \tag{2.30}$$

We now rephrase Proposition 2 as follows. Let h be of signature (p, q). We fix a non-degenerate symmetric matrix h_0 of this signature and with $|\det h_0| = 1$. The identity (2.10) shows that we can choose a frame so that $h = h_0$ at each point. Then it is possible to define a bundle

$$P = \{\, e \in \mathcal{F}\,;\ h = h_0\,,\ \omega_0^0 + \omega_{n+1}^{n+1} = 0\,,\ L = 0\,\}. \tag{2.31}$$

If $p \neq q$, then $\lambda\nu = 1$ and the structure group is equal to the isotropy group H of the quadric Q^n with respect to h_0 (see §1.). When $p = q$, assuming the hypersurface is orientable, we can construct a subbundle with H also as the structure group.

2.5. Remarks

1. From (2.27), we can see the property

$$L_{ij} = \sum_k h_{ijk} a^k \quad \text{for some } a^k \tag{2.32}$$

is independent of frames. Also

$$L_{ij} = \sum h_{ijk} a^k \quad \text{and} \quad \gamma_i = -\frac{1}{2} \sum h_{ijk} a^j a^k. \tag{2.33}$$

These properties seem to have interesting geometric conclusion. (see §4 and §2 of Chapter 6).

2. Let us consider the case when h is degenerate and $\operatorname{rank} h = r$ is constant. We will see that the hypersurface is then ruled in the sense that it is foliated locally by linear subspaces of dimension $n - r$. Fix a frame and assume, for simplicity, h is constant so that

$$\omega_i^{n+1} = \begin{cases} 0 & i \geqq r+1 \\ \sum_{j \leq r} h_{ij} \omega^j & i \leq r. \end{cases}$$

The exterior derivation gives

$$\sum_{k \leq r} \omega_i^j \wedge h_{jk} \omega^k = 0 \quad \text{for } i \geqq r+1,$$

which yields

$$\omega_i^j = \sum_{k \leq r} a_{ik}^j \omega^k \quad \text{for} \quad i \geqq r+1$$

for some functions a^j_{ik}. On the other hand,

$$d\omega^j = \sum_{i\le r}\omega^i\wedge\omega^j_i + \sum_{i\geqq r+1}\omega^i\wedge\omega^j_i + \omega^0_0\wedge\omega^j.$$

Hence

$$d\omega^j = \sum_{i\le r}\alpha^j_i\wedge\omega^i \quad \text{for} \quad j\le r,$$

for some forms α^j_i. This shows that the equation $\omega^1 = \cdots = \omega^r = 0$ is integrable. Now assume $\omega^0_0 = 0$, which is always possible (§2) by preserving the condition on rank$h$. Let N be one leaf. Then we have

$$de_0|_N = \sum_{j\geqq r+1}\omega^j e_j.$$

Hence $e_{r+1},\cdots,e_n$ are tangent to N and they form a basis of TN. Furthermore

$$\begin{aligned} de_j|_N &= \sum\omega^k_j e_k + \omega^{n+1}_j e_{n+1} \\ &= 0 \qquad \text{for} \quad j\geqq r+1. \end{aligned}$$

This says that e_j, $j\geqq r+1$, are constant along N, i.e. N is a linear subspace of $\mathbf{P}^{n+1}$.

§3. Explicit expression of projective invariants

In this section we will explain the process in §2 geometrically for a non-degenerate hypersurface given by an equation

$$x^{n+1} = f(x^1,\ldots,x^n)$$

in affine coordinates $x^1,\ldots,x^n,x^{n+1}$. Denote derivatives of f by $f_i = \partial f/\partial x^i$, $f_{ij} = \partial^2 f/\partial x^i\partial x^j,\ldots$. Define a frame $e = (e_0,\ldots,e_{n+1})$ by

$$\begin{aligned} e_0 &= (1,x^1,\ldots,x^n,f) \\ e_1 &= (0,1,0,\ldots,0,f_1) \\ &\ \ \vdots \\ e_n &= (\ 0,\ldots,0,1,f_n) \\ e_{n+1} &= (\ 0,\ldots\ldots,0,1). \end{aligned}$$

With respect to this frame $de = \omega e$, where

$$\omega = \begin{pmatrix} 0 & dx^1 \cdots dx^n & 0 \\ 0 & & f_{1j}\, dx^j \\ \vdots & & \vdots \\ & 0 & f_{nj}\, dx^j \\ 0 & \cdots & 0 \end{pmatrix}.$$

By the definition of the fundamental form (2.7),

$$h_{ij} = f_{ij}.$$

Hence the non-degeneracy of h_{ij} is equivalent to that of the hessian matrix of f. Choose affine coordinates so that $f(0) = f_i(0) = 0$ and develop f formally into a series

$$f(x) = \frac{1}{2} a_{ij} x^i x^j + \sum_{d \geqq 3} f_d, \tag{3.1}$$

f_d being a homogeneous polynomial of degree d. The matrix (a_{ij}) is assumed non-degenerate. Define coefficients of f_d by

$$f_d = \frac{1}{d!} \sum a_{i_1 \cdots i_d} x^{i_1} \cdots x^{i_d}. \tag{3.2}$$

Put

$$\begin{aligned} \operatorname{Tr} f_3 &= \sum a^{ij} a_{ijk} \\ \operatorname{Tr}^2 f_4 &= \sum a^{ij} a^{k\ell} a_{ijk\ell}, \end{aligned} \tag{3.3}$$

where (a^{ij}) is the inverse matrix of (a_{ij}). Then we can see

Proposition 4.3. (1) *At each point of a non-degenerate hypersurface, there exists a projective change of coordinates such that the hypersurface is represented by a function with property*

$$\operatorname{Tr} f_3 = \operatorname{Tr}^2 f_4 = 0. \tag{3.4}$$

(2) *Fix a_{ij}. Then every projective change of coordinates for which the hypersurface is represented by a function f satisfying this property and with $f_{ij}(0) = a_{ij}$ belongs to the isotropy subgroup at the origin of the quadric $x^{n+1} = \frac{1}{2}a_{ij}x^i x^j$.*

Note that the property (3.4) corresponds to the conditions (2.18) and (2.24).

We next normalize the frame e by defining a new frame $\overline{e}$ by

$$\begin{cases} \overline{e}_0 = e_0 \\ \overline{e}_i = \alpha e_i \\ \overline{e}_{n+1} = -\dfrac{\ell}{2}e_0 + \alpha c^i e_i + \alpha^{-n} e_{n+1}, \end{cases} \tag{3.5}$$

where

$$\begin{aligned} \alpha &= (\det\, f_{ij})^{-1/n(n+2)} \\ c^i &= n\alpha^{-n-2} f^{ij}\alpha_j \\ \ell &= \alpha^{-n-2}\big(\alpha\alpha_{ij} - (n+1)\alpha_i\alpha_j - \alpha\alpha_k f^{k\ell} f_{\ell ij}\big) f^{ij}. \end{aligned} \tag{3.6}$$

The associated form $\overline{\omega}$ is

$$\overline{\omega} = \begin{pmatrix} 0 & \omega^i & 0 \\ \frac{\ell}{2}\omega_j^{n+1} & \delta_j^i d\,\log\alpha - c^i\omega_j^{n+1} & \omega_j^{n+1} \\ \frac{1}{2}d\ell & -\frac{\ell}{2}\omega^i + d\,c^i + c^i d\,\log\alpha & 0 \end{pmatrix}, \tag{3.7}$$

where

$$\omega^i = \alpha^{-1}d\,x^i, \quad \omega_j^{n+1} = \alpha^{n+2} f_{jk}\omega^k. \tag{3.8}$$

Then we have

$$\begin{aligned} h_{ij} &= \alpha^{n+2} f_{ij}, \\ h_{ijk} &= \alpha^{n+2}(\alpha f_{ijk} + n(\alpha_k f_{ij} + \alpha_i f_{jk} + \alpha_j f_{ki})). \end{aligned} \tag{3.9}$$

Assume $f_{ij}(0) = \delta_{ij}$ for simplicity. Assume also the property (3.4): $\sum_i f_{ij}(0) = \sum_{ij} f_{iijj}(0) = 0$. Then, in particular, $\alpha_j(0) = 0$. This shows, at the origin,

$$h_{ij}(0) = \delta_{ij}\,, \quad h_{ijk}(0) = f_{ijk}(0)\,, \quad F(0) = \sum f_{ijk}(0)^2. \tag{3.10}$$

The quantities $L_{ij}(0)$ and $\gamma_j(0)$ are computed by the definitions (2.21) and (2.29):

$$
\begin{aligned}
L_{ij}(0) &= \frac{1}{n+2}\sum_{k,\ell}(f_{ik\ell}f_{jk\ell}) - \sum_k f_{ijkk} - \frac{1}{n(n+2)}F\delta_{ij}, \\
\gamma_i(0) &= \frac{-1}{2n(n+2)}\sum_{j,k}\left(\begin{aligned} f_{ijjkk} - 2\sum_\ell (f_{ijk\ell}f_{jk\ell} + f_{ijk}f_{jk\ell\ell}) \\ +3\sum_{\ell,m} f_{ijk}f_{k\ell m}f_{j\ell m}\end{aligned}\right).
\end{aligned} \tag{3.11}
$$

§4. Remarks on affine description of a hypersurface

We will briefly recall the affine description of a hypersurface in relation with the normalization in §2.

Let $\mathbf{A}^{n+1}$ be an $(n+1)$-dimensional affine space. The unimodular affine group G_a is defined by

$$
G_a = \left\{ \begin{pmatrix} 1 & u \\ 0 & a \end{pmatrix}; \quad a \in SL_{n+1}, \quad u \in \mathbf{A}^{n+1} \right\},
$$

which is a subgroup of $G = PSL_{n+2}$. Denote by (ω_α^β) the restriction of the Maurer-Cartan form of G to G_a. We have now $\omega_\alpha^0 = 0$. Let $f_0 \colon M^n \to \mathbf{A}^{n+1}$ be an immersion and attach to each point a set of independent vectors $(f_1, \ldots, f_{n+1})$ such that

$$
\det(f_1, \ldots, f_{n+1}) = 1,
$$

where det is a volume form invariant under G_a. Then a set $(f_0, f_1, \ldots, f_{n+1})$, called a *unimodular affine frame field*, can be seen a section of the canonical projection $G_a \to \mathbf{A}^{n+1}$ and satisfies

$$
d\,f_a = \omega_\alpha^\beta f_\beta.
$$

Considering frame fields with $\omega_0^{n+1} = 0$, we can define h_{ij} and φ_2 similarly as in §2. Furthermore, under the assumption of nondegeneracy of $h = (h_{ij})$, it is possible to choose a frame field by a transformation belonging to G_a so that $|\det h| = 1$ and $\omega_{n+1}^{n+1} = 0$. It is known that, for such a frame, the form φ_2 is uniquely defined and called the *affine metric*. Now the form ω looks like

$$\begin{pmatrix} 0 & \omega^j & 0 \\ 0 & \omega_i^j & \omega_i^{n+1} \\ 0 & \omega_{n+1}^j & 0 \end{pmatrix}. \tag{4.1}$$

A cubic form φ_3 is defined by the same way. The tensor ℓ_{ij} analogous to L_{ij} is given by $h_{ij}\omega_{n+1}^j = \ell_{ij}\omega^j$. This is called the *affine mean curvature tensor* and $\ell = \frac{1}{n}h^{ij}\ell_{ij}$ is called the *affine mean curvature*. The operator associated with the tensor $\ell_{ij}h^{jk}$ is called the *affine shape operator*. It is known that the last vector f_{n+1} of a frame is affinely invariant (up to ± 1 in case index $h = 0$) and it is called an *affine normal*. (cf. [CH]).

We next perform a change of frame by a transformation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_n & 0 \\ -\frac{\ell}{2} & 0 & 1 \end{pmatrix}$$

to get a frame satisfying the conditions in Proposition 2 of §2. The new form is given by

$$\begin{pmatrix} 0 & \omega^j & 0 \\ \frac{\ell}{2}\omega_i^{n+1} & \omega_i^j & \omega_i^{n+1} \\ -\frac{1}{2}d\ell & -\frac{\ell}{2}\omega^j + \omega_{n+1}^j & 0 \end{pmatrix}. \tag{4.2}$$

Then we see

$$\begin{aligned} L_{ij} &= \ell_{ij} - \ell\, h_{ij}, \\ \gamma_j &= \frac{1}{2}\ell_j \quad \text{where} \quad d\ell = \ell_j\omega^j. \end{aligned} \tag{4.3}$$

In the affine theory of hypersurfaces, this form L_{ij} has a special interest: the condition

$$\ell_{ij} - \ell\, h_{ij} = 0 \tag{4.4}$$

is known to define a surface called *affine hypersphere*. See [BL], [CAL], [S2]. A typical example is a quadratic hypersurface, whose invariants h_{ijk}, L_{ij} and γ_j all vanish; cf. §3 and §5.

Let us here recall the condition (2.32) in §2.5. In the above affine terminology it is written as

$$\ell_{ij} - \ell\, h_{ij} = \sum_k h_{ijk} a^k$$

for some vector (a^k). Since affine spheres satisfy this condition for $a^k = 0$, any projective transformation of an affine sphere satisfies (2.32) (and also (2.33)). But examples show that (2.32) or (2.33) defines a broader class of surfaces. See Chapter 6.

§5. A characterization of quadratic hypersurfaces

In this section we prove

Theorem 4.4. *Let M be a connected non-degenerate hypersurface in $\mathbf{P}^{n+1} (n \geqq 2)$. If the cubic form vanishes everywhere, then M is a part of a quadratic hypersurface.*

When $n = 2$, this was proved by Wilczynski [W1] and Pick [P]. Berwald [BER] generalized it for $n \geqq 3$.

Lemma 4.5. *Consider a connected non-degenerate hypersurface M. Assume there exists a frame field e with the property that the coframe ω, $de = \omega e$, takes values in $\mathfrak{o}(Q)$. Then M is a part of a quadratic hypersurface.*

Proof. Let $e^0 = (e^0_0, e^0_1, \ldots, e^0_{n+1})$ be a standard frame: $e^0_\alpha = (0, \ldots, 0, \overset{\alpha}{1}, 0, \ldots, 0)$. Define a G_0-valued function g on M by $e = g\, e^0$. Then $\omega = dg\, g^{-1}$. Since ω takes values in $\mathfrak{o}(Q)$, there exists a constant matrix a and an $O(Q)$-valued matrix h so that $g = h\, a$. Hence $e_\alpha = h^\beta_\alpha a^\gamma_\alpha e^0_\gamma$. In particular $e_0 = (\ldots, h^\beta_0 a^\gamma_\beta, \ldots)$. Then, defining $Q^a = a^{-1} Q\, {}^t a^{-1}$, we have

$$e_0 Q^a \, {}^t e_0 = h_0^\alpha Q_{\alpha\beta} h_0^\beta = Q_{00} = 0$$

. This shows M is contained in a quadratic hypersurface defined by Q^a.

Proof of Theorem. Choose a frame field belonging to the bundle P defined in §2. It satisfies

$$\omega_0^0 + \omega_{n+1}^{n+1} = 0 \quad \text{and} \quad L = 0. \tag{5.1}$$

We take $(\varepsilon_i \delta_{ij})$, $\varepsilon_i = \pm 1$, as a reference matrix h_0. The assumption $\varphi_3 = 0$ is then equivalent to

$$\varepsilon_i \omega_i^j + \varepsilon_j \omega_j^i = 0. \tag{5.2}$$

(Do not take summation in this proof unless $\sum$ is used.) Recall notations in §2:

$$\omega_i^{n+1} = \varepsilon_i \omega^i, \quad \varepsilon_i \omega_{n+1}^i - \omega_i^0 = \sum_j L_{ij} \omega^j, \quad \omega_{n+1}^0 = -\sum \gamma_j \omega^j. \tag{5.3}$$

For the proof it is enough to see $L_{ij} = \gamma_j = 0$ in view of Lemma 5. Take the exterior derivation of (5.2):

$$\begin{aligned}
\varepsilon_i d\omega_i^j + \varepsilon_j d\omega_j^i &= \varepsilon_i \Big(\omega_i^{n+1} \wedge \omega_{n+1}^j + \omega_i^0 \wedge \omega^j + \textstyle\sum_k \omega_i^k \wedge \omega_k^j\Big) \\
&\quad + \varepsilon_j \Big(\omega_j^{n+1} \wedge \omega_{n+1}^i + \omega_j^0 \wedge \omega^i + \textstyle\sum_k \omega_j^k \wedge \omega_k^i\Big) \\
&= \varepsilon_i (\varepsilon_i \omega^i \wedge \omega_{n+1}^j + \omega_i^0 \wedge \omega^j) + \varepsilon_j (\varepsilon_j \omega^j \wedge \omega_{n+1}^i + \omega_j^0 \wedge \omega^i) \\
&\quad + \textstyle\sum_k (\varepsilon_i \omega_i^k \wedge \omega_k^j + \varepsilon_j \omega_j^k \wedge \omega_k^i).
\end{aligned}$$

The last term vanishes by (5.2). Hence

$$\begin{aligned}
0 &= \omega^i \wedge (\omega_{n+1}^j - \varepsilon_j \omega_j^0) + \omega^j \wedge (\omega_{n+1}^i - \varepsilon_i \omega_i^0) \\
&= \varepsilon_j \sum L_{jk} \omega^i \wedge \omega^k + \varepsilon_i \sum L_{ik} \omega^j \wedge \omega^k.
\end{aligned}$$

Consider the case $i = j$ to see $L_{ik} = 0$ for $k \neq i$. When $i \neq j$, look at the term $\omega^i \wedge \omega^j$. Then $\varepsilon_i L_{ii} = \varepsilon_j L_{jj}$. Hence $L_{ij} = c\,\varepsilon_i \delta_{ij}$ for a scalar function c. On the other hand we know $h^{ij}L_{ij} = 0$. This proves $c = 0$ and $L_{ij} = 0$. Next take the derivation of $\varepsilon_i \omega^i_{n+1} - \omega^0_i = 0$:

$$\begin{aligned}
\varepsilon_i d\omega^i_{n+1} - d\omega^0_i &= \varepsilon_i \Big(\omega^0_{n+1} \wedge \omega^i + \sum \omega^k_{n+1} \wedge \omega^i_k + \omega^{n+1}_{n+1} \wedge \omega^i_{n+1} \Big) \\
&\qquad - \Big(\omega^0_i \wedge \omega^0_0 + \sum \omega^k_i \wedge \omega^0_k + \omega^{n+1}_i \wedge \omega^0_{n+1} \Big) \\
&= (\varepsilon_i \omega^{n+1}_{n+1} \wedge \omega^i_{n+1} - \omega^0_i \wedge \omega^0_0) \\
&\qquad + \sum (\omega^k_i \wedge \omega^0_k - \varepsilon_i \omega^k_{n+1} \wedge \omega^i_k) \\
&\qquad + (\omega^{n+1}_i + \varepsilon_i \omega^i) \wedge \omega^0_{n+1}.
\end{aligned}$$

Since the first and the second terms vanish by (5.1), by (5.2) and by the identity $\omega^0_i = \varepsilon_i \omega^i_{n+1}$, we have by (5.3)

$$\varepsilon_i \omega^i \wedge \omega^0_{n+1} = 0.$$

Hence $\omega^0_{n+1} = 0$ and this proves the theorem.

Theorem 4.4 gives the following

Corollary 4.6. *Let M be a closed locally strongly convex smooth hypersurface in $\mathbf{P}^{n+1}$, $n \geqq 2$. Then the connected component of the group of projective transformations which leave M invariant is compact unless M is a quadratic hypersurface.*

This is a weak form of a theorem proved by J. Bénzecri [BEN], which assumes weak convexity throughoute, and strong convexity and smoothness at one point. The proof of Corollary is given if we show the cubic form vanishes. Since the hypersurface is locally strongly convex, it is enough to see that the Fubini-Pick invariant F vanishes. Here note that φ_2 gives a Riemannian metric on M and that each projective automorphism is conformal with respect to φ_2. Then a theorem of M. Obata [OB] says that φ_2 is conformorphic to the standard metric of the unit sphere. Moreover $F\varphi_3$ is an invariant $(0,3)$-tensor. The same theorem says that such a tensor does not exist. Hence we have $F = 0$.

§6. Fundamental theorem of a hypersurface

The aim of this section is to construct a normal conformal connection associated with the fundamental form φ_2 and, by use of this connection, to formulate the fundamental theorem of hypersurfaces in $\mathbf{P}^{n+1}$. We assume the dimension $n \geqq 3$ throughout this section.

We first try to find a $\mathfrak{h}$-valued 1-form π satisfying the curvature condition (1.5), which defines a normal conformal connection on the bundle P. Since a general process of obtaining π is well-known (see [K, p.135-136]), it is simply necessary to relate π with ω. Assume π has the following form with unknown 1-form τ:

$$\pi = \omega + \tau. \tag{6.1}$$

The curvature form Ω of π is defined by

$$\Omega = d\pi - \pi \wedge \pi. \tag{6.2}$$

We want to determine τ so that $\Omega_0^0 = 0$ and that Ω_i^j is written as

$$\Omega_i^j = \frac{1}{2} C^j_{ik\ell} \pi_0^k \wedge \pi_0^\ell\,, \quad C^j_{ik\ell} + C^j_{i\ell k} = 0, \tag{6.3}$$

with the property

$$\sum_j C^j_{ij\ell} = 0. \tag{6.4}$$

From now on, the rule of raising and lowering indices with respect to the matrix $h = (h_{ij})$ will be used. We write π^i for π_0^i. First define

$$\begin{aligned} &\tau_0^\alpha = \tau_\alpha^{n+1} = 0\,, \quad \tau_{n+1}^0 = -\omega_{n+1}^0 \\ &\tau_i^j = \frac{1}{2} h^{jk} h_{ik\ell} \omega^\ell = \frac{1}{2} h_i{}^j{}_\ell \omega^\ell. \end{aligned} \tag{6.5}$$

Then

$$\begin{aligned} &\pi^i = \omega^i, \\ &dh_{ij} - h_{ik}\pi_j^k - h_{ij}\pi_i^k = 0. \end{aligned} \tag{6.6}$$

The other components of τ are tentatively supposed to have a form

$$\begin{aligned} \tau_i^0 &= M_{ij}\omega^j + L_{ij}\omega^j\,, \quad M_{ij} = M_{ji} \\ \tau_{n+1}^i &= h^{ij}M_{jk}\omega^k. \end{aligned} \tag{6.7}$$

Then π has values in $\mathfrak{h}$ relative to h. We can see

$$\Omega_0^\beta = \Omega_\alpha^{n+1} = \Omega_{n+1}^0 = 0. \tag{6.8}$$

These are because of symmetry of h_{ij}, h_{ijk}, L_{ij} and M_{ij}. In fact $\Omega_0^0 = 0$ is shown as follows:

$$\begin{aligned} \Omega_0^0 &= d\pi_0^0 - \pi_0^\alpha \wedge \pi_\alpha^0 \\ &= d\omega_0^0 - \omega_0^i \wedge (\omega_i^0 + L_{ij}\omega^j + M_{ij}\omega^j) \\ &= -\omega^i \wedge (L_{ij} + M_{ij})\omega^j \qquad (2.2) \\ &= 0 \qquad (\text{symmetry of } L_{ij} \text{ and } M_{ij}). \end{aligned}$$

The form Ω_i^j is by definition

$$\begin{aligned} \Omega_i^j &= d\pi_i^j - \pi_i^\alpha \wedge \pi_\alpha^j \\ &= d\tau_i^j - \tau_i^k \wedge \omega_k^j - \omega_i^k \wedge \tau_k^j - \tau_i^k \wedge \tau_k^j \\ &\quad - \omega_i^{n+1} \wedge \tau_{n+1}^j - (L_{ik} + M_{ik})\omega^k \wedge \omega^j. \end{aligned}$$

The definitions (2.16) and (2.21) show

$$\begin{aligned} -d(h_{ijk}\omega^k) = h_{ik\ell}\omega^\ell \wedge \omega_j^k + h_{jk\ell}\omega^\ell \wedge \omega_i^k + \\ + L_{jk}\omega_i^{n+1} \wedge \omega^k + L_{ik}\omega_j^{n+1} \wedge \omega^k, \end{aligned}$$

by which one can compute $d\tau_i^j$ and get

$$\begin{aligned} C_{ik\ell}^j = \frac{1}{4}(h_{ikm}h_\ell{}^{jm} - h_{i\ell m}h_k{}^{jm}) \\ + \Big(M_{i\ell} + \frac{1}{2}L_{i\ell}\Big)\delta_k^j - \Big(M_{ik} + \frac{1}{2}L_{ik}\Big)\delta_\ell^j \\ + \Big(M_k^j + \frac{1}{2}L_k^j\Big)h_{i\ell} - \Big(M_\ell^j + \frac{1}{2}L_\ell^j\Big)h_{ik}. \end{aligned}$$

Hence,

$$C^j_{ij\ell} = \frac{1}{4}K_{i\ell} + (n-2)\Big(M_{i\ell} + \frac{1}{2}L_{i\ell} + M^j_j h_{i\ell}\Big), \tag{6.9}$$

where we have put

$$K_{i\ell} = h_{ijk}h_\ell{}^{jk}. \tag{6.10}$$

Therefore the condition (6.4) is satisfied only when

$$M_{ij} = -\frac{1}{4(n-2)}K_{ij} - \frac{1}{2}L_{ij} + \frac{F}{8(n-2)(n-1)}h_{ij}. \tag{6.11}$$

Introduce a new invariant f_{ij} by

$$f_{ij} = -\frac{1}{4(n-2)}K_{ij} + \frac{F}{8(n-2)(n-1)}h_{ij}. \tag{6.12}$$

Then the definition (6.7) is written as

$$\begin{aligned} \tau^0_i &= \Big(f_{ij} + \frac{1}{2}L_{ij}\Big)\omega^j \\ \tau^j_{n+1} &= h^{j\ell}\Big(f_{\ell k} - \frac{1}{2}L_{\ell k}\Big)\omega^k. \end{aligned} \tag{6.7'}$$

Proposition 4.7. *Let $\widetilde{\tau}$ denote the form τ with respect to the frame $g\,e$. Then*

$$\widetilde{\tau} = g\tau g^{-1}.$$

Proof. By the definition (6.12) and by (2.9)

$$\widetilde{f}_{ij}\widetilde{\omega}^i\widetilde{\omega}^j = f_{ij}\omega^i\omega^j. \tag{6.13}$$

Using this identity and putting $\zeta_j = h_{jk\ell}c^\ell\omega^k$ for a moment, we can see following formulae from (2.27) and (2.30):

$$\widetilde{\tau}^0_i = \lambda^{-1}a^j_i\tau^0_j - \frac{1}{2}a^j_i\zeta_j$$

$$\widetilde{\tau}^j_{n+1} = \nu A^j_k \tau^k_{n+1} + \frac{1}{2} A^j_k \zeta^k$$
$$\widetilde{\tau}^0_{n+1} = \lambda^{-1}\nu\tau^0_{n+1} + \lambda^{-1}c^i + L_{ij}\omega^i - \frac{1}{2}(\lambda\nu)^{-1}c^j\zeta_j.$$

The definition of τ^j_i gives a formula

$$\widetilde{\tau}^j_i = a^k_i \tau^\ell_k A^j_\ell.$$

It is now immediate to see that these formulae together imply the result.

This proposition leads a transformation formula for π

(6.14) $$\widetilde{\pi} = dg\, g^{-1} + g\, \pi\, g^{-1} \quad \text{for} \quad g \in H.$$

Let now P be the principal H-bundle defined in §2.4. The formula (6.14) shows that the form π is defined on P and satisfies the conditions (1.4) in §1. The form τ also can be seen a basic 1-form on P by Proposition 4.7. Summarizing we have

Theorem 4.8. *Let M be a non-degenerate oriented hypersurface of type (p, q). Then the pair (p, π) defines a normal conformal connection of type (p, q).*

The form τ, whose components are defined by use of h_{ijk}, L_{ij} and γ_i, is as a whole an invariant of an immersed hypersurface. This plays an analogous role as the second fundamental form in the euclidean case. The *Gauss equation* which expresses the curvature tensor in terms of τ is given as follows.

Proposition 4.9. *Let* $\Omega^\beta_\alpha = \frac{1}{2}C^\beta_{\alpha k\ell}\pi^k\wedge\pi^\ell$ *and* $C^\beta_{\alpha k\ell}+C^\beta_{\alpha\ell k}=0$. *Then*

$$\begin{aligned}
(1)\quad C_{ijk\ell} &= h_{im}C^i_{jk\ell}\\
&= \frac{1}{4}(h_{i\ell m}h_{kj}{}^m - h_{ikm}h_{j\ell}{}^m)\\
&\qquad + \frac{1}{4(n-2)}(h_{jk}K_{i\ell} - h_{j\ell}K_{ik} + h_{i\ell}K_{jk} - h_{ik}K_{j\ell})\\
&\qquad + \frac{1}{4(n-1)(n-2)}(h_{ik}h_{j\ell} - h_{i\ell}h_{jk})F\\
(2)\quad C_{ijk} &:= C^0_{ijk}\\
&= f_{ik,j} - f_{ij,k} + \frac{1}{4}(h_{ij}{}^\ell L_{lk} - h_{ik}{}^\ell L_{\ell j})\\
(3)\quad \Omega^j_{n+1} &= h^{ji}\Omega^0_i
\end{aligned}$$

where

$$f_{ij,k}\pi^k = df_{ij} - f_{ik}\pi^k_j - f_{jk}\pi^k_i + 2f_{ij}\pi^0_0. \tag{6.15}$$

Proof. (1) follows directly from (6.8) and (6.11). (3) is obvious by definition. As for (2) recall

$$\Omega^0_i = d\tau^0_i - \tau^j_i\wedge\tau^0_j - \tau^0_i\wedge\pi^0_0 - \omega^{n+1}_i\wedge\tau^0_{n+1} - \omega^j_i\wedge\tau^0_j - \tau^j_i\wedge\omega^0_j.$$

First show

$$\begin{aligned}
d\tau^0_i = \frac{1}{2}\big(&h_{ijk}\omega^k\wedge\omega^j_{n+1} - 2\omega^{n+1}_i\wedge\omega^0_{n+1} + \omega^j_i\wedge L_{jk}\omega^k + L_{ik}\omega^k\wedge\omega^0_0\big)\\
&+ df_{ik}\wedge\omega^k - f_{ij}\omega^j_k\wedge\omega^k + f_{ik}\omega^0_0\wedge\omega^k,
\end{aligned}$$

then insert this to the above formula. Several cancellations by use of identities defining τ and L will prove (2).

We next define a higher order invariant: take covariant derivation of τ by

$$D\tau = d\tau - \tau\wedge\pi - \pi\wedge\tau. \tag{6.16}$$

It satisfies

$$D\widetilde{\tau} = g\,D\tau g^{-1} \tag{6.17}$$

for a frame change. In order to write down $D\tau$ explicitly, we will define

(6.18)
$$\begin{aligned}
h_{ijk,\ell}\pi^\ell &= dh_{ijk} - h_{\ell jk}\pi_i^\ell - h_{i\ell k}\pi_j^\ell - h_{ij\ell}\pi_k^\ell + h_{ijk}\pi_0^0,\\
L_{ij,k}\pi^k &= dL_{ij} - L_{kj}\pi_i^k - L_{ik}\pi_j^k + 2L_{ij}\pi_0^0 + h_{ij}{}^k\pi_k^0,\\
\gamma_{i,j}\pi^j &= d\gamma_i - \gamma_j\pi_i^j + 3\gamma_i\pi_0^0 - L_{ij}\pi_{n+1}^j.
\end{aligned}$$

Because of the transformation rule for τ, the right hand sides of these definitions are again basic forms. Then a calculation shows

$$D\tau = \begin{pmatrix} 0 & 0 & 0 \\ (D\tau)_i^0 & (D\tau)_i^j & 0 \\ (D\tau)_{n+1}^0 & (D\tau)_{n+1}^j & 0 \end{pmatrix} \tag{6.19}$$

where

(6.20)
$$\begin{aligned}
2h_{ik}(D\tau)_j^k &= \big(h_{ijk,\ell} + 2(f_{i\ell}h_{jk} + f_{jk}h_{i\ell}) + (L_{ik}h_{i\ell} + L_{jk}h_{i\ell})\big)\omega^\ell \wedge \omega^k,\\
(D\tau)_i^0 &= \Big(f_{ik,\ell} + \frac{1}{2}L_{ik,\ell} + h_{ik}\gamma_\ell\Big)\omega^\ell \wedge \omega^k,\\
h_{ik}(D\tau)_{n+1}^k &= \Big(f_{ik,\ell} - \frac{1}{2}L_{ik,\ell} + h_{ik}\gamma_\ell\Big)\omega^\ell \wedge \omega^k,\\
(D\tau)_{n+1}^0 &= \gamma_{k,\ell}\omega^\ell \wedge \omega^k.
\end{aligned}$$

Making use of this invariant $D\tau$, the curvature form is given by

$$\Omega = D\tau + \tau \wedge \tau. \tag{6.21}$$

We call this identity the Codazzi-Minardi equation.

Proposition 4.10. *The equation (6.21) is equivalent to the symmetry of h_{ijk} and L_{ij} and to the equations*

$$h_{ijk,\ell} - h_{ij\ell,k} = l_{i\ell}h_{jk} - L_{ik}h_{j\ell} + L_{j\ell}h_{ik} - L_{jk}h_{i\ell}, \tag{1}$$

$$L_{ij,k} - L_{ik,j} = h_{ij}{}^{\ell} f_{\ell k} - h_{ik}{}^{\ell} f_{\ell j} + 2(h_{ik}\gamma_j - h_{ij}\gamma_k) \tag{2}$$

$$\gamma_{i,j} - \gamma_{j,i} = L_{j\ell}f_i^{\ell} - L_{i\ell}f_j^{\ell}. \tag{3}$$

Proof. The $(0,0)$-th component of the right hand side of (6.21) is

$$\begin{aligned} d\tau_0^0 - \tau_0^\alpha \wedge \pi_\alpha^0 - \pi_0^\alpha \wedge \tau_\alpha^0 + \tau_0^\alpha \wedge \tau_\alpha^0 \\ = -\pi^i \wedge \tau_i^0 \\ = -\Big(f_{ij} + \frac{1}{2}L_{ij}\Big)\pi^i \wedge \pi^j. \end{aligned}$$

Similarly the $(n+1, n+1)$-, $(0,i)$- and $(j, n+1)$-th components are $-(f_{ij} - \frac{1}{2}L_{ij})\pi^i \wedge \pi^j$, $-\frac{1}{2}h^i{}_{jk}\pi^j \wedge \pi^k$ and $-\frac{1}{2}h_{ijk}\pi^i \wedge \pi^k$ respectively. So the vanishing of Ω_0^α and Ω_β^{n+1} implies the symmetry. The equations (1) to (3) follow from identities (6.20) and Proposition 4.9.

Corollary 4.11. (1) $L_{ij} = -\dfrac{1}{n}h_{ijk,}{}^{k}$.
(2) $\gamma_i = \dfrac{1}{2(n-1)}L_{ij}{}^{j} + \dfrac{1}{8(n-1)(n-2)}h_i{}^{jk}K_{jk}$.

Proof. Contracting (6.18) relative to h_{ij} and using the apolarity condition (2.18) and the trace condition (2.24), we get

$$h^{ij}h_{ijk,\ell} = h^{ij}L_{ij,k} = 0.$$

Then the contraction of (1) and (2) in Proposition 4.10 gives the result.

The fundamental theorem of a hypersurface in the projective case can now be stated as follows.

Theorem 4.12. *Let M be an n-dimensional manifold ($n \geqq 3$) with a normal conformal connection (p, π). Let τ be a $\mathfrak{g}_{n+2}$-valued basic 1-form on the structure bundle P whose components are given as illustrated in (6.5) and (6.7) with symmetric coefficients. Assume τ satisfies the covriant relations in Proposition 4.10 and the curvature tensor of π satisfies the relation in (6.8) and in Proposition 4.9. Then, for a given point p of M, there exists a neighborhood of p which can be embedded as a non-degenerate hypersurface in a projective space of dimension $n+1$ so that π and τ are the connection and the invariant induced by this embedding as described above. This embedding is unique up to a projective transformation.*

Proof. Given π and τ, define $\omega = \pi - \tau$. All assumptions together imply $d\omega = \omega \wedge \omega$. Therefore we can solve the equation $de = \omega e$ locally around p and we have the theorem. The ambiguity depends on initial conditions.

Remark. The theorem by Pick-Berwald in §5 follows from this theorem in case $n \geqq 3$. If $\varphi_3 = 0$, then $\Omega = 0$ and $\tau = 0$ by Proposition 4.9 and Corollary 4.11. Hence the uniqueness of the theorem says that if $\varphi_3 = 0$, then it is projectively equivalent to a quadratic hypersurface because $\tau = 0$ also for a quadratic hypersurface.

The *Bianchi identity* is as usual given by differentiating the equation of the curvature form.

$$d\Omega = \pi \wedge \Omega - \Omega \wedge \pi. \tag{6.22}$$

We here define covariant derivatives of the curvature tensor by

$$\begin{aligned} C_{ijk\ell,m}\pi^m &= dC_{ijk\ell} - C_{mjk\ell}\pi_i^m - C_{imk\ell}\pi_j^m \\ &\qquad - C_{ijm\ell}\pi_k^m - C_{ijkm}\pi_\ell^m + 2C_{ijk\ell}\pi_0^0, \\ C_{ijk,\ell}\pi^\ell &= dC_{ijk} - C_{\ell jk}\pi_i^\ell - C_{i\ell k}\pi_j^\ell - C_{ij\ell}\pi_k^\ell \\ &\qquad + 3C_{ijk}\pi_0^0 + C_{ijk}^\ell \pi_\ell^0. \end{aligned} \tag{6.23}$$

Proposition 4.13. *The Bianchi identity* (6.22) *implies*
(1) $\mathcal{S}(jk\ell)C_{ijk\ell} = 0, \quad \mathcal{S}(ijk)C_{ijk} = 0,$

(2) $\mathcal{S}(k\ell m)(C_{ijk\ell,m} - h_{im}C_{jk\ell} + h_{jk}C_{ik\ell}) = 0, \quad \mathcal{S}(jk\ell)C_{ijk,\ell} = 0,$ *where $\mathcal{S}(ijk)$ means an operation of taking a cyclic summation for i, j, k.*

Proof. The identity (6.22) for indices $(0,0)$ and $(0,i)$ implies

$$\pi^i \wedge \Omega_i^0 = \pi^j \wedge \Omega_j^i = 0.$$

This is (1). Components with idices $(n+1, n+1)$ and $(i, n+1)$ give the same result. The (i,j)-th component and the $(i,0)$-th component are respectively

$$d\Omega_i^j - \pi_i^k \wedge \Omega_k^j + \Omega_i^k \wedge \pi_k^j - \pi_i^{n+1} \wedge \Omega_{n+1}^j + \Omega_i^0 \wedge \pi^j = 0,$$
$$d\Omega_i^0 - \pi_i^j \wedge \Omega_j^0 + \Omega_i^j \wedge \pi_j^0 - \pi_i^{n+1} \wedge \Omega_{n+1}^0 + \Omega_i^0 \wedge \pi_0^0 = 0.$$

Then (6.23) implies (2).

By taking contractions of (1) and (2), we have

Corollary 4.14. (1) $(n-3)C_{ijk} = C_{\ell ijk},{}^{\ell}$,
(2) *when* $n \geqq 4$, $\Omega_i^j = 0$ *implies* $\Omega = 0$.

Remark. It is interesting to find a geometric characterization of a non-degenerate hypersurface which is conformally flat. In the euclidean case an elegant description of compact conformally flat hypersurfaces with respect to the induced Riemannian metric is known by U. Pinkall.

§7. Surfaces in $\mathbf{P}^3$

When the dimenson n of a hypersurface is two, we cannot follow the argument from (6.11) on. However a similar reasoning is possible relying on the next

Lemma 4.15. *Assume $n = 2$. Then* $K_{ij} = \dfrac{F}{2} h_{ij}$.

Proof. It is enough to show this identity when (h_{ij}) is diagonal because of the invariance of both sides. Let $h_{ij} = \varepsilon_i \delta_{ij}$, $\varepsilon = \pm 1$. Then the apolarity (2.18) leads to

$$\varepsilon_1 h_{11k} + \varepsilon_2 h_{22k} = 0\,, \quad k = 1, 2\,, \tag{7.1}$$

and consequently

$$F = 4(\varepsilon_1 (h_{111})^2 + \varepsilon_2 (h_{112})^2).$$

The tensor K_{ij} is by the definition (6.10)

$$K_{ij} = \sum\nolimits_{k,\ell} \varepsilon_k \varepsilon_\ell h_{ik\ell} h_{jk\ell}.$$

Hence, for example,

$$\begin{aligned} K_{11} &= (h_{111})^2 + 2\varepsilon_1\varepsilon_2 (h_{112})^2 + (h_{122})^2 \\ &= 2((h_{111})^2 + \varepsilon_1\varepsilon_2 (h_{112})^2) \\ &= \frac{F}{2}\varepsilon_1. \end{aligned}$$

Similarly $K_{12} = 0$ and $K_{22} = \dfrac{F}{2}\varepsilon_2$.

Recall the equation (6.9) for $n = 2$:

$$C^j_{ij\ell} = \frac{F}{8} h_{i\ell} + M^j_j h_{i\ell}.$$

So, if we put

$$f_{ij} = -\frac{F}{16} h_{ij} \quad \text{and} \quad M_{ij} = -\frac{1}{2} L_{ij} + f_{ij}, \tag{7.2}$$

then $C^j_{ij\ell} = 0$ and τ^0_i and τ^j_3 are defined by (6.7). With this definition Proposition 4.7 also holds, because the proof has depended solely on the transformation rules of h_{ijk}, L_{ij} and γ_i and on the fact that $f_{ij}\omega^i\omega^j$ is independent of frames (see (6.13)). The last property is now equivalent to the invariance of $F\varphi_2$, that is shown in Proposition 4.1. Hence we can define a form π by $\pi = \omega + \tau$. Then, similarly as Proposition 4.9, we have

Proposition 4.16. (1) $\Omega_i^j = 0$.
(2) $C_{ijk} = \frac{1}{4}(h_{ij}{}^{\ell}L_{\ell k} - h_{ik}{}^{\ell}L_{\ell j}) - \frac{1}{16}(F_j h_{ik} - F_k h_{ij})$,
where

$$F_k \pi^k = dF + 2F\pi_0^0. \tag{7.3}$$

(3) $\Omega_3^j = h^{ji}\Omega_i^0$.

The covariant derivations of h_{ijk}, L_{ij} and γ_i are also defined by (6.18). Then

$$F_i = 2h^{jk\ell}h_{jk\ell,i}. \tag{7.4}$$

The Codazzi-Minardi equation is

Proposition 4.17.
(1) $h_{ijk,\ell} - h_{ij\ell,k} = L_{i\ell}h_{jk} - L_{ik}h_{j\ell} + L_{j\ell}h_{ik} - L_{jk}h_{i\ell}$,
(2) $L_{ij,k} - L_{ik,j} = 2(h_{ik}\gamma_j - h_{ij}\gamma_k)$,
(3) $\gamma_{i,j} - \gamma_{j,i} = 0$.

Corollary 4.18. (1) $L_{ij} = -\frac{1}{2}h_{ijk,}{}^{k}$, (2) $\gamma_i = \frac{1}{2}L_{ij,}{}^{j}$.

Remark With these modifications we have a similar statement as in Theorem 4.12 for the case $n = 2$.

The case $n = 2$ has another feature that a complex structure is associated with the conformal structure. We will give some formulae in this point of view. To make notations simpler, assume

$$h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then each element of the group H with respect to h is written as

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ b_1 & \cos\theta & \sin\theta & 0 \\ b_2 & -\sin\theta & \cos\theta & 0 \\ \mu & c^1 & c^2 & \lambda^{-1} \end{pmatrix},$$

to which we associate a matrix in $GL_2(\mathbf{C})$

$$\begin{pmatrix} z & 0 \\ t & 1 \end{pmatrix}$$

where

$$\begin{aligned} z &= \lambda e^{-i\theta} \\ t &= \lambda(c^1 - i\,c^2). \end{aligned} \tag{7.5}$$

Then H is isomorphic to the affine transformation group of $\mathbf{C}^1$. Let

$$s = \lambda(b_1 + i\,b_2). \tag{7.6}$$

We have

$$\begin{aligned} s &= \bar{t}\,z \\ \mu &= \frac{1}{2|z|} t\,\bar{t}. \end{aligned} \tag{7.7}$$

Next, define complex invariants by

$$\begin{aligned} C &= h_{111} - i\,h_{222} \\ K &= L_{11} - i\,L_{12}, \\ \delta &= \gamma_1 - i\,\gamma_2. \end{aligned} \tag{7.8}$$

Then transformation rules are given by

$$\begin{aligned} \gamma^4 \widetilde{C} &= z^3 C \\ z^2 \widetilde{K} &= K + \bar{t} C \\ \gamma^2 z \widetilde{\delta} &= \delta + \bar{t} K - \frac{1}{2} \bar{t}^2 C. \end{aligned} \tag{7.9}$$

We define further

$$\begin{aligned} \xi &= \pi^1 + i\,\pi^2, \\ \tau_{(1)} &= \tau_1^1 - i\,\tau_1^2, \\ \tau_{(2)} &= \tau_1^0 + i\,\tau_2^0, \\ \tau_{(3)} &= \tau_3^1 + i\,\tau_3^2, \end{aligned} \tag{7.10}$$

then

$$\bar{\xi} = z\xi \tag{7.11}$$

and

$$\begin{aligned} \tau_{(1)} &= \frac{1}{2}C\xi, \\ \tau_{(2)} &= -\frac{F}{16}\xi + \frac{1}{2}\overline{K\xi}, \\ \tau_{(3)} &= -\frac{F}{16}\xi - \frac{1}{2}\overline{K\xi}. \end{aligned} \tag{7.12}$$

The Gauss equation is reformulated as follows:

$$\begin{aligned} d\xi &= \eta \wedge \xi \\ d\eta &= \xi \wedge \xi \\ d\zeta &= \zeta \wedge \eta + \Omega, \end{aligned} \tag{7.13}$$

where η, ζ, and Ω are defined by

$$\begin{aligned} \eta &= \pi_0^0 - i\,\pi_1^2 \\ \zeta &= \pi_1^0 - i\,\pi_2^0 \\ \Omega &= \Omega_1^0 - i\,\Omega_2^0. \end{aligned} \tag{7.14}$$

§8. Projective metric

When the Fubini-Pick invariant F vanishes nowhere, the projective metric $F\varphi_2$ becomes a true pseudo-Riemannian metric. We will summarize formulae in this case.

Recall first a transformation rule $\lambda\nu^{-1}\widetilde{F} = F$ (see Proposition 4.1.). So, if $F \neq 0$, then we can find a frame so that F is a constant, say 1. This means a restriction of frame change to $\lambda = \nu = 1$. Next recall ω_0^0 changes as $\widetilde{\omega}_0^0 = \omega_0^0 - b_i A_j^i \omega^j$, (2.9). Then we can assume $\omega_0^0 = 0$ and consequently a frame change is restricted to

$$g = \begin{pmatrix} 1 & & \\ & a & \\ & & 1 \end{pmatrix}.$$

This says that the quadratic form φ_2 defines globally a pseudo-Riemannian metric. The metric connection is given by a pair (ω^i, π^i_j), $\pi^i_j = \omega^i_j + \frac{1}{2}h^j_{ik}\omega^k$. The transformation rules by the above g are given by

$$
\begin{aligned}
\widetilde{\omega}^j &= \omega^k A^j_i \\
\widetilde{\omega}^{n+1}_i &= a^k_i \omega^{n+1}_k \\
\widetilde{\omega}^j_i &= da^k_i A^j_k + a^\ell_i \omega^k_\ell A^j_k \\
\widetilde{\omega}^0_i &= a^j_i \omega^0_j \\
\widetilde{\omega}^j_{n+1} &= \omega^k_{n+1} A^j_k \\
\widetilde{\omega}^0_{n+1} &= \omega^0_{n+1}.
\end{aligned}
\tag{8.1}
$$

We define new invariants $p = (p_{ij})$ and $q = (q_{ij})$ by

$$
\omega^0_i = p_{ij}\omega^j \quad \text{and} \quad h_{ij}\omega^j_{n+1} = q_{ij}\omega^j. \tag{8.2}
$$

These invariants together with $\gamma = (\gamma_j)$ defined by $\omega^0_{n+1} = -\gamma_j\omega^j$ transform as

$$
\widetilde{p} = a\, p\, {}^t a, \quad \widetilde{q} = a\, q\, {}^t a, \quad \gamma = a\gamma. \tag{8.3}
$$

The invariant L_{ij} is now equal to $q_{ij} - p_{ij}$. Symmetrically put

$$
U_{ij} = q_{ij} + p_{ij}. \tag{8.4}
$$

Then these satisfy the relations that will be called the Codazzi-Minardi equation.

Proposition 4.18. (1) *q_{ij}, p_{ij}, L_{ij}, and U_{ij} are symmetric tensors.*

(2) $$h_{ijk,\ell} - h_{ij\ell,k} = h_{ik}L_{j\ell} - h_{i\ell}L_{jk} + h_{jk}L_{i\ell} - h_{i\ell}L_{ik}$$

(3) $$\begin{aligned} p_{ij,k} - p_{ik,j} &= \frac{1}{2}(h_{ij}{}^{\ell}p_{\ell k} - h_{ik}{}^{\ell}p_{\ell j}) + (h_{ij}\gamma_k - h_{ik}\gamma_j), \\ q_{ij,k} - q_{ik,j} &= -\frac{1}{2}(h_{ij}{}^{\ell}q_{\ell k} - h_{ik}{}^{\ell}q_{\ell j}) - (h_{ij}\gamma_k - h_{ik}\gamma_j), \\ L_{ij,k} - L_{ik,j} &= -\frac{1}{2}(h_{ij}{}^{\ell}U_{\ell k} - h_{ik}{}^{\ell}U_{\ell j}) - 2(h_{ij}\gamma_k - h_{ik}\gamma_j), \\ U_{ij,k} - U_{ik,j} &= -\frac{1}{2}(h_{ij}{}^{\ell}L_{\ell k} - h_{ik}{}^{\ell}L_{\ell j}), \\ \gamma_{i,j} - \gamma_{j,i} &= p_{jk}L^k_i - p_{ik}L^k_j. \end{aligned}$$

(4) $$\begin{aligned} h_{ijk,}{}^{k} &= -n\, L_{ij}, \\ L^j_{ij} &= -\frac{1}{2}h_i{}^{jk}U_{jk} + 2(n-1)\gamma_i. \end{aligned}$$

These relations are consequences of the integrability condition $d\omega = \omega \wedge \omega$. We will not reproduce here. The Riemannian curvature tensor is given as follows:

Proposition 4.19. (1) $R_{ijk\ell} = \frac{1}{2}(U_{i\ell}h_{jk} - U_{ik}h_{j\ell} + U_{jk}h_{i\ell} - U_{j\ell}h_{ik}) + \frac{1}{4}(h_{jk}{}^{m}h_{i\ell m} - h_{ik}{}^{m}h_{j\ell m})$.
(2) *The Ricci tensor R_{ij} and the scalar curvature R is given by*

$$\begin{aligned} R_{ij} &= -\frac{1}{2}(n-2)U_{ij} - \frac{1}{2}\mathrm{Tr}(U)h_{ij} + \frac{1}{4}K_{ij}, \\ R &= -(n-1)\mathrm{Tr}(U) + \frac{1}{4}F. \end{aligned}$$

The last vector e_{n+1} of a frame is uniquely determined in the present case. It may be called the *projective normal*, cf. [BOL, vol. 2, p. 35]. The next proposition gives a relation connecting e_{n+1} and e_0; this is a projective analogue of the relation in the affine geometry (see [FL]).

Proposition 4.20. *Let Δ be the Laplacian of the metric h_{ij}. Then*

$$\Delta e_0 = n\, e_{n+1} + \mathrm{tr}(p)e_0.$$

Proof. Since $de_0 = \omega^i e_i$, the covariant derivation of e_0 is e_i. The derivation of e_i is

$$de_i - e_j\pi_i^j = \omega_i e_0 + \omega_i^{n+1} e_{n+1} - \frac{1}{2} h_{ik}{}^j \omega^k e_j.$$

Hence $e_{ij} = p_{ij} e_0 - \frac{1}{2} h_{ij}{}^k e_k + h_{ij} e_{n+1}$. Taking traces, we have the result.

5. Systems of linear differential equations and hypersurfaces

The purpose of this chapter is to apply the geometric formulation of a non-degenerate hypersurface to the study of a system of linear differential equations of n variables. Let $z\colon M^n \to \mathbf{P}^{n+1}$ be an immersion of an n-manifold. Choose local coordinates (x^i) on M and put $z_i = \partial z/\partial x^i$, $z_{ij} = \partial^2 z/\partial x^i \partial x^j, \ldots$ We assume vectors $\{z, z_1, \ldots, z_n, z_{1n}\}$ are linearly independent. Then z_{ij}'s are expressible by linear combinations of these vectors such as

$$z_{ij} = g_{ij} z_{1n} + \sum A_{ij}^k z_k + A_{ij}^0 z. \tag{0.1}$$

Here coefficients are scalar functions and symmetric with respect to subindices and

$$g_{1n} = 1\,, \quad A_{1n}^k = A_{1n}^0 = 0. \tag{0.2}$$

Since the immersion z is, as we have seen in the previous chapter, determined by the induced conformal connection and the invariant τ, the coefficients of (0.1) are expected to have relations with these geometric quantities. In fact, the system (0.1) can be written in a Pfaffian form, which is geometrically a differential equation satisfied by a projective frame. This will be made clear in §1 for $n \geqq 3$. The §2 deals with a special case where the image of z lies in a quadratic hypersurface. The case $n = 2$ is treated in §3. In §4 the notion of dual immersions and dual systems will be defined. This chapter is mostly based on [SY1] and [SY2].

§1. Systems of linear differential equations defining a hypersurface

Let $z\colon M^n \to \mathbf{P}^{n+1}$ be a non-degenerate immersion. Assume $n \geqq 3$. Let (x^i) be a local coordinate system and take a frame field with $\omega^i = dx^i$. Let Γ_{ik}^j be the Christoffel symbol of the tensor h_{ij} with respect to this coordinate system:

$$\Gamma_{ik}^j = \frac{1}{2} h^{j\ell}(h_{i\ell,k} + h_{k\ell,i} - h_{ik,\ell})\,, \quad dh_{i\ell} = h_{i\ell,k} dx^k.$$

Then the conformal connection form is $\pi_i^j = \Gamma_{ik}^j \omega^k$ by the requirement of (6.6) of Chapter 4. Let $R_{ik\ell}^j$ denote the Riemannian curvature tensor:

$$d\pi_i^j - \pi_i^k \wedge \pi_k^j = \frac{1}{2} R_{ik\ell}^j \omega^k \wedge \omega^\ell.$$

The Ricci and the scalar curvatures are denoted by R_{ij} and by R respectively:

$$R_{ij} = R_{i\ell j}^\ell\,, \quad R = h^{ij} R_{ij}.$$

Put

$$\pi_i^0 = -S_{ik}\omega^k,$$

then the conformal curvature tensor $C_{ik\ell}^j$ is given by

$$C_{ik\ell}^j = R_{ik\ell}^j - S_{ik}\delta_\ell^j + S_{i\ell}\delta_k^j - h_{ik}h^{jm}S_{m\ell} + h_{i\ell}h^{jm}S_{mk}.$$

The requirement (6.4) of Chapter 4 shows

$$S_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{R}{2(n-1)}h_{ij}\right). \tag{1.1}$$

The tensor S_{ij} is called the *Schouten tensor* relative to the tensor h_{ij}. Now we have

$$\pi = \begin{pmatrix} 0 & \pi^j & 0 \\ \pi_i^0 & \pi_i^j & \pi_i^{n+1} \\ 0 & \pi_{n+1}^j & 0 \end{pmatrix} = \begin{pmatrix} 0 & dx^j & 0 \\ -S_{ik}dx^k & \Gamma_{ik}^j dx^k & h_{ij}dx^k \\ 0 & -h^{j\ell}S_{\ell k}dx^k & 0 \end{pmatrix}.$$

We continue the argument in the introduction. The third-order derivatives of z are also linear combinations of $\{z, z_1, \ldots, z_n, z_{1n}\}$, among which we need

$$z_{1jn} = G_j z_{1n} + \sum B_j^k z_k + B_j^0 z\,, \quad 1 \le j \le n, \tag{1.2}$$

where G_j, B_j^k and B_j^0 are certain scalar functions. Define a function θ by

$$e^\theta = |\det(z, z_1, \ldots, z_n, z_{1n})|. \tag{1.3}$$

We call θ the *normalizing factor* of the system (0.1). Define a frame field $e = (e_0, \ldots, e_{n+1})$ by

$$e_0 = z\,, \quad e_i = z_i\,, \quad e_{n+1} = e^{-\theta} z_{1n}. \tag{1.4}$$

Then the system (0.1) combined with (1.2) is written in a Pfaffian form

$$d\,e = \omega e \tag{1.5}$$

where

$$\omega = \begin{pmatrix} 0 & dx^j & 0 \\ A^0_{ik}\, dx^k & A^j_{ik}\, dx^k & e^\theta g_{ik}\, dx^k \\ e^{-\theta} B^0_k\, dx^k & e^{-\theta} B^j_k\, dx^k & (G_k - \theta_k)\, dx^k \end{pmatrix}.$$

The induced conformal tensor h_{ij} is by definition $e^\theta g_{ij}$. We apply a process of normalization to the frame e: to find a transformation g so that, with respect to the frame $e' = g\,e$,

$$|\det h'_{ij}| = 1\,, \quad \omega'^0_0 + \omega'^{n+1}_{n+1} = 0 \quad \text{and} \quad L' = 0, \tag{1.6}$$

where

$$\omega' = dg\, g^{-1} + g\omega g^{-1}.$$

Since ω' is decomposed into the sum of the connection form π associated with h'_{ij} and the invariant form τ of the embedding z, we have

$$\omega = dh\, h^{-1} + h(\pi - \tau)h^{-1} \tag{1.7}$$

for $h = g^{-1}$. This equality shows that ω is represented by geometric invariants in the right side. And, consequently, the coefficients of (0.1) is written in terms of invariants of the hypersurface z. Assume

$$|\det(e^\theta g_{ij})| = 1 \tag{1.8}$$

and take a transformation g of the form

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_n & 0 \\ \mu & c & 1 \end{pmatrix}.$$

Then it will be seen that the condition (0.2) determines g uniquely.

Proposition 5.1. *Assume* (1.8). *Then the coefficients A_{ik} are given by*

$$A^j_{ik} = (\Gamma^j_{ik} - g_{ik}\Gamma^j_{1n}) - \frac{1}{2}({h_{ik}}^j g_{ik} {h_{1n}}^j)$$
$$A^0_{ik} = -(S_{ik} - g_{ik}S_{1n}) - \Big(f_{ik} + \frac{1}{2}L_{ik} - g_{ik}\Big(f_{1n} + \frac{1}{2}L_{1n}\Big)\Big),$$

where Γ^j_{ik} and S_{ik} are the Christoffel symbol and the Schouten tensor of $e^\theta g_{ij}$. The h^j_{ik}, L_{ik} and f_{ik} are components of the form τ with respect to the frame e'.

Proof. By the assumption $h'_{ij} = h_{ij} = e^\theta g_{ij}$, the form τ has the following form

$$\tau = \begin{pmatrix} 0 & 0 & 0 \\ (f_{ik} + \frac{1}{2}L_{ik})\,dx^k & \frac{1}{2}h^j_{ik}\,dx^k & 0 \\ -\omega^0_{n+1} & h^{j\ell}(f_{\ell k} - \frac{1}{2}L_{\ell k})\,dx^k & 0 \end{pmatrix}.$$

We see $\pi^0_0 = \pi^{n+1}_{n+1} = 0$ because ${\omega'}^0_0 = \omega^0_0 = 0$. Then (1.7) turns out to be

$$\omega = \begin{pmatrix} 0 & \pi^j & 0 \\ \pi^0_i - \tau^0_i + \mu\pi^{n+1}_i & \pi^j_i - \tau^j_i + c^j\pi^{n+1}_i & \pi^{n+1}_i \\ \begin{matrix} -\,d\mu - \tau^0_{n+1} \\ -\,c^i(\pi^0_i - \tau^0_i + \mu\pi^{n+1}_i) \end{matrix} & \begin{matrix} \pi^j_{n+1} - \tau^j_{n+1} - dc^j - \mu\pi^j \\ -c^i(\pi^j_i - \tau^j_i + c^j\pi^{n+1}_i) \end{matrix} & c^i\pi^{n+1}_i \end{pmatrix}.$$

Hence

$$A^j_{ik} = \Gamma^j_{ik} - \frac{1}{2}{h_{ik}}^j + c^j h_{ik}$$
$$A^0_{ik} = -S_{ik} - \Big(f_{ik} + \frac{1}{2}L_{ik}\Big) + \mu h_{ik}.$$

The requirement (0.2) shows

$$c^j = -e^{-\theta}\Bigl(\Gamma^j_{1n} - \frac{1}{2}h^j_{1n}\Bigr), \quad \mu = e^{-\theta}\Bigl(S_{1n} + f_{1n} + \frac{1}{2}L_{1n}\Bigr).$$

Hence we have the formula.

If the condition (1.8) is not satisfied, then by multiplying a suitable function to the unknown z, one can transform the system (0.1), without changing the hypersurface nor the coefficients g_{ij}, into a system satisfying this condition. Then the other coefficients are obtained by the following lemma, which is known as the transformation rule of connection forms under a conformal change of metric (cf. [G]).

Lemma 5.2. *If the unknown z is transformed into a new unknown w by $w = e^{\alpha}z$, then the system (0.1) changes into*

$$w_{ik} = g_{ik}w_{1n} + P^j_{ik}w_j + P^0_{ik}w, \tag{1.9}$$

where

(1.10)
$$\begin{aligned} P^j_{ik} &= A^j_{ik} + \alpha_i\delta^j_k + \alpha_k\delta^j_i - g_{ik}(\alpha_1\delta^j_n + \alpha_n\delta^j_1) \\ P^0_{ik} &= A^0_{ik} + (\alpha_{ik} - \alpha_i\alpha_k) + A^j_{ik}\alpha_j - g_{ik}(\alpha_{1n} - \alpha_1\alpha_n). \end{aligned}$$

The new normalization factor is $e^{\theta+(n+2)\alpha}$.

§2. Systems of linear differential equations defining maps into quadratic hypersurfaces

We consider in this section an n-manifold with a conformally flat structure. Such a manifold has a mapping called the developing map, which is defined on the universal cover of M into a quadric. Since a quadric is embedded in $\mathbf{P}^{n+1}$ as a quadratic hypersurface, this map defines a (multi-valued) immersion of M into $\mathbf{P}^{n+1}$.

Definition. The system (0.1) is said to satisfy the *quadric condition* if the image of z is contained in a quadratic hypersurface. In view of the theorem in §5, Chapter 4, this is equivalent to say the cubic form of the immersed hypersurface by z vanishes identically. Then the invariant form τ vanishes and the connection form π is flat. Therefore we have, as a corollary of Proposition 5.1,

Theorem 5.3. *Assume the system* (0.1) *satisfy the condition* (1.8) *and the quadric condition. Then the coefficients A_{ij} are expressed as rational functions of g_{ij} and of their derivatives:*

$$A^j_{ik} = \Gamma^j_{ik} - g_{ik}\Gamma^j_{1n}, \qquad (2.1)$$
$$A^0_{ik} = -S_{ik} + g_{ik}S_{1n}.$$

A converse of this theorem holds.

Theorem 5.4. *Assume $n \geqq 3$. Let $g_{ij}dx^idx^j (g_{1n} = 1)$ be a non-degenerate symmetric tensor which is conformally flat. Define θ so that $|\det(e^\theta g_{ij})| = 1$ and define A^k_{ij} and A^0_{ij} by* (2.1) *with respect to the tensor $e^\theta g_{ij}$. Then the number of independent solutions of the system (called the rank) $z_{ij} = g_{ij}z_{1n} + A^k_{ij}z_k + A^0_{ij}z$ is $n+2$ and this system satisfies the quadric condition. Its normalization factor is e^θ.*

Proof. Put $h_{ij} = e^\theta g_{ij}$. Since by assumption h_{ij} is conformally flat, the associated normal conformal connection π is integrable. Apply Theorem 4.12 by putting $\tau = 0$. The Gauss and the Codazzi-Minardi equations are trivially satisfied so that there is an unique immersion z of x-space into $\mathbf{P}^{n+1}$ such that the induced conformal tensor is h_{ij} and the invariant form is zero. Let

$$z_{ij} = g'_{ij}z_{1n} + A'^k_{ij}z_k + A'^0_{ij} \qquad (2.2)$$

be the system with z as solutions and with the normalizing factor $e^{\theta'}$. The induced conformal metric is $e^{\theta'}g'_{ij}$, which coincides with $e^\theta g_{ij}$. Hence $e^\theta = e^{\theta'}$ and $g_{ij} = g'_{ij}$. Then Theorem 5.3 shows the conclusion.

This theorem can be formulated in a more symmetric way.

Theorem 5.4′. *Assume $n \geqq 3$. Let $\sigma_{ij}dx^i dx^j$ be a non-degenerate symmetric tensor which is conformally flat. Put*

$$Z_{ij} = z_{ij} - \Gamma^k_{ij} z_k + \frac{1}{n-2} R_{ij} z.$$

Then the system

$$\sigma_{ij} Z_{k\ell} = \sigma_{k\ell} Z_{ij} \tag{2.3}$$

is of rank $n+2$ and satisfies the quadric condition. Here Γ^k_{ij} and R_{ij} stand for the Christoffel symbol and the Ricci tensor with respect to σ_{ij}.

Proof. Assume first $e^\eta := \sigma_{1n} \neq 0$ and put $g_{ij} = e^{-\eta}\sigma_{ij}$ and $|\det g_{ij}| = e^{-2n\rho}$. Define $h_{ij} = e^{2\rho} g_{ij}$ so that $|\det h_{ij}| = 1$. We have only to combine Theorem 5.4 and Lemma 5.2 as well as the transformation formulae of the Christoffel symbol and the Ricci tensor for h_{ij} into those for σ_{ij}:

$$\begin{aligned} \Gamma^j_{ik}(\sigma) &= \Gamma^j_{ik}(h) + \alpha_i \delta^j_k + \alpha_k \delta^j_i - h_{ik} h^{jp} \alpha_p \\ R_{ik}(\sigma) &= R_{ik}(h) - (n-2)(\alpha_{ik} - \alpha_i \alpha_k - \alpha_j \Gamma^j_{ik}(h)) \\ &\qquad - \{\Delta_h \alpha + (n-2) h^{j\ell} \alpha_j \alpha_\ell\} h_{ik} \end{aligned}$$

where $\alpha = \frac{1}{2}\eta + \rho$ and Δ_h is the laplacian of h_{ij} (see [G, p. 115]). When $\sigma_{ij} = 0$ for $i \neq j$, change coordinates by $y_1 = x_1 + x_n$ and $y_i = x_i$ for $i \geqq 2$. Define S_{ij} by $S_{ij} dy^i dy^j = \sigma_{ij} dx^i dx^j$. Then $S_{1n} \neq 0$ and we can apply the above case. In fact, let γ^k_{ij} and r_{ij} stand for the Christoffel symbol and Ricci tensor for (y^i, S_{ij}). Put $W_{ij} = \partial^2 z/\partial y^i \partial y^j - \gamma^k_{ij} \partial z/\partial x^k + r_{ij}/(n-2)z$. Then, since the transformation is linear, we can see easily the identity

$$S_{ij} W_{k\ell} - S_{k\ell} W_{ij} = \frac{\partial x^p}{\partial y^i}\frac{\partial x^q}{\partial y^j}\frac{\partial x^r}{\partial y^k}\frac{\partial x^s}{\partial y^\ell}(\sigma_{ij} Z_{k\ell} - \sigma_{k\ell} Z_{ij}).$$

This completes the proof.

Example. Let $M^n = \mathbf{R}^n$ with the standard metric $\sum (dx^i)^2$. The corresponding system (2.3) is

$$z_{ij} = 0 \quad i \neq j, \quad z_{ii} = z_{jj}.$$

A set of independent solutions is $\{1, x^1, \dots, x^n, (x^1)^2 + \cdots + (x^n)^2\}$ which defines a mapping from $\mathbf{R}^n$ into a paraboloid. This mapping is projectively equivalent to the inverse map of the stereographic projection from $S^n - \{\infty\}$ into $\mathbf{R}^n$.

Example. The following system is an example satisfying the quadric condition:

$$
\begin{aligned}
&w_{ii} + \left(\frac{1}{x^i} + \frac{1}{x^i - 1} + \frac{1}{2}\left(\frac{1}{x^i - x^j} + \frac{1}{x^i - x^k}\right)\right) w_i \\
&- \frac{x^j(x^j - 1)}{2x^i(x^i - 1)(x^i - x^j)} w_j - \frac{x^k(x^k - 1)}{2x^j(x^i - 1)(x^i - x^k)} w_k + \frac{1}{x^i(x^i - 1)} w = 0, \\
&(x^k - x^i)x^j(x^j - 1)\left(\begin{aligned} 2w_{ij} &+ \left(\frac{1}{x^j - x^k} + \frac{1}{x^j - x^i}\right) w_i \\ &+ \left(\frac{1}{x^i - x^k} + \frac{1}{x^i - x^j}\right) w_j \\ &+ \frac{1}{(x^k - x^i)(x^k - x^j)} w \end{aligned}\right) \\
&\quad = (x^i - x^j)x^k(x^k - 1)\left(\begin{aligned} 2w_{ik} &+ \left(\frac{1}{x^k - x^j} + \frac{1}{x^k - x^i}\right) w_i \\ &+ \left(\frac{1}{x^i - x^k} + \frac{1}{x^i - x^j}\right) w_k \\ &+ \frac{1}{(x^j - x^i)(x^j - x^k)} w \end{aligned}\right)
\end{aligned}
$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$, and $w_i = \partial w / \partial x^i$, $w_{ij} = \partial^2 w / \partial x^i \partial x^j$.

This system is defined on $\mathbf{P}^3$ and x^i are affine coordinates. A set of independent solutions defines a multi-valued mapping into the Siegel upper half space $\mathcal{H}_2$ of degree 2, contained in its compact dual, a quadratic hypersurface of dimension 3. The map from $\mathcal{H}_2$ onto $\mathbf{P}^3$ is given by taking the quotient with respect to the Siegel modular group. For another example when $n = 4$ refer [MSY].

§3. Systems of linear differential equations in two variables

E. J. Wilczynski has studied the surfaces in $\mathbf{P}^3$ as the integrating surface of a system of linear differential equations:

$$\left\{\begin{aligned} z_{xx} &= \ell\, z_{xy} + a\, z_x + b\, z_y + p\, z \\ z_{yy} &= m\, z_{xy} + c\, z_x + d\, z_y + q\, z, \end{aligned}\right. \tag{3.1}$$

(x, y) being local coordinates. We here summarize some basic facts about this system in our point of view.

Let us start with an example which shows the difference between the case $n = 2$ and the case $n \geqq 3$. Consider a system

$$\left\{\begin{aligned} z_{xx} &= p\, z \\ z_{yy} &= q\, z, \end{aligned}\right. \tag{3.2}$$

which is assumed to have four independent solutions. The integrability condition of this system is

$$p_y = q_x = 0.$$

Hence both equations of (3.2) are ordinary differential equations with respect to x and with respect to y respectively. So both have two independent solutions; say z_1, z_2 and w_1, w_2. Then $\{\, z_i w_j;\ 1 \le i, j \le 2 \,\}$ is a set of fundamental solutions of (3.2) and the map

$$(x, y) \mapsto (z_1 w_1, z_1 w_2, z_2 w_1, z_2 w_2) \in \mathbf{P}^3$$

defines an immersion into a quadric defined by $X^1 X^4 = X^2 X^3$, (X^i) denoting homogeneous coordinates. In the terminology of the previous section the system (3.2) satisfies the quadric condition. However the coefficients p and q are arbitrary, contrary to the case $n \geqq 3$ where all coefficients are determined by the conformal tensor.

Assume now (3.1) has four independent solutions which define an immersion z of a surface. Define a frame field $e = (e_0, e_1, e_2, e_3)$ by

$$e_0 = z\,, \quad e_1 = z_x\,, \quad e_2 = z_y\,, \quad e_3 = e^{-\theta} z_{xy}$$

where θ is defined by

$$e^{\theta} = |\det(z, z_x, z_y, z_{xy})|. \tag{3.3}$$

Then the coframe ω is computed to be

$$\omega = \begin{pmatrix} 0 & dx & dy & 0 \\ p\,dx & a\,dx & b\,dx & e^{\theta}(\ell dx + dy) \\ q\,dy & c\,dy & d\,dy & e^{\theta}\left(dx + m\,dy\right) \\ e^{-\theta}\begin{pmatrix} B^0 dx \\ + C^0 dy \end{pmatrix} & e^{-\theta}\begin{pmatrix} B^1 dx \\ + C^1 dy \end{pmatrix} & e^{-\theta}\begin{pmatrix} B^2 dx \\ + C^2 dy \end{pmatrix} & -a\,dx - d\,dy \end{pmatrix}. \tag{3.4}$$

Hence, the fundamental form φ_2 is given by

$$\varphi_2 = \ell\,dx^2 + 2dx\,dy + m\,dy^2. \tag{3.5}$$

The non-degeneracy of the associated surface is equivalent to

$$1 - \ell\,m \neq 0.$$

Coefficients B^i, C^i are given as follows:

$$\begin{aligned}
B^0 &= \{p_y + bq + \ell(q_x + cp)\}/(1 - \ell m) \\
B^1 &= (A + \ell q)/(1 - \ell m) \\
B^2 &= (B + p)/(1 - \ell m) \\
B^3 &= \{\ell_y + a + bm + \ell(m_x + d + c\ell)\}/(1 - \ell m) \\
C^0 &= \{q_x + cp + m(p_y + bq)\}/(1 - \ell m) \\
C^1 &= (C + q)/(1 - \ell m) \\
C^2 &= (D + mp)/(1 - \ell m) \\
C^3 &= \{m_x + d + c\ell + m(\ell_y + a + bm)\}/(1 - \ell m) \\
A &= a_y + bc + \ell(c_x + ac) \qquad B = b_y + bd + \ell(dx + bc) \\
C &= c_x + ac + m(a_y + bc) \qquad D = d_x + bc + m(b_y + bd).
\end{aligned} \tag{3.6}$$

Taking derivation of (3.3), we see

$$\theta_x = a + B^3 \qquad (3.7)$$
$$\theta_y = d + C^3.$$

The integrability condition $d\omega - \omega \wedge \omega = 0$ consists of above equations (3.6) and the following four equations

(3.8)
$$\begin{aligned}
&(a + B^3)_x = (d + C^3)_y \\
&\ell q_y - 2q_x - m\,p_y - (\ell\xi_y - \xi_x - 2\ell_y)q = R^1 \\
&m\,p_x - 2p_y - \ell q_x - (m\xi_x - \xi_y - 2m_x)p = R^2 \\
&p_{yy} - q_{xx} - m\,p_{xy} + \ell q_{xy} \\
&\quad = c\,p_x - b\,q_y + (d + 2m_x + \xi_y - m\xi_x)p_y - (a + 2\ell_y + \xi_x - \ell\xi_y)q_x \\
&\qquad + \{m\,a_y + 2c_x - 2c\,\ell_y - \ell\,c_y - c(\xi_x - \ell\xi_y)\}p \\
&\qquad - \{\ell\,dx + 2b_y - 2b\,m_x - m\,b_x - b(\xi_y - m\,\xi_x)\}q,
\end{aligned}$$

where

$$\begin{aligned}
R^1 &= (C^3 + \xi_y)A - (B^3 - a + \xi_x)C - cB + C_x - A_y \\
R^2 &= (B^3 + \xi_x)D - (C^3 - d + \xi_y)B - bC + B_y - D_x \\
\xi &= \log|1 - \ell\,m|.
\end{aligned}$$

We next consider the case $\ell\,m - 1 < 0$, i.e. φ_2 is of indefinite type. Recall the definition of asymptotic curves defined in §2, Chapter 3. In the present case such a curve $(x(t), y(t))$ is defined by

$$\ell\left(\frac{dx}{dx}\right)^2 + 2\frac{dx}{dt}\frac{dy}{dt} + m\left(\frac{dy}{dt}\right)^2 = 0.$$

Since $\ell m - 1 < 0$, we have a set of two curves through each point. Therefore, if we take these curves as coordinate curves, then $\varphi_2 = 2\,dx\,dy$, i.e. $\ell = m = 0$. Now the system (3.1) is simplified to

$$\begin{cases} z_{xx} = a\,z_x + b\,z_y + p\,z \\ z_{yy} = c\,z_x + d\,z_y + q\,z. \end{cases}$$

For this system, (3.7) implies

$$\theta_x = 2a\,, \quad \theta_y = 2d.$$

So, by putting $w = e^{\theta/2} z$, this system is further simplified to

$$\begin{cases} w_{xx} = b\, w_y + p\, w \\ w_{yy} = c\, w_x + q\, w, \end{cases}$$

with new coefficients. This is called a *canonical form* of (3.1) and intensively studied by Wilczynski and others. Define a new frame $e = (e_0, e_1, e_2, e_3)$ by

$$e_0 = z\,, \quad e_1 = z_x\,, \quad e_2 = z_y\,, \quad e_3 = z_{xy} - \frac{1}{2} bc\, z.$$

Since the coframe ω has the form

$$\omega = \begin{pmatrix} 0 & dx & dy & 0 \\ p\,dx + \frac{1}{2}bc\,dy & 0 & b\,dx & dy \\ q\,dy + \frac{1}{2}bc\,dx & c\,dy & 0 & dx \\ \begin{array}{l}(bq + p_y)\,dx \\ \;+ (cp + q_x)\,dy - \frac{1}{2}d(bc)\end{array} & \begin{array}{l}\frac{1}{2}bc\,dx \\ \;+ (q + c_x)\,dy\end{array} & \begin{array}{l}(p + b_y)\,dx \\ \;+\frac{1}{2}bc\,dy\end{array} & 0 \end{pmatrix}.$$

This frame is normalized in the sense of Proposition 4.2.

Proposition 5.5. (1) *The cubic form h_{ijk} is given by*

$$h_{111} = -2b\,, \quad h_{222} = -2c, \quad h_{112} = h_{122} = 0. \tag{3.9}$$

In particular, $F = 8bc$. (2) *The quantities L_{ij} and γ_j with respect to the above frame e are given by*

$$(L_{ij}) = \begin{pmatrix} b_y & 0 \\ 0 & c_x \end{pmatrix}, \tag{3.10}$$

$$\gamma_1 = \frac{1}{2}(bc)_x - bq - p_y \tag{3.11}$$
$$\gamma_2 = \frac{1}{2}(bc)_y - cp - q_x.$$

Proof. Recall $h = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The apolarity condition (2.18) of Chapter 4 says $0 = h^{ij}h_{ijk} = 2h_{12k}$. The definition of the cubic form gives

$$h_{11k}\omega^k = -2h_{12}\omega_1^2 = -2b\omega^1$$
$$h_{22k}\omega^k = -2h_{21}\omega_2^1 = -2c\omega^2.$$

So we have seen (1). Since $L_{ij}\omega^i = h_{ij}\omega_3^j - \omega_i^0$,

$$L_{ij}\omega^j = \omega_3^2 - \omega_1^0 = b_y\omega^1$$
$$L_{2j}\omega^j = \omega_3^1 - \omega_2^0 = c_x\omega^2.$$

Similarly from the definition $\omega_3^0 = -\lambda_j\omega^j$ we have (2).

The integrability condition (3.8) simplifies to

$$\begin{aligned} 2p_y &= (b\,c)_x + b\,c_x - b_{yy} \\ 2q_x &= (b\,c)_y + c\,b_y - c_{xx} \\ p_{yy} - q_{xx} &= (c\,p)_x - (b\,q)_y + p\,c_x - q\,b_y. \end{aligned} \tag{3.12}$$

We will conclude this section by giving a characterization of non-degenerate ruled surfaces in terms of the Fubini-Pick invariant. Recall that a ruled surface is defined by a map

$$(x, y) \mapsto z(x, y) = u(x) + y\,v(x),$$

where $u(x)$ and $v(x)$ are generating curves in $\mathbf{P}^3$ (§1, Chapter 3). We have seen that curves u and v and local coordinates (x, y) are so chosen that they satisfy a system of ordinary differential equations

$$\begin{cases} u_{xx} = \alpha u + \beta v \\ v_{xx} = \gamma u + \delta v. \end{cases}$$

It is easy to see that z satisfies the equations

$$
\left\{
\begin{aligned}
z_{xx} &= (\alpha + \gamma y)z + \{\beta + (\delta - \alpha)y - \gamma\, y^2\} z_y \\
z_{yy} &= 0,
\end{aligned}
\right.
\tag{3.13}
$$

which is of canonical form. The Fubini-Pick invariant vanishes. Conversely, let $M \subset \mathbf{P}^3$ be a non-degenerate surface of indefinite type with $F = 0$. Then the associated system is written as

$$
\left\{
\begin{aligned}
z_{xx} &= b\, z_y + p\, z \\
z_{yy} &= \qquad\quad q\, z
\end{aligned}
\right.
$$

for some coordinates (x, y). The integrability condition is

$$
2p_y = -b_{yy}\,, \quad q_x = 0 \quad p_{yy} = -(b\, q)_y - q\, b_y; \tag{3.14}
$$

see (3.12). Let φ be a non-zero solution of the second equation and define a new coordinate y' and a new unknown variable w by

$$
y' = \int \varphi^{-2} dy\,, \quad w = \varphi^{-1} z.
$$

Then a calculation shows

$$
\left\{
\begin{aligned}
w_{xx} &= b\, w_{y'} + p' w \\
w_{y'y'} &= 0.
\end{aligned}
\right.
\tag{3.15}
$$

Again from (3.14), we can see

$$
\begin{aligned}
p' &= \alpha + \gamma y' \\
b &= -\gamma {y'}^2 + \delta' y' + \beta
\end{aligned}
$$

for some functions α, β, γ and δ' of x. So the system (3.15) is defining a ruled surface, see (3.13). We have proved

Proposition 5.6. *A non-degenerate surface of indefinite type in* $\mathbf{P}^3$ *is a ruled surface if and only if the Fubini-Pick invariant vanishes.*

§4. Dual immersions and dual systems.

Let us recall the situation in Chapter 4: M is a non-degenerate immersed hypersurface and $e = (e_0, \ldots, e_{n+1})$ is a projective frame field satisfying a system of equations

$$de = \omega\, e. \tag{4.1}$$

The components e_α are vectors in $V = \mathbf{R}^{n+2}$. The first vector e_0 has a special meaning that it gives the immersion. Define E^α by

$$E^\alpha = (-1)^\alpha e_0 \wedge \cdots \wedge \check{e}_\alpha \wedge \cdots \wedge e_{n+1}, \tag{4.2}$$

which are vectors in the space $\bigwedge^{n+1} V$. If we identify $\bigwedge^{n+2} V$ with $\mathbf{R}$, then $\bigwedge^{n+1} V$ is a dual space of V and by a canonical pairing $\langle\ ,\ \rangle$ we have

$$\langle e_\alpha, E^\beta \rangle = \delta_\alpha^\beta. \tag{4.3}$$

Notice that the vector E^{n+1} is determined up to a non-zero multiple independently of the choice of frames and hence it defines a mapping from M to $\mathbf{P}^{n+1*}$, the dual of $\mathbf{P}^{n+1}$. If this is an immersion, we call E^{n+1} the *dual* immersion. The set $E = \{E^0, E^1, \ldots, E^{n+1}\}$ is a frame field along E^{n+1}. From (4.1), we can see that E satisfies

$$dE = -E\omega \tag{4.4}$$

(here E is considered as a row vector of E^α). In fact

$$\begin{aligned}
dE^\alpha &= \sum_\beta (-1)^\alpha e_0 \wedge \cdots \wedge de_\beta \wedge \cdots \wedge \hat{e}_\alpha \wedge \cdots \wedge e_{n+1} \\
&= \sum_{\beta,\gamma} (-1)^\alpha \omega^\gamma_\beta e_0 \wedge \cdots \wedge \overset{\beta}{\smile}{\!\!\!\!e}^{\,\beta}_{\gamma} \wedge \cdots \wedge \hat{e}_\alpha \wedge \cdots \wedge e_{n+1} \\
&= \Big(\sum_{\beta\neq\alpha} \omega^\beta_\beta\Big) E^\alpha + \sum_{\beta\neq\alpha} (-1)^\alpha \omega^\alpha_\beta e_0 \wedge \cdots \wedge \overset{\beta}{\smile}{\!\!\!\!e}^{\,\beta}_{\alpha} \wedge \cdots \wedge \hat{e}_\alpha \wedge \cdots \wedge e_{n+1} \\
&= \Big(\sum_{\beta\neq\alpha} \omega^\beta_\beta\Big) E^\alpha + \sum_{\beta\neq\alpha} (-1)^{\beta+1} \omega^\alpha_\beta e_0 \wedge \cdots \wedge \hat{e}_\beta \wedge \cdots \wedge e_{n+1} \\
&= -\omega^\alpha_\beta E^\beta \qquad (\text{note} \quad \omega^\alpha_\alpha = 0).
\end{aligned}$$

If we define a column vector $\check{E}$ by

$$\check{E} = (E^{n+1}, E^1, \ldots, E^n, E^0),$$

then

$$d\check{E} = \Omega \check{E} \tag{4.5}$$

where

$$-\Omega = \begin{pmatrix}
\omega^{n+1}_{n+1} & \omega^{n+1}_1 & \ldots & \omega^{n+1}_n & 0 \\
\omega^1_{n+1} & \omega^1_1 & \ldots & \omega^1_n & \omega^1 \\
\vdots & \vdots & & \vdots & \vdots \\
\omega^n_{n+1} & \omega^n_1 & \ldots & \omega^n_n & \omega^n \\
\omega^0_{n+1} & \omega^0_1 & \ldots & \omega^0_n & \omega^0_0
\end{pmatrix}. \tag{4.6}$$

Hence $\check{E}$ is a projective frame along E^{n+1}. Since $\Omega^i = -\omega^{n+1}_i$ and $\Omega^{n+1}_i = -\omega^i$, we have $\Omega^{n+1}_i = \sum_i h^{ij}\Omega^i$; the fundamental tensor h^*_{ij} of the dual immersion is equal to h^{ij}. Similarly we have

Lemma 5.7. *Denote invariants of the dual immersion by attaching "$*$". Then*

$$\begin{array}{llll} h^*_{ij} = h^{ij} & h^*_{ijk} = h^{ijk} & L^*_{ij} = -L^{ij} & \gamma^*_i = \gamma^i \\ F^* = F & \varphi^*_2 = \varphi_2 & \varphi^*_3 = \varphi_3. & \end{array}$$

In particular, if e is normalized, then $\check{E}$ is also normalized.

Consider now a system of linear differential equations of n variables of rank $n+2$, which can be written in a form (4.1). Then we obtain another system (4.4) called the *dual* system. Assume here both systems are known explicitly. Then the identity (4.3) gives a relation among solutions of both systems. In order to clarify this phenomenon we show an

Example. Let $E_2(\alpha, \beta, \beta', \gamma, \gamma')$ be the system of equations given by

$$(4.7)\quad \begin{cases} x(1-x)z_{xx} = xy\, z_{xy} + \{\alpha+\beta+1)x - \gamma\} z_x + \beta y\, z_y + \alpha\,\beta\, z \\ y(1-y)z_{yy} = xy\, z_{xy} + \{(\alpha+\beta'+1)x - \gamma'\} z_y + \beta' x\, z_x + \alpha\,\beta' z \end{cases}$$

which is called Appell's E_2 (see [ERD]). The function

$$F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) = \sum_{m,n} \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m)(\gamma', n)(1, m)(1, n)} x^m y^n$$

is a solution of this system. We can prove the dual system is equivalent to $F_2(1-\alpha, 1-\beta, 1-\beta', 2-\gamma, 2-\gamma')$. Then the identity (4.3) gives a formula

$$\begin{pmatrix} F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) \cdot F_2(1-\alpha, 1-\beta, 1-\beta', 2-\gamma, 2-\gamma'; x, y) \\ \quad - F_2(\alpha+1-\gamma, \beta+1-\gamma, \beta', 2-\gamma, \gamma'; x, y) \\ \qquad \cdot F_2(\gamma-\alpha, 1-\beta, 1-\beta', \gamma, 2-\gamma'; x, y) \\ \quad - F_2(\alpha+1-\gamma', \beta, \beta'+1-\gamma', \gamma, 2-\gamma'; x, y) \\ \qquad \cdot F_2(\gamma'-\alpha, 1-\beta, \gamma'-\beta', 2-\gamma, \gamma'; x, y) \\ \quad + F_2(\alpha+2-\gamma-\gamma', \beta+1-\gamma, \beta'+1-\gamma', 2-\gamma, 2-\gamma'; x, y) \\ \qquad \cdot F_2(\gamma+\gamma'-\alpha-1, \gamma-\beta, \beta, \gamma'-\beta', 2-\gamma, \gamma'; x, y) \end{pmatrix} = 0.$$

This formula reflects the symmetry possessed by the system (4.7). As for this and for other examples, refer [SY3]. Refer also §5, Chapter 2.

6. Projectively minimal hypersurfaces

In §2 of Chapter 4 we have seen that the quadratic form $F\varphi_2$ is an absolute invariant of a hypersurface, which is called the projective metric provided that the hypersurface is non-degenerate. The associated volume form is $|F^n \det h|^{1/2}\omega^1 \wedge \cdots \wedge \omega^n$ once we fix an orientation. Define an area functional P by

$$P(C) = \int_C |F^n \det h|^{1/2}\omega^1 \wedge \cdots \wedge \omega^n$$

for C a relatively compact domain. A non-degenerate hypersurface is called *projectively minimal* if this functional is critical for any infinitesimal deformation of the hypersurface. In §1 we will show how to derive a differential equation defining projectively minimal hypersurfaces and give some examples of such hypersurfaces for dimension $n = 2$ and also for $n \geqq 3$. In §2 we will define a transformation of surfaces called the Demoulin transform and then in §3 discuss its relation with projectively minimal surfaces.

§1. Variational formula and examples

We first notice that the volume form cannot be always positive. It vanishes where $F = 0$. In particular every quadratic hypersurface is projectively minimal. However, to avoid the differentiability problem at $F = 0$, we assume for the moment that φ_2 is positive definite and consider the functional where $F \neq 0$.

Let M be such a hypersurface and $e = (e_0, \ldots, e_{n+1})$ a projective frame field, for which

$$\det H = 1 \quad \text{and} \quad \omega_0^0 + \omega_{n+1}^{n+1} = 0 \tag{1.1}$$

hold. A deformation M_t of M is given by a vector

$$e_{0t} = e_0 + a^i(t)e_i + \nu(t)e_{n+1}, \tag{1.2}$$

where t is a deformation parameter and $a^i(t)$ and $\nu(t)$ are functions on M with parameter t such that

$$a^i(0) = 0\,, \quad \nu(0) = 0.$$

These functions are assumed to have compact supports in a domain C; and C is contained in the domain where the frame is defined. Let P_t be the value of P for M_t over C. Let δa denote the value $da/dt|_t = 0$ for a function a of t. Then the hypersurface is projectively minimal if and only if

$$\delta P = 0.$$

Since the deformation is defined by functions a^i and ν, we can expect to get a formula such as

$$\delta P = \int_C \{A_k \delta a^k + A\delta\nu\}\omega^1 \wedge \cdots \wedge \omega^n. \tag{1.3}$$

In fact the computation has been carried out in [SA3], where we took one affinely normalized frame field explained in §4 of Chapter 4 as a reference frame. Denote by "; "the formal covariant derivations with respect to forms $\pi_i^j = \omega_i^j + \frac{1}{2}h_{ik}{}^j\omega^k$ introduced in §6 of the same chapter: for quantities with suffices from 1 to n, say for a_i^j, define formally $a^j_{i;k}\omega^k = d\,a_i^j - a_k^j\pi_i^k + a_i^k\pi_k^j$ as in usual tensor analysis. This is compatible with h_{ij} because $h_{ij;k} = 0$. Then we can see that

$$A_k = 0 \tag{1.4}$$

and that the minimality is defined by

$$A = 0,$$

where

$$\begin{aligned} -\frac{2}{n}A = {} & F^{n/2-1}K^{ij}L_{ij} + \big((F^{n/2-1})_j K^{ji}\big)_{;i} \\ & - \big((F^{n/2-1}h_{ijk})_{;k}h^{ij\ell}\big)_{;\ell} - (n+2)\big(F^{n/2-1}h^{ijk}L_{ij}\big)_{;k} \\ & - 2(F^{n/2-1}h^{ijk})_{;k}L_{ij} - \frac{1}{2}(F^{n/2-1}h^{ijk}K_{ij})_{;k} \\ & + 2(F^{n/2-1}h^{ijk})_{;kji}, \end{aligned} \tag{1.5}$$

see p. 247 of [SA3]. The summation convention and the raising-and-lowering rule by h_{ij} are used. Although the expression (1.5) is not projectively invariant because of tentative use of ";", the definition of covariant derivations (6.18) of Chapter 4 enables us to rewrite this expression in an invariant form, which will be shown later for $n = 2$.

Before applying the formula (1.5) we want to redefine the projective minimality. This formula of A contains a term with fourth-order derivatives of $F^{n/2}$ unless $n = 2$, 4 or 6. However, if we put $A' = A$ for $n = 2$, 4 or 6 and $A' = F^{-n/2+4}A$ otherwise, then A' turns out to be finite even where $F = 0$ and has meaning also for the case when h is indefinite. So we pose a

Definition. A non-degenerate hypersurface is said to be *projectively minimal* if it satisfies the differential equation

$$A' = 0.$$

Remark. When h is definite it is seen that $F^{-n/2+2}A$ is finite. As far as we are concerned with the part where $F \neq 0$, this definition is of course the same as the previous.

We now find some special solutions. Consider an affine hypersurface which is defined by the condition $L_{ij} = \ell_{ij} - \ell\, h_{ij} = 0$. Since $\omega_0^0 = \pi_0^0 = 0$ for affine frames, we see $h_{ijk;k} = 0$ by (6.18) and Corollary 4.11 of Chapter 4. This makes the equation simple so that

$$4(F^{n/2-1}h^{ijk})_{;kji} - (F^{n/2-1}K_{ij}h^{ijk})_{;k} = 0. \tag{1.6}$$

Assume the metric φ_2 is Einstein. Since the Ricci tensor of this metric φ_2 is known to be

$$R_{ij} = -\frac{1}{2}(n-2)\ell_{ij} - \frac{1}{2}\ell\, h_{ij} + \frac{1}{4}K_{ij}$$

(see f.ex. [SA3]), we have $K_{ij} = K\, h_{ij}$. Hence the second term of (1.6) vanishes by the apolarity. The first term also vanishes if we assume that the hypersphere is homogeneous under a unimodular transformation group; because F is invariant under such transformations. We have seen

Proposition 6.1. *Let M be an affine hypersphere in $\mathbf{A}^{n+1}$. Assume M is homogeneous under a unimodular transformation group and the affine metric φ_2 is Einstein. Then M and every surface projectively equivalent to M are projectively minimal.*

Example. Let V be a non-degenerate convex cone in $\mathbf{A}^{n+1}$ and let χ be the characteristic function of V defined by $\chi(x) = \int_{V^*} e^{-\langle x,\xi\rangle} d\xi$, where V^* is the dual cone and $\langle\ ,\ \rangle$ is a dual pairing. If V is affinely homogeneous, then the hypersurface $\{\chi = 1\}$ is an affine hypersphere. The assumptions in Proposition 6.1 are satisfied when $V = \{\,(x^1, \ldots, x^n) \in \mathbf{A}^{n+1}\,;\, x^i > 0\,\}$ or when V is an irreducible self-dual cone.

When $n = 2$, the formula (1.5) becomes simpler because of the identity $K_{ij} = Fh_{ij}/2$, that was shown in Lemma 4.15. In fact we have

$$h^{ij}\ell_{;ij} = L^{ij}L_{ij}. \tag{1.7}$$

This formula shows in particular

Proposition 6.2. *Every affine sphere in $\mathbf{A}^3$ is projectively minimal.*

A converse of this fact in a global sense is given by

Theorem 6.3. *A compact strongly convex projective minimal surface in $\mathbf{A}^3$ is a quadratic surface.*

For a proof, first integrate (1.7) over the surface to see it is an affine sphere $L_{ij} = 0$ and next use the theorem that a compact strongly convex affine sphere is a quadratic surface, due to Calabi [CAL1] and Pogorelov [PO].

The formula (1.7) will now be rewritten in projective terminology. Recall that every affine frame is normalized projectively in the form (4.2) of Chapter 4. Hence $\gamma_i = \frac{1}{2}\ell_i$. On the other hand, the covariant derivation of γ_i has been given by (6.18) of Chapter 4:

$$\gamma_{i,j}\omega^j = d\gamma_i - \gamma_j\pi_i^j + 3\gamma_i\pi_0^0 - L_{ij}\pi_{n+1}^j. \tag{1.8}$$

Since $\pi_0^0 = 0$ and $\pi_{n+1}^j = \omega_{n+1}^j + \tau_{n+1}^j = h^{jk}(\ell_{ki} + f_{ki} - \frac{1}{2}L_{ki})\omega^k$ where $f_{ki} = -\dfrac{F}{16}h_{ki}$ (§8, Chapter 4), we have

$$\gamma_{i,j} = \gamma_{i;j} - L_{ik}h^{k\ell}\Big(\ell_{j\ell} - \frac{F}{16}h_{j\ell} - \frac{1}{2}L_{j\ell}\Big).$$

Therefore

$$h^{ij}\gamma_{i,j} = h^{ij}\gamma_{i;j} - \frac{1}{2}L_{ij}L^{ij}.$$

We have

Theorem 6.4. *A non-degenerate surface in* $\mathbf{P}^3$ *is projectively minimal if and only if*

$$h^{ij}\gamma_{i,j} = 0. \tag{1.9}$$

Proof. Since this holds for affinely normalized frames, it is enough to see that this condition is independent of frames. This will be shown in the next lemma.

Lemma 6.5. *Consider a frame change* $\widetilde{e} = g\,e$ *by*

$$g = \begin{pmatrix} \lambda & & \\ b & a & \\ \mu & c & \nu \end{pmatrix}, \qquad \begin{cases} |\lambda\nu| = 1, \\ b = \nu^{-1}a\,h^t c \\ \mu = \frac{1}{2}\nu^{-1}c\,h^t c. \end{cases}$$

A denotes the inverse of a. Then

(1) $$\begin{aligned} \lambda^3\widetilde{\gamma}_{p,q}A_i^pA_j^q = {} & \nu\gamma_{i,j} + L_{ik,j}c^k - \frac{1}{2}\nu^{-1}h_{ik\ell,j}c^kc^\ell \\ & - h_{ik}c^k\Big(\gamma_j + \nu^{-1}L_{j\ell}c^\ell - \frac{1}{2}\nu^{-2}h_{j\ell m}c^\ell c^m\Big) \\ & + h_{ij}c^k\Big(\gamma_k + \nu^{-1}L_{k\ell}c^\ell - \frac{1}{2}\nu^{-2}h_{k\ell m}c^\ell c^m\Big) \\ & - 3\gamma_i h_{jk}c^k - 2\nu^{-1}h_{jk}L_{i\ell}c^kc^\ell + \frac{1}{2}\nu^{-1}h_{jk}c^kh_{i\ell m}c^\ell c^m \\ & - \mu(L_{ij} - \nu^{-1}h_{ijk}c^k). \end{aligned}$$

(2) *Put* $\Gamma = h^{ij}\gamma_{i,j}$. *Then*

$$\lambda^4\widetilde{\Gamma} = \begin{cases} \Gamma & \textit{for } n = 2, \\ \Gamma - \dfrac{1}{4(n-2)}\nu^{-1}h_{ijk}K^{ij}c^k & \\ \quad + (n-2)\{3\nu^{-1}\gamma_i c^i + \dfrac{3}{2}\nu^{-2}L_{ij}c^i c^j - \dfrac{1}{2}\nu^{-3}h_{ijk}c^i c^j c^k\} & \textit{for } n \geqq 3. \end{cases}$$

The proof is given by a routine calculation and omitted.

§2. Demoulin transforms of surfaces

We assume $n = 2$ and treat normalized frames:

(2.1) $$\omega_0^0 + \omega_3^3 = 0\,, \quad |H| = 1\,, \quad L = 0.$$

Recall transformation rules for such frames.

(2.2) $$\begin{aligned} \lambda^2\nu\widetilde{h}_{ijk} &= h_{pqr}a_i^p a_j^q a_k^r \\ \lambda^2\widetilde{L}_{ij} &= (L_{k\ell} - \nu^{-1}h_{k\ell m}c^m)a_i^k a_j^\ell \\ \lambda^2\nu^{-1}\widetilde{\gamma}_j &= \Big(\gamma_j + \nu^{-1}L_{kj}c^k - \frac{1}{2}\nu^{-2}h_{jk\ell}c^k c^\ell\Big)a_i^j. \end{aligned}$$

We try to find a frame so that $\gamma_i = 0$. To simplify notations we consider the case $h = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We have seen already $h_{112} = h_{122} = 0$ and $L_{12} = 0$ in this case (cf. Chapter 4). Put

$$h_{111} = -2a\,, \quad h_{222} = -2b.$$

Then $\widetilde{\gamma}_i = 0$ if we can solve

(2.3) $$\begin{cases} a(c^1)^2 + L_{11}c^1 + \gamma_1 = 0 \\ b(c^2)^2 + L_{22}c^2 + \gamma_2 = 0 \end{cases}$$

to define a frame change. Let us denote by Δ_1 and Δ_2 the discriminants of these equations.

$$\Delta_1 = L_{11}^2 - 4\gamma_1 a, \qquad \Delta_2 = L_{22}^2 - 4\gamma_2 b.$$

Consider a frame change by

$$\begin{pmatrix} \lambda & & & \\ c^2 & \alpha & & \\ c^1 & & \beta & \\ * & c^1 & c^2 & \nu \end{pmatrix}.$$

Then (2.2) yields

$$\lambda^4\alpha^{-4}\widetilde{\Delta}_1 = \Delta_1\,, \quad \lambda^4\beta^{-4}\widetilde{\Delta}_2 = \Delta_2.$$

Since the remaining freedom of choice of frames is the exchange of e_1 and e_2, the condition

$$\Delta_1 \geqq 0\,, \quad \Delta_2 \geqq 0 \tag{2.4}$$

is independent of frames.

Assume this condition and the condition

$$a \neq 0\,, \quad b \neq 0. \tag{2.5}$$

Then (2.3) has solutions, the number of which is generally 4. Let $(c^1, c^2) = (t^1, t^2)$ be one solution and $\widetilde{e}$ be the corresponding new frame:

$$\begin{aligned} &\widetilde{e}_0 = e_0\,, \quad \widetilde{e}_1 = t^2 e_0 + e_1\,, \quad \widetilde{e}_2 = t^1 e_0 + e_2, \\ &\widetilde{e}_3 = t^1 t^2 e_0 + t^1 e_1 + t^2 e_2 + e_3. \end{aligned}$$

New $\widetilde{L}_{ij}$'s are

$$\widetilde{L}_{11} = L_{11} + 2a\,t^1\,, \quad \widetilde{L}_{22} = L_{22} + 2b\,t^2.$$

The equation (2.3) is for this frame

$$\begin{cases} a(c^1)^2 + \widetilde{L}_{11}c^1 = 0 \\ b(c^2)^2 + \widetilde{L}_{22}c^2 = 0. \end{cases}$$

Hence, as is easily seen, any solution of this system defines an interchange among $\widetilde{e}$'s. We have seen

Proposition 6.6. *Consider a surface of indefinite type. Then under the assumptions* (2.4) *and* (2.5) *there exists a normalized frame e which satisfy the condition* $\omega_3^0 = 0$*. The set of the last vector* e_3 *of such frames is uniquely determined up to multiplication and consists of at most four elements.*

Definition. We call such a frame a *Demoulin frame.* When the last vector e_3 of a Demoulin frame defines a surface, we call e_3 a *Demoulin transform* of the original surface. (cf. [BOL, vol. 2, §120])

For a Demoulin frame put

$$h_{ij}\omega_3^j = q_{ij}\omega^j\,, \quad \omega_i^0 = p_{ij}\omega^j. \tag{2.6}$$

(Since transformation among Demoulin frames are restricted so that c^i take only finite possible values, we can think of p_{ij} and q_{ij} as relative invariants.) The condition for e_3 to define a surface is that ω_3^1 and ω_3^2 are linearly independent because $d\,e_3 = \omega_3^3 e_3 + \omega_3^1 e_1 + \omega_3^2 e_2$. Namely

$$\det q \neq 0\,, \quad q = (q_{ij}). \tag{2.7}$$

In this case a set $\overline{e} = (e_3, e_1, e_2, e_0)$ in this order defines a projective frame of e_3 and the coframe $\overline{\omega}$ is

$$\overline{\omega} = \begin{pmatrix} \omega_3^3 & \omega_3^1 & \omega_3^2 & 0 \\ \omega_1^3 & \omega_1^1 & \omega_1^2 & \omega_1^0 \\ \omega_2^3 & \omega_2^1 & \omega_2^2 & \omega_2^0 \\ 0 & \omega^1 & \omega^2 & \omega_0^0 \end{pmatrix} \tag{2.8}$$

Therefore the associated fundamental form $\overline{\varphi}_2$ is $\omega_3^1 \cdot \omega_1^0 + \omega_3^2 \cdot \omega_2^0$, which is non-degenerate when

$$\det p \neq 0\,, \quad p = (p_{ij}). \tag{2.9}$$

Since $L_{ij} = q_{ij} - p_{ij}$ satisfies the condition $L = \mathrm{tr} L_{ij} = 0$,

$$p_{12} = q_{12}\,, \quad p_{21} = q_{21}, \tag{2.10}$$

from which we have

$$\overline{\varphi}_2 = p_{21}(p_{11}+q_{11})\omega^1\omega^1 + p_{12}(p_{22}+q_{22})\omega^2\omega^2 + (2p_{12}p_{21} + p_{11}q_{22} + q_{11}p_{22})\omega^1\omega^2. \tag{2.11}$$

Moreover $\omega_3^0 = 0$ implies $\omega_3^i \wedge \omega_i^0 = 0$. Hence

$$p_{11}q_{22} - p_{22}q_{11} = 0. \tag{2.12}$$

Denote by $\overline{h}$, $\overline{p}$, and $\overline{q}$ the quantities with respect to $\overline{e}$. Let $Q = q^{-1}$. Then a computation shows

$$\overline{h} = pQh\,, \quad \overline{p} = hQh\,, \quad \overline{q} = pQhQh. \tag{2.13}$$

Definition. (O. Mayer [MAY]). We call a surface of indefinite type with the condition (2.5) a *Demoulin surface* if $\Delta_1 = \Delta_2 = 0$, or equivalently if it has only one Demoulin transform.

In §2 of Chapter 4 we posed a problem to characterize hypersurfaces with the property that

$$L_{ij} = h_{ijk}a^k\,, \quad \gamma_i = \frac{1}{2}h_{ijk}a^ia^j \tag{2.14}$$

for some vector a^k. In case $n = 2$ we have an answer:

Proposition 6.7. *Assume* (2.5). *Then the condition* (2.14) *is equivalent to that the surface is Demoulin.*

Proof. Since $h_{111} \neq 0$ and $h_{222} \neq 0$, the condition $\gamma_i = -\frac{1}{2}h_{ijk}a^ja^k$ for a Demoulin frame is $a^i = 0$. Hence $L_{ij} = 0$, which implies $\Delta_i = 0$ and vice versa.

Let us consider a system of differential equations

$$\begin{cases} z_{xx} = b\,z_y + u\,z \\ z_{yy} = c\,z_x + v\,z \end{cases}$$

with $b \neq 0$ and $c \neq 0$. Referring a formula of the associate normalized frame (§3 of Chapter 5), we see that Demoulin transforms w of the surface z are given by

$$w = \Big(c^1 c^2 - \frac{1}{2} b\, c\Big) z + c^1 z_x + c^2 z_y + z_{xy}, \tag{2.15}$$

where

$$c^1 = \frac{-b_y \pm \sqrt{\Delta_1}}{2b}, \quad c^2 = \frac{-c_x \pm \sqrt{\Delta_2}}{2c}, \tag{2.16}$$

and

$$\begin{cases} \Delta_1 = (b_y)^2 + 4b \Big(b\, v + u_y - \dfrac{1}{2}(bc)_x\Big) \\ \Delta_2 = (c_x)^2 + 4c \Big(c\, u + v_x - \dfrac{1}{2}(bc)_y\Big). \end{cases}$$

Example. Assume $u = v = 0$. Then the surface z is equivalent to the surface $\{\, (x^2 + y^2) z = 1 \,\}$ in affine coordinates. Its Demoulin transform $w = z_{xy} - \frac{1}{2} z$ also satisfy the same system.

Remark. The condition (2.4) is not of course necessary for complex coefficients. The case when h is definite is similarly treated. Use the identity (7.19) of Chapter 4:

$$\lambda^2 z \widetilde{\delta} = \delta + K \bar{t} - \frac{1}{2} c \bar{t}^2.$$

If $C = h_{111} + i\, h_{112} \neq 0$, i.e. if $F \neq 0$, then this is always solvable with respect to $\bar{t}$ and we have a frame with $\omega_3^0 = 0$.

§3. Demoulin transforms of projectively minimal surfaces

Assume $n = 2$ and consider a Demoulin frame, i.e. $\omega_3^0 = 0$. By the formula (1.8) this case yields

$$h^{ij} \gamma_{i,j} = -\frac{1}{2} L^{ij} (p_{ij} + q_{ij}).$$

Namely

Proposition 6.8. *A non-degenerate surface is projectively minimal if and only if*

$$L^{ij}(p_{ij} + q_{ij}) = 0, \tag{3.1}$$

for a Demoulin frame.

Assume the surface is of indefinite type. Then, by the conditions (2.10) and (2.12), the condition (3.1) is equivalent to one of the following equivalent conditions:

$$\det p = \det q\,, \tag{3.2}$$

$$L_{22}(p_{11} + q_{11}) = 0\,, \tag{3.3}$$

$$L_{11}(p_{22} + q_{22}) = 0. \tag{3.4}$$

Theorem 6.9. ([MAY]). *Let M be a non-degenerate surface of indefinite type satisfying* (2.4). *Assume conditions* (2.5), (2.7) *and* $\det L_{ij} \neq 0$ *for a Demoulin frame. Then* (1) *if M is projectively minimal, then the conformal structure of a Demoulin transform of M is the same as the conformal structure of M; i.e. a Demoulin transform is Weingarten.* (2) *Conversely, if a Demoulin transform is Weingarten, then the original surface is projectively minimal or a surface with the property $p_{12}p_{21} = 0$.*

Proof. Since $\det L_{ij} = L_{11}L_{22} \neq 0$, the conditions (3.3) and (3.4) shows $\overline{\varphi}_2$ is conformal to $\varphi_2 = \omega^1\omega^2$ by (2.11). The converse is also immediate.

Remark. The surface with $p_{12}p_{21} = 0$ is called Q-surface in [BOL, p. 326].

Theorem 6.10. ([MAY]) *Let M be a non-degenerate projectively minimal surface of indefinite type. Assume* (2.5) *and* (2.7). *Then any Demoulin transform is again projectively minimal. One of Demoulin transforms of a Demoulin transform M is M itself.*

Proof. Let $\overline{e}$ the frame defined in §2. From the expression (2.8) of $\overline{\omega}$, $\overline{\omega}^0_0 + \overline{\omega}^3_3 = 0$. The identity $\det p = \det q$ shows $\det \overline{h} = 1$ by (2.13) and the identity (2.13) shows $\overline{h}^{ij}\overline{L}_{ij} = 0$. Hence $\overline{e}$ is normalized and a Demoulin frame because $\overline{\omega}_3^0 = \omega_0^3 = 0$. Now (2.13) again shows $\det \overline{p} = \det \overline{q}$, proving the first part. The second statement is immediate.

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