# Isomonodromic problems on elliptic curve, rigid tops and reflection equations *

M.A.Olshanetsky and A.V.Zotov

**Abstract**

We consider two types of the Isomonodromy problem on an elliptic curve. The first problem is related to the elliptic Calogero-Moser system, while the second type is related to the integrable Euler-Arnold top. The both types of the monodromy preserving equations are the non-autonomous Hamiltonian system with respect to the modular parameter of the elliptic curve, that plays the role of time. There exists a symplectic map from the first system to the second one. The rank two model corresponds to the one-parameter case of the Painlevè VI equation. For the general PVI we consider two-particle Calogero-Inozemtsev system. This form of PVI was introduced by Painlevè at 1906. We determine the corresponding Lax operator. The same symplectic map leads to the Zhukovsky-Volterra gyrostat. In this way we present the new form of the PVI equation. Its Lax operator satisfies classical reflection equations. The quantization of the autonomous version is achieved by the reflection equation. The corresponding quadratic algebra generalizes the Sklyanin algebra, corresponding to the Euler top. In this way we construct a new integrable XYZ spin chain with boundaries.

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# 1 Introduction

We present a new form of the monodromy preserving equations as Euler-Arnold tops on the group SL$(N,\mathbb{C})$ with the time depending inertia tensor. The role of time is played by the modular parameter of an elliptic curve. The corresponding linear system is related to the flat bundles of degree one over this curve. In the case $N = 2$ instead of the Euler-Arnold top we consider SL$(2,\mathbb{C})$ gyrostat. It will be referred to as the Zhukovsky-Volterra gyrostat (ZVG) [1, 2]. The equation of motion of the non-autonomous version of the gyrostat is equivalent to PVI. The lectures are based on the works [3, 4, 5, 6, 7].

The isomonodromy problems on curves with an arbitrary genus was considered in [3, 8]. The genus one curves can be considered in great details likewise the case of rational curves. The simplest case of the one point Fuchs singularity on the elliptic curve is defined by the Lax operator of the N-particle elliptic Calogero-Moser system. The two-particle case corresponds to the degenerate case of the elliptic form of the famous PainlevèVI equation (PVI). In 1906 Painlevè gave the elliptic form of PVI [9] (see, also, [10]). It depends on four free parameters (the coupling constants), while the degenerate case is its one-parametric subfamily. The Lax operator of the former system was obtained in [6]. It is related to the two-particle Calogero-Inozemtsev system [11].

In [4] we defined the symplectic map (symplectic Hecke correspondence) from the elliptic N-particle systems to integrable Euler-Arnold tops on a coadjoint orbit of group SL$(N,\mathbb{C})$. This map works in the non-autonomous case as well. For the two-particle Calogero-Inozemtsev system we come to the ZVG on GL$(2,\mathbb{C})$ depending on coupling constants. The Lax operator satisfies the classical reflection equation, while its components generate a new quadratic Poisson algebra. This algebra generalizes the classical Sklyanin algebra by the additional linear terms depending on the coupling constants of the Calogero-Inozemtsev system. In this way we present two descriptions of PVI by the linear brackets on SL$(2,\mathbb{C})$ and the quadratic Hamiltonians, or by the new quadratic brackets and a linear Hamiltonian. The quantization of this construction leads to the quantum reflection equation and to the generalization of the quantum Sklyanin algebra [12]. Its solutions allows us to describe the boundary conditions of the XYZ model on a finite lattice that generalize [13].

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# 2 Isomonodromic deformations and Elliptic Calogero-Moser System

We consider differential equations related to the integrable Elliptic Calogero-Moser system with spin [14, 15, 16]. They are defined as the monodromy preserving equations of some linear system on an elliptic curve. It is a Hamiltonian non-autonomous system that describes dynamics of $N$ particles with complex coordinates in a time-depending potential. The particles have internal degrees of freedom (the so-called "spin"). For the brevity we call the system ICMS.

## 2.1 Phase space of ICMS system with spin

Let $\Sigma_\tau = \mathbb{C}/\mathbb{Z}^2_\tau$, $\mathbb{Z}^2_\tau = \mathbb{Z} + \tau\mathbb{Z}$, $(\Im m\tau > 0)$ be the elliptic curve. The coordinates of the particles lie in $\Sigma_\tau$:

$$\mathbf{u} = (u_1, \dots, u_N)\,, \quad u_j \in \Sigma_\tau$$

with the constraint on the center of mass $\sum u_j = 0$. Let $\mathbb{Z}^2_\tau \ltimes W_N$ be the semi-direct product of the two-dimensional lattice group and the permutation $W_N$. The coordinate part of the phase space is the quotient

$$\Lambda = (\mathbb{C}^N/(\mathbb{Z}^2_\tau \ltimes W_N))/\Sigma_\tau\,.$$

The last quotient respects the constraint on the center of mass. Let

$$\mathbf{v} = (v_1, \dots, v_N)\,, \quad v_j \in \mathbb{C}\,, \ \sum v_j = 0\,.$$

be the momentum vector dual to $\mathbf{u}$. The pair $(\mathbf{v}, \mathbf{u})$ describes the "spinless" part of the phase space.

The additional phase variables, describing the internal degrees of freedom of the particles are the matrix elements of the $N$-order matrix $\mathbf{p}$. More exactly, we consider $\mathbf{p}$ as an element of the Lie coalgebra $\mathrm{sl}(N, \mathbb{C})^*$. The linear (Lie-Poisson) brackets on $\mathrm{sl}(N, \mathbb{C})^*$ for the matrix elements have the form

$$\{p_{jk}, p_{mn}\} = p_{jn}\delta_{km} - p_{mk}\delta_{jn}\,. \tag{2.1}$$

Let $\mathcal{O}$ be a coadjoint orbit

$$\mathcal{O} = \{\mathbf{p} \in \mathrm{sl}(N,\mathbb{C}) \mid \mathbf{p} = h^{-1}\mathbf{p}^0 h,\ h \in \mathrm{SL}(N,\mathbb{C})\,, \mathbf{p}^0 \in D\}\,, \tag{2.2}$$

where $D$ is the diagonal subgroup of $\mathrm{SL}(N,\mathbb{C})$. In fact, we assume that $\mathbf{p}$ belongs to the symplectic quotient

$$\tilde{\mathcal{O}} = \mathcal{O}//D \tag{2.3}$$

with respect to the action of $D$. It implies the following constraints:
i) the moment constraint $p_{jj} = 0$,
ii) the gauge fixing, for example, as $p_{j,j+1} = p_{j+1,j}$.
*Example.* Let $\mathbf{p}^0 = \mathrm{diag}(N-1, 1, \ldots, 1)$. Then $\dim \mathcal{O} = 2N-2$. It is the most degenerate non-trivial orbit. It leads to the spinless model, since in this case $\dim \tilde{\mathcal{O}} = 0$. We should represent $\mathbf{p}^0$ in the special form that takes into account the moment constraint (i):

$$\mathbf{p}^0 = \mathbf{J}^C = \nu \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}. \tag{2.4}$$

For $N = 2$ these orbits are generic.

In this way we come to the phase space of the ICMS

$$\mathcal{R}^{CM_N} = \{T^*(\Lambda) \cup \tilde{\mathcal{O}}\}\,, \tag{2.5}$$

Note that

$$\dim(\mathcal{R}^{CM_N}) = 2N - 2 + \dim \mathcal{O} - 2\dim(D) = \dim \mathcal{O}\,. \tag{2.6}$$

It is a symplectic manifold with the symplectic form

$$\omega = \langle D\mathbf{v} \wedge D\mathbf{u}\rangle - \langle \mathbf{p}^0 Dhh^{-1} Dhh^{-1}\rangle\,, \tag{2.7}$$

where the brackets stand for the trace.

## 2.2 Equations of motion and Painlevé VI

The ICMS Hamiltonian has the form

$$H^{CM,spin} = \frac{1}{2}\sum_{j=1}^{N} v_j^2 + \sum_{j>k} p_{jk}p_{kj}E_2(u_j - u_k;\tau)\,, \tag{2.8}$$

where $E_2(x;\tau) = \wp(x;\tau) + 2\eta_1(\tau)$ is the second Eisenstein function (A.4) and $\tau$ plays the role of time. For the orbit, corresponding to (2.4) the spinless Hamiltonian is

$$H^{CM} = \frac{1}{2}\sum_{j=1}^{N} v_j^2 + \nu^2 \sum_{j>k} E_2(u_j - u_k;\tau)\,. \tag{2.9}$$

For the non-autonomous Hamiltonian systems is convenient to work with the extended phase space $\mathcal{R}^{ext} = (\mathcal{R}^{\mathcal{CM}_N}, \mathcal{T})$, where $\mathcal{T} = \{\tau \,|\, \Im m\tau > 0\}$. Equip it with the degenerate two-form

$$\omega^{ext} = \omega - \frac{1}{\kappa} dH^{CM}(\mathbf{v}, \mathbf{u}, \tau) \wedge d\tau \,,$$

where $\kappa \geq 0$ is the so-called *classical level*. Note that $\omega^{ext}$ is invariant with respect the modular transformations $\mathrm{SL}(2,\mathbb{Z})$ of $\mathcal{T}$. It means that $\omega^{ext}$ can be restricted on the moduli space $\mathcal{M} = \mathcal{T}/\mathrm{SL}(\in,\mathbb{Z})$. The vector field

$$\sum_j \left(\frac{\partial H}{\partial u_j}\partial_{v_j} - \frac{\partial H}{\partial v_j}\partial_{u_j}\right) + \kappa\partial_\tau$$

annihilates $\omega$ and defines the equations of motion of ICMS.

$$\kappa\partial_\tau u_j = v_j \,, \tag{2.10}$$

$$\kappa\partial_\tau v_n = -\sum_{j\neq n} p_{jk}p_{kj}\partial_{u_n} E_2(u_j - u_n; \tau) \,, \tag{2.11}$$

$$\kappa\partial_\tau \mathbf{p} = 2[\mathbf{J_u}(\tau)\cdot\mathbf{p}, \mathbf{p}] \,, \tag{2.12}$$

where the operator $\mathbf{J_u}\cdot\mathbf{p}$ is the diagonal action

$$\mathbf{J_u}(\tau)\cdot\mathbf{p} \;:\; p_{jk} \;\rightarrow\; E_2(u_j - u_k; \tau)p_{jk} \,. \tag{2.13}$$

## 2.3 Elliptic form of Painlevé VI

For $N = 2$ we put $u_1 = -u_2 = u$, $v_1 = -v_2 = v$ and come to the second order equation

$$\partial_\tau^2 u = -\nu^2 \partial_u E_2(2u) \,. \tag{2.14}$$

It is a particular case of the elliptic form of the Painlevé VI equation.

The original form of Painlevé VI ($\mathrm{PVI}_{\alpha,\beta,\gamma,\delta}$) is

$$\frac{d^2X}{dt^2} = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)\frac{dX}{dt} +$$
$$+\frac{X(X-1)(X-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right). \tag{2.15}$$

The equation depends on four free parameters $(\alpha, \beta, \gamma, \delta)$. It can be transformed to the elliptic form [9, 10] that we will use here. Let $\omega_0 = 0$, and $\omega_{1,2}$ are the half periods of the elliptic curve $\Sigma_\tau$, $\omega_3 = \omega_1 + \omega_2$ and

$$\nu_0 = \alpha \,,\; \nu_1 = -\beta \,,\; \nu_2 = \gamma \,,\; \nu_3 = \frac{1}{2} - \delta \,.$$

Then PVI (2.15) takes the following form

$$\partial_\tau^2 u = -\sum_{j=0}^{3} \nu_j^2 \partial_u E_2'(u + \omega_j) \,, \tag{2.16}$$

where the variables are replaced as

$$(u,\tau)\to\left(X=\frac{E_2(u|\tau)-e_1}{e_2-e_1}\,,\ t=\frac{e_3-e_1}{e_2-e_1}\right)\,,\quad e_j=E_2(\omega_j)\,.$$

In the case $\nu_j^2=\frac{\nu^2}{4}$ (2.16) coincides with (2.14), because $4E_2(2u)=\sum_{j=0}^{3}E_2(u+\omega_j)$.

## 2.4 Lax representation

The goal of this subsection is the Lax representation of (2.10) – (2.12).

### 2.4.1 Deformation of elliptic curve

Let $T^2=\{(x,y)\in\mathbb{R}\,|\,x,y\in\mathbb{R}/\mathbb{Z}\}$ be a torus. Complex structure on $T^2$ is defined by the complex coordinates

$$\Sigma_{\tau_0}=\{z=x+\tau_0 y\,,\ \bar z=x+\bar\tau_0 y\}\,,\quad \Im m\tau_0>0\,,\quad \Sigma_{\tau_0}\sim\mathbb{C}/(\mathbb{Z}+\tau_0\mathbb{Z})\,.$$

Let $\chi(z,\bar z)$ be the characteristic function of a neighborhood of $z=0$. For two neighborhoods $\mathcal{U}'\supset\mathcal{U}$ of the point $z=0$ $\chi(z,\bar z)$ is defined as a smooth function

$$\chi_a(z,\bar z)=\left\{\begin{array}{ll}1, & z\in\mathcal{U}\\ 0, & z\in\Sigma_\tau\setminus\mathcal{U}'.\end{array}\right. \tag{2.17}$$

Define the chiral deformation of the complex coordinates

$$\left\{\begin{array}{l}w=z-\frac{\tau-\tau_0}{\rho}(\bar z-z)(1-\chi(z,\bar z))\,,\\ \bar w=\bar z\end{array}\right.\qquad(\rho=\tau_0-\bar\tau_0)\,.$$

In this way we come to the deformed elliptic curve $\Sigma_\tau=\{w,\bar w\}$.

In the new coordinates the partial derivatives assume the form

$$\left\{\begin{array}{l}\partial_w=\partial_z\,,\\ \partial_{\bar w}=\partial_{\bar z}+\mu\partial_z\,,\end{array}\right.\quad \mu=\frac{\tau-\tau_0}{\tau-\bar\tau_0}(1-\chi(z,\bar z))\,,$$

where $\mu$ – is the *Beltrami differential.*

### 2.4.2 Flat bundles of degree zero over $\Sigma_\tau$

Let $E_N^0$ be a flat vector bundle of rank $N$ and degree 0 over the deformed elliptic curve $\Sigma_\tau$. Consider the connections

$$\left\{\begin{array}{l}\kappa\partial+L^{(0)}(w,\bar w,\tau)\,,\\ \bar\partial+\bar L^{(0)}(w,\bar w,\tau)\,,\end{array}\right.$$

$$L^{(0)}(w,\bar w,\tau)\,,\ \bar L^{(0)}(w,\bar w,\tau)\in\mathrm{sl}(N,\mathbb{C})\,,\quad(\partial=\partial_w\,,\ \bar\partial=\partial_{\bar w})\,.$$

The flatness of the bundle $E_N^0$ means

$$\bar\partial L^{(0)}-\kappa\partial\bar L^{(0)}+[L^{(0)},\bar L^{(0)}]=0\,. \tag{2.18}$$

By the gauge transformations $f(w,\bar w) \in C^\infty\mathrm{Map}\,(\Sigma_\tau \to \mathrm{GL}(N,\mathbb{C}))$

$$\bar L^{(0)} \to f^{-1}\partial_{\bar w} f + f^{-1}\bar L^{(0)} f$$

the connections of the bundles of degree zero can be choose in the following form.
1. $\bar L^{(0)} = 0$. Then from the flatness (2.18) we have

$$\bar\partial L^{(0)}(w,\bar w) = 0\,.$$

2. $\deg(E_N^0) = 0$ means the quasi-periodicity of the Lax matrix:

$$L^{(0)}(w+1) = L^{(0)}(w)\,,\quad L^{(0)}(w+\tau) = \mathbf{e}(\mathbf{u})L^{(0)}(w)\mathbf{e}(-\mathbf{u})\,,$$

where the diagonal elements of $\mathbf{e}(\mathbf{u}) = \mathrm{diag}(\exp(2\pi i u_1),\ldots,\exp(2\pi i u_N))$ define *the moduli of holomorphic bundles*. We identify $\mathbf{u}$ with the coordinates of particles. In fact, $u_j$, $j = 1,\ldots,N$ belong to the dual to $\Sigma_\tau$ elliptic curve (*the Jacobian*). It is isomorphic to $\Sigma_\tau$.
3. Assume that $L^{(0)}$ has a simple pole at $w = 0$

$$Res|_{w=0}\, L^{(0)}(w) = \mathbf{p}\,.$$

The conditions $1, 2, 3$ fix $L^{(0)}$ up to a diagonal matrix $P$

$$L^{(0)} = P + X\,, \tag{2.19}$$

$$P = 2\pi i\frac{\tau-\tau_0}{\rho}\mathrm{diag}(v_1,\ldots,v_N)\,,$$
$$X = \{X_{jk}\}\,,\ (j\neq k)\,,\quad X_{jk} = p_{jk}\phi(u_j-u_k,w)\,.$$

The function $\phi$ is determined by (A.7). We do not explain here why the free parameters of $P$ can be identified with the momenta. Details can be found in [3].

The flatness of the bundle upon the gauge transform amounts the consistency of the system

$$\left\{\begin{array}{ll} i. & (\kappa\partial + L^{(0)}(w,\tau))\Psi = 0\,, \\ ii. & \bar\partial\Psi = 0\,. \end{array}\right. \tag{2.20}$$

To come to a dynamical system, namely to the monodromy preserving equation for this system, we assume that the Baker-Akhiezer function $\Psi$ satisfies the third equation

$$iii.\quad (\kappa\partial_\tau + M^{(0)}(w))\Psi = 0\,. \tag{2.21}$$

It has the following meaning. Let $\mathcal{Y}$ be a monodromy matrix of the system (2.20) corresponding to homotopically non-trivial cycles $\Psi \to \Psi\mathcal{Y}$. The equation (2.21) means that $\partial_\tau\mathcal{Y} = 0$, and thereby the monodromy is independent on the complex structure of $T^2$. The consistency of i. and iii. is the monodromy preserving equation

$$\partial_\tau L^{(0)} - \partial_w M^{(0)} - \frac{1}{\kappa}[L^{(0)}, M^{(0)}] = 0\,. \tag{2.22}$$

In contrast with the standard Lax equation it has additional term $\partial_w M^{(0)}$.

**Proposition 2.1** *The equation (2.22) is equivalent to the monodromy preserving equations (2.10), (2.11), (2.12) for $L^{(0)}$ (2.19) and*

$$M^{(0)} = -D + Y\,, \quad D = \frac{1}{\kappa}\mathrm{diag}(d_1, \dots, d_N)\,, \quad Y = \{Y_{jk}\}\,,\ (j \neq k)\,,$$

$$d_j = \sum_{i \neq j} E_2(u_i - u_j)\,,$$

$$Y_{jk} = p_{jk} f(u_j - u_k, w)\,, \quad f(u, w) = \partial_u \phi(u, w)\,.$$

Proof is based on the Calogero functional equation (A.16) and the heat equation (A.10).

## 2.5 Isomonodromic deformations and integrable systems

We can consider the isomonodromy preserving equations as a deformation (*Whitham quantization*) of integrable equations. The level $\kappa$ plays the role of the deformation parameter [17]. Here we investigate the integrable limit of the vector generalization of PVI (2.10) – (2.12) [3].

Consider the limit $\kappa \to 0$ and introduce the independent time $t$ as $\tau = \tau_0 + \kappa t$ for some fixed $\tau_0$. It means that $t$ plays the role of a local coordinate in an neighborhood of the point $\tau_0$ in the moduli space of elliptic curves $\mathcal{M}$. Then we come to the elliptic Calogero-Moser integrable system with spin related to the curve $\Sigma_{\tau_0}$:

$$\partial_t u_j = v_j\,,$$
$$\partial_t v_n = -\sum_{j \neq n} p_{jk} p_{kj} \partial_{u_n} E_2(u_j - u_n; \tau_0)\,, \tag{2.23}$$
$$\partial_t \mathbf{p} = 2[\mathbf{J_u} \cdot \mathbf{p}, \mathbf{p}]\,,$$

The linear problem for this system is obtained from the linear problem for the Isomonodromy problem (2.20), (2.21) by the analog of the quasi-classical limit in Quantum Mechanics. Represent the Baker-Akhiezer function in the WKB form

$$\Psi = \Phi \exp(\frac{\mathcal{S}^{(0)}}{\kappa} + \mathcal{S}^{(1)})\,, \tag{2.24}$$

where $\Phi$ is a group valued function and $\mathcal{S}^{(0)}$, $\mathcal{S}^{(1)}$ are diagonal matrices. Substitute (2.24) in the linear system (2.20), (2.21). If $\frac{\partial}{\partial \bar{w}_0}\mathcal{S}^{(0)} = 0$ and $\frac{\partial}{\partial t}\mathcal{S}^{(0)} = 0$, then the terms of order $\kappa^{-1}$ vanish. In the quasi-classical limit we put $\partial \mathcal{S}^{(0)} = \lambda$. In the zero order approximation we come to the linear system

$$\left\{ \begin{array}{ll} i. & (\lambda + L^{(0)}(w, \tau))\Psi = 0\,, \\ ii. & \bar{\partial}\Psi = 0\,, \\ iii. & (\partial_t + M^{(0)}(w))\Psi = 0\,. \end{array} \right.$$

The consistency of this linear system is equivalent to the Calogero-Moser equations (2.23). The Baker-Akhiezer function $Y$ takes the form

$$Y = \Phi e^{t \frac{\partial}{\partial \tau_0} \mathcal{S}^{(0)}}\,.$$

The inverse procedure from the autonomous integrable equations to the non-autonomous monodromy preserving equation is an example of the Whitham quantization.

The same quasi-classical limit can be applied for the monodromy preserving equations that will be considered below.

# 3 Isomonodromic deformations and Elliptic Top

## 3.1 Abstract Euler-Arnold top (EAT).

Let $G$ be a Lie group, $\mathfrak{g}$ ($\mathfrak{g}^*$) its Lie algebra (coalgebra). $\mathfrak{g}^*$ is a Poisson space with the linear Lie-Poisson brackets

$$\{S_\alpha, S_\beta\}_1 = C^\gamma_{\alpha\beta} S_\gamma \,,$$

where $C^\gamma_{\alpha\beta}$ are the structure constants of $\mathfrak{g}$. Note that the Lie-Poisson brackets are degenerated on $\mathfrak{g}^*$ and their symplectic leaves are coadjoint orbits of $G$. To descend to a particular coadjoint orbit $\mathcal{O}$ one should fix the values of the Casimirs for the linear bracket.

The equation of motion on $\mathfrak{g}^*$ is defined by the Hamiltonian functional:

$$\partial_t \mathbf{S} = \{H, \mathbf{S}\}_1 := \mathrm{ad}^*_{\nabla H} \mathbf{S} \,, \tag{3.1}$$

where $\nabla H \in \mathfrak{g}$ and $\mathrm{ad}^*$ is the coadjoint action.

The Hamiltonian of EAT has a special form. Define invertible symmetric operator

$$\mathbf{J} \,:\, \mathfrak{g}^* \to \mathfrak{g} \,.$$

The inverse operator is called *the inertia tensor*. The Hamiltonian of EAT is the quadratic functional of the form

$$H = -\frac{1}{2}\langle \mathbf{S}, \mathbf{J}(\mathbf{S})\rangle \,, \quad \mathbf{S} \in \mathfrak{g}^* \,,$$

where $\langle\ ,\ \rangle$ stands for the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. The equations of motion of EAT assume the form

$$\partial_t \mathbf{S} = \{\mathbf{S}, \mathbf{J}(\mathbf{S})\}_1 \,.$$

The phase space of the system (3.1) is a coadjoint orbit

$$\mathcal{R}^{EAT} = \{\mathbf{S} \in \mathfrak{g}^* \,|\, \mathbf{S} = g\mathbf{j}_0 g^{-1} \,,\ g \in \mathrm{SL}(N, \mathbb{C}) \,,\ \mathbf{j}_0 \in \mathfrak{g}^*\} \,. \tag{3.2}$$

## 3.2 Non-autonomous Elliptic top (NAET).

For $G = \mathrm{SL}(N, \mathbb{C})$ define the inverse inertia tensor in the following way. Let

$$E_2(\alpha) = E_2(\frac{\alpha_1 + \alpha_2\tau}{N}|\tau) \,, \quad \alpha = (\alpha_1, \alpha_2) \in \tilde{\mathbb{Z}}^{(2)}_N \,,$$

where $\tilde{\mathbb{Z}}^{(2)}_N$ is defined by (B.3)

$$\mathbf{J}(\tau, \mathbf{S}) \ :\ S_\alpha \to E_2(\alpha) S_\alpha \,, \quad \mathbf{J}(\tau, \mathbf{S}) = \mathbf{J}(\tau) \cdot \mathbf{S} \,.$$

Then the Hamiltonian of EAT assumes the form

$$H^{ET}(\mathbf{S},\tau)=\frac{2\pi^2}{N^2}\langle\mathbf{S},\mathbf{J}(\tau)\cdot\mathbf{S}\rangle=-\frac{1}{2}\sum_{\gamma\in\mathbb{Z}_N^{(2)}}S_\gamma E_2(\gamma)S_{-\gamma}\,.$$

The equation of motion of NAET can be read off from (3.1)

$$\kappa\partial_\tau\mathbf{S}=[\mathbf{J}(\tau)\cdot\mathbf{S},\mathbf{S}]\,,\quad(\,\kappa\partial_\tau S_\alpha=\{H^{ET},S_\alpha\}_1\,)\,. \tag{3.3}$$

## 3.3 Lax representation

### 3.3.1 Flat bundles of degree one

Let $E_N^1$ be a flat bundle over the deformed elliptic curve $\Sigma_\tau$ of rank $N$ and degree 1 with the connections

$$\left\{\begin{array}{l}\kappa\partial_w+L^{(1)}(w,\bar{w},\tau)\,,\\ \partial_{\bar{w}}+\bar{L}^{(1)}(w,\bar{w},\tau)\,.\end{array}\right. \tag{3.4}$$

For flat bundles of degree one the connections can be choose in the form
1. $\bar{L}^{(0)}=0$. From the flatness one has

$$\bar{\partial}L^{(0)}=0$$

2. The Lax matrix satisfies the quasi-periodicity conditions

$$\begin{array}{c}L^{(1)}(w+1)=QL^{(1)}(w)Q^{-1}\,,\\ L^{(1)}(w+\tau)=\Lambda L^{(1)}(w)\Lambda^{-1}+\frac{2\pi i\kappa}{N}\,,\end{array}$$

for $Q,\Lambda$ (B.1), (B.2). It means that there are no moduli parameters for $E_N^1$.
3. All degrees of freedom come from the residue

$$Res|_{w=0}\,L^{(1)}(w)=\mathbf{S}\,.$$

**Lemma 3.1** *The connection assumes the form*

$$L^{(1)}(w)=-\frac{\kappa}{N}\partial_w\ln\vartheta(w;\tau)Id+\sum_\gamma S_\alpha\varphi_\alpha(w)T_\alpha\,. \tag{3.5}$$

*where $\varphi(\alpha,w)$ is defined by (B.10), and $T_\alpha$ are the basis elements (B.4).*

Fixing the connections we come from (3.4) to the linear system on $\Sigma_\tau$

$$\left\{\begin{array}{ll}i. & (\kappa\partial_w+L^{(1)}(w))\Psi=0\,,\\ ii & \partial_{\bar{w}}\Psi=0\,.\end{array}\right. \tag{3.6}$$

As above, the independence of the monodromy of (3.6) means that the Baker-Akhiezer function satisfies the additional linear equation

$$iii.\ (\kappa\partial_\tau+M^{(1)})\Psi=0\,. \tag{3.7}$$

**Lemma 3.2** *The equation of motion of the non-autonomous top (3.3) is the monodromy preserving equation for (3.6) with the Lax representation*

$$\partial_\tau L^{(1)} - \partial_w M^{(1)} + \frac{1}{\kappa}[L^{(1)}, M^{(1)}] = 0\,, \tag{3.8}$$

*where $L^{(1)}$ is defined by (3.5),*

$$M^{(1)} = -\frac{\kappa}{N}\partial_\tau \ln \vartheta(w;\tau) Id + \sum_\gamma S_\gamma f_\gamma(w) T_\gamma\,,$$

*and $f_\gamma(w)$ is defined by (B.11).*

The proof of the equivalence of (3.3) and (3.8) is based on the same addition formula (A.16) and the same heat equation (A.10) as in the case of ICMS.

In the quasi-classical limit $\kappa \to 0$ we come to the integrable top on $\mathrm{SL}(N,\mathbb{C})$ [18].

# 4 Symplectic Hecke correspondence

We will construct a map from the phase space of ICMS (2.5) to the phase space of IET (3.2)

$$\Xi^+ \,:\, \mathcal{R}^{CM} \to \mathcal{R}^{EAT}\,, \quad ((\mathbf{v},\mathbf{u},\mathbf{p}) \mapsto \mathbf{S})\,,$$

such that $\Xi^+$ is the symplectic map

$$\Xi^{+*}\omega(\mathbf{S}) = \omega(\mathbf{v},\mathbf{u},\mathbf{p})\,.$$

To construct it we define the map of the sheaves of sections $\Gamma(E_N^{(0)}) \to \Gamma(E_N^{(1)})$ such that it is an isomorphism on the complement to $w$ and it has one-dimensional cokernel at $w \in \Sigma_\tau$ :

$$0 \to \Gamma(E_N^{(0)}) \stackrel{\Xi^+}{\to} \Gamma(E_N^{(1)}) \to \mathbb{C}|_w \to 0\,. \tag{4.1}$$

It is the so-called *upper modification* of the bundle $E_N^{(0)}$ at the point $w$. On the complement to the point $w$ consider the map

$$\Gamma(E_N^{(1)}) \stackrel{\Xi^-}{\leftarrow} \Gamma(E_N^{(0)})\,,$$

such that $\Xi^-\Xi^+ =$Id. It defines *the lower modification* at the point $w$. We will construct $\Xi^+$ at $w = 0$.

The upper modification of $E_N^{(0)}$ at $w = 0$ is performed by the gauge transform

$$L^{(1)} = \Xi^+\kappa\partial\Xi^{+-1} + \Xi^+ L^{(0)}\Xi^{+-1}\,. \tag{4.2}$$

Thereby, $\Xi^+$ is the symplectic map.

The upper modification $\Xi^+(z)$ according with its definition satisfies the following properties:

• *Quasi-periodicity*:

$$\Xi(z+1,\tau) = Q \times \Xi(z,\tau)\,, \tag{4.3}$$

$$\Xi(z+\tau,\tau) = \tilde{\Lambda}(z,\tau) \times \Xi(z,\tau) \times \mathrm{diag}(\mathbf{e}(u_j))\,, \quad \tilde{\Lambda}(z,\tau) = \mathbf{e}(-(z-\frac{\tau}{2})\Lambda \tag{4.4}$$

•• Let $\mathbf{r}_i$ be an eigen-vector of the orbit matrix $\mathbf{p} \in \tilde{\mathcal{O}}$ (2.3), $\mathbf{pr}_i = p_i^0 \mathbf{r}_i$. Then $\Xi\mathbf{r}_i = 0$. The former condition provides that the quasi-periods of the transformed Lax matrix corresponds to the bundle of degree one. The latter condition implies that $L^{(1)}$ has only a simple pole at $z = 0$. The residue at the pole is identified with $\mathbf{S}$.

We construct first $(N \times N)$- matrix $\tilde{\Xi}(z,\mathbf{u};\tau)$ that satisfies (4.3) and (4.4) but has a special one-dimensional kernel:

$$\tilde{\Xi}_{ij}(z,\mathbf{u};\tau) = \theta\left[\begin{array}{c} \frac{i}{N} - \frac{1}{2} \\ \frac{N}{2} \end{array}\right](z - Nu_j, N\tau), \tag{4.5}$$

where $\theta\left[\begin{array}{c} a \\ b \end{array}\right](z,\tau)$ is the theta function with a characteristic

$$\theta\left[\begin{array}{c} a \\ b \end{array}\right](z,\tau) = \sum_{j\in\mathbb{Z}} \mathbf{e}\left((j+a)^2\frac{\tau}{2} + (j+a)(z+b)\right)\,.$$

It can be proved that the kernel of $\tilde{\Xi}$ at $z = 0$ is generated by the following column-vector :

$$\left\{(-1)^l \prod_{j<k;j,k\neq l} \vartheta(u_k - u_j,\tau)\right\}, \quad l = 1,2,\cdots,N\,.$$

Then the matrix $\Xi(z,\mathbf{u},\mathbf{r}_i)$, $(\mathbf{r}_i = (r_{1,i},\ldots < r_{N,i}))$ assumes the form

$$\Xi(z,\mathbf{u},\mathbf{r}_i) = \tilde{\Xi}(z) \times \mathrm{diag}\left(\frac{(-1)^l}{r_{l,i}} \prod_{j<k;j,k\neq l} \vartheta(u_k - u_j,\tau)\right). \tag{4.6}$$

It leads to the map $\mathcal{R}^{CM_N} \to \mathcal{R}^{EAT}$.

For the most degenerate orbit (Example in subsection 2.1) in the spinless ICMS, defined by the coupling constant $\nu^2$, this transformation leads to the degenerate orbit of the NAET.

Note that equation for the spin variables of ICMS (2.12) reminds the equation of motion for the NAET with the coordinate-dependent operator $\mathbf{J}_\mathbf{u}$ (2.13). The only difference is the structure of the phase spaces $\mathcal{R}^{CM_N}$ (2.5) and $\mathcal{R}^{EAT}$ (3.2). The gauge transform $\Xi$ carries out the pass from $\mathcal{R}^{CM_N}$ to $\mathcal{R}^{EAT}$. It depends only on the part of variables on $\mathcal{R}^{CM_N}$, namely on $\mathbf{u}$ and $\mathbf{p}$ through the eigenvector $\mathbf{r}_j$.

For the rank two bundles it is possible to write down the explicit dependence $\mathbf{S}(u,v,\nu)$. We postpone this example to the general case of PVI. It should be mentioned that the analogous transformation was used in [21, 22] for other purposes.

# 5 Quadratic brackets and NAET

## 5.1 r-matrix structure

Define the classical $r$-matrix as an element of $\mathrm{End}(E_N^{(1)})\otimes\mathrm{End}(E_N^{(1)})$ [19] by

$$r(w) = \sum_\gamma \varphi_\gamma(w) T_\gamma \otimes T_{-\gamma}\,, \tag{5.7}$$

where $\varphi_\gamma$ is defined by (B.10). It satisfies the classical Yang-Baxter equation

$$[r^{(12)}(z-w), r^{(13)}(z)] + [r^{(12)}(z-w), r^{(23)}(w)] \\ +[r^{(13)}(z), r^{(23)}(w)] = 0\,. \tag{5.8}$$

By means of the $r$-matrix one can define the linear brackets.

**Proposition 5.1** *The Lie-Poisson brackets on* $\mathrm{sl}(N,\mathbb{C})$

$$\{S_\alpha, S_\beta\}_1 = \mathbf{C}(\alpha,\beta)S_{\alpha+\beta}$$

*in terms of the Lax operator* $L^{(1)}(\mathbf{S},w)$ *are equivalent to the following relation for the Lax operator*

$$\{L_1^{(1)}(w), L_2^{(1)}(w')\}_1 = [r(w-w'), L^{(1)}(w)\otimes Id + Id\otimes L^{(1)}(w')]$$
$$L_1^{(1)} = L^{(1)}\otimes Id\,,\quad L_2^{(1)} = Id\otimes L^{(1)}\,.$$

The proof is based on the Fay three-section formula (A.15).

## 5.2 Quadratic Poisson algebra

In addition to the $N^2-1$ variables $\mathbf{S} = \{S_\alpha,\ \alpha\in\tilde{\mathbb{Z}}_N^{(2)}\}$ introduce a new variable $S_0$ and the $\mathrm{GL}(N,\mathbb{C})$-valued Lax operator

$$\tilde{L} = -S_0 Id + L^{(1)}(\mathbf{S},w)\,.$$

It satisfies the classical exchange algebra:

$$\{\tilde{L}_1(w), \tilde{L}_2(w')\}_2 = [r(w-w'), \tilde{L}_1(w)\otimes\tilde{L}_2(w')] \tag{5.9}$$

These brackets are Poisson, since the Jacobi identity is provided by the classical Yang-Baxter equation.

**Proposition 5.2** *The quadratic Poisson algebra (5.9) in the coordinates* $(S_0,\mathbf{S})$ *takes the form*

$$\{S_\alpha, S_0\}_2 = \sum_{\gamma\neq\alpha} S_{\alpha-\gamma}S_\gamma(\wp_\theta(\gamma)-\wp(\alpha-\gamma))\mathbf{C}(\alpha,\gamma)\,, \tag{5.10}$$

$$\{S_\alpha, S_\beta\}_2 = S_0 S_{\alpha+\beta}\mathbf{C}(\alpha,\beta) + \sum_{\gamma\neq\alpha,-\beta} S_{\alpha-\gamma}S_{\beta+\gamma}\mathbf{f}(\alpha,\beta,\gamma)\mathbf{C}(\gamma,\alpha-\beta)\,,$$

*where*

$$\mathbf{f}(\alpha,\beta,\gamma) = E_1(\gamma) + E_1(\beta-\alpha+\gamma) - E_1(\beta+\gamma) + E_1(\alpha-\gamma)\,.$$

It is the classical Sklyanin-Feigin-Odesski (SFO) algebra [12, 20]. These brackets are extracted from (5.9) by means of (A.17), (A.18).

Two Poisson structures are called *compatible* (or, form *a Poisson pair*) if their linear combinations are Poisson structures as well. It turns out that the linear and quadratic

Poisson brackets on $\mathrm{gl}(N,\mathbb{C})$ are compatible, namely, there exists the one-parametric family of the Poisson brackets

$$\{\mathbf{S},\mathbf{S}\}_\lambda = \{\mathbf{S},\mathbf{S}\}_2 + \lambda\{\mathbf{S},\mathbf{S}\}_1\,.$$

In the case of integrable hierarchy of the Elliptic Top these compatible brackets provide the hierarchy with the bihamiltonian structure [5]. The hierarchies of the monodromy preserving equations are more intricate [23] and we do not consider here the hierarchy of NAET. However, we have the following manifestation of the bihamiltonian structure.

**Proposition 5.3** *In terms of the quadratic brackets the equation of motion of NAET (3.3) assumes the form*

$$\kappa\partial_\tau S_\alpha = \{S_0, S_\alpha\}_2\,.$$

The proof follows immediately from (5.10). Thus, we replace the linear brackets on quadratic but the quadratic Hamiltonian $H^{ET}$ on the linear $S_0$.

# 6 PVI as a non-autonomous Zhukovsky-Volterra gyrostat

We have already defined the degenerate case of PVI as NAET. The main goal of this Section is the rigid top form of $\mathrm{PVI}_{\alpha,\beta,\gamma,\delta}$ (2.15) with the arbitrary four parameters.

Let

$$\tilde\nu_a = \frac{1}{4}I_{ab}\nu_b\,,\quad \nu'_\alpha = -\tilde\nu_\alpha\left(\frac{\vartheta(\alpha)}{\vartheta'(0)}\right)^2\,,$$

$$I = \begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}\,,$$

and $\nu' = \sum \nu'_\alpha\sigma_\alpha$. The following statement holds

**Proposition 6.1** *The PVI (2.15) (or equivalently (2.16)) can be put in the form*

$$\kappa\partial_\tau\mathbf{S} = \{H,\mathbf{S}\}_1\,, \qquad (6.11)$$

*for the Hamiltonian*

$$H = -\frac{\pi^2}{2}\langle\mathbf{S},(\mathbf{J}(\tau)\cdot\mathbf{S}+\nu')\rangle\,, \qquad (6.12)$$

*and the Lie-Poisson $SL_2$-brackets.*

The Hamiltonian has additional linear in $\mathbf{S}$ term.

We will prove below that the same equation can be written in terms of the quadratic Poisson algebra.

## 6.1 Elliptic form of Painlevé VI equation

We remind the Hamiltonian

$$H^{PVI} = \frac{1}{2}v^2 + \sum_{a=0}^{3} \nu_a^2 E_2(u+\omega_a)\,, \quad \omega_a = \frac{a_1 + a_2\tau}{2}\,,\ a_{1,2} = (0,1) \tag{6.13}$$

that leads to PVI (2.16).

**Proposition 6.2 ([6])** *The equation (2.16) has the Lax form*

$$\partial_\tau L^{PVI} - \partial_w M^{PVI} + \frac{1}{\kappa}[L^{PVI}, M^{PVI}] = 0 \tag{6.14}$$

*with*

$$L^{PVI} = P + X\,, \quad P = \mathrm{diag}(v, -v)\,, \tag{6.15}$$

$$X_{12}(u,z) = \sum_a \tilde{\nu}_a \phi(2u, z+\omega_a)\,, \quad X_{21}(u,z) = X_{12}(-u,z)\,,$$

$$M^{PVI} = Y_{jk}\,, \quad (j \neq k\,,\ j,k = 1,2) \tag{6.16}$$

$$Y_{12}(u,z) = \sum_a \tilde{\nu}_a f(2u, z+\omega_a)\,, \quad Y_{21}(u,z) = Y_{12}(-u,z)\,.$$

## 6.2 Lax form of ZVG

The equation of motion of ZVG (6.11) can be rewritten in the form

$$\kappa\partial_\tau \mathbf{S} = [\mathbf{S}, \mathbf{J}(\tau)\cdot\mathbf{S} + \nu'] \tag{6.17}$$

**Proposition 6.3 ([7])** *The equation of motion of ZVG has the Lax form with*

$$L^{ZVG}(z) = -\frac{\kappa}{2}\partial_z \ln\vartheta(z;\tau)\sigma_0 + \sum_\alpha (S_\alpha\varphi_\alpha(z) + \nu_\alpha\varphi_\alpha(z-\omega_\alpha))\sigma_\alpha\,. \tag{6.18}$$

$$M^{ZVG} = -\frac{\kappa}{2}\partial_\tau \ln\vartheta(z;\tau)\sigma_0 + \sum_\alpha S_\alpha \frac{\varphi_1(z)\varphi_2(z)\varphi_3(z)}{\varphi_\alpha(z)}\sigma_\alpha - E_1(z)L^{ZVG}(z,\kappa=0)\,. \tag{6.19}$$

As in the degenerate case the same upper modification (4.2) transforms $L^{PVI}$ to $L^{ZVG}$. It leads to the following relations between the phase spaces

$$\begin{gathered}
S_1 = -v\frac{\theta_2(0)}{\vartheta'(0)}\frac{\theta_2(2u)}{\vartheta(2u)} - \frac{\kappa}{2}\frac{\theta_2(0)}{\vartheta'(0)}\frac{\theta_2'(2u)}{\vartheta(2u)} + \\
\tilde{\nu}_0\frac{\theta_2^2(0)}{\theta_3(0)\theta_4(0)}\frac{\theta_3(2u)\theta_4(2u)}{\vartheta^2(2u)} + \tilde{\nu}_1\frac{\theta_2^2(2u)}{\vartheta^2(2u)} + \tilde{\nu}_2\frac{\theta_2(0)}{\theta_4(0)}\frac{\theta_2(2u)\theta_4(2u)}{\vartheta^2(2u)} + \tilde{\nu}_3\frac{\theta_2(0)}{\theta_3(0)}\frac{\theta_2(2u)\theta_3(2u)}{\vartheta^2(2u)} \\
iS_2 = v\frac{\theta_3(0)}{\vartheta'(0)}\frac{\theta_3(2u)}{\vartheta(2u)} + \frac{\kappa}{2}\frac{\theta_3(0)}{\vartheta'(0)}\frac{\theta_3'(2u)}{\vartheta(2u)} - \\
\tilde{\nu}_0\frac{\theta_3^2(0)}{\theta_2(0)\theta_4(0)}\frac{\theta_2(2u)\theta_4(2u)}{\vartheta^2(2u)} - \tilde{\nu}_1\frac{\theta_3(0)}{\theta_2(0)}\frac{\theta_3(2u)\theta_2(2u)}{\vartheta^2(2u)} - \tilde{\nu}_2\frac{\theta_3(0)}{\theta_4(0)}\frac{\theta_3(2u)\theta_4(2u)}{\vartheta^2(2u)} - \tilde{\nu}_3\frac{\theta_3^2(2u)}{\vartheta^2(2u)} \\
S_3 = -v\frac{\theta_4(0)}{\vartheta'(0)}\frac{\theta_4(2u)}{\vartheta(2u)} - \frac{\kappa}{2}\frac{\theta_4(0)}{\vartheta'(0)}\frac{\theta_4'(2u)}{\vartheta(2u)} + \\
\tilde{\nu}_0\frac{\theta_4^2(0)}{\theta_2(0)\theta_3(0)}\frac{\theta_2(2u)\theta_3(2u)}{\vartheta^2(2u)} + \tilde{\nu}_1\frac{\theta_4(0)}{\theta_2(0)}\frac{\theta_2(2u)\theta_4(2u)}{\vartheta^2(2u)} + \tilde{\nu}_2\frac{\theta_4^2(2u)}{\vartheta^2(2u)} + \tilde{\nu}_3\frac{\theta_4(0)}{\theta_3(0)}\frac{\theta_4(2u)\theta_3(2u)}{\vartheta^2(2u)}
\end{gathered} \tag{6.20}$$

# 7 Reflection equation and generalized Sklyanin algebra

In this Section we present another Hamiltonian form of PVI (6.11). It is based on the quadratic Poisson brackets. The quantization of these brackets is described by quantum reflection equation.

## 7.1 Quantum reflection equation

Let $R^-$ be the quantum vertex R-matrix, that arises in the XYZ model. We introduce also the matrix $R^+$

$$R^{\pm}(z,w) = 1 \otimes 1\phi^{\frac{\hbar}{2}}(z \pm w) + \sum_{\alpha} \sigma_\alpha \otimes \sigma_\alpha \varphi_\alpha^{\frac{\hbar}{2}}(z \pm w) \tag{7.1}$$

where $\varphi_a^{\frac{\hbar}{2}}$ is defined by (B.13). Define the quantum Lax operator

$$\hat{L}(z) = \hat{S}_0 \phi^{\hbar}(z)\sigma_0 + \sum_{\alpha}(\hat{S}_\alpha \varphi_\alpha^{\hbar}(z) + \nu_\alpha \varphi_\alpha^{\hbar}(z - \omega_\alpha))\sigma_\alpha \,. \tag{7.2}$$

**Proposition 7.1** *The Lax operator satisfies the quantum reflection equation*

$$R^-(z,w)\hat{L}_1(z)R^+(z,w)\hat{L}_2(w) = \hat{L}_2(w)R^+(z,w)\hat{L}_1(z)R^-(z,w)\,. \tag{7.3}$$

*iff its components $S_a$ generate the associative algebra with relations*

$$[\nu_\alpha, \nu_\beta] = 0\,, \quad [\nu_\alpha, \hat{S}_a] = 0\,, \tag{7.4}$$

$$i[\hat{S}_0, \hat{S}_\alpha]_+ = [\hat{S}_\beta, \hat{S}_\gamma]\,, \tag{7.5}$$

$$[\hat{S}_\gamma, \hat{S}_0] = i\frac{K_\beta - K_\alpha}{K_\gamma}[\hat{S}_\alpha, \hat{S}_\beta]_+ + 2i\frac{1}{K_\gamma}(\nu_\alpha \rho_\alpha \hat{S}_\beta - \nu_\beta \rho_\beta \hat{S}_\alpha)\,, \tag{7.6}$$

*where*

$$K_\alpha = E_1(\hbar + \omega_\alpha) - E_1(\hbar) - E_1(\omega_\alpha)\,, \quad \rho_\alpha = \mathbf{e}(\omega_\alpha \partial_\tau \omega_\alpha)\phi(\omega_\alpha + \hbar, -\omega_\alpha)\,.$$

If all $\nu_\alpha = 0$ (7.4-7.6) the algebra coincides with the Sklyanin algebra. Therefore, the algebra (7.4) – (7.6) is a three parametric deformation of the Sklyanin algebra [12].

## 7.2 Classical reflection equations

Consider (7.3) in the limit $\hbar \to 0$. The classical $r^{\pm}$-matrices are defined from the expansion

$$R^{\pm}(z,w) = \left(\frac{\hbar}{2}\right)^{-1} \sigma_0 \otimes \sigma_0 + r^{\pm}(z,w) + O(\hbar)\,,$$

$$r^{\pm}(z,w) = \sum_{\alpha} \varphi_\alpha(z \pm w)\sigma_\alpha\,.$$

For the Lax operator one has

$$\hat{L}_(z) = \hbar^{-1}\hat{S}_0\sigma_0 + \sum_{\alpha}(\hat{S}_\alpha\varphi_\alpha(z) + \nu_\alpha\varphi_\alpha(z-\omega_\alpha))\sigma_\alpha + O(\hbar)\,.$$

Define the classical variables

$$\hat{S}_\alpha \to S_\alpha\,,\quad \hat{S}_0 \to \hbar S_0\,,$$

and the corresponding classical Lax operator

$$\tilde{L}(z) = S_0\sigma_0 + \sum_{\alpha=1}^{3}(S_\alpha\varphi_\alpha(z) + \nu_\alpha\varphi_\alpha(z-\omega_\alpha))\sigma_\alpha\,.$$

Then, taking into account that $[L_1, L_2] = \hbar\{\tilde{L}_1, \tilde{L}_2\} + O(\hbar^2)$ one finds the classical reflection equation in the first order of $\hbar^{-1}$

$$\{\tilde{L}_1(z), \tilde{L}_2(w)\}_2 = \frac{1}{2}[\tilde{L}_1(z)\tilde{L}_2(w), r^-(z,w)]+ \tag{7.7}$$

$$\frac{1}{2}\tilde{L}_2(w)r^+(z,w)\tilde{L}_1(z) - \frac{1}{2}\tilde{L}_1(z)r^+(z,w)\tilde{L}_2(w)\,.$$

by passing from the group-valued element $\tilde{L}$ to the Lie-algebraic element $L$

$$\tilde{L}(z) \to L(z) = \sum_{\alpha=1}^{3}(S_\alpha\varphi_\alpha(z) + \nu_\alpha\varphi_\alpha(z-\omega_\alpha))\sigma_\alpha$$

we come to the linear brackets

$$\{L_1(z), L_2(w)\}_1 = -\frac{1}{2}[r^-(z,w), L_1(z) + L_2(w)] + \frac{1}{2}[r^+(z,w), L_1(z) - L_2(w)]\,. \tag{7.8}$$

**Proposition 7.2** *The classical reflection equations (7.7), (7.8) leads to the two compatible Poisson algebra*

$$\{S_\alpha, S_\beta\}_2 = 2i\varepsilon_{\alpha\beta\gamma}S_0S_\gamma\,, \tag{7.9}$$

$$\{S_0, S_\alpha\}_2 = \varepsilon_{\alpha\beta\gamma}S_\beta S_\gamma(E_2(\omega_\beta) - E_2(\omega_\gamma)) + \varepsilon_{\alpha\beta\gamma}S_\beta\nu'_\gamma\,,$$

*and $sl_2$ Lie-Poisson algebra*

$$\{S_\alpha, S_\beta\}_1 = 2i\varepsilon_{\alpha\beta\gamma}S_\gamma\,. \tag{7.10}$$

The equation of motion of PVI (6.11) is written in terms of the linear brackets. The straightforward calculations shows that (6.11) can be written in the form

$$\partial_\tau S_\alpha = \{H_0, S_\alpha\}_2\,,\quad H_0 = S_0\,. \tag{7.11}$$

In this way for the generic form of PVI we have the same analog of the bihamiltonian property as in the degenerate case (Proposition 5.4).

## 7.3 Spin chain with boundaries

Quantum reflection equation allows us to define XYZ model on a finite lattice with boundary conditions [13]. The Lax operator (7.2) can be considered as a new solution of the reflection equation.

As it is known from [13], if there is a pair of matrices $K^{\pm}(z)$ with Poisson brackets

$$\{K_1^{\pm}, K_2^{\pm}\} = [K_1^{\pm}(z)K_2^{\pm}(w), r(z-w)] + K_2^{\pm}(w)r(z+w)K_1^{\pm}(z) - K_1^{\pm}(z)r(z+w)K_2^{\pm}(w) \tag{7.12}$$

and $L^i(z),\ i = 1...N$ with brackets

$$\{L_1^i(z), L_2^j(w)\} = \delta^{ij}[r(z-w), L_1^i(z)L_2^j(w)]\,, \tag{7.13}$$

then

$$h(z) = tr\left[K^+(z)L^N(z)...L^1(z)K^-(z)\left(L^1(-z)\right)^{-1}...\left(L^N(-z)\right)^{-1}\right] \tag{7.14}$$

is the generating function of hamiltonians since

$$\{h(z), h(w)\} = 0\,. \tag{7.15}$$

Choosing $K^{\pm}(z) = L(\mathbf{S}^{\pm}, \nu^{\pm}, z)$ we construct a spin chain with boundaries. But in (7.7) we have a factor $\frac{1}{2}$ which comes from (7.1) on the quantum level. Thus, we should put the brackets on boundaries for $L^{ZVG_{\pm}}(z)$ to be two times more than in (7.9). In other words $R$-matrices for $\hat{L}^{ZVG_{\pm}}(z)$ should depend on the same Planck constant $\hbar$ as for all other $\hat{L}^i(z)$.

**Proposition 7.3** *Spin chain involving $N$ internal vertices $L^i(z)$ with boundaries $L^{ZVG_{\pm}}(z)$ is integrable for the case when*

$$\begin{array}{c} \{S_\alpha^{\pm}, S_\beta^{\pm}\} = 4\sqrt{-1}\varepsilon_{\alpha\beta\gamma}S_0^{\pm}S_\gamma^{\pm}\,, \\ \{S_0^{\pm}, S_\alpha^{\pm}\} = 2\sqrt{-1}\varepsilon_{\alpha\beta\gamma}(S_\beta^{\pm}S_\gamma^{\pm}\,, (\wp(\omega_\beta) - \wp(\omega_\gamma)) + S_\beta^{\pm}\nu'_\gamma) \end{array} \tag{7.16}$$

*and for $i, j = 1..N$*

$$\begin{array}{c} \{S_\alpha^i, S_\beta^j\} = 2\delta^{ij}\sqrt{-1}\varepsilon_{\alpha\beta\gamma}S_0^iS_\gamma^i\,, \\ \{S_0^i, S_\alpha^j\} = \delta^{ij}\sqrt{-1}\varepsilon_{\alpha\beta\gamma}S_\beta^iS_\gamma^i(\wp(\omega_\beta) - \wp(\omega_\gamma))\,. \end{array} \tag{7.17}$$

*The nearest-neighbour interaction is described by the Hamiltonian:*

$$\begin{array}{c} H = \sum_\alpha S_\alpha^- S_\alpha^1(C - \wp(\omega_\alpha)) + (\nu')_\alpha^- S_\alpha^1 + S_\alpha^N S_\alpha^+(C - \wp(\omega_\alpha)) + (\nu')_\alpha^+ S_\alpha^N + \\ S_0^- S_0^1 + S_0^N S_0^+ + \sum_{i=1}^{N-1}\left(S_0^i S_0^{i+1} + \sum_\alpha S_\alpha^i S_\alpha^{i+1}(C - \wp(\omega_\alpha))\right)\,, \end{array} \tag{7.18}$$

*where $C$ is a constant which is equal to the fraction of the values of the Casimir functions for Sklyanin bracket (7.17).*

It should be supposed that the quantum Hamiltonian is of the same form.

# 8 Appendix

## 8.1 Appendix A. Elliptic functions.

We assume that $q = \exp 2\pi i\tau$, where $\tau$ is the modular parameter of the elliptic curve $E_\tau$.

The basic element is the theta function:

$$\vartheta(z|\tau) = q^{\frac{1}{8}} \sum_{n\in\mathbf{Z}} (-1)^n e^{\pi i(n(n+1)\tau+2nz)} = \tag{A.1}$$

$$q^{\frac{1}{8}} e^{-\frac{i\pi}{4}} (e^{i\pi z} - e^{-i\pi z}) \prod_{n=1}^{\infty} (1-q^n)(1-q^n e^{2i\pi z})(1-q^n e^{-2i\pi z}) .$$

*The Eisenstein functions*

$$E_1(z|\tau) = \partial_z \log\vartheta(z|\tau), \quad E_1(z|\tau) \sim \frac{1}{z} - 2\eta_1 z, \tag{A.2}$$

where

$$\eta_1(\tau) = \frac{3}{\pi^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{1}{(m\tau+n)^2} = \frac{24}{2\pi i} \frac{\eta'(\tau)}{\eta(\tau)} , \tag{A.3}$$

where

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n>0} (1-q^n) .$$

is the Dedekind function.

$$E_2(z|\tau) = -\partial_z E_1(z|\tau) = \partial_z^2 \log\vartheta(z|\tau), \quad E_2(z|\tau) \sim \frac{1}{z^2} + 2\eta_1 . \tag{A.4}$$

*Relation to the Weierstrass functions*

$$\zeta(z,\tau) = E_1(z,\tau) + 2\eta_1(\tau) z , \tag{A.5}$$

$$\wp(z,\tau) = E_2(z,\tau) - 2\eta_1(\tau) . \tag{A.6}$$

The next important function is

$$\phi(u,z) = \frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)} . \tag{A.7}$$

It has a pole at $z = 0$ and

$$\phi(u,z) = \frac{1}{z} + E_1(u) + \frac{z}{2}(E_1^2(u) - \wp(u)) + \dots , \tag{A.8}$$

and

$$\phi(u,z)^{-1} \partial_u \phi(u,z) = E_1(u+z) - E_1(u) . \tag{A.9}$$

*Heat equation*

$$\partial_\tau \phi(u,w) - \frac{1}{2\pi i} \partial_u \partial_w \phi(u,w) = 0 . \tag{A.10}$$

*Quasi-periodicity*

$$\vartheta(z+1) = -\vartheta(z)\,, \quad \vartheta(z+\tau) = -q^{-\frac{1}{2}}e^{-2\pi i z}\vartheta(z)\,, \tag{A.11}$$

$$E_1(z+1) = E_1(z)\,, \quad E_1(z+\tau) = E_1(z) - 2\pi i\,, \tag{A.12}$$

$$E_2(z+1) = E_2(z)\,, \quad E_2(z+\tau) = E_2(z)\,, \tag{A.13}$$

$$\phi(u+1,z) = \phi(u,z)\,, \quad \phi(u+\tau,z) = e^{-2\pi i z}\phi(u,z)\,. \tag{A.14}$$

*The Fay three-section formula:*

$$\phi(u_1,z_1)\phi(u_2,z_2) - \phi(u_1+u_2,z_1)\phi(u_2,z_2-z_1) - \phi(u_1+u_2,z_2)\phi(u_1,z_1-z_2) = 0\,. \tag{A.15}$$

Particular cases of this formula is the Calogero functional equation

$$\phi(u,z)\partial_v\phi(v,z) - \phi(v,z)\partial_u\phi(u,z) = (E_2(v) - E_2(u))\phi(u+v,z)\,, \tag{A.16}$$

Another important relation is

$$\phi(v,z-w)\phi(u_1-v,z)\phi(u_2+v,w) - \phi(u_1-u_2-v,z-w)\phi(u_2+v,z)\phi(u_1-v,w) = \phi(u_1,z)\phi(u_2,w)f(u_1,u_2,v)\,, \tag{A.17}$$

where

$$f(u_1,u_2,v) = \zeta(v) - \zeta(u_1-u_2-v) + \zeta(u_1-v) - \zeta(u_2+v)\,. \tag{A.18}$$

One can rewrite the last function as

$$f(u_1,u_2,v) = -\frac{\vartheta'(0)\vartheta(u_1)\vartheta(u_2)\vartheta(u_2-u_1+2v)}{\vartheta(u_1-v)\vartheta(u_2+v)\vartheta(u_2-u_1+v)\vartheta(v)}\,. \tag{A.19}$$

## 8.2 Appendix B. Lie algebra $\mathrm{sl}(N,\mathbb{C})$

Introduce the notation $\mathbf{e}(z) = \exp(\frac{2\pi i}{N}z)$, and two matrices

$$Q = \mathrm{diag}(\mathbf{e}(1),\ldots,\mathbf{e}(m),\ldots,1) \tag{B.1}$$

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{B.2}$$

Let

$$\mathbb{Z}_N^{(2)} = (\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z})\,, \quad \tilde{\mathbb{Z}}_N^{(2)}) = \mathbb{Z}_N^{(2)} \setminus (0,0) \tag{B.3}$$

be the two-dimensional of order $N^2$ and $N^2-1$ correspondingly. The matrices $Q^{a_1}\Lambda^{a_2}$, $a = (a_1,a_2) \in \mathbb{Z}_N^{(2)}$ generate a basis in $\mathrm{GL}(N,\mathbb{C})$, while $Q^{\alpha_1}\Lambda^{\alpha_2}$, $\alpha = (\alpha_1,\alpha_2) \in \tilde{\mathbb{Z}}_N^{(2)}$ generate a basis in $\mathrm{sl}(N,\mathbb{C})$. Consider the projective representation of $\mathbb{Z}_N^{(2)}$ in $\mathrm{GL}(N,\mathbb{C})$

$$a \to T_a = \frac{N}{2\pi i}\mathbf{e}(\frac{a_1a_2}{2})Q^{a_1}\Lambda^{a_2}\,, \tag{B.4}$$

$$T_aT_b = \frac{N}{2\pi}\mathbf{e}(-\frac{a\times b}{2})T_{a+b}\,, \quad (a\times b = a_1b_2 - a_2b_1) \tag{B.5}$$

Here $\frac{N}{2\pi}\mathbf{e}(-\frac{a\times b}{2})$ is a non-trivial two-cocycle in $H^2(\mathbb{Z}_N^{(2)}, \mathbb{Z}_{2N}$. It follows from (B.5) that

$$[T_\alpha, T_\beta] = \mathbf{C}(\alpha,\beta)T_{\alpha+\beta}\,, \tag{B.6}$$

where $\mathbf{C}_\theta(\alpha,\beta) = \frac{N}{\pi}\sin\frac{\pi}{N}(\alpha\times\beta)$ are the structure constants of $\mathrm{sl}(N,\mathbb{C})$.

Introduce the following constants on $\tilde{\mathbb{Z}}(2)$:

$$\vartheta(\gamma) = \vartheta(\frac{\gamma_1+\gamma_2\tau}{N})\,, \tag{B.7}$$

$$E_1(\gamma) = E_1(\frac{\gamma_1+\gamma_2\tau}{N})\,, \quad E_2(\gamma) = E_2(\frac{\gamma_1+\gamma_2\tau}{N})\,, \tag{B.8}$$

and the quasi-periodic functions on $\Sigma_\tau$

$$\phi_\gamma(z) = \phi(\frac{\gamma_1+\gamma_2\tau}{N}, z)\,, \tag{B.9}$$

$$\varphi_\gamma(z) = \mathbf{e}(\gamma_2 z)\phi(\gamma, z)\,, \tag{B.10}$$

$$f_\gamma(z) = \mathbf{e}(\gamma_2 z)\partial_u\phi(u,z)|_{u=\frac{\gamma_1+\gamma_2\tau}{N}}\,. \tag{B.11}$$

It follows from (A.7) that

$$\varphi_\gamma(z+1) = \mathbf{e}(\gamma_2)\varphi_\theta(\gamma,z)\,, \quad \varphi_\gamma(z+\tau) = \mathbf{e}(-\gamma_1)\varphi_\theta(\gamma,z)\,. \tag{B.12}$$

*Deformed functions*

$$\varphi_a^\eta(z) = \mathbf{e}(a_2 z)\phi(\frac{a_1+a_2\tau}{N}+\eta, z)\,, \ a\in\mathbb{Z}_N^{(2)}\,. \tag{B.13}$$

it follows from (B.12) that $\varphi_a^\eta(z)$ is well defined on $\mathbb{Z}_N^{(2)}$:

$$\varphi_{a+c}^\eta(z) = \varphi_a^\eta(z)\,, \text{ for } c_{1,2}\in\mathbb{Z} \bmod N\,. \tag{B.14}$$

$\mathrm{SL}(2,\mathbb{C})$ *case*

For $\mathrm{SL}(2,\mathbb{C})$ instead of $T_\alpha$ we use the basis of sigma-matrices

$$\sigma_0 = Id\,, \quad \sigma_1 = \pi T_{0,1}\,, \quad \sigma_2 = \pi T_{1,1}\,, \quad \sigma_3 = \pi T_{1,0}\,, \tag{B.15}$$

$\sigma = \{\sigma_a\} = \{\sigma_0, \sigma_\alpha\}\,, (a = 0,\alpha)\,, (\alpha = 1,2,3)$.

The standard theta-functions with the characteristics are

$$\theta_{0,0} = \theta_3\,, \quad \theta_{1,0} = \theta_2\,, \quad \theta_{0,1} = \theta_4\,, \quad \theta_{1,1} = \theta_1\,. \tag{B.16}$$

Thus we have the following table of correspondences in the notations

| $\alpha$ | (1,0) | (0,1) | (1,1) |
|---|---|---|---|
| $\sigma_\alpha$ | $\sigma_3$ | $\sigma_1$ | $\sigma_2$ |
| half-periods | $\omega_1 = \frac{1}{2}$ | $\omega_2 = \frac{\tau}{2}$ | $\omega_3 = \frac{1+\tau}{2}$ |
| $\varphi_\alpha(z)$ | $\frac{\theta_2(z)\theta_1'(0)}{\theta_2(0)\theta_1(z)}$ | $\frac{\theta_4(z)\theta_1'(0)}{\theta_4(0)\theta_1(z)}$ | $\frac{\theta_3(z)\theta_1'(0)}{\theta_3(0)\theta_1(z)}$ |

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M.A.Olshanetsky
Institute of Theoretical and Experimental Physics, Moscow, Russia,
e-mail `olshanet@itep.ru`

A.V.Zotov
Institute of Theoretical and Experimental Physics, Moscow, Russia,
e-mail `zotov@itep.ru`