# Form factors and vertex operators in the XYZ antiferromagnet*

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**Abstract**

This article summarizes my talk on form factors in the reflectionless eight-vertex model, given in the conference 'Elliptic Integrable Systems', held as a part of RIMS Project, Department of Mathematics, Kyoto University, November 8-11, 2004. It is shown that the form factors of $\widehat{\sigma^z}$ in the eight-vertex model at the so-called reflectionless points can be expressed in terms of the sum of theta functions without any integrals.

## 1 Introduction

The eight-vertex model can be specified by two parameters $q$ and $p$, where $p=(-q)^{2r}$. For general $r>1$ case, the bosonization recipe to obtain the eight-vertex form factor was given by Lashkevich [1]. For rational $r>1$, integral representations for the vertex operators in the eight-vertex model were conjectured by using certain representations of deformed W-algebras by Shiraishi [2]. As far as I know, the connection between the eight-vertex model and deformed W-algebras is not clear. As a trial to clarify that connection, in this article we consider form factors in the eight-vertex model at the reflectionless points. Related topics were also discussed in [3].

Let $r=r_N:=1+\frac{1}{N}$ $(N=1,2,3,\cdots)$. Then the eight-vertex model is at a reflectionless point. In particular, the eight-vertex model at $r=2$ $(N=1)$ reduces to the double Ising model. When $r=r_N$, $S$-matrix $S(u)=-R(u;r-1,\epsilon)$ becomes (anti-)diagonal. Thus, we expect that the form factor formulae will be simple at reflectionless points. Form factors are originally defined as matrix elements of local operators. In what follows, we consider the case

$$\mathcal{O}=\widehat{\sigma^z}=\sum_{\varepsilon=\pm}\varepsilon\Phi_{-\varepsilon}(u-1)\Phi_{\varepsilon}(u).$$

## 2 Basic definitions

The $R$-matrix of the eight-vertex model in the principal regime is given as follows:

$$R(u)=\frac{1}{\kappa(u)}\begin{bmatrix} a(u) & & & d(u) \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ d(u) & & & a(u) \end{bmatrix} \tag{2.1}$$

*Partly based on a joint work with Michael Lashkevich.

$$
\begin{aligned}
\tilde{\kappa}(u) &= \zeta^{\frac{r-1}{r}}\frac{g(z)}{g(z^{-1})}, \ (z=\zeta^2=x^{2u}, x=e^{-\epsilon}) \\
g(z) &= \frac{(x^2z;x^4,x^{2r})_\infty (x^{2r+2}z;x^4,x^{2r})_\infty}{(x^4z;x^4,x^{2r})_\infty (x^{2r}z;x^4,x^{2r})_\infty}, \\
a(u) &= g^{--}_{--}(u), \quad b(u)=g^{+-}_{+-}(u), \quad c(u)=g^{-+}_{+-}(u), \quad d(\zeta) = -g^{++}_{--}(u), \\
g^{\varepsilon_1\varepsilon_2}_{\varepsilon_3 -}(u) &= \frac{h_{\varepsilon_1}(u)h_{\varepsilon_2}(1)}{h_{\varepsilon_3}(1-u)h_-(0)}, \qquad h_\varepsilon(u) := \begin{cases} \theta_1(\frac{u}{2r};\frac{\pi\sqrt{-1}}{2\epsilon r}) & (\varepsilon>0) \\ \theta_2(\frac{u}{2r};\frac{\pi\sqrt{-1}}{2\epsilon r}), & (\varepsilon<0), \end{cases}
\end{aligned}
$$

where $0<u<1$, $r>1$, $\epsilon>0$.

The nonzero Boltzmann weights of the eight-vertex SOS model in Regime III are given as follows:

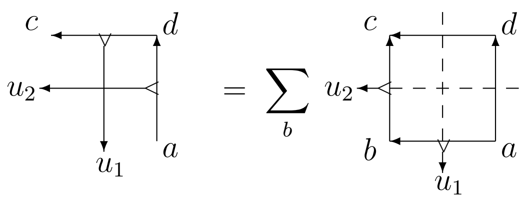
$$
\begin{aligned}
W\left[\begin{array}{cc} k\pm 2 & k\pm 1 \\ k\pm 1 & k \end{array}\middle| u\right] &= \frac{1}{\kappa(u)}, \\
W\left[\begin{array}{cc} k & k\pm 1 \\ k\pm 1 & k \end{array}\middle| u\right] &= \frac{1}{\kappa(u)}\frac{[1][k\pm u]}{[1-u][k]}, \\
W\left[\begin{array}{cc} k & k\pm 1 \\ k\mp 1 & k \end{array}\middle| u\right] &= -\frac{1}{\kappa(u)}\frac{[u][k\pm 1]}{[1-u][k]}.
\end{aligned}
\tag{2.2}
$$

Here,

$$
\begin{aligned}
[u] &= x^{\frac{u^2}{r}-u}\Theta_{x^{2r}}(x^{2u}), \\
\Theta_p(z) &= (z;p)_\infty(pz^{-1};p)_\infty(p;p)_\infty = \sum_{n\in\mathbb{Z}} p^{n(n-1)/2}(-z)^n.
\end{aligned}
$$

The local state $k\in\mathbb{Z}+\delta$ ($\delta$ is an irrational number), and the difference of adjoining sites should be equal to 1. Then we have the so-called vertex-face correspondence [4]:

$$
R(u_1-u_2)t^c_d(u_0-u_1)\otimes t^d_a(u_0-u_2) = \sum_b W\left[\begin{array}{cc} c & d \\ b & a \end{array}\middle| u_1-u_2\right] t^b_a(u_0-u_1)\otimes t^c_b(u_0-u_2). \tag{2.3}
$$



Introduce the following basic bosons:

$$[\beta_m, \beta_n] = m\frac{[m]_x[(r-1)m]_x}{[2m]_x[rm]_x}\delta_{m+n,0}, \qquad [Q,P] = \sqrt{-1},$$

where

$$[m]_x := \frac{x^m - x^{-m}}{x - x^{-1}}.$$

Let $\mathcal{F}_{l,k} := \mathbb{C}[\beta_{-1}, \beta_{-2}, \cdots]|l,k\rangle$ be the Fock space with the highest weight $|l,k\rangle$ for $k, l \in \mathbb{Z} + \delta$ ($\delta \notin \mathbb{Q}$) such that

$$\beta_n|l,k\rangle = 0 \ (n > 0), \qquad P|l,k\rangle = (\alpha_1 k + \alpha_2 l)|l,k\rangle.$$

Here, $\alpha_1 < \alpha_2$ are two roots of the following quadratic equation:

$$t^2 - \frac{\alpha_0}{2}t - \frac{1}{2} = (t-\alpha_1)(t-\alpha_2), \qquad \alpha_0 = \sqrt{\frac{2}{r(r-1)}}. \tag{2.4}$$

Introduce $\varphi_j$ ($j = 1, 2, 0$) as

$$\begin{aligned} \varphi_1(z) &:= \alpha_1(\sqrt{-1}Q + P\log z) - \sum_{m\neq 0}\frac{\beta_m}{m}z^{-m}, \qquad (2.5)\\ \varphi_2(z) &:= \alpha_2(\sqrt{-1}Q + P\log z) + \sum_{m\neq 0}\frac{\beta_m}{m}\frac{[rm]_x}{[(r-1)m]_x}(-z)^{-m}\\ \varphi_0(z) &:= -\alpha_0(\sqrt{-1}Q + P\log(-z)) - \sum_{m\neq 0}\frac{\beta_m}{m}\frac{[2m]_x}{[(r-1)m]_x}z^{-m}. \end{aligned}$$

Let

$$\Phi_+(u) = z^{\frac{r-1}{4r}} : \exp(\varphi_1(z)) :, \qquad A(v) = w^{\frac{r-1}{r}} : \exp(-\varphi_1(xw) - \varphi_1(x^{-1}w)) :, \tag{2.6}$$

where $z = x^{2u}$, $w = x^{2v}$. Then the type I vertex operator in the eight-vertex SOS model on $\mathcal{F}_{l,k}$:

$$\Phi_k^{k+1}(u) = \frac{1}{[k]}\Phi_+(u), \qquad \Phi_k^{k-1}(u) = \frac{1}{[k]}\Phi_+(u)X(u) = -\frac{1}{[k]}Y(u)\Phi_+(u), \tag{2.7}$$

where

$$\begin{aligned} X(u) &= \frac{1}{\lambda}\oint_C \frac{dw}{2\pi\sqrt{-1}w}A(v)\frac{[v-u+\frac{1}{2}-k]}{[v-u-\frac{1}{2}]},\\ Y(u) &= \frac{1}{\lambda}\oint_C \frac{dw}{2\pi\sqrt{-1}w}A(v)\frac{[v-u+\frac{3}{2}-k]}{[v-u+\frac{1}{2}]}. \end{aligned} \tag{2.8}$$

Here, the contour $C$ is chosen such that the poles from the factor $[v-u-\frac{1}{2}]$ at $w = x^{1+2nr}z$ are inside if $n \in \mathbb{Z}_{\geqslant 0}$ (outside if $n \in \mathbb{Z}_{<0}$) and the poles resulting from the normal ordering of $\Phi_+(u)A(v)$ at $w = x^{-1-2n'r}z$ ($n' \in \mathbb{Z}_{\geqslant 0}$) are outside. The factor $\lambda$ can be determined from the inversion property of $\Phi$ and $\Phi^*$:

$$\Phi^{*k'}_{\ k}(u) = [k]\Phi_k^{k'}(u-1), \qquad \lambda = \frac{x^{\frac{1-r}{2r}}}{[1]}(x^{2r-2}; x^{2r})_\infty\frac{(x^4; x^4, x^{2r})_\infty(x^{2r+4}; x^4, x^{2r})_\infty}{(x^2; x^4, x^{2r})_\infty(x^{2r+2}; x^4, x^{2r})_\infty} \tag{2.9}$$

Let

$$\Psi^*_+(u) = z^{\frac{r}{4(r-1)}} : \exp(\varphi_2(z)) :, \qquad B(v) = w^{\frac{r}{r-1}} : \exp(-\varphi_2(xw) - \varphi_2(x^{-1}w)) : . \tag{2.10}$$

Then the type II vertex operator in the eight-vertex SOS model on $\mathcal{F}_{l,k}$:

$$\Psi^{*}{}_{l}^{l+1}(u) = \Psi^*_+(u), \qquad \Psi^{*}{}_{l}^{l-1}(u) = \Psi^*_+(u) X'(u), \tag{2.11}$$

where

$$X'(u) = \frac{1}{\lambda'} \oint_{C'} \frac{dw}{2\pi\sqrt{-1}w} B(v) \frac{[v-u-\frac{1}{2}+l]'}{[v-u+\frac{1}{2}]'}, \tag{2.12}$$

and $[u]' = [u]|_{r\to r-1}$. Here, the contour $C'$ is chosen such that the poles from the factor $[v-u+\frac{1}{2}]'$ at $w = x^{-1+2n(r-1)}z$ are inside if $n \in \mathbb{Z}_{\geqslant 0}$ (outside if $n \in \mathbb{Z}_{<0}$) and the poles resulting from the normal ordering of $\Psi^*_+(u)B(v)$ at $w = x^{1-2n'(r-1)}z$ ($n' \in \mathbb{Z}_{\geqslant 0}$) are outside. The factor $\lambda'$ can be determined from the inversion property of $\Psi^*$ and $\Psi$.

$$\Psi_l^{l'}(u) = \frac{1}{[l]'} \Psi^{*}{}_{l}^{l'}(u-1), \qquad \lambda' = \frac{x^{\frac{r}{2(r-1)}}}{[1]'} \frac{(x^{2r}; x^{2r-2})_\infty}{(x^{2r-2}; x^{2r-2})_\infty} \frac{(x^2; x^4, x^{2r-2})_\infty (x^{2r}; x^4, x^{2r-2})_\infty}{(x^4; x^4, x^{2r-2})_\infty (x^{2r-2}; x^4, x^{2r-2})_\infty} \tag{2.13}$$

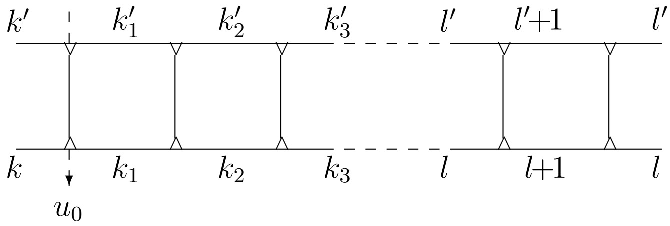
Let us introduce one more vertex operator:

$$W_-(u) = W(u - \tfrac{r-1}{2}), \qquad W(u) = (-z)^{\frac{1}{r(r-1)}} : \exp(\varphi_0(z)). \tag{2.14}$$

The objects of the eight-vertex model can be expressed in terms of those of the eight-vertex SOS model and a certain nonlocal operator, the tail operator [5, 1]:

$$\Lambda_{lk}^{l'k}(u_0) = T^{l'k'}(u_0) T_{lk}(u_0) \tag{2.15}$$

$=$




From vertex-face correspondence (2.3) we get the tail operators

$$\Lambda_{lk}^{lk'}(u_0) = (-)^s \frac{[k']}{[k]} X^s(u_0), \tag{2.16}$$

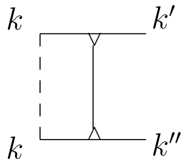
for $k' = k - 2s$ and $l' = l$; and

$$\Lambda_{lk}^{l'k'}(u_0) = D_{lk}^{l'k'} X'^{t-1}(u_0 + \Delta u_0) W_-(u_0) Y^{s-1}(u_0), \qquad \Delta u_0 = -\frac{1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}, \tag{2.17}$$

where $k' = k - 2s$, $l' = l - 2t$, and $D_{lk}^{l'k'}$ is some number. Later we use

$$D_{lk}^{l-2k-2} = \frac{1}{\lambda\lambda'(x - x^{-1})} \frac{[l]'}{[1]'} \frac{[k-1][k-2]}{[k]\partial[0]}. \tag{2.18}$$

Furthermore, we note that the inversion relation between the intertwining vector and its dual




$= \delta^{k'}_{k''}$ implies $\Lambda^{kl'}_{kl} = \delta^{l'}_{l}$, and $\Lambda^{l'k'}_{lk} = 0$ if $k > k'$, $l < l'$ or $k < k'$, $l > l'$.

In what follows we assume that $k \geqslant k'$ and consequently $l \geqslant l'$.

Let $\rho^{(i)}$ be the product of four corner transfer matrices of the eight-vertex model, and let $\rho^{(i)}_{lk}$ be that of the eight-vertex SOS model. Character identity [5]

$$\sum_{k\equiv l+i} [k]\chi^{(i)}_{lk} = [l]'\chi^{(i)} \tag{2.19}$$

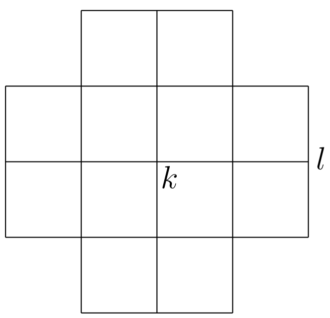
suggests

$$\rho^{(i)} = \sum_{k\equiv l+i\,(2)} \frac{1}{[l]'}\rho^{(i)}_{lk} = \sum_{k\equiv l+i\,(2)} T_{lk}(u_0)\frac{\rho^{(i)}_{lk}}{[l]'}T^{lk}(u_0), \tag{2.20}$$

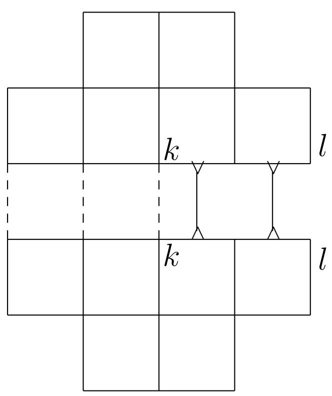
where

$$\rho^{(i)}_{lk} = [k]x^{4H_{lk}}, \qquad H_{lk} = \frac{P^2}{2} + \sum_{m>0}\frac{[2m]_x[rm]_x}{[m]_x[(r-1)m]_x}\beta_{-m}\beta_m. \tag{2.21}$$

$$\rho^{(i)} = \frac{1}{[l]'}\sum_{k\equiv l+i}$$




$$= \frac{1}{[l]'}\sum_{k\equiv l+i}$$



# 3 Vertex operators at the reflectionless points

In general, the eight-vertex model form factors can be obtained from the trace of product of $\Phi_\pm$, $\Psi^*_\pm$, $\Lambda^{l'k'}_{lk}$ and $\rho^{(i)}_{lk}$, by using vertex-face transformation [1].

Let $r = 1 + \frac{1}{N}$. Then $[1]' = 0$ and therefore $1/\lambda' = 0$. Recall

$$\begin{aligned}\Psi^{*l-1}_{l}(u) &= \frac{1}{\lambda'}\oint_{C'}\frac{dw}{2\pi\sqrt{-1}w}\Psi^*_+(u)B(v)\frac{[v-u-\frac{1}{2}+l]'}{[v-u+\frac{1}{2}]'} \\ &= \frac{1}{\lambda'}\oint_{C'}\frac{dw}{2\pi\sqrt{-1}w} : \Psi^*_+(u)B(v) : z^{-\frac{r}{r-1}}\frac{(x^{2r-1}w/z;x^{2r-2})_\infty}{(x^{-1}w/z;x^{2r-2})_\infty}\frac{[v-u-\frac{1}{2}+l]'}{[v-u+\frac{1}{2}]'}.\end{aligned} \tag{3.1}$$

The contour $C'$ encircles the poles at $w = x^{-1+2n(r-1)}z$ ($v = u - \frac{1}{2} + n(r-1)$) with $n \in \mathbb{Z}_{\geqslant 0}$ but not the poles at $w = x^{1-2n'(r-1)}z$ ($v = u + \frac{1}{2} - n'(r-1)$) with $n' = 0, 1, \cdots, N$. Thus, the pinching occurs when $n + n' = N$. Hence we have

$$\Psi^{*l-1}_{l}(u) = [l]'\sum_{\mu=0}^{N} c_\mu : \Psi^*_+(u)B(u+\tfrac{1}{2}-\mu(r-1)) :, \tag{3.2}$$

where

$$c_\mu = \frac{(-)^{\mu+1}x^{\mu r}}{x^{\frac{r}{2(r-1)}}}\frac{[1]_x[1-1/N]_x\cdots[1-(\mu-1)/N]_x}{[1/N]_x[2/N]_x\cdots[\mu/N]_x}\frac{(x^4;x^4,x^{2r-2})_\infty(x^{2r-2};x^4,x^{2r-2})_\infty}{(x^2;x^4,x^{2r-2})_\infty(x^{2r};x^4,x^{2r-2})_\infty}, \tag{3.3}$$

and

$$\begin{aligned} &: \Psi^*_+(u)B(u+\tfrac{1}{2}-\mu(r-1)) := x^{\frac{r}{r-1}-2\mu r}z^{\frac{5r}{4(r-1)}} \\ \times\ &: \exp\left(-\alpha_2(\sqrt{-1}Q + P\log x^{4\mu(r-1)-2}z) - \sum_{m\neq 0}\frac{\beta_m}{m}\gamma_m(-z)^{-m}\right) :,\end{aligned} \tag{3.4}$$

with

$$\gamma_m = \frac{[rm]_x}{[(r-1)m]_x}(x^{2m(\mu(r-1)-1)} + x^{2m\mu(r-1)} - 1).$$

When $N = 1$ ($r = 2$), these are simplified as follows:

$$c_{0,1} = \mp x^{\mp 1}, \tag{3.5}$$

and

$$\begin{aligned} &: \Psi^*_+(u)B(u\pm\tfrac{1}{2}) := x^{\pm 2}z^{\frac{5}{2}} \\ \times\ &: \exp\left(-(\sqrt{-1}Q + P\log x^{\mp 2}z) - \sum_{m\neq 0}\frac{\beta_m}{m}\frac{[2m]_x}{[m]_x}(-x^{\pm 2}z)^{-m}\right) :,\end{aligned} \tag{3.6}$$

For $\mathcal{O} = \widehat{\sigma^z} = \sum_{\varepsilon=\pm}\varepsilon\Phi_{-\varepsilon}(u-1)\Phi_\varepsilon(u)$, tail operator $\Lambda^{k'l'}_{kl}$ with $k' = k, k\pm 2$ are needed.

When $k' = k$,

$$\Lambda^{kl'}_{kl}(u_0) = \delta^{l'}_{l}.$$

For $k' = k - 2$,

$$\Lambda^{l-2t\,k-2}_{lk}(u_0) = D^{l-2t\,k-2}_{lk}X'^{t-1}(u_0+\Delta u_0)W_-(u_0).$$

Since

$$B(v)W_-(u) = w^{\frac{2}{r-1}} \frac{(-x^{-2}z/w; x^{2r-2})_\infty}{(-x^2 z/w; x^{2r-2})_\infty} : B(v)W_-(u) :,$$

no pinching occurs, and therefore $\Lambda_{lk}^{l-2t\,k-2} = 0$ if $t > 1$.

Thus, the only nontrivial tail operator is $\Lambda_{lk}^{l-2k-2}(u_0) \propto W_-(u_0)$. Hence, $\Psi^{*l+1}_{\ l}$ and $\Lambda_{lk}^{l-2k-2}$ can be expressed by the product of rational functions (zero modes parts) and exponentiated bosons, and $\Psi^{*l-1}_{\ l}$ is the sum of such product. For $\mathcal{O} = \widehat{\sigma^z}$, the form factors are therefore expressed in terms of the sum of theta function without any integrals.

# 4 Form factors at the reflectionless points

The $2m$-particle form factors ($\zeta_j = x^{u_j}$) are given as follows:

$$\begin{aligned} F_m^{(i)}(\zeta_1, \cdots, \zeta_{2m})_{\mu_1 \cdots \mu_{2m}} &= \frac{1}{\chi^{(i)}} \mathrm{Tr}_{\mathcal{H}_i} (\Psi^*_{\mu_{2m}}(\zeta_{2m}) \cdots \Psi^*_{\mu_1}(\zeta_1) \mathcal{O} \rho^{(i)}) \\ &= \sum_{l_1 \cdots l_{2m}} F_m^{(lk)}(\zeta)_{l_1 \cdots l_{2m} l} \prod_{j=1}^{2m} t'^{*l_j}_{\ l_{j+1}}(u_j - u_0 - \Delta u_0)_{\mu_j}, \end{aligned} \tag{4.1}$$

where $l_{2m+1} = l$. From the generalized ice condition,

$$F_m^{(i)}(\zeta)_{\mu_1 \cdots \mu_{2m}} = 0, \qquad \text{unless } \sharp\{j | \mu_j > 0\} \equiv m \ (\mathrm{mod}\ 2).$$

Note that from (4.2), nonzero terms in the sum on (4.1) results from the case $l_1 = l, l \pm 2$. When $l_1 = l$ (4.1) has an integral. On the other hand, (4.1) for $l_1 \neq l$ can be expressed in terms of the sum of theta function (without integrals), otherwise is equal to 0. Since

$$2^{2m} - \binom{2m}{m} \geqslant 2^{2m-1},$$

we can obtain no-integral formulae for $\widehat{\sigma^z}$ form factors, in principle.

For $\mathcal{O} = \widehat{\sigma^z}$, we have

$$\begin{aligned} F_m^{(lk)}(\zeta)_{l_1 \cdots l_{2m} l} &= \frac{1}{\chi^{(i)}} \sum_{\varepsilon} \sum_{k \equiv l+i\,(2)} \varepsilon t_{-\varepsilon}(u - u_0 - 1)^k_{k_2} t_\varepsilon (u - u_0)^{k_2}_{k_1} \\ &\times \ \mathrm{Tr}_{\mathcal{H}^{(i)}_{lk}} (\Psi^{*l}_{l_{2m}}(\zeta_{2m}) \cdots \Psi^{*l_2}_{l_1}(\zeta_1) \Phi^k_{k_2}(u-1) \Phi^{k_2}_{k_1}(u) \Lambda^{l_1 k_1}_{lk}(u_0) \frac{\rho_{lk}}{[l]'}). \end{aligned} \tag{4.2}$$

In what follows, for simplicity, we set $m = 2$ and $u_0 = u - \frac{\pi\sqrt{-1}}{2\epsilon}$ so that the terms for the case $k_2 = k$ vanish. The quantity $F_{l' l_2 l_3 l_4 l}$ for $l' \neq l$ has a non-integral expression.

For example, let us calculate $F_{l-2\,l-1\,l\,l+1\,l}$. Simple calculation shows

$$\Psi^{*l}_{l+1}(\zeta_4)\Psi^{*l+1}_{l}(\zeta_3)\Psi^{*l}_{l-1}(\zeta_2)\Psi^{*l-1}_{l}(\zeta_1)\Phi^{k}_{k-1}(u-1)\Phi^{k-1}_{k}(u)\Lambda^{l-2\,k-2}_{lk}(u_0)\frac{\rho_{lk}}{[l]'} = \frac{[l+1]'(x-x^{-1})^{-1}}{\lambda\lambda'[1]'\partial[0]}$$

$$\times \sum_{\mu=0}^{N} c_\mu : \Psi^*_+(u_4)\Psi^*_+(u_3)\Psi^*_+(u_2)\Psi^*_+(u_1)B(u_4^{(\mu)})\Phi_+(u-1)\Phi_+(u)W_-(u_0) : x^{4H_{lk}}$$

$$\times \prod_{j<k} z_k^{\frac{r}{2(r-1)}} \frac{(z_j/z_k;x^4,x^{2r-2})_\infty (x^{2r+2}z_j/z_k;x^4,x^{2r-2})_\infty}{(x^2z_j/z_k;x^4,x^{2r-2})_\infty (x^{2r}z_j/z_k;x^4,x^{2r-2})_\infty} \prod_{j=1}^{3} w^{-\frac{r}{r-1}} \frac{(x^{(2r-2)(1+\mu)}z_j/z_4;x^{2r-2})_\infty}{(x^{(2r-2)\mu-2}z_j/z_4;x^{2r-2})_\infty}$$

$$\times \prod_{j=1}^{4} \frac{xz_j}{1+x^{-1}z/z_j} \times 2(x^{-1}z)^{\frac{1}{r}} \frac{(-x^{2r};x^{2r})_\infty}{(-x^4;x^{2r})_\infty}(x^{1-(2r-2)\mu}z_4+z)(x^{1-(2r-2)\mu}z_4+x^{-2}z)$$

$$\times \prod_{j=1}^{4} z_j^{-\frac{1}{r-1}} \frac{(xz/z_j;x^{2r-2})_\infty}{(x^{-1}z/z_j;x^{2r-2})_\infty}.$$

Here, $u_4^{(\mu)} = u_4 + \frac{1}{2} - \mu(r-1)$, and note that $\lambda'[1]'$ is a constant. Let

$$: \Psi^*_+(u_4)\Psi^*_+(u_3)\Psi^*_+(u_2)\Psi^*_+(u_1)B(u_4^{(\mu)})\Phi_+(u-1)\Phi_+(u)W_-(u_0) := U_0 : \exp\left(\sum_{m\neq 0} A_m\beta_m\right) :,$$

where $U_0$ is the zero-mode part. Then the trace over oscillator modes

$$\mathrm{Tr}_*\left( : \exp\left(\sum_{m\neq 0} A_m\beta_m\right) : x^{4H_*}\right) = \prod_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{\langle l,k|\beta_m^n e^{A_{-m}\beta_{-m}} e^{A_m\beta_m}\beta_{-m}^n|l,k\rangle}{\langle l,k|\beta_m^n\beta_{-m}^n|l,k\rangle} x^{4mn} \tag{4.3}$$

can be calculated as follows. Note that

$$e^{A_{-m}\beta_{-m}} e^{A_m\beta_m}\beta_{-m}^n|l,k\rangle = e^{A_{-m}\beta_{-m}}\left(\beta_{-m} + m\frac{[m]_x[(r-1)m]_m}{[2m]_x[rm]_x}A_m\right)^n |l,k\rangle. \tag{4.4}$$

Multiply $x^{4mn}$ by the coeeficient of $\beta_{-m}^n|l,k\rangle$ on (4.4), and sum up with respect to $n$, we get

$$\begin{aligned} &\sum_{n=0}^{\infty} x^{4mn} \sum_{s=0}^{n} \frac{{}_nC_s}{s!}\left(m\frac{[m]_x[(r-1)m]_m}{[2m]_x[rm]_x}A_{-m}A_m\right)^s \\ =\ & \sum_{s=0}^{\infty}\frac{1}{s!}\left(m\frac{[m]_x[(r-1)m]_m}{[2m]_x[rm]_x}A_{-m}A_m\right)^s \sum_{n=s}^{\infty} {}_nC_s x^{4mn} \\ =\ & \frac{1}{1-x^{4m}}\exp\left(\frac{x^{4m}}{1-x^{4m}}m\frac{[m]_x[(r-1)m]_m}{[2m]_x[rm]_x}A_{-m}A_m\right). \end{aligned} \tag{4.5}$$

Thus, we have

$$(4.3) = \frac{1}{(x^4;x^4)_\infty}\exp\left(\sum_{m=1}^{\infty}\frac{x^{4m}}{1-x^{4m}}m\frac{[m]_x[(r-1)m]_m}{[2m]_x[rm]_x}A_{-m}A_m\right). \tag{4.6}$$

Since

$$\begin{aligned} mA_m &= \frac{[rm]_x}{[(r-1)m]_x}\sum_{j=1}^{4}(-z_j)^{-m} - \frac{[2m]_x[rm]_x}{[m]_x[(r-1)m]_x}(-x^{1-2(r-1)\mu}z_4)^{-m} \\ &- z^{-m}\left(1+x^{2m}+(-x)^{m(r-1)}\frac{[2m]_x}{[(r-1)m]_x}\right), \end{aligned}$$

we obtain

$$\begin{aligned} (4.3) &= \frac{d_\mu}{(x^4;x^4)_\infty}\prod_{j\neq k}\frac{(x^4z_j/z_k;x^4,x^4,x^{2r-2})_\infty(x^{2r+6}z_j/z_k;x^4,x^4,x^{2r-2})_\infty}{(x^6z_j/z_k;x^4,x^4,x^{2r-2})_\infty(x^{2r+4}z_j/z_k;x^4,x^4,x^{2r-2})_\infty} \\ &\times \prod_{j=1}^{4}\frac{(x^{2r+4-2(r-1)\mu}z_4/z_j;x^4,x^{2r-2})_\infty(x^{2r+2+2(r-1)\mu}z_j/z_4;x^4,x^{2r-2})_\infty}{(x^{2r+4-2(r-1)\mu}z_4/z_j;x^4,x^{2r-2})_\infty(x^{2+2(r-1)\mu}z_j/z_4;x^4,x^{2r-2})_\infty} \\ &\times \prod_{j=1}^{4}\frac{1}{(-x^3z/z_j;x^4)_\infty(-x^5z_j/z;x^4)_\infty}\frac{(x^5z/z_j;x^4,x^{2r-2})_\infty(x^{2r+3}z_j/z;x^4,x^{2r-2})_\infty}{(x^5z/z_j;x^4,x^{2r-2})_\infty(x^{2r+1}z_j/z;x^4,x^{2r-2})_\infty} \\ &\times (-x^{1+2(r-1)\mu}z/z_4;x^2)_\infty(-x^{5-2(r-1)\mu}z_4/z;x^2)_\infty \\ &\times (x^{1+2(r-1)\mu}z/z_4;x^{2r-2})_\infty(x^{1-2(r-1)(\mu-1)}z_4/z;x^{2r-2})_\infty, \end{aligned} \tag{4.7}$$

where $d_\mu$ is some constant. Contribution from the trace over zero modes is equal to:

$$\begin{aligned} & x^{\frac{r}{r-1}-\frac{r+1}{2r}-2\mu r+\frac{r}{2(r-1)}(u_1+u_2+u_3+5u_4)+(\frac{r-1}{r}+\frac{2}{r(r-1)})u+\frac{r-1}{r}k^2-2kl+\frac{r}{r-1}l^2} \\ \times\ & x^{-k(2\mu(r-1)+\frac{2(r-1)}{r}-1+u_1+u_2+u_3-u_4-2u)+l(2\mu r+2-\frac{r}{(r-1)}+\frac{r}{r-1}(u_1+u_2+u_3-u_4)-\frac{2ru}{r-1})}. \end{aligned} \tag{4.8}$$

Multiply (4.8) by $[k-1]$ (resulting from $\sum_{\varepsilon}\varepsilon t_{-\varepsilon}(u-u_0-1)^k_{k-1}t_\varepsilon(u-u_0)^{k-1}_{k-2}$) and sum up with respect to $k\equiv l+i$ (mod 2). Then we have

$$F^{(i)}_{l-2\,l-1\,l\,l+1\,l}(u_1,u_2,u_3,u_4;u) = \sum_{\mu=0}^{N}C_\mu(u_1,u_2,u_3,u_4;u) \tag{4.9}$$

$$\begin{aligned} &\times\Bigg((-)^\mu\theta_3\left(\frac{u_1+u_2+u_3-u_4}{2}-u+\mu(r-1);\frac{\pi\sqrt{-1}}{\epsilon}\right)\theta_1\left(\frac{l+\frac{u_1+u_2+u_3-u_4}{2}-u}{r-1};\frac{\pi\sqrt{-1}}{\epsilon(r-1)}\right) \\ &+(-)^{1-i}\,\theta_2\left(\frac{u_1+u_2+u_3-u_4}{2}-u+\mu(r-1);\frac{\pi\sqrt{-1}}{\epsilon}\right)\theta_4\left(\frac{l+\frac{u_1+u_2+u_3-u_4}{2}-u}{r-1};\frac{\pi\sqrt{-1}}{\epsilon(r-1)}\right)\Bigg), \end{aligned}$$

where $C_\mu(u_1,u_2,u_3,u_4;u)$ is some function, which is almost equal to the product of $[l+1]'c_\mu$ and (4.7).

# 5 Summary and discussion

In this article, it is shown that the type II vertex operators and the type II part ($X'$) of tail operators can be bosonized without any integrals at reflectionless points. Consequently, the form factors of $\mathcal{O}=\widehat{\sigma^z}$ in the eight-vertex model can be expressed in terms of the sum of theta functions. We wish to report explicit expressions for $\widehat{\sigma^z}$ form factors of the eight-vertex model at reflectionless points in a separate paper.

Concerning generic local operator $\mathcal{O}$, the tail operator $\Lambda_{kl}^{k'l'}$ with $k' = k \pm 2s$ $(s > 1)$ are needed. Nevertheless, there is no pinching for $l' = l \pm 2t$ with $t > 1$. Thus, the type II part of $\Lambda_{kl}^{k'l'}$ is always written without integrals. On the other hands, the type I part $(Y)$ of tail operators has an integral representation. Furthermore, generic local operator itself has an integral representation. Note that these integrals result from type I parts.

Let us remind Shiraishi's observation [2], where the type II vertex operators in the eight-vertex model (NOT SOS) at reflectionless points can be expressed in terms of certain representations of deformed $W$-algebra $\mathcal{D}_{N+1}$. Such non-integral structure is very close to our results here. However, Shiraishi's formulae of the type I vertex operators are different from ours. It is therefore a very important subject to find a connection between these two schemes.

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