Recurrences for elliptic hypergeometric integrals

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Abstract

In recent work on multivariate elliptic hypergeometric integrals, the author generalized a conjectural integral formula of van Diejen and Spiridonov to a ten parameter integral provably invariant under an action of the Weyl group E_7. In the present note, we consider the action of the affine Weyl group, or more precisely, the recurrences satisfied by special cases of the integral. These are of two flavors: linear recurrences that hold only up to dimension 6, and three families of bilinear recurrences that hold in arbitrary dimension, subject to a condition on the parameters. As a corollary, we find that a codimension one special case of the integral is a tau function for the elliptic Painlevé equation.

1 Introduction

In [10], we studied the following hypergeometric integral (generalizing the "Type II" integral of [5]), defined for $|p|, |q|, |t| < 1$, $t_0, \dots, t_7 \in \mathbb{C}^*$:

$$\begin{aligned} &II^{(n)}_{t;p,q}(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7) \\ &\quad := \frac{(p;p)^n (q;q)^n}{2^n n!} \int_{C^n} \prod_{1\le i<j\le n} \frac{\Gamma_{p,q}(t z_i^{\pm 1} z_j^{\pm 1})}{\Gamma_{p,q}(z_i^{\pm 1} z_j^{\pm 1})} \prod_{1\le i\le n} \frac{\prod_{0\le r\le 7} \Gamma_{p,q}(t_r z_i^{\pm 1})}{\Gamma_{p,q}(z_i^{\pm 2})} \frac{dz_i}{2\pi\sqrt{-1} z_i}, \end{aligned} \tag{1.1}$$

where $(p;p) = \prod_{1\le i}(1-p^i)$, $\Gamma_{p,q}$ is the elliptic Gamma function (see below), and C is a suitable choice of contour (which may be taken to be the unit circle when all parameters are inside the unit circle). We found that if the parameters satisfied the following "balancing" condition:

$$t^{2n-2} t_0 t_1 t_2 t_3 t_4 t_5 t_6 t_7 = p^2 q^2, \tag{1.2}$$

then the integral satisfied a certain transformation which, together with the obvious permutation symmetry of the arguments, generated an action of the Weyl group $W(E_7)$. More precisely, assuming balanced parameters, the renormalized integral

$$\tilde{II}^{(n)}_{t;p,q}(t_0, t_1, \dots, t_7) := \prod_{0\le r<s\le 7} \Gamma^+_{t,p,q}(t t_r t_s)\ II^{(n)}_{t;p,q}(t^{1/2} t_0, t^{1/2} t_1, \dots, t^{1/2} t_7) \tag{1.3}$$

is invariant under this action, which we now explain.

We first observe that we can view the above integral (given the balancing condition) as a function on an algebraic torus, the maximal torus $\mathrm{Hom}(\Lambda_{E_8}, \mathbb{C}^*)$ of the complex Lie group E_8 (where Λ_{E_8} is the root lattice). Indeed, we first observe that the integral is invariant under the symmetry

$$(t_0, t_1, \dots, t_7) \mapsto (-t_0, -t_1, \dots, -t_7), \tag{1.4}$$

simply by negating the z variables; as a result, it is only a function of the pairwise products and ratios $t_r^{\pm 1} t_s^{\pm 1}$. In other words, it is a function on the maximal torus $\mathrm{Hom}(\Lambda_{D_8}, \mathbb{C}^*)$. But the balancing condition forces a choice of square root

$$\sqrt{t_0 t_1 \dots t_7} = pq/t^{n+1} \tag{1.5}$$

and thus the parameters in fact determine a homomorphism from the lattice Λ_{E_8} to $\mathbb{C}^*$, mapping $\omega := (1/2, 1/2, \dots, 1/2)$ to pq/t^{n+1}.

If $\phi : \Lambda_{E_8} \to C^*$ is a homomorphism such that

$$\frac{pq}{t\phi(\omega)} = t^n \tag{1.6}$$

for some (uniquely determined) integer n, we define

$$\tilde{I\!I}_{t;p,q}(\phi) \tag{1.7}$$

as follows. If $n < 0$, then

$$\tilde{I\!I}_{t;p,q}(\phi) = 0; \tag{1.8}$$

otherwise, we set

$$\tilde{I\!I}_{t;p,q}(\phi) = \tilde{I\!I}^{(n)}_{t;p,q}(\phi(e_0), \dots, \phi(e_7)) \tag{1.9}$$

where $e_0, \dots, e_7$ are the coordinate vectors and we have chosen an extension of ϕ to $\Lambda^*_{D_8}$ (which as remarked above does not affect the value of the integral).

Theorem 1.1. *[10] Suppose* $\phi \in \mathrm{Hom}(\Lambda_{E_8}, \mathbb{C}^*)$ *Then for any element* $g \in W(E_8) = \mathrm{Aut}(\Lambda_{E_8})$ *such that*

$$\langle \omega, g\omega \rangle \in \{1, 2\}, \tag{1.10}$$

we have

$$\tilde{I\!I}_{t;p,q}(\phi) = \tilde{I\!I}_{t;p,q}(g^*\phi) \tag{1.11}$$

whenever

$$\frac{pq}{t\phi(\omega)}, \frac{pq}{t\phi(g\omega)} \in t^{\mathbb{Z}}, \tag{1.12}$$

so that both sides are defined.

Note that if $\langle \omega, g\omega \rangle = 2$, then $g\omega = \omega$. In other words, g is in the stabilizer $W(E_7)$ of ω, and the statement becomes that $\tilde{I\!I}_{t;p,q}(\phi)$ is invariant under $W(E_7)$ whenever it is defined.

In addition to the natural action of the finite Weyl group $W(E_8)$ on $\mathrm{Hom}(\Lambda_{E_8}, \mathbb{C}^*)$, there is a nearly natural action of the affine Weyl group. To be precise, if $v \in \Lambda_{E_8}$, we define a shift operator τ_v by

$$(\tau_v(\phi))(w) = \phi(w) q^{\langle v, w \rangle}, \tag{1.13}$$

for all $\phi \in \mathrm{Hom}(\Lambda_{E_8}, \mathbb{C}^*)$, $w \in \Lambda_{E_8}$. (It will be notationally convenient to extend this definition to $v \in \Lambda_{E_8} \otimes \mathbb{Q}$ by fixing a consistent family of mth roots of q.) The price of enlarging the group is that we no longer have invariance; instead, the most we can expect is that $\tilde{I\!I}_{t;p,q}$ should satisfy recurrences with respect to different shifts.

The purpose of the present note is to show that in certain special cases, such recurrences do indeed arise. These come in two main flavors. The first set of recurrences arises from the observation that certain shifts (by coordinate vectors, say) have the effect of multiplying the integrand by a relatively simple function; in low dimensions ($n \le 6$), these functions must be linearly dependent, and thus give rise to a linear recurrence.

The other set of recurrences are somewhat more subtle. The above integral can be viewed as a generalization of the Selberg integral, which suggests that the speical cases $t \in \{q^{1/2}, q, q^2\}$ should be particularly nice. Indeed, it turns out that in those cases the integral can be expressed (in many ways) as a determinant or pfaffian of one- or two-dimensional integrals. In particular, we can arrange for several minors of said determinant/pfaffian to themselves be special cases of our integral, with the result that the Plücker relations give rise to recurrences of our integral. Since the Plücker relations are bilinear, the resulting recurrences are also bilinear; for $t = q$ (the determinantal case), we obtain a three-term bilinear recurrence, while for $t = q^{1/2}, q^2$ (pfaffian cases), we obtain a four-term bilinear recurrence. The significance of these recurrences is perhaps underscored by the fact that the recurrence for $t = q$ has arisen in the theory of Sakai's elliptic Painlevé equation [12, 7].

The plan of the paper is as follows. After defining some notation for generalized q-symbols and theta functions, we proceed in section 2 to prove some theta function identities needed in the derivation of our recurrences. In section 3, we use these to give the aforementioned linear recurrences in low dimensions. Section 4 describes a general setting in which Plücker relations give rise to bilinear relations of integrals, which is then specialized in section 5 to give our bilinear Painlevé-type recurrences.

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Notation

Aside from the integral itself, most of the functions that appear in the sequel are most simply expressed as infinite products; as a result, we will need a shorthand notation for certain such products. Here p, q, t are complex numbers inside the open unit disc.

$$\theta_p(x) := \prod_{0\le k}(1-p^{k+1}/x)(1-p^k x) \tag{1.14}$$

$$\Gamma_{p,q}(x) := \prod_{0\le j,k}(1-p^{j+1}q^{k+1}/x)(1-p^j q^k x)^{-1} \tag{1.15}$$

$$\Gamma^+_{p,q,t}(x) := \prod_{0\le i,j,k}(1-p^{i+1}q^{j+1}t^{k+1}/x)(1-p^i q^j t^k x). \tag{1.16}$$

The first function is simply a version of Jacobi's theta function, while the second function is Ruijsenaars' elliptic Gamma function [11]. As these are generalized q-symbols (indeed, $\Gamma_{0,q}(x)^{-1}$ is precisely the usual q-symbol), we take the standard convention that the presence of multiple arguments indicates a product; thus, for instance, in the above integral,

$$\Gamma_{p,q}(z_i^{\pm 1} z_j^{\pm 1}) = \Gamma_{p,q}(z_i z_j)\Gamma_{p,q}(z_i/z_j)\Gamma_{p,q}(z_j/z_i)\Gamma_{p,q}(1/z_i z_j). \tag{1.17}$$

The main properties of these functions are reflection symmetry:

$$\theta_p(p/x) = \theta_p(x) \tag{1.18}$$

$$\Gamma_{p,q}(pq/x) = \Gamma_{p,q}(x)^{-1} \tag{1.19}$$

$$\Gamma^+_{p,q,t}(pqt/x) = \Gamma_{p,q,t}(x), \tag{1.20}$$

and a functional equation:

$$\theta_p(px) = \frac{1-1/x}{1-x}\theta_p(x) = -x^{-1}\theta_p(x) \tag{1.21}$$

$$\Gamma_{p,q}(qx) = \theta_p(x)\Gamma_{p,q}(x) \tag{1.22}$$

$$\Gamma^+_{p,q,t}(tx) = \Gamma_{p,q}(x)\Gamma^+_{p,q,t}(x), \tag{1.23}$$

with similar identities following by the symmetry of $\Gamma_{p,q}$ and $\Gamma^+_{p,q,t}$ in the parameters.

We recall that a (p-)theta function (in multiplicative notation) is a holomorphic function $f(x)$ on $\mathbb{C}^*$ such that

$$f(px) = C(-x)^{-m} f(x) \tag{1.24}$$

for some constant C (the multiplier), and some integer m (the degree). The canonical example of this is the function $\theta_p(x/a)$; indeed, any p-theta function is proportional to a function of the form

$$x^k \prod_{1\le i\le m} \theta_p(x/a_i), \tag{1.25}$$

with multiplier

$$p^k \prod_{1\le i\le m} a_i, \tag{1.26}$$

and thus the multiplier of a theta function is determined up to powers of p by its zeros. A meromorphic theta function is a ratio of holomorphic theta functions.

Similarly, a BC_n-symmetric theta function of degree m is defined to be a function on $(\mathbb{C}^*)^n$ invariant under permutations and inversions of its variables, and such that as a function of each variable it is a theta function of degree $2m$ with multiplier p^{-m}. Since the quotient of the elliptic curve $\mathbb{C}^*/\langle p\rangle$ by $x \mapsto 1/x$ is a projective line, it follows that the space of BC_1-symmetric theta functions of degree m is $m+1$-dimensional.

2 Theta function relations

Define a function $\psi_p(x,y)$ on $\mathbb{C}^* \times \mathbb{C}^*$ as follows:

$$\psi_p(x,y) = x^{-1}\theta_p(xy)\theta_p(x/y). \tag{2.1}$$

This is readily seen to satisfy the relations

$$\psi_p(x,y) = \psi_p(x,1/y) = -\psi_p(y,x) \tag{2.2}$$

and

$$\psi_p(x, py) = (py^2)^{-1}\psi_p(x, y). \tag{2.3}$$

Somewhat less trivial is the following:

Lemma 2.1. *For $x, y, z, w \in \mathbb{C}^*$,*

$$\psi_p(x, y)\psi_p(z, w) - \psi_p(x, z)\psi_p(y, w) + \psi_p(x, w)\psi_p(y, z) = 0. \tag{2.4}$$

Proof. Consider the skew-symmetric 4×4 matrix

$$A = \begin{pmatrix} \psi_p(x, x) & \psi_p(x, y) & \psi_p(x, z) & \psi_p(x, w) \\ \psi_p(y, x) & \psi_p(y, y) & \psi_p(y, z) & \psi_p(y, w) \\ \psi_p(z, x) & \psi_p(z, y) & \psi_p(z, z) & \psi_p(z, w) \\ \psi_p(w, x) & \psi_p(w, y) & \psi_p(w, z) & \psi_p(w, w) \end{pmatrix} \tag{2.5}$$

The functions $\psi_p(x, \_)$, $\psi_p(y, \_)$, $\psi_p(z, \_)$, $\psi_p(w, \_)$ all lie in the 2-dimensional space of BC_1-symmetric theta functions of degree 1, and thus any three of them satisfy a linear relation. In particular, it follows that the matrix A has rank at most 2, and thus has pfaffian 0; this is precisely the desired identity. □

Remark. This, of course, is simply the addition law for elliptic theta functions in disguise.

Proposition 2.2. *We have the following Cauchy-type determinant:*

$$\det_{1\le i,j\le n}\left(\frac{1}{\psi_p(x_i, y_j)}\right) = (-1)^{n(n-1)/2}\frac{\prod_{1\le i<j\le n}\psi_p(x_i, x_j)\psi_p(y_i, y_j)}{\prod_{1\le i,j\le n}\psi_p(x_i, y_j)} \tag{2.6}$$

Proof. ¿From the lemma, we can write

$$\frac{1}{\psi_p(x_i, y_j)} = \frac{\psi_p(z, w)}{\psi_p(z, x_i)\psi_p(w, y_j) - \psi_p(z, y_j)\psi_p(w, x_i)} \tag{2.7}$$

$$= \frac{\psi_p(z, w)}{\psi_p(w, x_i)\psi_p(w, y_j)}\frac{1}{(\psi_p(z, x_i)/\psi_p(w, x_i)) - (\psi_p(z, y_j)/\psi_p(w, y_j))} \tag{2.8}$$

for arbitrary z, w. The result thus follows immediately from the usual Cauchy determinant. □

Remark. That this identity is a special case of the usual Cauchy determinant is no accident: any function ψ satisfying the above identity can be written in the form

$$\psi(x, y) = \frac{\psi(z, x)\psi(w, y) - \psi(z, y)\psi(w, x)}{\psi(z, w)} \tag{2.9}$$

using the $n = 2$ instance of the identity.

Corollary 2.3. *For generic $x_1,\dots,x_{n+2}$, $y_1,\dots,y_n$,*

$$\sum_{1\le k\le n+2}\frac{\prod_{1\le j\le n}\psi_p(x_k, y_j)}{\prod_{i\ne k}\psi_p(x_k, x_i)} = 0. \tag{2.10}$$

Proof. Expand the $n+1$ dimensional instance of the above determinant along the last row, set $y_{n+1} = x_{n+2}$, then simplify. Alternatively, observe that some such relation must hold by dimensionality, and deduce the constants by setting $y_j = x_k$ for various choices of j, k. $\square$

Fix $p \in \mathbb{C}$, $t \in \mathbb{C}^*$, and define for u_0, u_1, u_2, u_3, $u_4 \in \mathbb{C}^*$ a function $g^{(n)}_{u_0,u_1,u_2,u_3,u_4}$ on $(\mathbb{C}^*)^n$ by

$$g^{(n)}_{u_0,u_1,u_2,u_3,u_4}(\dots z_i \dots) = \prod_{1\le i\le n} (1+R(z_i)) \frac{\prod_{0\le r\le 4} \theta_p(u_r z_i)\theta_p(z_i/t^{n-1}u_0u_1u_2u_3u_4)}{z_i^2\theta_p(z_i^2)} \prod_{1\le i<j\le n} \frac{\theta_p(tz_iz_j)}{\theta_p(z_iz_j)}, \tag{2.11}$$

where $R(z_i)$ is the operator $z_i \mapsto 1/z_i$. We also define a function

$$f^{(n)}_{u_0}(\dots z_i \dots) = \prod_{1\le i\le n} \theta_p(u_0 z_i, u_0/z_i). \tag{2.12}$$

The following lemma shows that this is a special case of the first family.

Lemma 2.4. *We have the identity*

$$g^{(n)}_{u_0,u_1,u_2,u_3,1/u_0}(\dots z_i \dots) = f^{(n)}_{u_0}(\dots z_i \dots) \prod_{1\le i\le n} \frac{\theta_p(t^{n-i}u_1u_2, t^{n-i}u_1u_3, t^{n-i}u_2u_3)}{t^{n-1}u_0u_1u_2u_3}. \tag{2.13}$$

Proof. If we divide both sides by $f^{(n)}_{u_0}(\dots z_i \dots)$, the result is simply Lemma 6.2 of [10]. $\square$

Theorem 2.5. *For $u_0, u_1, u_2, u_3, v_0, \dots, v_{n+1} \in \mathbb{C}^*$, and $\mathbf{z} \in (\mathbb{C}^*)^n$,*

$$\sum_{0\le i\le n+1} \frac{g^{(n)}_{u_0,u_1,u_2,u_3,v_i}(\mathbf{z})}{\prod_{r\ne i} v_r^{-1}\theta_p(v_r/v_i, t^{n-1}u_0u_1u_2u_3v_iv_r)} = 0. \tag{2.14}$$

Proof. If we pull the sum inside the symmetrization operation, we find that the result would follow from the identity

$$\sum_{0\le k\le n+1} \frac{\prod_{1\le i\le n} z_i^{-1}\theta_p(v_kz_i, z_i/t^{n-1}u_0u_1u_2u_3v_k)}{\prod_{r\ne k} v_r^{-1}\theta_p(v_r/v_k, t^{n-1}u_0u_1u_2u_3v_kv_r)} = 0. \tag{2.15}$$

But this is the special case of Corollary 2.3 with

$$x_k = v_{k-1}\sqrt{t^{n-1}u_0u_1u_2u_3} \quad y_k = \frac{z_k}{\sqrt{t^{n-1}u_0u_1u_2u_3}}. \tag{2.16}$$

$\square$

If we set one of the variables in $g^{(n)}_{u_0u_1u_2u_3u_4}$ equal to u_0, half of the terms vanish, and we thus find

$$g^{(n)}_{u_0u_1u_2u_3u_4}(u_0,\mathbf{z}) = \frac{\theta_p(1/t^{n-1}u_1u_2u_3u_4)\prod_{1\le r\le 4}\theta_p(u_0u_r)}{u_0^2} g^{(n-1)}_{tu_0,u_1,u_2,u_3,u_4}(\mathbf{z}). \tag{2.17}$$

This in some cases allows us to deduce relations between these functions. We concentrate on the case $n=4$, as this seems to be the primary source of identities between functions $f^{(n)}$ and $g^{(n)}$ not contained in Corollary 2.3 or Theorem 2.5; all other such identities we have been able to find are obtained by specializing the variables.

Proposition 2.6. *For any parameters u_0, u_1, u_2, u_3, $u_4 \in \mathbb{C}$, $\mathbf{z} \in \mathbb{C}^4$,*

$$g^{(4)}_{u_0,u_1,u_2,u_3,u_4}(\mathbf{z}) = \prod_{0\le i<j\le 4} \theta_p(u_iu_j, tu_iu_j) \sum_{0\le r\le 4} \prod_{i\ne r} \frac{\theta_p(u_i/t^3u_0u_1u_2u_3u_4)}{u_i^2\theta_p(u_r/u_i, u_ru_i, tu_ru_i)} f^{(4)}_{u_r}(\mathbf{z}) \tag{2.18}$$

Proof. The five functions $f^{(4)}_{u_r}(\mathbf{z})$ span the space of BC_4-symmetric theta functions of degree 1, so it remains only determine the coefficients of the expansion. If we evaluate $g^{(4)}_{u_0,u_1,u_2,u_3,u_4}(\mathbf{z})$ at the point $\mathbf{z} = (u_1, u_2, u_3, u_4)$, only the $f^{(4)}_{u_0}(\mathbf{z})$ term survives, and we thus can solve for its coefficient; the other coefficients are symmetrical. □

There is a sort of inverse to the above expansion, expressing $f^{(4)}_{u_0}(\mathbf{z})$ in terms of the five functions

$$g^{(4)}_{u_0,u_2,u_3,u_4,u_5}(\mathbf{z}), g^{(4)}_{u_0,u_1,u_3,u_4,u_5}(\mathbf{z}), \dots, g^{(4)}_{u_0,u_1,u_2,u_3,u_4}(\mathbf{z}).$$

Proposition 2.7. *For any parameters u_0, u_1, u_2, u_3, u_4, $u_5 \in \mathbb{C}$, $\mathbf{z} \in \mathbb{C}^4$,*

$$f^{(4)}_{u_0}(\mathbf{z}) = \frac{\prod_{1\le i\le 5}\theta_p(u_0u_i/t^3U)}{\prod_{1\le i<j\le 5}\theta_p(u_iu_j, tu_iu_j)} \sum_{1\le r\le 5} \frac{g_{u_0,\dots,\widehat{u_r},\dots}(\mathbf{z})}{\theta(u_0u_i/t^3U)} \prod_{0\le i\le 3} \frac{\theta_p(t^iu_0u_r)}{\theta_p(t^{i-6}/U)} \prod_{1\le i\ne r} \frac{u_i^2\theta_p(u_iu_r, tu_iu_r)}{\theta_p(u_r/u_i)}, \tag{2.19}$$

where $U = u_0u_1u_2u_3u_4u_5$.

Proof. ¿From Proposition 2.6 above, we obtain six different identities expressing the six functions $g^{(4)}_{\dots,\widehat{u_r},\dots}(\mathbf{z})$ in terms of the six functions $f^{(4)}_{u_r}(\mathbf{z})$. It turns out, in fact, that up to rescaling of rows and columns, the resulting 6×6 matrix is antisymmetric, and thus the inverse matrix can be expressed via pfaffians. The closed forms for the desired pfaffians can be obtained via the special case $a = 1$, $b = t$, $c = 1/t^3U$ of the following identity. □

Theorem 2.8. *[9] For arbitrary parameters $u_0, \dots, u_{2n-1}, a, b, c \in C$, we have*

$$\begin{aligned}\mathrm{pf}_{0\le i,j<2n}&\left(\frac{u_j\theta_p(u_i/u_j, au_iu_j, bu_iu_j)}{\theta_p(cu_iu_j)}\right)\\ &= c^{n(n-1)}\theta_p(a/c, b/c)^{n-1}\theta_p(ac^{n-1}U, bc^{n-1}U) \prod_{0\le i<j<2n} \frac{u_j\theta_p(u_i/u_j)}{\theta_p(cu_iu_j)},\end{aligned} \tag{2.20}$$

where $U = \prod_{0\le i<2n} u_i$

Proof. We first consider both sides as functions in a; we find that they are both theta functions with the same multiplier. Moreover, if we set $a = c$ on the left, we obtain a matrix of rank 2, and thus the pfaffian must have a zero of order $n-1$ at that point. This accounts for all but one zero in a fundamental region, and the remaining zero can be determined from the multiplier. Arguing similarly for b, we conclude that the left-hand side is a multiple of

$$\theta_p(a/c, b/c)^{n-1}\theta_p(ac^{n-1}U, bc^{n-1}U). \tag{2.21}$$

Since the pfaffian also vanishes whenever $u_i = u_j$, and has at most simple poles at points with $cu_iu_j = 1$, it follows that the ratio of the two sides is in fact constant. The value of this constant can then be determined from the asymptotics as $u_{2i} \to 1/cu_{2i-1}$. □

Associated to this is the following analogue of Corollary 2.3.

Corollary 2.9. *For arbitrary parameters $a, b \in \mathbb{C}$, $u \in \mathbb{C}^{n+1}$,*

$$\sum_{0\le r\le n} \theta_p(au_r, bu_r, aU/u_r, bU/u_r) \prod_{0\le i\le n; i\ne r} \frac{\theta_p(u_iu_r)}{u_i\theta_p(u_r/u_i)} = \delta_{n\ even}\theta_p(a, b, aU, bU), \tag{2.22}$$

where $U = \prod_{0\le r\le n} u_r$.

Proof. The case n even can be obtained by setting $c = u_{2n-1} = 1$ and expanding the pfaffian along the last row; the case n odd then follows by setting $u_{2n-2} = 1$. □

Similarly, the fact that the pfaffians are nice gives rise to a relation between the functions

$$f^{(4)}_{u_0}(\mathbf{z}), f^{(4)}_{u_1}(\mathbf{z}), f^{(4)}_{u_2}(\mathbf{z}), g^{(4)}_{u_0u_1u_2u_3u_4}(\mathbf{z}), g^{(4)}_{u_0u_1u_2u_3u_5}(\mathbf{z}), g^{(4)}_{u_0u_1u_2u_4u_5}(\mathbf{z}). \tag{2.23}$$

If we also use the relation between $f^{(4)}_{u_r}(\mathbf{z})$, $0 \le r \le 5$ coming from Corollary 2.3, then we can obtain similar relations involving 4, 2, or 0 of the f functions; we omit the details.

Finally, we will also need the following pfaffian identity.

Theorem 2.10. *We have the pfaffian*

$$\mathrm{pf}_{1\le i<j\le 2n}\Big(\frac{z_i^{-1}\theta(z_iz_j^{\pm 1}; t^2)}{\theta(tz_iz_j^{\pm 1}; t^2)}\Big) = \frac{t^{n(n-1)}\prod_{1\le i<j\le 2n} z_i^{-1}\theta(z_iz_j^{\pm 1}; t^2)}{\theta(tz_iz_j^{\pm 1}; t^2)} \tag{2.24}$$

Proof. Both sides are BC_n-antisymmetric abelian functions with the same polar divisor, and are thus proportional. Multiplying both sides by

$$\prod_{1\le i\le n} \theta(tz_{2i-1}/z_{2i}; t^2) \tag{2.25}$$

and taking the limit $z_{2i} \to tz_{2i-1}$ shows that the constant is 1. □

3 Recurrences in low dimensions

We can obtain recurrences for low-dimensional instances of our integral by observing that there are two ways in which shifting the parameters corresponds to multiplying the integrand by a degree 1 theta function. If we multiply t_r by q, this simply multiplies the integrand by

$$f^{(n)}_{t_r}(\dots z_i \dots) = \prod_{1\le i\le n} \theta_p(t_rz_i^{\pm 1}) = t_r^n \prod_{1\le i\le n} \psi_p(t_r, z_i). \tag{3.1}$$

Somewhat more subtly, if $t^{2n-2}t_0t_1t_2t_3t_4t_5t_6t_7 = p^2q$, multiplying the integrand by

$$\prod_{0\le i<n} \frac{q^2p^3}{(t_5t_6t_7)^2t^{2n-2}\theta_p(pq/t^it_5t_6, pq/t^it_5t_7, pq/t^it_6t_7)} g^{(n)}_{t_0,t_1,t_2,t_3,t_4}(\dots z_i \dots) \tag{3.2}$$

simply has the effect of multiplying t_0 through t_4 by $\sqrt{q}$ and dividing t_5, t_6, t_7 by $\sqrt{q}$. Indeed, this follows immediately by an adjointness argument as in the second proof of Theorem 6.1 of [10].

As a result, any linear dependence between the 8 functions $f^{(n)}$ and the 56 functions $g^{(n)}$ gives rise to a relation of integrals. Thus in principle we would obtain recurrences all the way up to dimension 62 (since the space of degree 1 BC_n-symmetric theta functions has dimension $n+1$); in practice, however, the coefficients of such relations do not appear to have nice closed forms in general. There is, however, one special case in which the coefficients are nice. Each function corresponds to a vector (by which it shifts the parameters); if the difference of any two such vectors in the collection is a root of E_7, the corresponding relation has nice coefficients. This, however, greatly reduces the possible number of theta functions in the relation, with the result that we only obtain recurrences for $n \leq 6$.

The simplest case is the linear relations between the functions $f^{(n)}$ from Corollary 2.3, which gives the following recurrence.

Theorem 3.1. *For $1 \leq n \leq 6$, let $t_0,\ldots, t_7$, t, p, q be parameters such that $|p|, |q|, |t| < 1$. Then*

$$\sum_{0\leq i\leq n+1} \frac{t_i II^{(n)}_{t;p,q}(t_0,\ldots,qt_i,\ldots,t_7)}{\prod_{0\leq j\leq n+1; j\neq i} \theta_p(t_i t_j^{\pm 1})} = 0. \tag{3.3}$$

Another source of such recurrences is Theorem 2.5, especially in combination with Lemma 2.4. The upshot is that we obtain (relatively) nice relations between any $n+2$ of the 8 functions

$$f^{(n)}_{t_0}, f^{(n)}_{t_1}, f^{(n)}_{t_2}, f^{(n)}_{t_3}, g^{(n)}_{t_0,t_1,t_2,t_3,t_4}, g^{(n)}_{t_0,t_1,t_2,t_3,t_5}, g^{(n)}_{t_0,t_1,t_2,t_3,t_6}, g^{(n)}_{t_0,t_1,t_2,t_3,t_7}; \tag{3.4}$$

we simply apply Theorem 2.5 with $u_i = t_i$; $v_0,\ldots,v_{n+1} \in \{1/t_0, 1/t_1, 1/t_2, 1/t_3, t_4, t_5, t_6, t_7\}$. The coefficients of the resulting relations are, unfortunately, rather complicated (albeit products of theta functions). In fact, the resulting recurrences are simply images of the recurrence of Theorem 3.1 under the action of the Weyl group $W(E_7)$ (assuming, of course, that $t^{2n-2}t_0t_1t_2t_3t_4t_5t_6t_7 = p^2q$, so that $W(E_7)$ actually does act). Ideally, we would prefer to give a manifestly $W(E_7)$-invariant description of the recurrences; in the absence of such a description, we leave the details to the reader, rather than list all of the superficially different recurrences arising in this way.

Another $W(E_7)$-orbit of recurrences arises from Propositions 2.6 and 2.7 above. For instance, from Proposition 2.6, we obtain the following.

Theorem 3.2. *For $n = 4$, let $t_0,\ldots, t_7$, t, p, q be parameters such that $|p|, |q|, |t| < 1$, $t^6t_0t_1t_2t_3t_4t_5t_6t_7 = p^2q$. Then*

$$\begin{aligned} &II^{(4)}_{t;p,q}(q^{1/2}t_0, q^{1/2}t_1, q^{1/2}t_2, q^{1/2}t_3, q^{1/2}t_4, q^{-1/2}t_5, q^{-1/2}t_6, q^{-1/2}t_7) \\ &\quad = \frac{t^{24}(t_0t_1\ldots t_4)^8 \prod_{0\leq i<j\leq 4} \theta_p(t_it_j, tt_it_j)}{p^4 \prod_{0\leq i<4} \theta_p(pq/t^it_5t_6, pq/t^it_5t_7, pq/t^it_6t_7)} \\ &\quad \times \sum_{0\leq r\leq 4} \prod_{0\leq i\leq 4; i\neq r} \frac{\theta_p(t_i/t^3t_0t_1t_2t_3t_4)}{t_i^2\theta_p(t_r/t_i, t_rt_i, tt_rt_i)} II^{(4)}_{t;p,q}(t_0,\ldots,qt_r,\ldots,t_7) \end{aligned} \tag{3.5}$$

In fact, the $W(E_7)$-images of this identity and those of Theorem 3.1 include every linear recurrence in which the differences of any two shifts is a root of E_7.

4 Generalized Fay identities

Suppose $\psi(x,y)$, $\psi'(x,y)$ are antisymmetric measurable functions on X^2 for some space X that satisfy the identity of Proposition 2.2; for instance, $\psi(x,y)=\psi_p(x,y)$. There is a natural family of multidimensional integrals attached to these functions in such a way that the Plücker relations between minors of a matrix translate into bilinear identities satisfied by integrals.

We define, for any measure μ

$$\tau^{(n)}(\mu;\psi,\psi') = \frac{1}{n!}\int_{X^n} \prod_{1\le i<j\le n}\psi(x_i,x_j)\prod_{1\le i<j\le n}\psi'(x_i,x_j)\prod_{1\le i\le n}\mu(dx_i), \tag{4.1}$$

assuming this integral converges. In addition, for notational convenience, we define

$$\begin{aligned}&\tau^{(n)}(\mu[a_1,\dots,a_k][b_1,\dots,b_l]';\psi,\psi')\\ &\quad:= \prod_{1\le i<j\le k}\psi(a_i,a_j)\prod_{1\le i<j\le l}\psi'(b_i,b_j)\tau^{(n)}\Bigl(\prod_{1\le i\le k}[a_i]\prod_{1\le i\le l}[b_i]'\mu;\psi,\psi'\Bigr)\end{aligned} \tag{4.2}$$

where $[a_i](x)=1/\psi(a_i,x)$, $[b_i]'(x)=1/\psi'(b_i,x)$. Since we will for the most part be fixing ψ,ψ', we will suppress them from the notation when no confusion will result.

Using the fact that ψ, ψ' satisfy Cauchy-type identities, we find that $\tau^{(n)}$ is the integral of a product of two determinants, and thus by the integral analogue of Cauchy-Binet, is itself a determinant of univariate integrals, and can be written as such a determinant in many different ways.

Theorem 4.1. *Assuming all integrals are defined,*

$$\tau^{(n)}(\mu[a_1,\dots,a_k][b_1,\dots,b_l]') = \det_{1\le i,j\le n}\tau^{(1)}(\mu[a_i][b_j]'). \tag{4.3}$$

Proof. We have

$$\prod_{1\le i<j\le n}\psi(x_i,x_j) = (-1)^{n(n-1)/2}\frac{\prod_{1\le i,j\le n}\psi(a_i,x_j)}{\prod_{1\le i<j\le n}\psi(a_i,a_j)}\det_{1\le i,j\le n}\Bigl(\frac{1}{\psi(a_i,x_j)}\Bigr) \tag{4.4}$$

and similarly

$$\prod_{1\le i<j\le n}\psi'(x_i,x_j) = (-1)^{n(n-1)/2}\frac{\prod_{1\le i,j\le n}\psi'(b_i,x_j)}{\prod_{1\le i<j\le n}\psi'(b_i,b_j)}\det_{1\le i,j\le n}\Bigl(\frac{1}{\psi'(b_i,x_j)}\Bigr) \tag{4.5}$$

and thus

$$\begin{aligned}\tau^{(n)}(\mu[a_1,\dots,a_n][b_1,\dots,b_n]') &= \int_{X^n}\det_{1\le i,j\le n}\Bigl(\frac{1}{\psi(a_i,x_j)}\Bigr)\det_{1\le i,j\le n}\Bigl(\frac{1}{\psi'(b_i,x_j)}\Bigr)\mu(dx_i) && (4.6)\\ &= \det_{1\le i,j\le n}\int_X \frac{1}{\psi(a_i,x)\psi'(b_j,x)}\mu(dx). && (4.7)\end{aligned}$$

□

Now, any minor of the above determinant is itself a determinant of the same form. As a consequence, any polynomial equation satisfied by minors of a general matrix translates immediately into a relation satisfied by our family of integrals. The ideal of such relations is known to be generated by a family of bilinear equations, known as the Plücker relations. For our purposes, we restrict our attention to the simplest such identities.

For any matrix M, we let $\det_{S,T}(M)$ denote the determinant of the submatrix of M with coordinates $i \in S, j \in T$. (This has a sign ambiguity which we can eliminate by fixing an ordering on the coordinates.) These determinants satisfy the following three identities:

$$\det_{S\cup\{a,b\},T\cup\{c,d\}}(M)\det_{S,T}(M) - \det_{S\cup\{a\},T\cup\{c\}}(M)\det_{S\cup\{b\},T\cup\{d\}}(M) + \det_{S\cup\{a\},T\cup\{d\}}(M)\det_{S\cup\{b\},T\cup\{c\}}(M) = 0, \tag{4.8}$$

where $|S| = |T|$ and the coordinates are ordered so that $S < a < b; T < c < d$,

$$\det_{S\cup\{a\},T\cup\{c,d\}}(M)\det_{S,T\cup\{b\}}(M) - \det_{S\cup\{a\},T\cup\{b,d\}}(M)\det_{S,T\cup\{c\}}(M) + \det_{S\cup\{a\},T\cup\{b,c\}}(M)\det_{S,T\cup\{d\}}(M) = 0, \tag{4.9}$$

where $|S| = |T| + 1$ and the coordinates are ordered so that $S < a; T < b < c < d$, and

$$\det_{S,T\cup\{a,b\}}(M)\det_{S,T\cup\{c,d\}}(M) - \det_{S,T\cup\{a,c\}}(M)\det_{S,T\cup\{b,d\}}(M) + \det_{S,T\cup\{a,d\}}(M)\det_{S,T\cup\{b,c\}}(M) = 0, \tag{4.10}$$

where $|S| = |T| + 2$ and the coordinates are ordered so that $T < a < b < c < d$.

Applying these identities to the matrix with entries

$$\tau^{(1)}(\mu[a_i][b_j]'), \tag{4.11}$$

and rescaling μ, we obtain the following identities.

Theorem 4.2. *Assuming the integrals in question are all defined, we have the following identities.*

$$\tau^{(n+1)}(\mu[a,b][c,d]')\tau^{(n-1)}(\mu) - \tau^{(n)}(\mu[a][c]')\tau^{(n)}(\mu[b][d]') + \tau^{(n)}(\mu[a][d]')\tau^{(n)}(\mu[b][c]') = 0 \tag{4.12}$$

$$\tau^{(n-1)}(\mu[b])\tau^{(n)}(\mu[c,d][a]') - \tau^{(n-1)}(\mu[c])\tau^{(n)}(\mu[b,d][a]') + \tau^{(n-1)}(\mu[d])\tau^{(n)}(\mu[c,d][a]') = 0 \tag{4.13}$$

$$\tau^{(n)}(\mu[c,d])\tau^{(n)}(\mu[a,b]) - \tau^{(n)}(\mu[b,d])\tau^{(n)}(\mu[a,c]) + \tau^{(n)}(\mu[b,c])\tau^{(n)}(\mu[a,d]) = 0. \tag{4.14}$$

If we set $\tau^{(n)} = 0$ for $n < 0$, these identities remain valid for all integers n.

Proof. The Plücker identity argument immediately gives the first identity for $n \geq 1$ and the other identities for $n \geq 2$. Similarly, the first two identities are trivial for $n \leq 0$, and the third identity is trivial for $n \leq -1$. So it remains to show the second identity for $n = 1$ and the third identity for $n = 0$, $n = 1$. The second identity for $n = 1$ is a linear relation between univariate integrals that follows immediately from the relation

$$\psi(c,d)\psi(b,x) - \psi(b,d)\psi(d,x) + \psi(b,c)\psi(c,x). \tag{4.15}$$

The third identity for $n = 0$ is just the case $x = a$ of this identity. Finally, for $n = 1$, the third identity is the pfaffian of a 4×4 matrix which has rank 2 by the second identity. $\square$

Remark. When $\psi(x,y) = \psi'(x,y) = x - y$, these are instances of the generalized Fay identities of [2]. By the remark after Proposition 2.2, this can be used to obtain the general $\psi = \psi'$ case via a change of variables. Similarly, the case $\psi \neq \psi'$ can be obtained via a change of variables and a delta function limit from the identities of [1]. The above more elementary proof based on the Cauchy determinant appears to be new, however.

Similarly, if ϵ is an arbitrary antisymmetric function on X, define, for n even,

$$\tau^{(n)}_{1/2}(\mu;\epsilon;\psi) = \frac{1}{n!}\int \mathrm{pf}_{1\le i,j\le n}(\epsilon(x_i,x_j)) \prod_{1\le i<j\le n} \psi(x_i,x_j) \prod_{1\le i\le n} \mu(dx_i) \tag{4.16}$$

For n odd, we also need a univariate function ϕ, and then define

$$\tau^{(n)}_{1/2}(\mu;\phi,\epsilon;\psi) = \frac{1}{n!}\int \mathrm{pf}_{1\le i,j\le n}(\phi(x_i);\epsilon(x_i,x_j)) \prod_{1\le i<j\le n} \psi(x_i,x_j) \prod_{1\le i\le n} \mu(dx_i) \tag{4.17}$$

Here, for n odd,

$$\mathrm{pf}_{1\le i,j\le n}(\phi(x_i);\epsilon(x_i,x_j)) \tag{4.18}$$

represents the pfaffian of the $n+1\times n+1$ antisymmetric matrix obtained from the $n\times n$ matrix $\epsilon(x_i,x_j)$ by adjoining a row $\phi(x_i)$ and a column $-\phi(x_i)$. Again, we will fix ϕ,ϵ,ψ and suppress them from the notation; whether ϕ appears is determined from the parity of n. Similarly, we extend the notation to cover negative integers by setting $\tau^{(n)}_{1/2} = 0$ for $n < 0$.

It follows by an identity of de Bruijn [4] (also see [13] for a discussion in the context of Selberg integrals) that these pfaffian τ functions can be written as pfaffians.

Proposition 4.3. *Assuming the integrals are all defined, we have the following expressions. For n even,*

$$\tau^{(n)}_{1/2}(\mu[a_1,\dots,a_n]) = \mathrm{pf}_{1\le i,j\le n}\left(\tau^{(2)}_{1/2}(\mu[a_i,a_j])\right) \tag{4.19}$$

and for n odd,

$$\tau^{(n)}_{1/2}(\mu[a_1,\dots,a_n]) = \mathrm{pf}_{1\le i,j\le n}\left(\tau^{(1)}_{1/2}(\mu[a_i]);\tau^{(2)}_{1/2}(\mu[a_i,a_j])\right) \tag{4.20}$$

Similarly to the determinantal case, there are a number of bilinear identities satisfied by the pfaffian minors of an antisymmetric matrix. For our purposes, we will restrict our attention to the following pair of four-term identities.

$$\begin{aligned}\mathrm{pf}_S(A)\,\mathrm{pf}_{S\cup\{a,b,c,d\}}(A) - \mathrm{pf}_{S\cup\{a,b\}}(A)\,\mathrm{pf}_{S\cup\{c,d\}}(A)\\ + \mathrm{pf}_{S\cup\{a,c\}}(A)\,\mathrm{pf}_{S\cup\{b,d\}}(A) - \mathrm{pf}_{S\cup\{a,d\}}(A)\,\mathrm{pf}_{S\cup\{b,c\}}(A) = 0,\end{aligned} \tag{4.21}$$

where $|S|$ is even, $S < a < b < c < d$, and

$$\begin{aligned}\mathrm{pf}_{S\cup\{a\}}(A)\,\mathrm{pf}_{S\cup\{b,c,d\}}(A) - \mathrm{pf}_{S\cup\{b\}}(A)\,\mathrm{pf}_{S\cup\{a,c,d\}}(A)\\ + \mathrm{pf}_{S\cup\{c\}}(A)\,\mathrm{pf}_{S\cup\{a,b,d\}}(A) - \mathrm{pf}_{S\cup\{d\}}(A)\,\mathrm{pf}_{S\cup\{a,b,c\}}(A) = 0,\end{aligned} \tag{4.22}$$

where $|S|$ is odd, $S < a < b < c < d$. See [8] for these, and other such identities.

These give rise to identities between our pfaffian τ functions; since in the odd-dimensional case one row and column is special, we obtain a total of six such identities. However, the resulting identities turn out to behave the same for n odd and n even, giving us a total of three identities.

Theorem 4.4. *Assuming the integrals are all defined, we have the following identities for all integers n.*

$$\begin{aligned}\tau_{1/2}^{(n+4)}(\mu[a_1,a_2,a_3,a_4])\tau_{1/2}^{(n)}(\mu) - \tau_{1/2}^{(n+2)}(\mu[a_1,a_2])\tau_{1/2}^{(n+2)}(\mu[a_3,a_4])\\ + \tau_{1/2}^{(n+2)}(\mu[a_1,a_3])\tau_{1/2}^{(n+2)}(\mu[a_2,a_4]) - \tau_{1/2}^{(n+2)}(\mu[a_1,a_4])\tau_{1/2}^{(n+2)}(\mu[a_2,a_3]) = 0.\end{aligned} \tag{4.23}$$

$$\begin{aligned}\tau_{1/2}^{(n+3)}(\mu[a_2,a_3,a_4])\tau_{1/2}^{(n+1)}(\mu[a_1]) - \tau_{1/2}^{(n+3)}(\mu[a_1,a_3,a_4])\tau_{1/2}^{(n+1)}(\mu[a_2])\\ + \tau_{1/2}^{(n+3)}(\mu[a_1,a_2,a_4])\tau_{1/2}^{(n+1)}(\mu[a_3]) - \tau_{1/2}^{(n+3)}(\mu[a_1,a_2,a_3])\tau_{1/2}^{(n+1)}(\mu[a_4]) = 0.\end{aligned} \tag{4.24}$$

$$\begin{aligned}\tau_{1/2}^{(n+3)}(\mu[a_1,a_2,a_3])\tau_{1/2}^{(n)}(\mu) - \tau_{1/2}^{(n+2)}(\mu[a_2,a_3])\tau_{1/2}^{(n+1)}(\mu[a_1])\\ + \tau_{1/2}^{(n+2)}(\mu[a_1,a_3])\tau_{1/2}^{(n+1)}(\mu[a_2]) - \tau_{1/2}^{(n+2)}(\mu[a_1,a_2])\tau_{1/2}^{(n+1)}(\mu[a_3]) = 0.\end{aligned} \tag{4.25}$$

Proof. For $n \geq 0$, these identities are just relations between minors of the antisymmetric matrix

$$(\tau_{1/2}^{(1)}(\mu[a_i]); \tau_{1/2}^{(2)}(\mu[a_i,a_j])) \tag{4.26}$$

For $n \leq -1$, the first and third identities follow from Theorem 4.2 (as, when nontrivial, they relate univariate integrals and scalars). The only remaining nontrivial case is the instance $n = -1$ of the second identity. But this is a linear relation between bivariate integrals coming from a linear relation of the integrands. □

Remark. Again, these could be obtained via a change of variables from the pfaffian Fay identities of [3], but our elementary proof is new.

5 Painlevé recurrences

If we apply the generalized Fay identities to an integral of the form $\tilde{I\!I}_{q;p,q}$, we find that for suitable choices of a_i, the integrals that appear are of the same form, with shifted parameters. As a result, Theorem 4.2 gives rise to three special cases of the following recurrence.

Theorem 5.1. *Let v_0, v_1, $v_2 \in \frac{1}{2}\Lambda_{E_8}$ be unit vectors in a common coset of Λ_{E_8}, and let $\phi \in \operatorname{Hom}(\Lambda_{E_8}, \mathbb{C}^*)$ be such that $pq/(\tau_{v_0}\phi)(\omega) \in q^{\mathbb{Z}}$. Then*

$$\sum_{0\leq r\leq 2} \frac{\tilde{I\!I}_{q;p,q}(\tau_{v_r}\phi)\tilde{I\!I}_{q;p,q}(\tau_{-v_r}\phi)}{\prod_{s\neq r}\psi_p(\phi(v_r),\phi(v_s))} = 0. \tag{5.1}$$

Proof. First, we observe as remarked that Theorem 4.2 gives essentially three special cases (up to signed permutations within the triples and the natural action of S_8 on the coordinates), namely:

$$v_0=(\frac12,\frac12,\frac12,\frac12,0,0,0,0)\qquad v_1=(\frac12,\frac12,-\frac12,-\frac12,0,0,0,0)\quad v_2=(\frac12,-\frac12,-\frac12,\frac12,0,0,0,0) \tag{5.2}$$

$$v_0=(\frac12,-\frac12,\frac12,\frac12,0,0,0,0)\qquad v_1=(\frac12,\frac12,-\frac12,\frac12,0,0,0,0)\qquad v_2=(\frac12,\frac12,\frac12,-\frac12,0,0,0,0) \tag{5.3}$$

$$v_0=(\frac12,\frac12,-\frac12,-\frac12,0,0,0,0)\quad v_1=(\frac12,-\frac12,\frac12,-\frac12,0,0,0,0)\quad v_2=(\frac12,-\frac12,-\frac12,\frac12,0,0,0,0) \tag{5.4}$$

Now, we know that $\tilde{II}_{q;p,q}$ is invariant under the action of $W(E_7)$, so it suffices to prove the theorem for one (unsigned, unordered) triple from each $W(E_7)$-orbit. There are four such orbits, so we still have one orbit remaining to consider, one representative of which is:

$$v_0=(\frac12,\frac12,\frac12,\frac12,0,0,0,0)\quad v_1=(0,0,0,0,\frac12,\frac12,\frac12,\frac12)\quad v_2=(0,0,0,0,\frac12,\frac12,-\frac12,-\frac12). \tag{5.5}$$

Now, consider the following two representatives of the orbit of our first special case:

$$v_0=(\frac12,\frac12,\frac12,\frac12,0,0,0,0)\quad v_1=(0,0,0,0,\frac12,-\frac12,\frac12,-\frac12)\quad v_2=(0,0,0,0,\frac12,\frac12,-\frac12,-\frac12) \tag{5.6}$$

$$v_0=(0,0,0,0,\frac12,\frac12,\frac12,\frac12)\quad v_1=(0,0,0,0,\frac12,-\frac12,\frac12,-\frac12)\quad v_2=(0,0,0,0,\frac12,\frac12,-\frac12,-\frac12). \tag{5.7}$$

If we take a linear combination of these two bilinear identities in such a way as to eliminate the term corresponding to $(0,0,0,0,1/2,-1/2,1/2,-1/2)$, we find that the result is precisely the desired bilinear identity corresponding to the above representative of the missing orbit. □

As the above scheme of bilinear recurrences has appeared elsewhere in the literature [7, Theorem 5.2], we immediately obtain the following corollary. Compare also the results of [6] for the Selberg limit.

Corollary 5.2. *The function $\tilde{II}_{q;p,q}$ is a tau function for the elliptic Painlevé equation.*

Remark. We should mention in this context that Sakai's version of elliptic Painlevé [12] is geometrically described in terms of the blow-up of $\mathbb{P}^2$ at 9 points. This has a natural S_9 symmetry, but the S_8 symmetry it gives is *not* conjugate to the natural S_8 symmetry on our integral. This suggests that, at least from the integral perspective, the more natural geometric context for elliptic Painlevé is the blow-up of $\mathbb{P}^1\times\mathbb{P}^1$ at 8 points.

We next turn to the case $t=q^2$. Here, the cross term in the integral is

$$\prod_{1\le i<j\le n}\theta_p(z_i^{\pm1}z_j^{\pm1})\theta_p(qz_i^{\pm1}z_j^{\pm1}). \tag{5.8}$$

As it stands, this does not appear to be amenable to either generalized Fay identity. However, we can write this as

$$q^{n(n-1)} \prod_{1\le i<j\le n} \psi_p(q^{\pm 1/2}z_i, q^{\pm 1/2}z_j), \tag{5.9}$$

which can in turn be written as

$$q^{5n(n-1)/4} \prod_{1\le i\le n} \frac{1}{z_i^{-1}\theta_p(z_i^2)} \prod_{1\le i<j\le 2n} \psi_p(w_i, w_j), \tag{5.10}$$

where $w_{2i-1} = q^{1/2}z_i$, $w_{2i} = q^{-1/2}z_i$. But, aside from a factor of $(2n)!/2^n n!$, this corresponds to a limiting case of $\tau^{(2n)}_{1/2}$, taking

$$\epsilon(x_i, x_j) = \delta(x_i - qx_j) - \delta(x_j - qx_i). \tag{5.11}$$

As a result, we again obtain bilinear identities, this time with four terms each.

Theorem 5.3. *Let v_0, v_1, v_2, $v_3 \in \frac{1}{2}\Lambda_{E_8}$ be unit vectors in a common coset of Λ_{E_8}, and let $\phi \in \operatorname{Hom}(\Lambda_{E_8}, \mathbb{C}^*)$ such that $pq/(\tau_{2v_0}\phi)(\omega) \in q^{2\mathbb{Z}}$. Then*

$$\sum_{0\le r\le 3} \frac{\tilde{I\!I}_{q^2;p,q}(\tau_{2v_r}\phi)\tilde{I\!I}_{q^2;p,q}(\tau_{-2v_r}\phi)}{\prod_{s\ne r}\psi_p(\phi(v_r),\phi(v_s))} = 0. \tag{5.12}$$

Proof. Here, the generalized Fay identities give us two of the four $W(E_7)$-orbits we require:

$$\begin{aligned} v_0 &= (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0), & v_1 &= (\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},0,0,0,0),\\ v_2 &= (\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},0,0,0,0), & v_3 &= (\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},0,0,0,0), \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} v_0 &= (-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0), & v_1 &= (\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0),\\ v_2 &= (\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},0,0,0,0), & v_3 &= (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},0,0,0,0). \end{aligned} \tag{5.14}$$

But again, we can obtain the bilinear identities corresponding to the missing orbits as linear combinations of $W(E_7)$-images of the identities coming from the generalized Fay identities. The key point here is that if two of our identities have three monomials in common, and we take a linear combination to eliminate one of those monomials, the result is another of our identities. $\square$

For $t = q^{1/2}$, we apply Theorem 2.10 (as an identity of q-theta functions!) to write the integrand as an instance of $\tau^{(n)}_{1/2}$. The first two generalized Fay identities turn out to give us a recurrence strikingly like the one we have just seen; again, by taking linear combinations of $W(E_7)$-images, we can obtain the entire $W(E_8)$-orbit.

Theorem 5.4. *Let v_0, v_1, v_2, $v_3 \in \frac{1}{2}\Lambda_{E_8}$ be unit vectors in a common coset of Λ_{E_8}, and let $\phi \in \mathrm{Hom}(\Lambda_{E_8}, \mathbb{C}^*)$ such that $pq/(\tau_{v_0}\phi)(\omega) \in q^{\mathbb{Z}/2}$. Then*

$$\sum_{0\le r\le 3} \frac{\tilde{I\!I}_{q^{1/2};p,q}(\tau_{v_r}\phi)\tilde{I\!I}_{q^{1/2};p,q}(\tau_{-v_r}\phi)}{\prod_{s\ne r}\psi_p(\phi(v_r),\phi(v_s))} = 0. \tag{5.15}$$

The third Fay identity corresponds to a different $W(E_8)$-orbit. Unfortunately, we can no longer combine instances of the $W(E_7)$-orbit, and have in fact been unable to prove the presumable general form of the recurrence. We do, however, have the following.

Theorem 5.5. *Suppose the unordered, unsigned quadruple (v_0, v_1, v_2, v_3) is in the $W(E_7)$-orbit of*

$$((\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0,0),(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,0,0),(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},0,0,0,0,0),(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},0,0,0,0,0)). \tag{5.16}$$

Then equation (5.15) *still holds.*

Unlike the determinantal case, the recurrences corresponding to the pfaffian cases ($t \in \{q^2, q^{1/2}\}$) appear to be new, even in the Painlevé setting. It would be very interesting to know a geometric interpretation for these recurrences, and more generally to understand how they relate to the usual elliptic Painlevé equation.

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