ELLIPTIC INTEGRABLE SYSTEMS OF CALOGERO-MOSER TYPE: A SURVEY

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ABSTRACT. We survey the definitions of the Hamiltonians of A_{N-1}/BC_N-type, classical/quantum, NR(nonrelativistic)/REL(relativistic) models and survey eigenfunction literature in the quantum case.

1. INTRODUCTION

In recent years, various elliptic models have been researched, such as elliptic analogue of well-known Calogero-Sutherland model and its relativistic analogue called Ruijsenaars-Schneider model both in the classical and the quantum cases [43, 46, 52, 58]. Moreover these models are generalized to those associated with affine root systems. In this article, we survey the definitions of the Hamiltonians of A_{N-1} and BC_N-type, the classical and the quantum, nonrelativistic and relativistic models and eigenfunction literature in the quantum case.

In Section 2, we give a brief introduction to the notion of integrability. In the classical case, a system is called integrable if there are sufficiently many independent Poisson-commuting functions including the Hamiltonian. In addition, if the flows are complete, the global structure can be analyzed. We call such system Liouville integrable. In contrast with the classical case, quantum integrability is not established and includes ambiguity. We clarify its definition employed in this survey. Moreover we introduce the notion of Hilbert integrability, since the commutativity among the infinitesimal generators is not sufficient for that of the evolutions, which is required by quantum mechanics.

In Section 3 and 4, we investigate A_{N-1} and BC_N-type models, respectively. We present the Hamiltonian functions and the Hamiltonian operators of nonrelativistic and relativistic models. In the classical case, the main tool is the Lax matrix and we give some information about it. In the quantum case, though we have no general theory, we see that R-operators, elliptic quantum groups and Hecke algebras play some important roles.

In section 5, we survey eigenfunction literature in the quantum case. It is a quite important problem and there are many results. However we have not arrived at fully understanding.

This survey is based on the first lecture and the transparencies by Professor S. N. M. Ruijsenaars. The author adds some other related results and topics and the references, but they are not complete; the reader is referred to the original articles for proofs and for complete references. If this survey includes any errors, Y.K is responsible for them.

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2. Integrability

A-type systems of Calogero-Moser type are surveyed extensively in [53]. We sketch the notion of integrability both at the classical and at the quantum levels. For the general theory of classical mechanics, see [1], and for quantum mechanics, see [48, 50].

2.1. **Classical Regime.** We consider an N-particle system in a one-dimensional space and the position of N-particles is described by an element of an N-dimensional manifold M. Then the phase space is a $2N$-dimensional symplectic manifold $\langle \Omega, \omega \rangle$, where $\Omega = T^*M$ is the cotangent bundle of M and its standard symplectic form ω is defined as follows: on a local coordinate system (U, x) of M, we have corresponding local coordinate system $(\pi^{-1}(U) \simeq U \times \mathbb{R}^N, (x, p))$ of T^*M as $\sum_{i=1}^N p_i dx_i$ where $\pi : T^*M \to M$ is the natural projection, and 2-form $\sum_{i=1}^N dx_i \wedge dp_i$ globally defines ω. Since ω is nondegenerate 2-form, for a 1-form α, a vector field $X^{(\alpha)}$ is uniquely determined by the condition

$$\omega(X^{(\alpha)}, X) = \alpha(X), \tag{2.1}$$

for arbitrary vector fields X. For a smooth function f, let X_f be a vector field defined by $X^{(df)}$. Then the Poisson bracket is defined for f, g by

$$\{f, g\} = \omega(X_f, X_g) = df(X_g) = X_g f. \tag{2.2}$$

The Poisson bracket satisfies the following properties:

$$\{f, g\} = -\{g, f\}, \tag{2.3}$$

$$\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}, \tag{2.4}$$

$$\{fg, h\} = f\{g, h\} + g\{f, h\}, \tag{2.5}$$

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0, \tag{2.6}$$

where $\alpha, \beta \in \mathbb{R}$, and is given explicitly on a local coordinate system by

$$\{f, g\} = \sum_{j=1}^N \Big(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_j} \Big). \tag{2.7}$$

The motion of particles is governed by a smooth function called the Hamiltonian as

$$\frac{du}{dt} = X_H(u) = \{u, H\}. \tag{2.8}$$

Let $\exp(tX_H)$ or, for brevity, e^{tH} denotes the flow generated by H. Thus $u(t) = e^{tH}(u_0)$ for the initial point u_0.

If there exist functions I_j for $j = 1, \dots, N$ such that their differentials dI_j are linearly independent on dense open subset and they are involutive, $\{I_j, I_k\} = \{I_j, H\}$, then this system is called a *completely integrable system* or simply a *integrable system*.

By use of the relation

$$[X_f, X_g] = X_{-\{f,g\}}, \tag{2.9}$$

we obtain involutive vector fields $X_j = X_{I_j}$, i.e., $[X_j, X_k] = 0$. Hence for a point u_0 where $dI_j(u_0)$ are linearly independent, the map $u : U \to \Omega$, $u(t) = e^{t_1 I_1} \cdots e^{t_N I_N}(u_0)$ gives rise to a local diffeomorphism from some open neighborhood $U \subset \mathbb{R}^N$ of the origin to the submanifold of its image. Let $a \in \mathbb{R}^N$ be a regular value of the moment map $I = (I_1, \ldots, I_N)$. Then the level set $I^{-1}(a)$ is a Lagrangian submanifold. If all the flows generated by I_j are complete, for a connected component M_0 of the Lagrangian submanifold and $u_0 \in M_0$, one obtains $M_0 = \{u(t) \mid t \in \mathbb{R}^N\} \simeq \mathbb{T}^k \times \mathbb{R}^{N-k}$ for some $k \in \mathbb{N}$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Here we call such system *Liouville integrable* [53]. The term "Liouville integrability" is also used for "complete integrability" in other references.

In the A-type root system, we consider N-particle systems where no collision occurs and particles are on a circle $\mathbb{T}$. Thus M is one of the connected components of $\mathbb{T}^N \setminus \{x_i \equiv x_j\}$ and in particular we choose M as the image of $\tilde{M} = \{x \in \mathbb{R}^N \mid x_N < \cdots < x_1\}$ by the natural projection. Similarly in the BC-type root system, we set M to be one of the connected components of $\mathbb{T}^N \setminus \{x_i \equiv x_j, x_i \equiv -x_i\}$. Note that in a general root system, $M \subset \mathbb{R} \otimes_{\mathbb{Z}} Q^\vee / 2\pi Q^\vee$ is chosen as the image of the fundamental domain, where $Q^\vee$ is the coroot lattice. For the other possible phase spaces, see [53].

2.2. **Quantum Regime.** For our convenience, first we discuss the case of the A-type root system and thus assume that particles are on a circle $\mathbb{T}$ and let M be the same as in the previous section. Then the canonical quantization implies $p_j \to \hat{p}_j = -i\hbar\partial/\partial x_j$ where $\hbar$ is the Planck constant. $\hat{p}_j$ is a linear operator on the Hilbert space $V = L^2(\mathbb{T}^N, dx)$ of square-integrable complex-valued functions. A unit vector ψ of V is called a state, and $\int_A |\psi(x)|^2 dx$ is interpreted as the probability to find the particles in the region A. The symmetric or antisymmetric subspace, $L^2_s(\mathbb{T}^N, dx)$ and $L^2_a(\mathbb{T}^N, dx)$ respectively, is physically chosen, and both are uniquely determined by their restrictions to M. In fact, we have $L^2_s(\mathbb{T}^N, dx) \simeq L^2_a(\mathbb{T}^N, dx) \simeq L^2(M, dx)$. In this survey, we work with $L^2_s(\mathbb{T}^N, dx)$.

In quantum cases, the time evolution of a state ψ is governed by the Hamiltonian operator $\hat{H}$ as $\psi(t) = e^{-it\hat{H}}\psi_0$ for the initial state ψ_0. Hence ψ obeys

$$\frac{d\psi}{dt} = -i\hat{H}\psi, \tag{2.10}$$

if the initial state ψ_0 is in the domain of $\hat{H}$.

In contrast with the classical case, only commutativity does not make any sense, because if an operator A is selfadjoint, which is required by quantum mechanics, then all the spectral projections commute with A. Hence we have no direct quantum analogue of the integrability. We adopt the definition of the quantum integrability in an algebraic sense or in a formal sense. Namely, for a Hamiltonian of the form

$$\hat{H} = \sum_{j=1}^{N} \hat{p}_j^2 + V(x), \tag{2.11}$$

with V meromorphic and $\mathfrak{S}_N$-invariant function, it is said that $\hat{H}$ is *integrable* when there exist algebraically independent $\mathfrak{S}_N$-invariant partial differential operators (PDOs) over $\mathbb{C}$, $I_1 = \hat{H}$, $I_2(x, \hat{p})$, $\ldots$, $I_N(x, \hat{p})$ that commute pairwise. To be more precise, we interpret these operators as elements of the polynomials $(\mathcal{M}[\hat{p}])^{\mathfrak{S}_N}$ where $\mathcal{M}$ is a field of meromorphic functions on $(\mathbb{C}/2\pi\mathbb{Z})^N$.

From the view point of the Hilbert space theory, the definition above is not sufficient, since it is required that all I_j are essentially selfadjoint on the same core. Moreover in general, the commutativity among I_j does not imply that of evolutions $e^{iI_j t_j}$, where the latter condition is called strong commutativity. The strong commutativity is equivalent to the commutativity of spectral projections and in particular it is sufficient that all I_j are simultaneously diagonalized. Here we call such system *Hilbert integrable* [53].

In the relativistic case, as a quantum analogue of e^{ap}, we introduce an operator of the form $e^{a\hat{p}}$. In an obvious way, we interpret this operator as a difference (shift) operator

$$(e^{a\hat{p_j}}\psi)(x_1,\ldots,x_j,\ldots,x_N) = \psi(x_1,\ldots,x_j - i\hbar a,\ldots,x_N), \tag{2.12}$$

where ψ should be at least analytic and analytically continued to some strip including real axis. Similarly to differential cases, we regard gauge-transformed Hamiltonians as elements of $(\mathcal{M}[e^{\hat{p}}])^{\mathfrak{S}_N}$ in a formal sense, where the gauge function is an element of $\mathcal{M}^{1/2}$.

For the BC-type root system, we have only to employ the corresponding M and replace the symmetric group $\mathfrak{S}_N$ by $\mathfrak{S}_N \ltimes (\mathbb{Z}/2\mathbb{Z})^N$ in the above, where the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts on function spaces as sign flip of a variable. Note that in a general root system, the corresponding Weyl group plays the same role as the symmetric group and $\mathcal{M}$ is a field of meromorphic functions on $\mathbb{C} \otimes_{\mathbb{Z}} Q^\vee/2\pi Q^\vee$

3. The Defining A_{N-1} Hamiltonians

3.1. **Classical Regime.**

3.1.1. *Nonrelativistic case.* The Hamiltonian, the momentum and the Galilei boost are given by

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + g^2 \sum_{1\le j<k\le N} \wp(x_j - x_k), \tag{3.1}$$

$$P = \sum_{j=1}^{N} p_j, \tag{3.2}$$

$$B = -\sum_{j=1}^{N} x_j, \tag{3.3}$$

where $g \in \mathbb{R}$ is a coupling constant and $\wp(x)$ is Weierstrass elliptic function (6.8) with half periods ω_1, ω_2. We set $\omega_1 = \pi, i\omega_2 \in \mathbb{R}$ for $\wp(x)$ to be real on $\mathbb{T}$. Note that though the boost B is well defined only in the case that the phase space is $T^*\tilde{M}$, the vector field X_B descends to T^*M.

H, P and B satisfy a central extension of the Lie algebra of the Galilei group:

$$\{H, P\} = 0, \qquad \{P, B\} = N, \qquad \{H, B\} = P. \tag{3.4}$$

Calogero showed that this system has a Lax representation [7] and Olshanetsky/Perelomov showed that the system is completely integrable, i.e., all the conserved quantities are involutive, by showing the involutivity of the eigenvalues of the Lax matrix [43]. The following Lax representation with a spectral parameter is due to Krichever [34], which was found through the study of the relation between the KP equation and the Calogero model, and includes the Lax matrix L due to Calogero as a special case.

A Poisson-commuting family is given by the traces of the Lax matrix

$$H_k = \frac{1}{k} \operatorname{Tr} L_{\mathrm{NR}}^k, \qquad k = 1, \ldots, N, \tag{3.5}$$

$$(L_{\mathrm{NR}})_{jk} = \delta_{jk} p_j + ig(1 - \delta_{jk}) \frac{\sigma(x_j - x_k + \lambda)}{\sigma(\lambda)\sigma(x_j - x_k)}, \tag{3.6}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $H_1 = P$ and $H_2 = H - g^2 N(N-1)\wp(\lambda)/2$.

Another proof of the integrability using classical r-matrix structure was found by Sklyanin [62], and Braden/Suzuki [6] independently. Yet another proof is a direct consequence of its quantum version. See Section 3.2.1 and Appendix 6.4.1.

The Liouville integrability, i.e., the completeness of the flows is shown in [53].

3.1.2. *Relativistic case.* A relativistic variant of the elliptic Calogero-Sutherland model was introduced in [58]. The Hamiltonian, the momentum and the Lorentz boost are given by

$$H = \frac{1}{2\beta^2}(S_1 + S_{-1}) = \frac{1}{\beta^2} \sum_{j=1}^{N} \operatorname{ch}(\beta p_j) \prod_{k \neq j} v(x_j - x_k), \tag{3.7}$$

$$P = \frac{1}{2\beta}(S_1 - S_{-1}), \tag{3.8}$$

$$B = -\sum_{j=1}^{N} x_j, \tag{3.9}$$

where $\beta = 1/c$ is the inverse of the speed of light and $\mu = i\beta g$,

$$S_{\pm 1} = \sum_{j=1}^{N} e^{\pm \beta p_j} \prod_{k \neq j} v(x_j - x_k), \tag{3.10}$$

$$v(x) = \Big(\frac{\sigma(\mu + x)\sigma(\mu - x)}{\sigma(x)\sigma(-x)} \Big)^{1/2}. \tag{3.11}$$

H, P and B satisfy the Lie algebra of the Poincaré group,

$$\{H, P\} = 0, \qquad \{H, B\} = P, \qquad \{P, B\} = H/c^2. \tag{3.12}$$

From (6.12), we have the following expansion:

$$H_{\mathrm{rel}} = \frac{1}{\beta^2} \sum_{j=1}^{N} (1 + \frac{p_j^2}{2}\beta^2 + O(\beta^4)) \prod_{k \neq j} (1 + \frac{g^2}{2}\wp(x_j - x_k)\beta^2 + O(\beta^4)), \tag{3.13}$$

$$P_{\mathrm{rel}} = \frac{1}{\beta} \sum_{j=1}^{N} (p_j \beta + O(\beta^3)) \prod_{k \neq j} (1 + \frac{g^2}{2}\wp(x_j - x_k)\beta^2 + O(\beta^4)). \tag{3.14}$$

Thus when the speed of light goes to infinity, we have

$$\lim_{c\to\infty} H_{\mathrm{rel}} - Nc^2 = H_{\mathrm{nr}}, \tag{3.15}$$

$$\lim_{c\to\infty} P_{\mathrm{rel}} = P_{\mathrm{nr}}, \tag{3.16}$$

$$\lim_{c\to\infty} B_{\mathrm{rel}} = B_{\mathrm{nr}}. \tag{3.17}$$

A Poisson-commuting family is given in terms of S_k which are symmetric functions of the Lax matrix L for $k = 1, \ldots, N$ [52]:

$$\det(L + \alpha I) = \sum_{l=0}^{N} \alpha^l \Sigma_{N-l}, \qquad \Sigma_k = c_k S_k, \tag{3.18}$$

$$c_k = \frac{\sigma(\lambda-\mu)^{k-1}\sigma(\lambda+(k-1)\mu)}{\sigma(\lambda)^k}, \tag{3.19}$$

which directly follows from formula (6.7). Here λ is a spectral parameter and

$$(L_{\mathrm{REL}})_{jk} = d_j c_{jk}, \tag{3.20}$$

$$d_j = e^{\beta p_j} \prod_{k\neq j} v(x_j - x_k), \tag{3.21}$$

$$c_{jk} = \frac{\sigma(\mu)}{\sigma(\lambda)} \frac{\sigma(x_j - x_k + \lambda)}{\sigma(x_j - x_k + \mu)}, \tag{3.22}$$

$$S_{\pm l} = \sum_{\substack{I\subset\{1,\ldots,N\}\\ |I|=l}} \exp(\pm\beta \sum_{j\in I} p_j) \prod_{\substack{j\in I\\ k\notin I}} v(x_j - x_k). \tag{3.23}$$

Since

$$S_{-N} = S_N^{-1} = \exp(-\beta \sum_{j=1}^{N} p_j), \tag{3.24}$$

we have

$$S_{-j} = S_{N-j} S_N^{-1}. \tag{3.25}$$

For instance, the first two quantities are given by

$$S_1 = \operatorname{Tr} L(\beta), \tag{3.26}$$

$$S_2 = (\operatorname{Tr}(L(\beta)^2) - (\operatorname{Tr} L(\beta))^2)/(2\sigma(\mu)^2(\wp(\mu) - \wp(\lambda))). \tag{3.27}$$

Note that the Lax matrix L_{NR} of the nonrelativistic case is obtained as $L_{\mathrm{REL}}(\beta) = 1_N + \beta L_{\mathrm{NR}} + O(\beta^2), \qquad \beta \to 0.$

The integrability was originally shown in [52], and is also a direct consequence of its quantum version. See Section 3.2.2 and Appendix 6.4.2. In [40], M matrix is also given and the integrability is shown by use of quadratic r-matrix formalism.

3.2. **Quantum Regime.**

3.2.1. *Nonrelativistic case.* The Hamiltonian, the momentum and the Galilei boost operators are given by

$$\hat{H} = \frac{1}{2}\sum_{j=1}^{N}\hat{p}_j^2 + g(g-\hbar)\sum_{1\le j<k\le N}\wp(x_j - x_k), \tag{3.28}$$

$$\hat{P} = \sum_{j=1}^{N}\hat{p}_j, \tag{3.29}$$

$$\hat{B} = -\sum_{j=1}^{N}x_j. \tag{3.30}$$

These satisfy a central extension of the Lie algebra of the Galilei group. The integrability is due to Olshanetsky/Perelomov [44]. Explicit forms of the commuting PDOs $\hat{H}_k$ ($k = 1,\dots,N$) are given by Oshima/Ochiai/Sekiguchi [42,45,46].

So far, the Hilbert integrability is not known. The essential selfadjointness of the Hamiltonian is easily shown by perturbation of the trigonometric case with the same domain, because the Hamiltonian of the trigonometric model is diagonalized by Jack-Sutherland polynomials and the perturbation is a bounded multiplication operator, while the selfadjointness of the higher order operators are unknown.

3.2.2. *Relativistic case.* A relativistic analogue was introduced in [52]. The Hamiltonian, the momentum and the Lorentz boost operators are given by

$$\hat{H} = \frac{1}{2\beta^2}(\hat{S}_1 + \hat{S}_{-1}), \tag{3.31}$$

$$\hat{P} = \frac{1}{2\beta}(\hat{S}_1 - \hat{S}_{-1}), \tag{3.32}$$

$$\hat{B} = -\sum_{j=1}^{N}x_j. \tag{3.33}$$

$\hat{H}$ and $\hat{P}$ are obtained by reordering the classical counterparts (3.7), (3.8), and $\hat{H}$, $\hat{P}$ and $\hat{B}$ satisfy the Lie algebra of the Poincaré group. Here for $l = 1,\dots,N$,

$$\hat{S}_{\pm l} = \sum_{\substack{I\subset\{1,\dots,N\}\\ |I|=l}}\prod_{\substack{j\in I\\ k\notin I}}v_{\mp}(x_j - x_k)\cdot\exp(\pm\beta\sum_{j\in I}\hat{p}_j)\cdot\prod_{\substack{j\in I\\ k\notin I}}v_{\pm}(x_j - x_k), \tag{3.34}$$

$$v_\delta(x) = \Big(\frac{\sigma(x+\delta\mu)}{\sigma(x)}\Big)^{1/2}, \qquad \mu = i\beta g. \tag{3.35}$$

Similarly to the classical case, we have

$$\hat{S}_{-N} = \hat{S}_N^{-1} = \exp(-\beta\sum_{j=1}^{N}\hat{p}_j), \qquad \hat{S}_{-j} = \hat{S}_{N-j}\hat{S}_N^{-1}. \tag{3.36}$$

The commutativity is encoded in the sequence of functional equations for Weierstrass' sigma function:

$$\sum_{\substack{I\subset\{1,\ldots,N\}\\|I|=l}}\prod_{\substack{m\in I\\n\notin I}}\frac{\sigma(x_m-x_n-\mu)\sigma(x_m-x_n-\gamma+\mu)}{\sigma(x_m-x_n)\sigma(x_m-x_n-\gamma)}=(x\to -x), \tag{3.37}$$

For integer couplings, it can be shown in the context of elliptic quantum groups [15]. For other approaches, a completion of Hecke algebras [9], and R-matrix formalism [29,31] are utilized. [27].

A gauge-transformation of $\hat{S}_{\pm l}$ via elliptic gamma function (6.18) [54] yields

$$A_{\pm l}=\sum_{\substack{I\subset\{1,\ldots,N\}\\|I|=l}}\prod_{\substack{j\in I\\k\notin I}}\frac{\sigma(x_j-x_k\mp\mu)}{\sigma(x_j-x_k)}\exp(\pm\beta\sum_{j\in I}\hat{p}_j), \tag{3.38}$$

which is an elliptic generalization of Macdonald operators [38]

$$M_{\pm l}=\sum_{\substack{I\subset\{1,\ldots,N\}\\|I|=l}}\prod_{\substack{j\in I\\k\notin I}}\frac{1-t^{\mp 1}X_j/X_k}{1-X_j/X_k}\prod_{j\in I}T_j^{\pm 1}, \tag{3.39}$$

where $t=e^{2\pi i\mu}$, $X_j=e^{2\pi i x_j}$ and $(T_jf)(X_1,\ldots,X_j,\ldots,X_N)=f(X_1,\ldots,qX_j,\ldots,X_N)$, and $M_{\pm l}$ acts on $\mathbb{C}[X]^{\mathfrak{S}_N}$.

The Hilbert integrability is not known while in the trigonometric case, it is shown because the Macdonald operators are explicitly diagonalized by the Macdonald polynomials [38]. The essential selfadjointness of $\hat{S}_1$ is shown by use of perturbation from the free particle system $i\hbar\beta=\mu$ [26].

4. The Defining BC_N Hamiltonians

In the other root system than A_{N-1} type, the terms "nonrelativistic" and "relativistic" are meaningless. However we abuse these terms for the purpose of comparison to A_{N-1}-type models.

4.1. **Classical Regime.**

4.1.1. *Nonrelativistic case.* The Hamiltonian of Inozemtsev model is given by

$$H=\frac{1}{2}\sum_{j=1}^{N}p_j^2+g^2\sum_{1\le j<k\le N}\big(\wp(x_j-x_k)+\wp(x_j+x_k)\big)+\frac{1}{2}\sum_{t=0}^{3}g_t^2\sum_{j=1}^{N}\wp(x_j+\omega_t), \tag{4.1}$$

where $\omega_0=0$, $\omega_3=-\omega_1-\omega_2$. Since this model is not translationally invariant, the momentum is not a conserved quantity. A Lax matrix is known by Inozemtsev [22], which is a $3N\times 3N$ matrix and in $N=1$ case, a 2×2 Lax matrix is found by Zotov [72]. Thus there exist N independent conserved quantities, but it has not been shown yet that these are in involution. However the integrability can be independently shown by use of its quantum version. See Section 4.2.1 and Appendix 6.4.1. Note that a special case is dealt in a systematic way in [4, 5, 10].

4.1.2. *Relativistic case.* The Hamiltonian of van Diejen model [29,31,68] is given by

$$H=\sum_{j=1}^{N}2\,\mathrm{ch}(\beta p_j)v_e(x_j)\prod_{k\neq j}v(x_j-x_k)v(x_j+x_k)+\sum_{t=0}^{3}q_t\prod_{j=1}^{N}v_t(x_j), \tag{4.2}$$

$$v(x)=\left(\frac{\sigma(\mu+x)\sigma(\mu-x)}{\sigma(x)\sigma(-x)}\right)^{1/2} \tag{4.3}$$

$$v_e(x)=\left(\prod_{t=0}^{3}\frac{\sigma_t(\mu_t+x)\sigma_t(\mu_t-x)}{\sigma_t(x)\sigma_t(-x)}\frac{\sigma_t(\mu_t'+x)\sigma_t(\mu_t'-x)}{\sigma_t(x)\sigma_t(-x)}\right)^{1/2} \tag{4.4}$$

$$q_t=\frac{2}{\sigma(\mu)^2}\prod_{s=0}^{3}\sigma_s(\mu_{\pi_t(s)})\sigma_s(\mu'_{\pi_t(s)}),\qquad v_t(x)=\frac{\sigma_t(\mu+x)\sigma_t(\mu-x)}{\sigma_t(x)\sigma_t(-x)}, \tag{4.5}$$

where $\pi_0=\mathrm{id}$, $\pi_1=(01)(23)$, $\pi_2=(02)(13)$, $\pi_3=(03)(12)$.

No Lax matrix is known. For the trigonometric case, see [2,8]. However the integrability of this model was shown by van Diejen in two body case with the constraint $\sum_{t=0}^{3}\mu_t+\mu_t'=0$, i.e., the product of potentials is elliptic with respect to x. This fact follows directly from the quantum version of this model as in the A_{N-1}-type case [68]. Since the constraint on the parameter was removed with arbitrary number of particles in [29,31] in the quantum case, the integrability of the classical version follows. See Section 4.2.2 and Appendix 6.4.2.

Let $\mu=i\beta g$, $\mu_i=i\beta\tilde{g}_i$, $\mu_i'=i\beta\tilde{g}_i'$. Then the Hamiltonian has the following expansion:

$$\begin{aligned}H_{\mathrm{REL}}&=2\sum_{j=1}^{N}(1+\frac{p_j^2}{2}\beta^2+O(\beta^4))\\&\times\prod_{t=0}^{3}(1+\frac{\tilde{g}_t^2}{2}\wp(x_j+\omega_t)\beta^2+O(\beta^4))(1+\frac{\tilde{g}_t'^2}{2}\wp(x_j+\omega_t)\beta^2+O(\beta^4))\\&\times\prod_{k\neq j}(1+\frac{g^2}{2}\wp(x_j-x_k)\beta^2+O(\beta^4))(1+\frac{g^2}{2}\wp(x_j+x_k)\beta^2+O(\beta^4))\\&+\sum_{t=0}^{3}\Big(\frac{2\tilde{g}_t\tilde{g}_t'}{g^2}+c_t\beta^2+O(\beta^4)\Big)\prod_{j=1}^{N}(1+g^2\wp(x_j+\omega_t)\beta^2+O(\beta^4))\end{aligned} \tag{4.6}$$

$$=2N+\sum_{t=0}^{3}\frac{2\tilde{g}_t\tilde{g}_t'}{g^2}+(2H_{\mathrm{NR}}+c)\beta^2+O(\beta^4) \tag{4.7}$$

where c_t is a constant independent of x, $c=\sum_{t=0}^{3}c_t$ and $g_t=\tilde{g}_t+\tilde{g}_t'$. Hence one sees that this Hamiltonian is a five-parameter generalization of the Hamiltonian of the Inozemtsev model.

4.2. **Quantum Regime.**

4.2.1. *Nonrelativistic case.* The Hamiltonian of the quantum Inozemtsev model is given by

$$\hat{H} = \frac{1}{2}\sum_{j=1}^{N} \hat{p}_j^2 + g(g-\hbar) \sum_{1\le j<k\le N} \big(\wp(x_j - x_k) + \wp(x_j + x_k)\big) + \frac{1}{2}\sum_{t=0}^{3} g_t(g_t - \hbar)\sum_{j=1}^{N} \wp(x_j + \omega_t), \tag{4.8}$$

The conserved operators are explicitly given by Oshima/Ochiai/Sekiguchi in [42, 45, 46].

In [41], it was clarified that in $N = 2$ case, exceptional systems exist.

4.2.2. *Relativistic case.* The Hamiltonian of van Diejen model was introduced in [68], and their commutative operators are given in [29, 31]. The Hamiltonian reads

$$\hat{H} = \sum_{\substack{1\le j\le N\\ \epsilon=\pm1}} V_{\epsilon j}^{1/2} e^{-\epsilon\beta\hat{p}_j} V_{-\epsilon j}^{1/2} + U, \tag{4.9}$$

where

$$V_{\epsilon j} = v_e(\epsilon x_j)\prod_{k\ne j} v(\epsilon x_j - x_k)v(\epsilon x_j + x_k), \tag{4.10}$$

$$U = \sum_{t=0}^{3} q_t \prod_{j=1}^{N} v_t(x_j), \tag{4.11}$$

$$v(x) = \frac{\sigma(\mu + x)}{\sigma(x)}, \tag{4.12}$$

$$v_e(x) = \prod_{t=0}^{3} \frac{\sigma_t(\mu_t + x)}{\sigma_t(x)}\frac{\sigma_t(\mu_t' + \gamma + x)}{\sigma_t(x+\gamma)}, \tag{4.13}$$

$$q_t = \frac{2}{\sigma(\mu)\sigma(\mu - 2\gamma)}\prod_{s=0}^{3}\sigma_s(\mu_{\pi_t(s)} - \gamma)\sigma_s(\mu'_{\pi_t(s)}), \qquad v_t(x) = \frac{\sigma_t(\mu - \gamma + x)}{\sigma_t(-\gamma + x)}\frac{\sigma_t(\mu - \gamma - x)}{\sigma_t(-\gamma - x)}, \tag{4.14}$$

with $\gamma = i\beta\hbar/2$. This Hamiltonian does play an important role since it reduces to finite Toda chains and various A-type models in external fields [69].

A gauge-transformation yields

$$A = \sum_{\substack{1\le j\le N\\ \epsilon=\pm1}} V_{\epsilon j} e^{-\epsilon\beta\hat{p}_j} + U. \tag{4.15}$$

If the parameters satisfy the elliptic condition $\sum \mu_t + \mu_t' = 0$, then the operator reduces to the form

$$A = \sum_{\substack{1\le j\le N\\ \epsilon=\pm1}} V_{\epsilon j}(e^{-\epsilon\beta\hat{p}_j} - 1) + c, \tag{4.16}$$

with a certain constant c. Hence the operator (4.15) is regarded as an elliptic generalization of Macdonald-Koornwinder operator [33] with extra 4 parameters,

(4.17)
$$M=\sum_{\substack{1\le j\le N\\ \epsilon=\pm1}}\frac{(1-aX_j^\epsilon)(1-bX_j^\epsilon)(1-cX_j^\epsilon)(1-dX_j^\epsilon)}{(1-X_j^{2\epsilon})(1-qX_j^{2\epsilon})}\prod_{k\neq j}\frac{1-tX_j^\epsilon/X_k}{1-X_j^\epsilon/X_k}\frac{1-tX_j^\epsilon X_k}{1-X_j^\epsilon X_k}(T_j^\epsilon-1),$$

where M acts on $\mathbb{C}[X,X^{-1}]^{\mathfrak{S}_N\ltimes\mathbb{Z}/2\mathbb{Z}}$.

Exceptional systems corresponding to the nonrelativistic systems with $N=2$, can be obtained by use of the results in [28, 30].

It can be seen from [24] that the BC_1 system with a special eigenvalue is identified with the reduction to elliptic hypergeometric equation from elliptic difference Painlevé equation and thus have $E_7^{(1)}$ symmetry. The relation to elliptic hypergeometric equation can be also seen in [63]. A direct relation is clarified in [25] or Y.K's article in this volume.

It is also shown in [57] that the Hamiltonian itself possesses D_8 symmetry if $\hat{H}$ is rewritten with an additional constant as follows: Let $a_+,a_->0$ and μ, h_n ($n=0,\dots,7$) be coupling constants. We define

$$\hat{H}=\sum_{j=1}^{N}(\mathcal{V}_j(x)^{1/2}\exp(-ia_-\partial_{x_j})\mathcal{V}_j(-x)^{1/2}+(x\to -x))+\mathcal{V}(x), \tag{4.18}$$

where using functions in Appendix,

$$\mathcal{V}_j(x)=\frac{\prod_{n=0}^{7}R_+(x_j-h_n-ia_-/2)}{R_+(2x_j+ia_+/2)R_+(2x_j-ia_-+ia_+/2)}\times \prod_{k\neq j}\frac{R_+(x_j+x_k-\mu+ia_+/2)}{R_+(x_j+x_k+ia_+/2)}\frac{R_+(x_j-x_k-\mu+ia_+/2)}{R_+(x_j-x_k+ia_+/2)}, \tag{4.19}$$

$$\mathcal{V}(x)=\frac{1}{2R_+(\mu-ia_+/2)R_+(\mu-ia_--ia_+/2)}\sum_{t=0}^{3}p_t\left(\prod_{j=1}^{N}\mathcal{E}_t(x_j)-\mathcal{E}_t(z_t)^N\right), \tag{4.20}$$

$$\mathcal{E}_t(z)=\frac{R_+(z+\mu-ia-\omega_t)R_+(z-\mu+ia-\omega_t)}{R_+(z-ia-\omega_t)R_+(z+ia-\omega_t)},\quad t=0,\dots,3, \tag{4.21}$$

with constants

$$\omega_0 = 0, \quad \omega_1 = \pi/2r, \quad \omega_2 = ia_+/2, \quad \omega_3 = -\omega_1 - \omega_2, \tag{4.22}$$

$$z_0 = z_2 = \pi/2r, \quad z_1 = z_3 = 0, \quad a = (a_+ + a_-)/2, \tag{4.23}$$

$$p_0 = \prod_{n=0}^{7} R_+(h_n), \quad p_1 = \prod_{n=0}^{7} R_+(h_n - \pi/2r), \tag{4.24}$$

$$p_2 = \exp(-2ra_+) \prod_{n=0}^{7} \exp(-irh_n) R_+(h_n - ia_+/2), \tag{4.25}$$

$$p_3 = \exp(-2ra_+) \prod_{n=0}^{7} \exp(irh_n) R_+(h_n - \omega_3). \tag{4.26}$$

Then the Hamiltonian (4.18) is invariant under arbitrary permutations and even sign flips of h_n, hence has D_8 symmetry.

5. Eigenfunctions

In this section, we set $\hbar = 1$ for simplicity, and for $r, a > 0$, we set $\omega_1 = \pi/2r$, $\omega_2 = ia/2$. We treat eigenfunctions which may not possess desired symmetry or may not be square-integrable.

5.1. **Nonrelativistic case.**

5.1.1. *A_1 case.* We solve time-independent Schrödinger equation. We set $z = x_1 - x_2$ and consider

$$-F''(z) + g(g-1)\wp(z)F(z) = E\,F(z), \tag{5.1}$$

which appeared more than a century ago, and is called Lamé equation. For this eigenvalue problem, we have following approaches:

- general method: Iterate Volterra type integral equation to get 2-dimensional solution space. This method works for arbitrary g, E. However it uses only the fact that the problem is an ordinary differential equation and we have no explicit information.
- analytic method: We have power series solutions around analytic points, and fractional power series solutions around regular singularities. This method works for arbitrary g, E and uses the analyticity of the equation. Thus we have some information. For instance, solutions near $z = 0$ behave as z^g and z^{1-g}.
- 'Bethe Ansatz' works for $g = M + 1 \in \mathbb{N}^*$ which is due to Hermite [70], yielding 'explicit' solutions $F(\pm z, y)$ for generic E with

$$F(z, y) = \prod_{j=1}^{M} \frac{s(z + z_j)}{s(z)} \exp(izy), \qquad E = -(2M-1)\sum_j \wp(z_j), \tag{5.2}$$

  where $y = i\sum_j s'(z_j)/s(z_j)$ and $z_1, \dots, z_M$ satisfy

$$M\frac{s'(z_k)}{s(z_k)} + \sum_{j \neq k} \frac{s'(z_j - z_k)}{s(z_j - z_k)} - \sum_j \frac{s'(z_j)}{s(z_j)} = 0, \qquad (k = 1, \dots, M). \tag{5.3}$$

An integral representation is also obtained in [70].

- For $g = M + 1 \in \mathbb{N}^*$ and $(2M + 1)$ E-values, elliptic solutions exist, which are called Lamé polynomials (Lamé functions) and correspond to band edges in finite-gap picture in the system

$$-F''(z) + g(g-1)\wp(z+\omega_2)F(z) = E\,F(z), \tag{5.4}$$

with $L^2(\mathbb{R}, dx)$, cf. Chapter XIII.16 in [49].

5.1.2. A_{N-1} *case.*

- Explicit $\hat{H}$-eigenfunctions for $g = 2$ and $N = 3$ are constructed (Dittrich/Inozemtsev [11]).
- For $g \in \mathbb{N}^*$ 'Bethe Ansatz' $\hat{H}$-eigenfunctions are known (Felder/Varchenko [13, 16]); properties of joint eigenfunctions are unclear.
- For $g > 0$, perturbation theory in small parameter $1/a$ can be used to exploit known joint eigenfunctions (Jack polynomials) in trigonometric regime ($a = \infty$); when convergent, this yields joint eigenfunctions (Y.K/Takemura [32]). Open problems include:
  - region of convergence
  - analytic character of eigenfunctions
  - connection to Bethe Ansatz solutions for $g \in \mathbb{N}^*$
  - connection to Langmann's algorithm (formal solutions) in [36, 37]

5.1.3. BC_1 *case.*

$$-F''(z) + \sum_{t=0}^{3} g_t(g_t-1)\wp(z+\omega_t)F(z) = E\,F(z) \tag{5.5}$$

The eigenvalue problem of the Inozemtsev model is transformed into the Heun equation, which is a general form of second-order Fuchsian differential equations with four regular singular points.

- General and analytic methods apply again but with little information. In [39], the complete list of the 192 solutions of the Heun equation is given, which is analogous to Kummer's 24 solutions of the Gauss hypergeometric equation.
- For integer couplings we have extensive information (Treibich/Verdier [67], Gesztesy/Weikard [18], Takemura [66]). and 'Bethe Ansatz' (Takemura [65]).

5.1.4. BC_N *case.* With linear constraint on couplings, finite-dimensional $\hat{H}_j$-invariant subspaces are known at once for all $1 \le j \le N$ (Takemura [64], Madrid school [19], for the trigonometric case, Sasaki/Takasaki [59]).

5.2. **Relativistic case.**

5.2.1. A_1 *case.*

$$\frac{\sigma(z - i\beta g)}{\sigma(z)}F(z - i\beta) + \frac{\sigma(z + i\beta g)}{\sigma(z)}F(z + i\beta) = E\,F(z) \tag{5.6}$$

- Meromorphic solutions exist for arbitrary β, g, E. But very little is known about them.

- For $g \in \mathbb{N}^*$ Hermite's eigenfunctions can be generalized (Ruijsenaars [55,56], Krichever/Zabrodin [35,71], Felder/Varchenko [14]).
- More generally, Bethe Ansatz type eigenfunctions exist for a dense set in parameter space $\{a, \beta > 0, g \in \mathbb{R}\}$ (Ruijsenaars [55,56]).
- For $g \in \mathbb{N}^*/2$, generalizations of the Lamé polynomials were found by Sklyanin, yielding $(2g+1)$-dimensional representation space for Sklyanin algebra [51,60,61].

5.2.2. *A_{N-1} case.*

- The $g \in \mathbb{N}^*$ Bethe Ansatz eigenfunctions of Felder/Varchenko can be generalized (Billey [3]); properties for the higher order operators are not known.
- The $g \in \mathbb{N}^*$ Sklyanin A_1-solutions generalize to finite dimensional spaces of θ-functions invariant under $A_{\pm l}$ $(l = 1, \ldots, N)$ (Hasegawa [20])

5.2.3. *BC_1 case.*

$$v_e(z)F(z - i\beta) + v_e(-z)F(z + i\beta) + U(z)f(z) = E\,F(z), \tag{5.7}$$

where U and v_e are defined in (4.11) and (4.13). Note that U is independent of μ though U includes μ apparently.

- Meromorphic solutions exist for arbitrary β, $\mu_0, \ldots, \mu_3, \mu'_0, \ldots, \mu'_3, E$.
- When $\sum \mu_t + \mu'_t = 0$, one gets elliptic coefficients $v_e(\pm z)$ so that $U(z) = -v_e(z) - v_e(-z) + c$. Hence $F(z) = 1$ is an eigenfunction for A (4.16) with $E = c$, and a gauge function is an eigenfunction for $\hat{H}$ ('ground state'). Generally if $\sum \mu_t + \mu'_t = i\beta g$ for $g \in \mathbb{N}$, there exist a finite dimensional invariant subspace (Hikami/Y.K [21]).
- Elliptic hypergeometric series and integral solutions exist. Elliptic hypergeometric integral can be reparametrized so that they solve (5.7) for one E-value, provided μ, μ' satisfies ellipticity constraint $\sum \mu_t + \mu'_t = 0$ (balancing condition) and one further constraint. Hence one gets a 6-dimensional subfamily of μ_j, μ'_j. If we set one of μ_j or μ'_j to be an integer M, then the elliptic hypergeometric integral reduces to terminating elliptic hypergeometric series ${}_{12}V_{11}$ (${}_{10}E_9$ in some references). Hence for M fixed one gets a 5-dimensional subfamily of μ_j, μ'_j (for elliptic hypergeometric series, Kajiwara/Masuda/Noumi/Ohta/Yamada [24] and Spiridonov [63], for elliptic hypergeometric integrals, Y.K/Noumi [25]).

5.2.4. *BC_N case.*

- With linear constraint on μ_j, μ'_j, μ, there exist finite-dimensional spaces of θ-functions invariant under the commuting BC_N AΔOs (Hikami/Y.K [21]). This result is generalized to arbitrary root systems, where the reduced root system of type $A_{2N}^{(2)}$ corresponds to the root system of type BC_N and invariant spaces consist of theta functions of positive level associated to the corresponding affine Lie algebra [27].

6. Appendix

6.1. **Weierstrass' sigma function.** Let $\omega_1, \omega_2 \in \mathbb{C}$ be basic half periods such that $\omega_1/\omega_2 \notin \mathbb{R}$ and set $\omega_0 = 0$ and $\omega_3 = -\omega_1 - \omega_2$.

Weierstrass' sigma function is defined by the following:

$$\sigma(z;\omega_1,\omega_2)=\sigma(z)=z\prod_{\substack{m,n=-\infty\\(m,n)\neq(0,0)}}^{\infty}\Big[\Big(1-\frac{z}{\Omega_{mn}}\Big)\exp\Big(\frac{z}{\Omega_{mn}}+\frac{z^2}{2\Omega_{mn}^2}\Big)\Big], \tag{6.1}$$

where $\Omega_{mn}=2m\omega_1+2n\omega_2$.

$\sigma(z)$ is an entire function and satisfies the following periodicity:

$$\sigma(z+2\omega_1)=-e^{2\eta_1(z+\omega_1)}\sigma(z), \tag{6.2}$$

$$\sigma(z+2\omega_2)=-e^{2\eta_2(z+\omega_2)}\sigma(z). \tag{6.3}$$

with simple zeros at $2\omega_1\mathbb{Z}+2\omega_2\mathbb{Z}$, where $\eta_j=\zeta(\omega_j)$ and $\zeta(z)$ is Weierstrass' zeta function $\zeta(z)=\sigma'(z)/\sigma(z)$. The Taylor expansion of $\sigma(z)$ at the origin is

$$\sigma(z)=z+O(z^5) \tag{6.4}$$

Conversely, these properties determine the sigma function uniquely.

The cosigma functions are defined by

$$\sigma_j(z)=e^{-\eta_j z}\frac{\sigma(z+\omega_j)}{\sigma(\omega_j)},\qquad (j=1,2,3). \tag{6.5}$$

The cosigma functions are even functions and $\sigma_j(0)=1$ while the sigma function is an odd function. In addition, we set $\sigma_0(z)=\sigma(z)$.

By use of the following Cauchy type determinant formula

$$\det\left(\frac{\sigma(x_i-y_j+\lambda)}{\sigma(x_i-y_j)\sigma(\lambda)}\right)_{i,j=1}^{N}=\frac{\sigma(\lambda+\sum_{i=1}^N x_i-y_i)}{\sigma(\lambda)}\frac{\prod_{1\le i<j\le N}\sigma(x_i-x_j)\sigma(y_i-y_j)}{\prod_{1\le i,j\le N}\sigma(x_i-y_j)}, \tag{6.6}$$

we have

(6.7)

$$\det\left(\frac{\sigma(x_i-x_j+\lambda)\sigma(\mu)}{\sigma(x_i-x_j+\mu)\sigma(\lambda)}\right)_{i,j=1}^{N}=\frac{\sigma(\lambda-\mu)^{N-1}\sigma(\lambda+(N-1)\mu)}{\sigma(\lambda)^N}\frac{\prod_{1\le i<j\le N}\sigma(x_i-x_j)^2}{\prod_{1\le i<j\le N}\sigma(x_i-x_j+\mu)}.$$

The formula (6.6) was first obtained by Frobenius [17]. Later on, it showed up in several distinct contexts, giving rise to different proofs, cf. [12, 23, 47, 52].

6.2. **Weierstrass' $\wp$ function.** Weierstrass' $\wp$ function is an even elliptic function defined by

$$\wp(z;\omega_1,\omega_2)=\wp(z)=\frac{1}{z^2}+\sum_{\substack{m,n=-\infty\\(m,n)\neq(0,0)}}^{\infty}\Big[\frac{1}{(z-\Omega_{mn})^2}-\frac{1}{\Omega_{mn}^2}\Big], \tag{6.8}$$

whose Laurent expansion at the origin is given by,

$$\wp(z)=\frac{1}{z^2}+O(z^2). \tag{6.9}$$

We have the following relations between the functions $\wp$ and σ.

$$\wp(z_1) - \wp(z_2) = -\frac{\sigma(z_1 - z_2)\sigma(z_1 + z_2)}{\sigma(z_1)^2\sigma(z_2)^2}, \tag{6.10}$$

$$-\frac{d}{dz}\frac{\sigma_j'(z)}{\sigma_j(z)} = \frac{\sigma_j(z)\sigma_j''(z) - \sigma_j'(z)^2}{\sigma_j(z)^2} = \wp(z + \omega_j), \qquad (j = 0, 1, 2, 3). \tag{6.11}$$

It follows that

$$\frac{\sigma_t(i\beta g + x)\sigma_t(i\beta g - x)}{\sigma_t(x)\sigma_t(-x)} = 1 + g^2\wp(x + \omega_t)\beta^2 + O(\beta^4). \tag{6.12}$$

6.3. **Elliptic gamma and allied functions.** For the detail of this subsection, see [54]. Starting from Weierstrass sigma function $\sigma(z)$, introduce

$$s(r, a; z) = \exp(-\eta_1 z^2/2\omega_1)\sigma(z; \omega_1, \omega_2), \tag{6.13}$$

$$R(r, a; z) = c(r, a)e^{irz}s(r, a; z + ia/2), \qquad c(r, a) = -2ire^{-ar/2}\prod_{k=1}^{\infty}(1 - e^{-2rak})^2 \tag{6.14}$$

where $\omega_1 = \pi/2r$, $\omega_2 = ia/2$, for $r, a > 0$. The function $s(r, a; z)$ is odd function and π/r-antiperiodic, and satisfies

$$\lim_{a\to\infty} = \frac{\sin rz}{r}, \tag{6.15}$$

$$\lim_{r\to\infty} = \frac{\sinh \pi z/a}{\pi z/a}, \tag{6.16}$$

uniformly on compacts.

Now consider first order analytic difference equation:

$$\frac{G(z + ib/2)}{G(z - ib/2)} = R(r, a; z), \qquad r, a, b > 0. \tag{6.17}$$

The 'simplest' (=minimal) solution is elliptic gamma function:

$$G(r, a, b; z) = \prod_{m,n=0}^{\infty} \frac{1 - \exp(-(2m + 1)ra - (2n + 1)rb - 2irz)}{1 - \exp(-(2m + 1)ra - (2n + 1)rb + 2irz)}. \tag{6.18}$$

Noting $a \leftrightarrow b$ invariance, we work with $G(r, a_+, a_-; z)$ obeying

$$\frac{G(z + ia_{-\delta}/2)}{G(z - ia_{-\delta}/2)} = R_\delta(z), \qquad \delta = +, - \tag{6.19}$$

$$R_\delta(z) = R(r, a_\delta; z). \tag{6.20}$$

Likewise we define

$$s_\delta(z) = s(r, a_\delta; z). \tag{6.21}$$

Useful G-features are as follows:

$$G(r, a_+, a_-; z) = G(r, a_-, a_+; z), \tag{6.22}$$

$$G(\lambda^{-1} r, \lambda a_+, \lambda a_-; \lambda z) = G(r, a_-, a_+; z), \tag{6.23}$$

$$G(z + \pi/r) = G(z), \tag{6.24}$$

$$G(-z) = G(z), \tag{6.25}$$

$$G(z) = \exp\Big(i \sum_{n=1}^{\infty} \frac{\sin(2nrz)}{2n \operatorname{sh}(nra_+) \operatorname{sh}(nra_-)}\Big), \qquad |\Im z| < (a_+ + a_-)/2, \tag{6.26}$$

$$\lim_{a_- \downarrow 0} \frac{G(r, a_+, a_-; z - ia_-\kappa)}{G(r, a_+, a_-; z - ia_-\lambda)} = \exp[(\lambda - \kappa) \ln R(r, a_+; z)]. \tag{6.27}$$

Multiplication formulae:

$$G(2r, a_+, a_-; z) = G(r, a_+, a_-; z) G(r, a_+, a_-; z - \pi/2r) \tag{6.28}$$

$$G\Big(r, \frac{a_+}{M}, \frac{a_-}{N}; z\Big) = \prod_{j=1}^{M} \prod_{k=1}^{N} G\Big(r, a_+, a_-; z + ia_+ \frac{M+1-2j}{2M} + ia_- \frac{N+1-2k}{2N}\Big) \tag{6.29}$$

In particular, we get duplication formula

$$\begin{aligned} G(r, a_+, a_-; 2z) = \prod_{l,m=+,-} & G\Big(r, a_+, a_-; z + i\frac{la_+ + ma_-}{4}\Big) \\ & \cdot G\Big(r, a_+, a_-; z + i\frac{la_+ + ma_-}{4} - \pi/2r\Big) \end{aligned} \tag{6.30}$$

6.4. **Classical Limit.** The integrability of the classical models follows directly from that of the quantum counterparts. In this subsection, we give the proof of this assertion both in the nonrelativistic and the relativistic cases.

6.4.1. *Nonrelativistic case.*

Lemma 6.1. *Let $n_j \in \mathbb{N}^N$ $(j = 1, 2)$ be multi-indices and $\delta_k = (0, \ldots, \overset{k}{1}, \ldots, 0)$. Assume that $V_j(x, \hbar)$ $(j = 1, 2)$ are holomorphic in $\hbar$ around $\hbar = 0$, and holomorphic in x on U, where U is an open dense subset of $\mathbb{R}^N$. Let*

$$\hat{O}_j = V_j(x, \hbar) \hat{p}^{n_j}, \qquad (j = 1, 2), \tag{6.31}$$

$$O_j = V_j(x, 0) p^{n_j}, \qquad (j = 1, 2). \tag{6.32}$$

Then the brackets are of the forms

$$[\hat{O}_1, \hat{O}_2] = \sum_{k \le \max\{n_1, n_2\}} V_{[1,2],k}(x, \hbar) \hat{p}^{n_1 + n_2 - k}, \tag{6.33}$$

$$\{O_1, O_2\} = \sum_{j=1}^{N} V_{\{1,2\},j}(x) p^{n_1 + n_2 - \delta_j}, \tag{6.34}$$

and

$$\lim_{\hbar\to 0}\frac{V_{[1,2],k}(x,\hbar)}{i\hbar}=\begin{cases}V_{\{1,2\},j}(x), & \text{if } k=\delta_j.\\ 0, & \text{otherwise.}\end{cases}\tag{6.35}$$

In particular,

$$[\hat{O}_1,\hat{O}_2]=0 \quad\Longrightarrow\quad \{O_1,O_2\}=0.\tag{6.36}$$

Proof. The commutator is calculated as

$$\begin{aligned}[\hat{O}_1,\hat{O}_2]=\sum_{k\le n_1}V_1(x,\hbar)(-i\hbar)^{|k|}\binom{n_1}{k}V_2^{(k)}(x,\hbar)\hat{p}^{n_1+n_2-k}\\ -\sum_{k\le n_2}V_2(x,\hbar)(-i\hbar)^{|k|}\binom{n_2}{k}V_1^{(k)}(x,\hbar)\hat{p}^{n_1+n_2-k}.\end{aligned}\tag{6.37}$$

and the Poisson bracket as

$$\{O_1,O_2\}=\sum_{j=1}^{N}\big(n_2^{(j)}V_1^{(\delta_j)}(x,0)V_2(x,0)-n_1^{(j)}V_2^{(\delta_j)}(x,0)V_1(x,0)\big)p^{n_1+n_2-\delta_j},\tag{6.38}$$

where $n_k^{(j)}$ denotes the j-th index of n_k. One sees that the coefficient of $\hat{p}^{n_1+n_2}$ vanishes and

$$V_{[1,2],k}(x,\hbar)=\begin{cases}V_{\{1,2\},j}(x)i\hbar+O(\hbar^2), & \text{if } k=\delta_j.\\ O(\hbar^2), & \text{otherwise.}\end{cases}\tag{6.39}$$

□

6.4.2. *Relativistic case.*

Lemma 6.2 (van Diejen [68])**.** *Let $\kappa_j\in\mathbb{C}^N$ $(j=1,2)$. Assume that $V_j(x,\hbar)$ $(j=1,2)$ are holomorphic in $\hbar$ around $\hbar=0$, and holomorphic in x on $U+i\kappa_{3-j}\mathbb{R}$, where U is an open dense subset of $\mathbb{R}^N$. Let*

$$\hat{O}_j=V_j(x,\hbar)e^{-\kappa_j\cdot\hat{p}},\qquad (j=1,2),\tag{6.40}$$

$$O_j=V_j(x,0)e^{-\kappa_j\cdot p},\qquad (j=1,2).\tag{6.41}$$

Then the brackets are of the forms

$$[\hat{O}_1,\hat{O}_2]=V_{[1,2]}(x,\hbar)e^{-(\kappa_1+\kappa_2)\cdot\hat{p}},\tag{6.42}$$

$$\{O_1,O_2\}=V_{\{1,2\}}(x)e^{-(\kappa_1+\kappa_2)\cdot p},\tag{6.43}$$

and

$$\lim_{\hbar\to 0}\frac{V_{[1,2]}(x,\hbar)}{i\hbar}=V_{\{1,2\}}(x).\tag{6.44}$$

In particular,

$$[\hat{O}_1,\hat{O}_2]=0 \quad\Longrightarrow\quad \{O_1,O_2\}=0.\tag{6.45}$$

Proof. The functions $V_{[1,2]}(x,\hbar)$ and $V_{\{1,2\}}(x)$ are explicitly calculated as

$$V_{[1,2]}(x,\hbar) = V_1(x,\hbar)V_2(x+i\hbar\kappa_1,\hbar) - V_2(x,\hbar)V_1(x+i\hbar\kappa_2,\hbar), \tag{6.46}$$

$$V_{\{1,2\}}(x) = V_1(x,0)(\kappa_1\cdot\nabla V_2)(x,0) - V_2(x,0)(\kappa_2\cdot\nabla V_1)(x,0). \tag{6.47}$$

The Taylor expansion of (6.46) around $\hbar = 0$ yields

$$V_{[1,2]}(x,\hbar) = V_{\{1,2\}}(x)i\hbar + O(\hbar^2). \tag{6.48}$$

□

References

1. R. Abraham and J. E. Marsden, *Foundations of mechanics*, 2nd ed., Benjamin-Cummings, 1978.
2. J. Avan and G. Rollet, *Structures in BC_N Ruijsenaars-Schneider models*, J. Math. Phys. **43** (2001), no. 1.
3. E. Billey, *Algebraic nested Bethe ansatz for the elliptic Ruijsenaars model*, (1998), math.QA/9806068.
4. A. J. Bordner, E. Corrigan, and R. Sasaki, *Calogero-Moser models: A new formulation*, Progr. Theoret. Phys. **100** (1998), no. 6, 1107–1129, hep-th/9805106.
5. A. J. Bordner, R. Sasaki, and K. Takasaki, *Calogero-Moser models II: Symmetries and foldings*, Progr. Theoret. Phys. **101** (1998), no. 3, 487–518, hep-th/9809068.
6. H.W. Braden and T. Suzuki, *R-matrices for elliptic Calogero-Moser models*, Lett. Math. Phys. **30** (1994), 147–158.
7. F. Calogero, *Exactly solvable one-dimensional many-body systems*, Lett. Nuovo Cimento **13** (1975), 411–415.
8. K. Chen, B.-Y. Hou, and W.-L. Yang, *Integrability of the C_n and BC_n Ruijsenaars-Schneider models*, J. Math. Phys. **41** (2000), 8132–8147.
9. I. Cherednik, *Difference-elliptic operators and root systems*, Internat. Math. Res. Notices **1** (1995), 43–59.
10. E. D'Hoker and D. H. Phong, *Calogero-Moser Lax pairs with spectral parameter for general Lie algebras*, Nucl. Phys. **B530** (1998), 537–610.
11. J. Dittrich and V. I. Inozemtsev, *On the structure of eigenvectors of the multidimensional Lamé operator*, J. Phys. A **26** (1993), L753–L756.
12. J. D. Fay, *Theta functions on Riemann surfaces*, Springer-Verlag, 1973, Lecture Notes in Math. 352.
13. G. Felder and A. Varchenko, *Integral representation of solutions of the elliptic Knizhnik-Zamolodchikov-Bernard equations*, Internat. Math. Res. Notices (1995), no. 5, 221–233.
14. ______, *Algebraic Bethe ansatz for the elliptic quantum group $E_{\tau,\eta}(sl_2)$*, Nucl. Phys. **B480** (1996), 485–503.
15. ______, *Elliptic quantum groups and Ruijsenaars models*, J. Stat. Phys. **89** (1997), 963–980.
16. ______, *Three formulae for eigenfunctions of integrable Schrödinger operator*, Compositio Math. **107** (1997), no. 2, 143–175.
17. G. Frobenius, *über die elliptischen functionen zweiter art*, J. Reine und Angew. Math. **93** (1882), 53–68.
18. F. Gesztesy and R. Weikard, *Treibich-Verdier potentials and the stationary (m)KdV hierarchy*, Math. Z. **219** (1995), 451–476.
19. D. Gomez-Ullate, A. Gonzalez-Lopez, and M. A. Rodriguez, *Exact solutions of a new elliptic Calogero-Sutherland model*, Phys. Lett. B **511** (2001), 112–118.
20. K. Hasegawa, *Ruijsenaars' commuting difference operators as commuting transfer matrices*, Commun. Math. Phys. **187** (1997), 289–325.
21. K. Hikami and Y. Komori, *Integrability, fusion, and duality in the elliptic Ruijsenaars model*, Mod. Phys. Lett. A **12** (1997), 751–761.
22. V. I. Inozemtsev, *Lax representation with spectral parameter on a torus for integrable particle systems*, Lett. Math. Phys. **17** (1989), 11–17.

23. Y. Kajihara and M. Noumi, *Multiple elliptic hypergeometric series. an approach from the Cauchy determinant*, Indag. Math. (N.S.) **14** (2003), no. 3–4, 395–421.
24. K.Kajiwara, T.Masuda, M.Noumi, Y.Ohta, and Y.Yamada, ${}_{10}E_9$ *solution to the elliptic Painlevé equation*, J. Phys. A. **36** (2003), L263–L272, nlin.SI/0303032.
25. Y. Komori, *Elliptic Ruijsenaars operators and elliptic hypergeometric integrals*, RIMS kokyuroku, to appear.
26. ———, *Essential self-adjointness of the elliptic Ruijsenaars models*, J. Math. Phys. **42** (2001), 4523–4553.
27. ———, *Ruijsenaars' commuting difference operators and invariant subspace spanned by theta functions*, J. Math. Phys. **42** (2001), 4503–4522.
28. ———, *Elliptic Ruijsenaars operators and functional equations*, J. Math. Phys. **43** (2002), 5637–5653.
29. Y. Komori and K. Hikami, *Quantum integrability of the generalized elliptic Ruijsenaars models*, J. Phys. A **30** (1997), 4341–4364.
30. ———, *Affine R-matrix and the generalized elliptic Ruijsenaars models*, Lett. Math. Phys. **43** (1998), 335–346.
31. ———, *Conserved operators of the generalized elliptic Ruijsenaars models*, J. Math. Phys. **39** (1998), 6175–6190.
32. Y. Komori and K. Takemura, *The perturbation of the quantum Calogero-Moser-Sutherland system and related results*, Commun. Math. Phys. **227** (2002), 93–118.
33. T. H. Koornwinder, *Askey-Wilson polynomials for root systems of type BC*, Contemp. Math. **138** (1992), 189–204.
34. I. M. Krichever, *Elliptic solutions of the Kadomtsev-Petviashvili equation and integrable systems of particles*, Funct. Anal. Appl. **14** (1980), 282–290.
35. I. M. Krichever and A. Zabrodin, *Spin generalization of the Ruijsenaars-Schneider model, the nonabelian two-dimensionalized Toda lattice, and representations of the Sklyanin algebra*, Russian Math. Surveys **50** (1995), no. 6, 1101–1150.
36. E. Langmann, *An algorithm to solve the elliptic Calogero-Sutherland model*, math-ph/0401029.
37. ———, *Anyons and the elliptic Calogero-Sutherland model*, Lett. Math. Phys. **54** (2000), 279–289.
38. I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Univ. Press, Oxford, 1995.
39. R. S. Maier, *The 192 solutions of the Heun equation*, (2004), math.CA/0408317.
40. F. W. Nijhoff, V. B. Kuznetsov, E. K. Sklyanin, and O. Ragnisco, *Dynamical r-matrix for the elliptic Ruijsenaars-Schneider system*, J. Phys. A **29** (1996), L333–L340.
41. H. Ochiai and T. Oshima, *Commuting differential operators of type* B_2, Funkcialaj Ekvacioj **46** (2003), 297–336, preprint, UTMS 94–65, Dept. of Mathematical Science, University of Tokyo, 1994.
42. H. Ochiai, T. Oshima, and H. Sekiguchi, *Commuting families of symmetric differential operators*, Proc. Japan. Acad. **70** (1994), 62–66.
43. M. A. Olshanetsky and A. M. Perelomov, *Classical integrable finite-dimensional systems related to Lie algebras*, Phys. Rep. **71** (1981), 313–400.
44. ———, *Quantum integrable systems related to Lie algebras*, Phys. Rep. **94** (1983), 313–404.
45. T. Oshima, *Completely integrable systems with a symmetry in coordinates*, Asian J. Math. **2** (1998), 935–955, preprint, UTMS 94–6, Dept. of Mathematical Science, University of Tokyo, 1994.
46. T. Oshima and H. Sekiguchi, *Commuting families of differential operators invariant under the action of a Weyl group*, J. Math. Sci. Univ. Tokyo **2** (1995), 1–75, preprint, UTMS 93–43, Dept. of Mathematical Science, University of Tokyo, 1993.
47. A. K. Raina, *An algebraic geometry study of the b-c system with arbitrary twist fields and arbitrary statistics*, Commun. Math. Phys. **140(2)** (1991), 373–397.
48. M. Reed and B. Simon, *Fourier analysis and self-adjointness*, Methods of modern mathematical physics, vol. II, Academic Press, New York, 1975.
49. ———, *Analysis of operators*, Methods of modern mathematical physics, vol. IV, Academic Press, New York, 1978.

50. ———, *Functional analysis*, revised and enlarged ed., Methods of modern mathematical physics, vol. I, Academic Press, New York, 1980.
51. H. Rosengren, *Sklyanin invariant integration*, (2004), math.QA/0405072.
52. S. N. M. Ruijsenaars, *Complete integrability of relativistic Calogero-Moser systems and elliptic function identities*, Commun. Math. Phys. **110** (1987), 191–213.
53. ———, *Systems of Calogero-Moser type*, Particles and Fields, Springer-Verlag, 1994, pp. 251–352.
54. ———, *First-order analytic difference equations and integrable quantum systems.*, J. Math. Phys. **38** (1997), 1069–1146.
55. ———, *Generalized Lamé functions I. the elliptic case*, J. Math. Phys. **40** (1999), 1595–1626.
56. ———, *Relativistic Lamé functions: the special case $g = 2$*, J. Phys. A **32** (1999), 1737–1772.
57. ———, *Integrable BC_N analytic difference operators: hidden parameter symmetries and eigenfunctions*, New Trends in Integrability and Partial Solvability, Kluwer Academic, 2004, pp. 217–262.
58. S. N. M. Ruijsenaars and H. Schneider, *A new class of integrable systems and its relation to solitons*, Ann. Phys. **170** (1986), 370–405.
59. R. Sasaki and K. Takasaki, *Quantum Inozemtsev model, quasi-exact solvability and N-fold supersymmetry*, J. Phys. A **34** (2001), 9533–9554.
60. E. K. Sklyanin, *Some algebraic structures connected with the Yang-Baxter equation*, Funk. Anal. i Ego Pril. **16** (1982), 27–34.
61. ———, *Some algebraic structures connected with the Yang-Baxter equation. representation of quantum algebras*, Funk. Anal. i Ego Pril. **17** (1983), 34–48.
62. ———, *Dynamical r-matrices for elliptic Calogero-Moser model*, St. Petersburg J. Math. **6** (1994), 33–37.
63. V. P. Spiridonov, *Theta hypergeometric integrals*, Algebra i Analiz **15** (2003), no. 6, 161–215.
64. K. Takemura, *Quasi-exact solvability of Inozemtsev models*, J. Phys. A **35** (2002), 8867–8881, math.CA/0205274.
65. ———, *The Heun equation and the Calogero-Moser-Sutherland system I: the Bethe Ansatz method*, Commun. Math. Phys. **235** (2003), 467–494, math.CA/0103077.
66. ———, *The Heun equation and the Calogero-Moser-Sutherland system II: the perturbation and the algebraic solution*, Electron. J. Differential Equations (2004), no. 15, 1–30, math.CA/0112179.
67. A. Treibich and J.-L. Verdier, *Solitons elliptiques*, Progr. Math. **88** (1990), 437–480.
68. J. F. van Diejen, *Integrability of difference Calogero-Moser systems*, J. Math. Phys. **35** (1994), 2983–2998.
69. ———, *Difference Calogero-Moser systems and finite Toda chains*, J. Math. Phys. **36** (1995), 1299–1323.
70. E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 4th ed., Cambridge Univ. Press, Cambridge, 1927.
71. A. Zabrodin, *On the spectral curve of the difference Lamé operator*, Internat. Math. Res. Notices (1999), no. 11, 589–614.
72. A. Zotov, *Elliptic linear problem for the Calogero-Inozemtsev model and Painlevé VI equation*, Lett. Math. Phys. **67** (2004), no. 2, 153–165.

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