# ELLIPTIC INTEGRABLE SYSTEMS OF CALOGERO-MOSER TYPE: SOME NEW RESULTS ON JOINT EIGENFUNCTIONS

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ABSTRACT. We present results on special eigenfunctions for differences of elliptic Calogero-Moser type Hamiltonians. We show that these results have a bearing on the existence of joint Hilbert space eigenfunctions for the commuting Hamiltonians.

CONTENTS



## 1. PREAMBLE

The preceding lecture notes [1] contain a survey of the literature on elliptic $N$-particle systems of Calogero-Moser type. Here we report some hitherto unpublished results that open up a new avenue towards the goal of promoting the commuting quantum Hamiltonians to commuting self-adjoint operators on a suitable Hilbert space. Thus, using terminology explained in the survey (whose notation we also use without further explanation), the results under consideration have a bearing on the question whether the elliptic systems are not only integrable, but also Hilbert integrable. Ever since the various systems were introduced, this fundamental problem has remained unsolved, save for some quite special (rank-1) cases.

The material to be presented has two distinct faces, so to speak. One face can be observed in complete clarity, with every wrinkle in full sight. It consists of findings of a primarily calculational and algebraic character. They date back a couple of years, but we present them here in published form for the first time.

The second face has been invisible until half a year ago, and is still mostly dark. It involves a Hilbert space reinterpretation of the eigenfunction properties expounded in Section 2, with consequences that are to date mostly conjectural. As the change of viewpoint involved here was neither obvious to us nor to several audiences to which we have meanwhile spoken about it, we have opted for a somewhat unorthodox introduction, in the hope that this helps to appreciate better what is at issue. In particular, we describe in some detail how the conjectured scenario emerged.

To begin with, we should be more specific concerning the well-lit face. To this end we denote the defining Hamiltonian of one of the four classes of $N$-particle systems (differential/difference, $A_{N-1}/BC_N$) by $H(p;x)$, where $p$ denotes the parameters and $x$ belongs to $\mathbb{C}^N$. The principal result for the two $BC_N$ cases is then that there exists a quite simple function $\Psi(p;x,y)$ that satisfies

$$(H(p;x) - H(p';y))\Psi(p;x,y) = \sigma(p)\Psi(p;x,y), \qquad (BC_N). \tag{1.1}$$

Here, the parameters $p'$ are linear functions of the parameters $p$; moreover, for $N > 1$ there is one linear constraint on $p$.

For the two $A_{N-1}$ cases there exists a function satisfying

$$(H(p;x) - H(p;-y))\Psi(p;x,y) = 0, \qquad (A_{N-1}). \tag{1.2}$$

In contrast to the $BC_N$ cases, where we know only one nontrivial solution to (1.1), the special solution $\Psi(p;x,y)$ to (1.2) can be viewed as the simplest one in an infinite-dimensional solution space.

In case the Hamiltonian is an analytic difference operator (A$\Delta$O), the square of the function $\Psi$ equals a product of elliptic gamma functions. In the differential ('nonrelativistic') limit, $\Psi$ can be expressed in terms of the functions $s$ and $R$, cf. the Appendix in [1]. Since the elliptic gamma function is 'modular invariant' (i.e., invariant under the interchange of $a_+$ and $a_-$), $\Psi(p;x,y)$ is modular invariant in the relativistic cases. Therefore, (1.1) and (1.2) also hold for the 'modular duals' of $H$, namely the Hamiltonians $H'$ obtained by taking $a_+ \leftrightarrow a_-$ in $H$. Since one has $[H,H'] = 0$ in the sense of A$\Delta$Os, the function $\Psi$ can be viewed as a joint eigenfunction of two commuting differences of A$\Delta$Os.

In the two $A_{N-1}$ cases more can be shown: on the relativistic level (1.2) is also obeyed when $H$ is replaced by the higher commuting A$\Delta$Os $H_2, \ldots, H_N$ and their modular duals, and in the nonrelativistic limit this joint eigenfunction property persists. We surmise that in the $BC_N$ cases $\Psi(p;x,y)$ is also an eigenfunction of the differences of the higher Hamiltonians; since their structure is quite involved, it is not an easy matter to verify or refute this. (Cf. the papers by Inozemtsev [2], van Diejen [3] and Komori/Hikami [4], where the $BC_N$ systems were introduced.)

The results just sketched are detailed in Section 2. Admittedly, at face value (well-lit or not), they look somewhat bizarre. Moreover, at this point the reader has every reason to ask: Why does the existence of a *very special* eigenfunction of a *difference* of elliptic Calogero-Moser type Hamiltonians (as embodied in (1.1)–(1.2)) have any bearing on the problem of finding appropriate Hilbert space eigenfunctions for the defining Hamiltonian $H(p;x)$ and its commuting relatives?

For a long time we were not aware of any other answer to this question than a quite complicated (and already non-obvious) answer that Langmann has proposed some five years ago and explored ever since. To be more specific, we should first mention that Langmann already found the pertinent eigenfunction $\Psi(p;x,y)$ for the nonrelativistic (differential) $A_{N-1}$ case, and reported his finding at a conference in Rome (May 2001) we also attended. He found his result as a spin-off of previous work on anyonic quantum field theory, and has been using it as a starting point for a perturbative algorithm to construct eigenfunctions of the defining $A_{N-1}$ PDO in terms of Jack-Sutherland polynomials. (His various papers on this subject can be traced from the recent preprint [5].)

Just as Langmann, we obtained the results described in Section 2 as a spin-off of work in another direction. Specifically, they arose some two years ago from a failed attempt to find eigenfunctions for the elliptic $BC_1$ AΔO. At the time we thought that these results might at best be used to try and extend Langmann's program to the joint $A_{N-1}$, $BC_N$, and relativistic settings. However, this looked like a formidable undertaking, so we turned to unrelated and more urgent matters. We returned to our elliptic findings while preparing our RIMS Workshop lectures during a visit (October 2004) at the Max-Planck-Institute in Munich, with an eye on reporting them in Kyoto (as indeed we did). As a consequence of discussions with E. Seiler, we came to realize how the peculiar findings of the form (1.1)–(1.2) can be reinterpreted so as to throw considerable light on the issue of Hilbert integrability (if not resolve it, with due effort). The pertinent discussions concerned Appendix A in his joint paper with Niedermaier [6], which summarizes some results on harmonic analysis in $L^2(H)$, with $H$ the Poincaré upper half plane viewed in terms of $SL(2,\mathbb{R})$ theory.

The connection of this setting to the nonrelativistic rank-1 (reduced 2-particle) hyperbolic Calogero-Moser and Toda systems has been known ever since the seminal work by Olshanetsky and Perelomov [7]. In [6] a crucial role is played by a transfer matrix. For the case under consideration in Appendix A of [6], it amounts to an integral operator commuting with the $SL(2,\mathbb{R})$ generators. In particular, it therefore commutes with the Casimir operator, which in suitable coordinates reduces to Calogero-Moser or Toda Hamiltonians.

This state of affairs led to our reinterpretation of the function $\Psi$ that we now attempt to summarize in a few words: under suitable restrictions on the parameters, $\Psi(p;x,y)$ can be viewed as the kernel of a Hilbert-Schmidt integral operator on a Hilbert space $\mathcal{H} = L^2(F;dx)$ (with $F$ defined by (3.2)), and we conjecture that the functions of $x$ in its canonical singular value decomposition are the long sought-after joint Hilbert space eigenfunctions of $H(p;x)$ and its relatives. Now the Hilbert-Schmidt property of $\Psi(p;x,y)$, viewed as the kernel of an integral operator on $\mathcal{H}$, is quite easily verified. At this stage, however, the above hunch may still seem to come out of the blue.

Without providing far more detail, the precise statement of the conjectures, their plausibility and their consequences cannot easily be explained, so we present a more complete account in Section 3. Here we only sketch a 'converse' of this scenario, which can be readily understood, and then tie this in with the situation in [6].

To explain the $BC_N$ case, we assume that $\hat{H}_1$ and $\hat{H}_2$ are two self-adjoint Hamiltonians on $\mathcal{H}$ that act as differential operators or analytic difference operators $H_1(x)$ and $H_2(x)$ on orthonormal bases of eigenvectors $\{f_n\}$ and $\{g_n\}$ , such that

$$\hat{H}_1 f_n = E_n f_n, \qquad \hat{H}_2 g_n = (E_n - \sigma) g_n, \qquad n \in \mathbb{N}. \tag{1.3}$$

Moreover, we assume $H_1(x)$ and $H_2(x)$ commute with complex conjugation, so that we may and will choose $f_n(x)$ and $g_n(y)$ real-valued. Now consider any kernel of the form

$$\Psi(x,y) = \sum_{n=0}^{\infty} \lambda_n f_n(x) g_n(y), \tag{1.4}$$

where $\lambda_0$ , $\lambda_1$, ... are complex numbers satisfying

$$|\lambda_0| \geq |\lambda_1| \geq \cdots , \qquad \sum_{n=0}^{\infty} |\lambda_n|^2 < \infty. \tag{1.5}$$

Assuming $\Psi(x, y)$ is sufficiently regular, this entails

$$(H_1(x) - H_2(y))\Psi(x, y) = \sigma\Psi(x, y). \tag{1.6}$$

Obviously, the integral operator $\mathcal{I}$ with kernel given by (1.4)–(1.5) is Hilbert-Schmidt and yields an intertwining relation

$$\hat{H}_1\mathcal{I} = \mathcal{I}(\hat{H}_2 + \sigma). \tag{1.7}$$

In particular, if $H_1(x) = H_2(x) = H(x)$, we can take $f_n = g_n$, so that $\sigma = 0$ and (1.7) becomes

$$[\hat{H}, \mathcal{I}] = 0. \tag{1.8}$$

Comparing all this to our starting point (1.1), it will become clear that we are anticipating that $\sigma(p)$ is a *spectral shift* connecting two self-adjoint Hilbert space operators $\hat{H}(p)$ and $\hat{H}(p')$ associated to $H(p; x)$ and $H(p'; x)$. No such shift should occur for $p = p'$ (since the operators are then the same), and this is indeed the case for the operators occurring in Subsections 2.1–2.4.

Turning to the $A_{N-1}$ case, let $H$ be a self-adjoint Hamiltonian on $\mathcal{H}$ that acts as a differential operator or analytic difference operator $H(x)$ on an orthonormal base of eigenvectors $\{f_n\}$, such that

$$\hat{H}f_n = E_n f_n, \qquad n \in \mathbb{N}. \tag{1.9}$$

Then any kernel of the form

$$\Psi(x, y) = \sum_{n=0}^{\infty} \lambda_n f_n(x)\overline{f_n(y)}, \tag{1.10}$$

with $\{\lambda_n\}$ satisfying (1.5) yields a Hilbert-Schmidt operator that commutes with $\hat{H}$.

Returning to the afore-mentioned state of affairs in Appendix A of [6], the commutativity of the Casimir $\hat{H}$ and the transfer matrix $\mathcal{T}$ is present to begin with. In this case it can be readily checked that the reduced transfer matrix kernels satisfy

$$(H_R(x) - H_R(y))K_R(x, y) = 0, \qquad R = CM, T, \tag{1.11}$$

where $CM$ and $T$ denote the Calogero-Moser and Toda reductions. Here, however, the kernels do not yield Hilbert-Schmidt operators, since the $L^2(H)$ setting is not elliptic, but hyperbolic; similarly, the Hamiltonians $H_R$ have solely continuous spectrum (their eigenfunctions are non-normalizable scattering states).

Even so, it was the similarity of (1.11) and (1.2) that led to the reappraisal of $\Psi(p; x, y)$ and the corresponding Hilbert space scenario. As will be further explained in Section 3, the main problem to substantiate the latter for the $A_{N-1}$ case consists in showing that the Hilbert-Schmidt operator $\mathcal{I}$ derived from $\Psi$ *commutes* with a self-adjoint Hilbert space operator $\hat{H}$ associated to $H(p; x)$. Since $\mathcal{I}$ is self-adjoint in the $A_{N-1}$ cases, the connection between its eigenvectors and those of $\hat{H}$ will now be clear: by commutativity, $\hat{H}$ leaves the finite-dimensional eigenvalue subspaces of $\mathcal{I}$ invariant, so the $\mathcal{I}$-eigenvectors may be assumed to be $\hat{H}$-eigenvectors.

In the $BC_N$ cases one can check that $\sigma(p)$ vanishes whenever $p = p'$, in agreement with the eigenvector conjecture. For such special parameters $\mathcal{I}$ is again self-adjoint, and (1.1) should be once more promoted to a Hilbert space commutativity relation. When $p'$ is not equal to $p$, we are anticipating a spectral shift relation between the discrete spectra of well-defined self-adjoint Hamiltonians $\hat{H}(p)$ and $\hat{H}(p')$ on $\mathcal{H}$ associated to $H(p;x)$ and $H(p';x)$, a picture that would imply a hidden $E_8$ spectral symmetry in the difference setting. We discuss these matters in more detail in Section 3.

## 2. Special eigenfunctions for differences of Hamiltonians

In this section we detail the Hamiltonians and special function in (1.1)–(1.2), considering successively the $BC_1$, $BC_N$ and $A_{N-1}$ cases, where $N \geq 2$. It is convenient to consider first operators $A$ obtained from the Hamiltonians $H$ by a suitable similarity transformation,

$$A(p;x) = w(p;x)^{-1/2} H(p;x) w(p;x)^{1/2}, \tag{2.1}$$

where $w(x)$ is a weight function, cf. [1] and [8]. Indeed, the associated identities

$$(A(p;x) - A(p';y))\mathcal{S}(p;x,y) = \sigma(p)\mathcal{S}(p;x,y), \qquad \mathcal{S}(p;x,y) = \frac{\Psi(p;x,y)}{[w(p;x)w(p';y)]^{1/2}}, \tag{2.2}$$

$$(A(p;x) - A(p;-y))\mathcal{S}(p;x,y) = 0, \qquad \mathcal{S}(p;x,y) = \frac{\Psi(p;x,y)}{[w(p;x)w(p;y)]^{1/2}}, \tag{2.3}$$

are more easily verified than (1.1)–(1.2).

Even so, we do not present complete proofs of the identities (2.2)–(2.3), since the details are quite substantial. We also skip the calculations leading from the defining A$\Delta$Os $A$ and $H$ to their differential operator (nonrelativistic) counterparts. (These limits can be handled via the formulas in Section 6 of [8].) Finally, we point out that all of the results in this section have rather obvious hyperbolic counterparts. By contrast, trigonometric counterparts need far more work (due to the asymmetry that is inevitably introduced when relating the elliptic to the trigonometric gamma function [9]). We intend to return to these issues elsewhere.

### 2.1. The $(BC_1)_{\rm rel}$ case.

Here we start from the A$\Delta$O

$$A(h;x) = V(x)\exp(-ia_- d/dx) + V(-x)\exp(ia_- d/dx) + V_b(x), \tag{2.4}$$

where

$$V(x) = \frac{\prod_{n=0}^{7} R_+(x - h_n - ia_-/2)}{R_+(2x + ia_+/2)R_+(2x - ia_- + ia_+/2)}, \tag{2.5}$$

$$V_b(x) = \frac{1}{2R_+(\mu - ia_+/2)R_+(\mu - ia_- - ia_+/2)} \sum_{t=0}^{3} p_t[\mathcal{E}_t(x) - \mathcal{E}_t(z_t)], \tag{2.6}$$

cf. (4.19)–(4.26) in [1]. (The function $V_b(x)$ does not depend on $\mu$, cf. Lemma 3.2 in [8].) Now we introduce the special function

$$\mathcal{S}(h;x,y)=\prod_{\delta_1,\delta_2=+,-}G(\delta_1 x+\delta_2 y-ia+\phi),\quad a=(a_++a_-)/2,\quad \phi=-\frac{1}{4}\sum_{n=1}^{7}h_n. \tag{2.7}$$

The function $G(z)$ denotes the elliptic gamma function

$$G(z)=G(r,a_+,a_-;z), \tag{2.8}$$

cf. Subsection 6.3 in [1]; throughout this paper we assume

$$r,a_+,a_-\in(0,\infty). \tag{2.9}$$

Next, we define the reflection

$$J_R=\mathbf{1}_8-\frac{1}{4}\zeta\otimes\zeta,\qquad \zeta_n=1,\qquad n=0,\dots,7. \tag{2.10}$$

(Together with the $D_8$ reflections, $J_R$ generates the Weyl group of $E_8$; the vector $\zeta/2$ belongs to the $E_8$ root lattice, but not to the $D_8$ root lattice.) Then we have the eigenfunction identity (2.2) with

$$h'\equiv -h-\phi\zeta=-J_R h. \tag{2.11}$$

The validity of (2.2) is far from clear by inspection. But its proof is straightforward: If we divide the lhs of (2.2) by $\mathcal{S}(h;x,y)$, we can use the first order $G$-AΔE (6.19) in [1] with $\delta=+$ to obtain a function that is expressed solely in terms of $R_+$. This function can be verified to be a sum of terms that are elliptic in $x$ and $y$, with poles that are generically simple. Thus one need only verify that the residues at these poles cancel to deduce (2.2).

Even though each of the steps just sketched has a routine character, the necessary calculations are quite unpleasant, and so we skip them here. Once (2.2) is checked, it is obvious that $\mathcal{S}(h;x,y)$ is also an eigenfunction of the AΔO difference obtained from $A(h;x)-A(h';y)$ by interchanging $a_+$ and $a_-$. (Indeed, $\mathcal{S}(h;x,y)$ is invariant under $a_+\leftrightarrow a_-$, as is plain from its definition.)

The connection of the AΔO $A$ to the defining Hamiltonian $H$ is given by (2.1) with the weight function

$$w(h;x)=1/c(x)c(-x), \tag{2.12}$$

where $c(x)$ is the generalized Harish-Chandra $c$-function

$$c(x)=\frac{1}{G(2x+ia)}\prod_{n=0}^{7}G(x-h_n), \tag{2.13}$$

cf. Section 3 in [8]. Specifically, one has

$$H(h;x)=V(x)^{1/2}\exp(-ia_-d/dx)V(-x)^{1/2}+(x\to -x)+V_b(x). \tag{2.14}$$

We also point out the implication

$$h=h'\Rightarrow\sigma(h)=0. \tag{2.15}$$

Indeed, the function $\Psi(h;x,y)$ (given by (2.2) with (2.7) and (2.12) in effect) is invariant under $x\leftrightarrow y$ when $h'$ equals $h$. More generally, $\sigma(h)$ can be shown to vanish whenever

$h'$ can be rewritten as $\tau(h)$, with $\tau$ in the $D_8$ Weyl group. Since $H(h;x)$ is $D_8$-invariant, this also implies that $H(h';x)$ equals $H(h;x)$.

2.2. **The $(BC_1)_{\rm nr}$ case.** The nonrelativistic limit $a_- \to 0$ of the above quantities can be obtained by using pp. 251– 253 in [8]. We omit the details here, since the results can be checked independently of the relevant limit calculations. Dropping an unwieldy additive constant arising from the limit, we wind up with the differential operator

$$A(g;x) = -\frac{d^2}{dx^2} - 2\mathcal{L}(x)\frac{d}{dx} - \mathcal{L}(x)^2 + \sum_{t=0}^{3} g_t^2 \wp(x+\omega_t), \tag{2.16}$$

where

$$\mathcal{L}(x) = g_0\frac{s'(x)}{s(x)} + g_1\frac{s'(x+\omega_1)}{s(x+\omega_1)} + g_2\frac{R'(x)}{R(x)} + g_3\frac{R'(x+\omega_1)}{R(x+\omega_1)}. \tag{2.17}$$

(To ease the notation, we omit the +-subscripts in this subsection; thus we have e.g. $s(x) = s(r, a_+; x)$, cf. Appendix A in [1].) It is related to the defining $BC_1$ Hamiltonian

$$H(g;x) = -\frac{d^2}{dx^2} + \sum_{t=0}^{3} g_t(g_t - 1)\wp(x+\omega_t) \tag{2.18}$$

via (2.1), where

$$w(g;x) = p_w s(x)^{2g_0} s(x+\omega_1)^{2g_1} R(x)^{2g_2} R(x+\omega_1)^{2g_3}, \tag{2.19}$$

with $p_w$ a constant. (More precisely, this holds true up to an additive constant, cf. p. 252 in [8].)

The pertinent special function is now given by

$$\mathcal{S}(g;x,y) = \exp(-s\ln[R(x+y)R(x-y)]), \qquad s = \frac{1}{2}\sum_{t=0}^{3} g_t. \tag{2.20}$$

It arises from (2.7) via the limit (6.27) in [1]. Next, introducing

$$J_N = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}, \tag{2.21}$$

we set

$$g' \equiv J_N g. \tag{2.22}$$

Then (2.2) holds true. Once more this can be verified by first noting that when the lhs of (2.2) is divided by $\mathcal{S}$, one obtains a function that is elliptic in $x$ and $y$ with periods $\pi/r$ and $ia_+$. Comparing principal parts at the double and simple poles, one then sees that it is constant.

Finally, we point out the equivalence

$$g' = g \Leftrightarrow g_0 - g_1 + g_2 - g_3 = 0. \tag{2.23}$$

Thus equality of $g$ and $g'$ yields a 3-dimensional family of couplings. For this family one again deduces $\sigma(g) = 0$ by $x \leftrightarrow y$ invariance of $\mathcal{S}$ (2.20).

2.3. **The** $(BC_N)_{\rm rel}$ **case.** The $N>1$ generalization of (2.4) reads

$$A(h,\mu;x)=\sum_{j=1}^{N}(\mathcal{V}_j(x)\exp(-ia_-\partial_{x_j})+\mathcal{V}_j(-x)\exp(ia_-\partial_{x_j}))+\mathcal{V}(x), \tag{2.24}$$

cf. (4.19)–(4.26) in [1]. Here the special function is given by

$$\mathcal{S}(h;x,y)=\prod_{\delta_1,\delta_2=+,-}\prod_{j,k=1}^{N}G(\delta_1x_j+\delta_2y_k-ia+\phi), \tag{2.25}$$

where we have

$$\phi=-\frac{1}{4}(\zeta,h), \tag{2.26}$$

just as in the $N=1$ case. Also, (2.2) reads

$$(A(h,\mu_h;x)-A(h',\mu_h;y))\mathcal{S}(h;x,y)=\sigma(h)\mathcal{S}(h;x,y), \tag{2.27}$$

where we have again

$$h'\equiv -J_Rh, \tag{2.28}$$

whereas $\mu_h$ is fixed in terms of $h$:

$$\mu_h\equiv 2ia+\frac{1}{2}(\zeta,h). \tag{2.29}$$

The latter constraint ensures that when the lhs of (2.27) is divided by $\mathcal{S}$, one obtains a function that is elliptic in $x_1,\ldots,x_N$, $y_1,\ldots,y_N$. After verification of residue cancellation, one deduces (2.27).

The pertinent weight function is now defined by

$$w(h,\mu_h;x)=1/C(x)C(-x), \tag{2.30}$$

where

$$C(x)=\prod_{j=1}^{N}c(x_j)\cdot\prod_{1\le j<k\le N}\frac{G(x_j-x_k-\mu_h+ia)G(x_j+x_k-\mu_h+ia)}{G(x_j-x_k+ia)G(x_j+x_k+ia)}, \tag{2.31}$$

and $c(x_j)$ is given by (2.13). Then the similarity transformation (2.1) yields the van Diejen/Hikami/Komori Hamiltonian (4.18) in [1].

2.4. **The** $(BC_N)_{\rm nr}$ **case.** The $a_-\to 0$ limit can be handled along the lines sketched on p. 254 of [8]. The special function becomes

$$\mathcal{S}(g;x,y)=\prod_{j,k=1}^{N}\exp(-s\ln[R(x_j-y_k)R(x_j+y_k)]),\qquad s=\frac{1}{2}\sum_{t=0}^{3}g_t, \tag{2.32}$$

and the constraint (2.29) gives rise to a constraint between the coupling $\lambda$ and the external field couplings $g_t$ in the Inozemtsev Hamiltonian [2]

$$\begin{aligned} H(g,\lambda;x) &= -\sum_{j=1}^{N}\partial_{x_j}^2 + 2\lambda(\lambda-1)\sum_{1\le j<k\le N}(\wp(x_j-x_k)+\wp(x_j+x_k)) \\ &\quad +\sum_{t=0}^{3} g_t(g_t-1)\sum_{j=1}^{N}\wp(x_j+\omega_t). \end{aligned} \tag{2.33}$$

Specifically, we need

$$\lambda \equiv s = \frac{1}{2}\sum_{t=0}^{3} g_t. \tag{2.34}$$

Then we get

$$(H(g,s;x) - H(g',s;y))\Psi(g;x,y) = \sigma(g)\Psi(g;x,y), \tag{2.35}$$

where $g'$ is defined by (2.22). Also, $\Psi$ is obtained via (2.2) with

$$w(g,s;x) = p\prod_{j=1}^{N} w(g;x_j)\cdot\prod_{1\le j<k\le N}(s(x_j-x_k)s(x_j+x_k))^{2s}, \tag{2.36}$$

where $p$ is a constant and $w(g;x_j)$ is given by (2.19).

2.5. **The $(A_{N-1})_{\mathrm{rel}}$ case.** In this case we can work with the special function

$$\mathcal{S}(\mu;x,y) = \prod_{j,k=1}^{N}\frac{G(x_j-y_k-\mu/2)}{G(x_j-y_k+\mu/2)}, \tag{2.37}$$

and we have

$$(A_{k,\delta}(\mu;x) - A_{k,\delta}(\mu;-y))\mathcal{S}(\mu;x,y) = 0, \qquad k\in\{\pm1,\dots,\pm N\}, \qquad \delta\in\{+,-\}, \tag{2.38}$$

where

$$A_{l,\delta}(\mu;x) = \sum_{\substack{I\subset\{1,\dots,N\}\\ |I|=l}}\prod_{\substack{m\in I\\ n\notin I}}\frac{R_\delta(x_m-x_n-\mu+ia_\delta/2)}{R_\delta(x_m-x_n+ia_\delta/2)}\cdot\prod_{m\in I}\exp(-ia_{-\delta}\partial_{x_m}), \tag{2.39}$$

$$A_{-l,\delta}(\mu;x) = A_{l,\delta}(\mu;-x), \qquad l=1,\dots,N, \qquad \delta=+,-. \tag{2.40}$$

(Fixing $\delta$, these operators are proportional to the A$\Delta$Os [1] (3.38).)

In the present case the weight function that relates the commuting A$\Delta$Os (2.39)–(2.40) to the commuting Hamiltonians

$$\begin{aligned} H_{\pm l,\delta}(\mu;x) &= \sum_{\substack{I\subset\{1,\dots,N\}\\ |I|=l}}\prod_{\substack{m\in I\\ n\notin I}}\left(\frac{R_\delta(x_m-x_n\mp\mu\pm ia_\delta/2)}{R(x_m-x_n\pm ia_\delta/2)}\right)^{1/2}\cdot\prod_{m\in I}\exp(\mp ia_{-\delta}\partial_{x_m}) \\ &\quad \cdot\prod_{\substack{m\in I\\ n\notin I}}\left(\frac{R_\delta(x_m-x_n\pm\mu\mp ia_\delta/2)}{R(x_m-x_n\mp ia_\delta/2)}\right)^{1/2}, \quad l=1,\dots,N,\ \delta=+,-, \end{aligned} \tag{2.41}$$

is given by

$$w(\mu;x)=1/c_\mu(x)c_\mu(-x), \tag{2.42}$$

where

$$c_\mu(x)=\prod_{1\le j<k\le N}\frac{G(x_j-x_k-\mu+ia)}{G(x_j-x_k+ia)}. \tag{2.43}$$

We proceed to study the eigenfunction identities (2.38). Dividing the lhs by $\mathcal{S}$ and using the G-AΔEs [1] (6.19), we deduce that they are equivalent to the functional equations

$$\sum_{\substack{I\subset\{1,\ldots,N\}\\|I|=l}}\prod_{\substack{m\in I\\ n\notin I}}\frac{R(x_m-x_n-\mu+\omega_2)}{R(x_m-x_n+\omega_2)}\cdot\prod_{\substack{m\in I\\ n=1,\ldots,N}}\frac{R(x_m-y_n+\mu)}{R(x_m-y_n)}=(x\leftrightarrow -y),\qquad l=1,\ldots,N, \tag{2.44}$$

where

$$R(z)=R(r,a;z),\quad \omega_2=ia/2,\quad \mu\in\mathbb{C}. \tag{2.45}$$

Let us first consider the special case

$$y_j=x_j+\gamma-\omega_2,\qquad \gamma\in\mathbb{C},\qquad j=1,\ldots,N, \tag{2.46}$$

of (2.44). Using proportionality of $R(z+\omega_2)$ to $e^{cz}s(z)$ (cf. [1] (6.14)), this specialization can be written

$$\sum_{\substack{I\subset\{1,\ldots,N\}\\|I|=l}}\prod_{\substack{m\in I\\ n\notin I}}\frac{s(x_m-x_n-\mu)}{s(x_m-x_n)}\cdot\prod_{\substack{m\in I\\ n=1,\ldots,N}}\frac{s(x_m-x_n-\gamma+\mu)}{s(x_m-x_n-\gamma)}=(x\to -x),\qquad l=1,\ldots,N. \tag{2.47}$$

We now compare (2.47) to the functional equations that encode the commutativity of the AΔOs (2.39), cf. [1] (3.37). Using [1] (6.13), we see the latter can be rewritten as

$$\sum_{\substack{I\subset\{1,\ldots,N\}\\|I|=l}}\prod_{\substack{m\in I\\ n\notin I}}\frac{s(x_m-x_n-\mu)s(x_m-x_n-\gamma+\mu)}{s(x_m-x_n)s(x_m-x_n-\gamma)}=(x\to -x),\qquad l=1,\ldots,N. \tag{2.48}$$

For $l=1$ the two sequences of functional equations (2.47) and (2.48) are clearly equivalent. But for $l>1$ this is no longer true in general (as we erroneously reported in our Kyoto lecture). Indeed, the cancellations one needs to turn (2.47) into (2.48) occur only if $\mu$ equals $2\gamma$.

Nevertheless, our proof of (2.48) in [10] can be adapted to prove (2.44). The main difference is that the terms in (2.44) are $\pi/r$-periodic, but not $ia$-periodic in $x_j$, $y_j$, $j=1,\ldots,N$. But under the shift $x_1\to x_1+ia$ (say) we obtain the same multiplier $\exp(-2irl\mu)$ for all of the terms, so that we are again reduced to showing that the residues at all poles vanish. Just as in the proof of Theorem A2 in [10], this can be shown by pairing off and induction; in fact the details are simpler than in *loc. cit.*

The upshot is that the eigenfunction relations (2.38) are valid. Choosing $l=1$, they entail the $l=1$ equations (2.48), which encode relativistic invariance [10]; furthermore, for the special $\mu$-values $2ia_+$ and $2ia_-$ they imply the commutativity of all of the A$\Delta$Os (2.39).

We point out that basically the same functional identities (2.44) were found by Kajihara and Noumi, using the Frobenius determinant identity (cf. [1] (6.6)) to prove them (see Theorem 1.3 in [11]). They use the identities as a starting point to obtain various results on multiple elliptic hypergeometric series, including Bailey type transformations.

Another related result in the literature concerns a trigonometric version of (2.38), which can be found in a monograph by Macdonald [12]. We were made aware of this by Noumi; together with Kirillov he has used the pertinent special function (an infinite $q$-product) as a tool to construct raising operators for the Macdonald polynomials [13].

We conclude this subsection by pointing out more general solutions to (2.38). To begin with, we may replace (2.37) by

$$\mathcal{S}_\eta(\mu;x,y)=\prod_{j,k=1}^{N}\frac{G(x_j-y_k+\eta+\mu/2)}{G(x_j-y_k+\eta-\mu/2)},\qquad \eta\in\mathbb{C}. \tag{2.49}$$

(To see that these functions still satisfy (2.38), notice that they arise from (2.37) by taking $y_j\to y_j-\eta$, $j=1,\ldots,N$, and that the A$\Delta$Os $A_{k,\delta}(\mu;-y)$ are invariant under this substitution.) This already yields an infinite-dimensional solution space. These more general solutions are parametrized by one complex number, but a much larger solution space arises upon multiplication of $\mathcal{S}_\eta(p;x,y)$ by

$$f\Big(\sum_{j=1}^{N}(x_j-y_j)\Big), \tag{2.50}$$

where $f$ is an arbitrary meromorphic function. (This is readily checked from the definitions.)

2.6. **The $(A_{N-1})_{\rm nr}$ case.** Setting

$$\mu=ia_-g \tag{2.51}$$

in (2.37) and taking $a_-$ to 0, we obtain the nonrelativistic special function

$$\mathcal{S}(g;x,y)=\prod_{j,k=1}^{N}R(x_j-y_k)^{-g},\qquad R(z)=R(r,a_+;z), \tag{2.52}$$

cf. [1] (6.27). More generally, from (2.49) we obtain

$$\mathcal{S}_\eta(g;x,y)=\prod_{j,k=1}^{N}R(x_j-y_k+\eta)^{-g}. \tag{2.53}$$

Likewise, the $a_-\to 0$ limit of (2.42) yields the nonrelativistic weight function

$$w(g;x)=C\prod_{1\le j<k\le N}s(x_j-x_k)^{2g},\qquad s(z)=s(r,a_+;z). \tag{2.54}$$

Taking suitable linear combinations of the A$\Delta$Os (2.39) and Hamiltonians (2.41) and letting $a_-$ tend to 0, one obtains their nonrelativistic counterparts (cf. Subsections 4.2 and 4.3 in [14]).

To be more specific, on the nonrelativistic level one may work with $N$ commuting PDOs of the form [15]

$$H_1 = -i\sum_{j=1}^{N}\partial_{x_j}, \tag{2.55}$$

$$H_2 = -\sum_{1\le j_1<j_2\le N}\partial_{x_{j_1}}\partial_{x_{j_2}} - g(g-1)\sum_{1\le j<k\le N}\wp(x_j-x_k), \tag{2.56}$$

$$H_l = (-i)^l\sum_{1\le j_1<\cdots<j_l\le N}\partial_{x_{j_1}}\cdots\partial_{x_{j_l}} + \text{l. o.},\qquad l=3,\ldots,N, \tag{2.57}$$

where l. o. denotes terms that are of lower order in the $x_j$-partials. Then one has

$$(H_k(g;x)-H_k(g;-y))\Psi_\eta(g;x,y)=0,\qquad k=1,\ldots,N, \tag{2.58}$$

with

$$\Psi_\eta(g;x,y)=C\left(\frac{\prod_{1\le j<k\le N}s(x_j-x_k)s(y_j-y_k)}{\prod_{j,k=1}^{N}R(x_j-y_k+\eta)}\right),\qquad \eta\in\mathbb{C}. \tag{2.59}$$

Clearly, for the defining Hamiltonian

$$H=\frac{1}{2}H_1^2-H_2=-\frac{1}{2}\sum_{j=1}^{N}\partial_{x_j}^2+g(g-1)\sum_{1\le j<k\le N}\wp(x_j-x_k), \tag{2.60}$$

the eigenfunction relation (2.58) holds true as well. Essentially the same formula was first obtained by Langmann in [16].

## 3. A reinterpretation of the eigenfunctions

As announced in Section 1, we now take a quite different look at the eigenfunction $\Psi(p;x,y)$ featuring in the general formulas (1.1)–(1.2): we interpret it as the kernel of an integral operator $\mathcal{I}$ on the Hilbert space

$$\mathcal{H}=L^2(F,dx), \tag{3.1}$$

where $F$ is given by

$$F=\begin{cases}[\,0,\pi/2r\,], & (BC_1),\\ \{x\in[\,0,\pi/2r\,]^N\mid x_N<\cdots<x_1\}, & (BC_N),\\ \{x\in[\,-\pi/2r,\pi/2r\,]^N\mid x_N<\cdots<x_1\}, & (A_{N-1}).\end{cases} \tag{3.2}$$

We may and will view $F$ as a fundamental domain for the action of the $BC_N$ and $A_{N-1}$ Weyl groups on the torus $[\,-\pi/2r,\pi/2r\,]^N$. Indeed, in the $BC_N$ cases all of the relevant objects are invariant under sign changes and permutations, cf. Subsections 2.1–2.4, and in the $A_{N-1}$ cases under permutations, cf. Subsections 2.5 and 2.6. (We should add that we are a bit cavalier with sets of measure zero at this point; a precise account can be found in Subsections 2.2 and 6.2 of [14].)

We now specialize to parameters for which the Hamiltonians $H(p;x)$ and $H(p';x)$ are at least *formally* self-adjoint on $\mathcal{H}$ . Thus we need

$$(3.3)\quad \Re(\zeta,h)=0\ (\mathrm{mod}\,2\pi/r),\quad \Re h_n=0\ (\mathrm{mod}\,\pi/2r),\quad n=0,\dots,7,\qquad (BC_N)_{\mathrm{rel}},$$

$$(3.4)\quad g_t\in\mathbb{R},\qquad t=0,1,2,3,\qquad (BC_N)_{\mathrm{nr}},$$

$$(3.5)\quad \mu\in i\mathbb{R},\qquad (A_{N-1})_{\mathrm{rel}},$$

$$(3.6)\quad g\in\mathbb{R},\qquad (A_{N-1})_{\mathrm{nr}}.$$

It is straightforward to verify that the above restrictions suffice for the weight functions to be nonnegative on $F$. Likewise, denoting complex conjugation by $C$, they also imply

$$(3.7)\quad CH(p;x)C \;=\; H(p;x),\qquad (BC_N),$$

$$(3.8)\quad CH(p;x_1,\dots,x_N)C \;=\; H(p;-x_N,\dots,-x_1),\qquad (A_{N-1}).$$

To ensure that the weight function integrals over $F$ are finite, however, we need further restrictions. It is sufficient to require

$$(3.9)\quad \Im(\zeta,h)\in(-4a,0),\quad \Im h_n,\Im h_n'\in(-a,a),\quad n=0,\dots,7,\qquad (BC_N)_{\mathrm{rel}},$$

$$(3.10)\quad \sum_{t=0}^{3} g_t\in(0,\infty),\quad g_0,g_1\in(-1/2,\infty),\quad |g_2-g_3|<1+g_0+g_1,\qquad (BC_N)_{\mathrm{nr}},$$

$$(3.11)\quad \mu\in i(0,2a),\qquad (A_{N-1})_{\mathrm{rel}},$$

$$(3.12)\quad g\in(0,\infty),\qquad (A_{N-1})_{\mathrm{nr}}.$$

These extra restrictions are chosen so that they also imply that the functions $\mathcal{S}(p;x,y)$ are nontrivial and bounded on $F\times F$.

Therefore, we may now conclude

$$(3.13)\quad \int_{F^2}|\Psi(p;x,y)|^2\,dx\,dy=\int_{F^2}w(p;x)|\mathcal{S}(p;x,y)|^2w(p';y)\,dx\,dy<\infty.$$

Hence the integral operators

$$(3.14)\quad (\mathcal{I}f)(x)=\int_F\Psi(p;x,y)f(y)\,dy,\qquad f\in\mathcal{H},$$

are Hilbert-Schmidt, as advertized in Section 1.

Specializing to the $BC_N$ cases, one readily checks that $\Psi(p;x,y)$ is real-valued on $F^2$. Thus $\mathcal{I}^*$ has kernel $\Psi(p;y,x)$. Now $\mathcal{S}(p;x,y)$ is manifestly invariant under $x\leftrightarrow y$. Hence we infer

$$(3.15)\quad p'=p\Rightarrow\mathcal{I}^*=\mathcal{I},\qquad (BC_N).$$

Assuming $p'=p$, the orthonormal $\mathcal{I}$-eigenvectors $f_0$, $f_1$, …corresponding to the ordered eigenvalues $|\lambda_0|\geq|\lambda_1|\geq\cdots$ may be chosen real-valued. Thus we deduce

$$(3.16)\quad \Psi(p;x,y)=\sum_{n=0}^{\infty}\lambda_nf_n(x)f_n(y),\lambda_n\in\mathbb{R},\qquad (p'=p),$$

where

$$(3.17)\quad (f_n,f_m)=\delta_{nm},\qquad n,m\in\mathbb{N}.$$

Our conjecture is now that the eigenvectors $f_n(x)$ occurring in (3.16) may be chosen to be eigenfunctions of the commuting Hamiltonians as well. More generally, for $p' = p$ we still have a singular value decomposition

$$\Psi(p;x,y) = \sum_{n=0}^{\infty} s_n f_n(x) g_n(y), \qquad s_0 \geq s_1 \geq \cdots \geq 0, \tag{3.18}$$

with

$$(f_n, f_m) = \delta_{nm}, \qquad (g_n, g_m) = \delta_{nm}, \qquad n, m \in \mathbb{N}, \tag{3.19}$$

and we conjecture that one may choose $f_n(x)$ and $g_n(y)$ to be joint eigenfunctions of the Hamiltonians $H(p;x)$ and $H(p';y)$, resp., and their commuting relatives.

Turning to the $A_{N-1}$ cases, we consider the family of integral operators

$$(\mathcal{I}_\eta f)(x) = \int_F \Psi_\eta(p;x,y) f(y)\, dy, \qquad f \in \mathcal{H}. \tag{3.20}$$

Here one readily checks that the Hilbert-Schmidt property (3.13) holds true for $\Psi = \Psi_\eta$ with $\Im\eta$ sufficiently small. Clearly, we have

$$(\mathcal{I}_\eta)^* = \mathcal{I}_{-\overline{\eta}}. \tag{3.21}$$

In particular, $\mathcal{I}_\eta$ is self-adjoint for imaginary $\eta$. It is also readily verified that we have

$$[\, \mathcal{I}_{\eta_1}, \mathcal{I}_{\eta_2}\, ] = 0. \tag{3.22}$$

Therefore, we obtain a commuting family of normal Hilbert-schmidt operators, which implies

$$\Psi_\eta(p;x,y) = \sum_{n=0}^{\infty} \lambda_n(\eta) f_n(x) \overline{f_n(y)}, \tag{3.23}$$

with $\{f_n\}$ orthonormal. In this case we conjecture that the joint eigenvectors $f_n(x)$ of the family $\mathcal{I}_\eta$ are also joint eigenvectors of the defining Hamiltonian $H(p;x)$ and its commuting relatives.

Both for the $BC_N$ cases and for the $A_{N-1}$ cases, it is plausible that the integral operators have a finite-dimensional kernel and that the eigenvalues of the Hamiltonians are real. Assuming the validity of all of these hunches (or, perhaps more appropriately, working hypotheses), it follows that we obtain well-defined commuting self-adjoint operators on $\mathcal{H}$ with an orthonormal base of joint eigenvectors; moreover, the action of these operators on the span of the eigenvectors coincides with the action of the Hamiltonians.

We proceed to sketch what is involved in substantiating the above scenario. To this end we begin by recalling that it is not possible to associate a *bounded* self-adjoint operator $\hat{H}$ to a formally self-adjoint differential operator or analytic difference operator $H$. Thus one should aim for an initial domain $\mathcal{D}$ (dense subspace) on which the operator $H$ is well defined and symmetric, and then study the existence and properties of self-adjoint extensions.

Let us now start from the eigenfunction relations (1.2) or (1.1) with $p' = p$, restricting the parameters as specified above. Then we should first try to find a symmetry domain

$\mathcal{D}$ for which

$$(Hf_1, \mathcal{I}f_2) = (f_1, \mathcal{I}Hf_2), \qquad \forall f_1, f_2 \in \mathcal{D}. \tag{3.24}$$

We continue to illustrate this crucial first step with a simple example. Consider the $(BC_1)_{\mathrm{nr}}$ case

$$g_0 = g_1 = k \in (0, \infty), \quad g_2 = g_3 = 0, \quad H(g; x) = -d^2/dx^2 + k(k-1)(\wp(x) + \wp(x + \omega_1)). \tag{3.25}$$

Thus we have $g' = g$, cf. (2.23), and

$$(\mathcal{I}f)(x) = cs(x)^k s(x+\omega_1)^k \int_0^{\omega_1} dy \left( \frac{s(y)s(y+\omega_1)}{R(x+y)R(x-y)} \right)^k f(y), \quad f \in L^2([\,0, \omega_1\,], dy). \tag{3.26}$$

We now choose the initial domain

$$\mathcal{D} = C_0^\infty([\,0, \omega_1\,]), \tag{3.27}$$

and consider

$$(H(g)f_1, \mathcal{J}f_2) = \int_0^{\omega_1} dx \int_0^{\omega_1} dy\, \overline{(H(g;x)f_1)(x)} \Psi(g; x, y) f_2(y), \qquad f_1, f_2 \in \mathcal{D}. \tag{3.28}$$

Integrating by parts twice, we see this equals

$$\int_0^{\omega_1} dx \int_0^{\omega_1} dy\, \overline{f_1(x)} f_2(y) H(g; x) \Psi(g; x, y). \tag{3.29}$$

Therefore we are in the position to use (1.1), obtaining

$$\int_0^{\omega_1} dx \int_0^{\omega_1} dy\, \overline{f_1(x)} f_2(y) H(g; y) \Psi(g; x, y). \tag{3.30}$$

Integrating by parts again, we now deduce (3.24).

More generally, we expect that a judicious choice of initial domain yields symmetry of $H(p; x)$ and (3.24); for the nonrelativistic cases (3.24) should be shown via integration by parts, whereas for the relativistic cases (3.24) should follow via contour shifts and Cauchy's theorem.

Now if $H$ were bounded, it would be obvious from (3.24) that $H$ and $\mathcal{J}$ commute, and hence have common eigenvectors. But for the unbounded $H$ under consideration, we can only infer from (3.24) that the subspace $\mathcal{I}(\mathcal{D})$ belongs to the domain of $H^*$. Returning to the above example, we have extensive additional information from the theory of Schrödinger operators. Choosing $k \geq 3/2$, the Hamiltonian $H(g; x)$ is essentially self-adjoint on the domain (3.27). Denoting its self-adjoint closure by $\hat{H}$, we obtain from (3.24) by taking limits

$$(\hat{H}f_1, \mathcal{I}f_2) = (f_1, \mathcal{I}\hat{H}f_2), \qquad f_1, f_2 \in \mathcal{D}(\hat{H}). \tag{3.31}$$

Therefore $\mathcal{I}$ leaves the domain of $\hat{H}$ invariant and we have

$$[\mathcal{I}, (\hat{H} + \lambda)^{-1}] = 0, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.32}$$

Since the $\hat{H}$-resolvent occurring in (3.32) is a bounded normal operator, it is now clear that $\hat{H}$ leaves the finite-dimensional eigenspaces of the self-adjoint Hilbert-Schmidt operator $\mathcal{I}$

invariant, so that common eigenvectors exist. In particular, we can conclude that $\hat{H}$ has solely discrete spectrum whenever the $\mathcal{I}$-kernel is finite-dimensional.

Of course, for the example at hand one can readily show discreteness of the spectrum by using Schrödinger operator theory. But for formally self-adjoint A$\Delta$Os very little is known regarding self-adjointness and spectral properties. Once one succeeds in reexpressing the key relations (1.1) (with $p = p'$) and (1.2) as a 'commutativity formula' (3.24), the latter should be of great help in clarifying the Hilbert space status of the commuting Calogero-Moser type elliptic A$\Delta$Os. For example, we expect to complete our Hilbert space results in [17] (most of which only hold for a dense set of couplings) by exploiting the pertinent Hilbert-Schmidt integral operators.

Turning to the $BC_N$ cases with $p \neq p'$, we should try to find symmetry domains $\mathcal{D}^{(\prime)}$ for $H(p^{(\prime)}; x)$ such that we have

$$(H(p)f_1, \mathcal{I}f_2) = (f_1, \mathcal{I}(H(p') + \sigma(p))f_2), \qquad \forall (f_1, f_2) \in \mathcal{D} \times \mathcal{D}'. \tag{3.33}$$

Once more, this relation should be shown to follow from (1.1) via integration by parts or contour shifts. We proceed to explain how (3.33) may imply the conjectured eigenvector relations. For this purpose we need to show that $\mathcal{D}$ and $\mathcal{D}'$ have been chosen large enough so that $H(p)$ and $H(p')$ are essentially self-adjoint. Then it follows as before that we have

$$(\hat{H}(p) + \lambda)^{-1}\mathcal{I} = \mathcal{I}(\hat{H}(p') + \sigma(p) + \lambda)^{-1}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.34}$$

Taking adjoints, we readily deduce

$$[(\hat{H}(p) + \lambda)^{-1}, \mathcal{I}\mathcal{I}^*] = 0, \qquad [(\hat{H}(p') + \sigma(p) + \lambda)^{-1}, \mathcal{I}^*\mathcal{I}] = 0. \tag{3.35}$$

Hence $\hat{H}(p)/\hat{H}(p')$ leave the finite-dimensional eigenspaces of $\mathcal{I}\mathcal{I}^*/\mathcal{I}^*\mathcal{I}$ associated with positive eigenvalues invariant, implying the existence of common eigenvectors. Again this entails the physically expected discrete spectrum property whenever the kernels of $\mathcal{I}$ and $\mathcal{I}^*$ have finite dimension.

It will be clear from the above sketch that much remains to be done to fill in the details of the expected scenario. But various encouraging results have already been obtained, and thus far all tests we could explicitly work out have been passed. We mention in particular one result that may be quite useful as a starting point for a perturbation theory approach. This result pertains to the special choice $\mu = ia_+$ in Subsection 2.5. Then the A$\Delta$Os (2.41) are 'free', and the corresponding joint eigenvector ONB for $\mathcal{H}$ consists of suitable linear combinations of plane waves. Now one can check that these eigenvectors are shared by the Hilbert-Schmidt operators arising from (2.49)–(2.50) by making a suitable choice for the function $f$. Moreover, the eigenvalues can be determined explicitly, and they are all nonzero. (We will present the details elsewhere.)

To conclude, we point out that the developments in this section have a certain resemblance to the theory of $Q$-operators, cf. especially [18] and references given there. More specifically, (3.22) should be compared to (1.2) in [18], and the expected commutativity of the integral operator family with the commuting Hamiltonians should be compared to (1.3) in [18]. (In fact, the integral operator family employed in [18] seems closely related to the trigonometric degeneration of (3.24).)

What is lacking here, however, is the so-called Baxter or separation equation (1.5) in [18]. (More precisely, the dependence of the $\mathcal{I}_\eta$-eigenvalues on $\eta$ is of the very simple form

$\exp(ic\eta)$, $c \in \mathbb{R}$, in all cases where it is known explicitly; this form seems inappropriate for separation purposes.) Indeed, the outlook of [18] and related papers is quite different: One uses the $Q$-operators to set up a separation of variables for joint eigenfunctions of Hamiltonians that are already known in more or less explicit form (such as the Jack or Macdonald polynomials). By contrast, our emphasis is on using the integral operators $\mathcal{I}$ to promote the formally self-adjoint Hamiltonians to commuting self-adjoint operators on $\mathcal{H}$ in those cases where this problem is to date wide open. Quite likely, this program hinges on establishing suitable properties of the $\mathcal{I}$-eigenfunctions, rather than constructing them in explicit (let alone separated) form. To be sure, the latter issue is of great interest in itself.

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## References

[1] Y. Komori and S. N. M. Ruijsenaars, *Elliptic integrable systems of Calogero-Moser type: A survey*, these Proceedings.

[2] V. I. Inozemtsev, *Lax representation with spectral parameter on a torus for integrable particle systems*, Lett. Math. Phys. **17** (1989), 11–17.

[3] J. F. van Diejen, *Integrability of difference Calogero-Moser systems*, J. Math. Phys. **35** (1994), 2983–2998.

[4] Y. Komori and K. Hikami, *Quantum integrability of the generalized elliptic Ruijsenaars models*, J. Phys. A **30** (1997), 4341–4364.

[5] E. Langmann, *An explicit solution of the (quantum) elliptic Calogero-Sutherland model*, math-ph/0407050.

[6] M. Niedermaier and E. Seiler, *Non-amenability and spontaneous symmetry breaking– The hyperbolic spin-chain*, to appear in Ann. de l'Inst. H. Poincaré.

[7] M. A. Olshanetsky and A. M. Perelomov, *Quantum integrable systems related to Lie algebras*, Phys. Rep. 94 (1983), 313–404.

[8] S. N. M. Ruijsenaars, *Integrable $BC_N$ analytic difference operators: Hidden parameter symmetries and eigenfunctions*, in Proceedings Cadiz 2002 NATO Advanced Research Workshop "New trends in integrability and partial solvability", NATO Science Series Vol. **132**, (A.B. Shabat, A. González-López, M. Mañas, L. Martínez Alonso, M.A. Rodríguez, Eds.), pp. 217–261, Kluwer, Dordrecht, 2004.

[9] S. N. M. Ruijsenaars, *First order analytic difference equations and integrable quantum systems*, J. Math. Phys. **38** (1997), 1069–1146.

[10] S. N. M. Ruijsenaars, *Complete integrability of relativistic Calogero-Moser systems and elliptic function identities*, Commun. Math. Phys. **110** (1987), 191–213.

[11] Y. Kajihara and M. Noumi, *Multiple elliptic hypergeometric series. An approach from the Cauchy determinant*, Indag. Math., N. S. **14** (2003), 395–421.

[12] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Univ. Press, Oxford, 1995.

[13] A. N. Kirillov and M. Noumi, *Affine Hecke algebras and raising operators for Macdonald polynomials*, Duke Math. J. **93** (1998), 1–39.

[14] S. N. M. Ruijsenaars, *Systems of Calogero-Moser type*, in Proceedings of the 1994 Banff summer school "Particles and fields", CRM series in mathematical physics, (G. Semenoff, L. Vinet, Eds.), pp. 251–352, Springer, New York, 1999.

[15] T. Oshima and H. Sekiguchi, *Commuting families of differential operators invariant under the action of a Weyl group*, J. Math. Sci. Univ. Tokyo **2** (1995), 1–75.

[16] E. Langmann, *Anyons and the elliptic Calogero-Sutherland model*, Lett. Math. Phys. **54** (2000), 279–289.

[17] S. N. M. Ruijsenaars, *Relativistic Lamé functions: Completeness vs. polynomial asymptotics*, Indag. Math., N. S. **14** (2003), 515–544.

[18] V. B. Kuznetsov, V. V. Mangazeev and E. K. Sklyanin, *Q-operator and factorized separation chain for Jack polynomials*, Indag. Math., N. S. **14** (2003), 451–482.

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