AN ELLIPTIC DETERMINANT TRANSFORMATION

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ABSTRACT. We prove a transformation formula relating two determinants involving elliptic shifted factorials. Similar determinants have been applied to multiple elliptic hypergeometric series.

1. INTRODUCTION

Determinant evaluations play an important role in mathematics, perhaps most notably in combinatorics, see Krattenthaler's surveys [K2] and [K3]. Many useful determinant evaluations are *rational* identities, which rises the question of finding generalizations to the *elliptic* level. In recent work with Schlosser [RS2], we gave an approach to elliptic determinant evaluations that encompasses most results in the literature, from the classical Frobenius determinant to the Macdonald identities for non-exceptional affine root systems.

As an example of an elliptic determinant evaluation, we mention Warnaar's determinant [W, Corollary 5.4], which we write as

$$\det_{1\leq j,k\leq n}\left(\frac{(bx_j,c/x_j)_{k-1}}{(a/bx_j,ax_j/c)_{k-1}}\right)\\ = c^{\binom{n}{2}}q^{\binom{n}{3}}\prod_{1\leq i<j\leq n}x_i^{-1}\theta(x_i/x_j)\theta(bx_ix_j/c)\prod_{j=1}^{n}\frac{(a/bc,aq^{j-2})_{j-1}}{(a/bx_j,ax_j/c)_{n-1}}. \quad (1)$$

Here, we use the notation

$$\theta(x)=\prod_{j=0}^{\infty}(1-p^jx)(1-p^{j+1}/x),$$

$$(a)_k=\theta(a)\theta(aq)\cdots\theta(aq^{k-1}),$$

$$(a_1,\dots,a_n)_k=(a_1)_k\cdots(a_n)_k,$$

where p and q are fixed parameters with $|p|<1$. In the trigonometric case, $p=0$, we recover the usual q-shifted factorials, which we denote

$$(a)_k^{\mathrm{trig}}=(1-a)(1-aq)\cdots(1-aq^{k-1}).$$

The case $p=0$ of (1) is a special case of a determinant evaluation due to Krattenthaler [K1, Lemma 34].

Warnaar used (1) to derive a multivariable extension of the elliptic Jackson summation. In the terminology of [DS], this is a "Schlosser-type" sum, which is obtained by taking the determinant of one-dimensional summations. In spite of its conceptual simplicity, Warnaar's sum is a key result for multiple elliptic hypergeometric series, since it can be used both to derive an "Aomoto–Itô–Macdonald-type" sum (conjectured in [W] and

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proved in [R1]) and a "Gustafson–Milne-type" sum (conjectured in [DS] and proved in [R2]). Alternative proofs of these summations were found by Rains [Ra1, Ra2]. See [S] for an application of (1) to elliptic hypergeometric integrals.

In [RS1, Eq. (7.27)], as a by-product of deriving transformation formulas for Schlosser-type series, we discovered the identity

$$\det_{1\le j,k\le n}\left(\frac{(z_j)_{k-1}^{\mathrm{trig}}}{(a_jz_j)_{k-1}^{\mathrm{trig}}}\right)=(-1)^{\binom{n}{2}}q^{\binom{n}{3}}\det_{1\le j,k\le n}\left(z_j^{k-1}\frac{(a_j)_{k-1}^{\mathrm{trig}}}{(a_jz_j)_{k-1}^{\mathrm{trig}}}\right). \tag{2}$$

Krattenthaler found a more natural proof of (2), based on the q-Chu–Vandermonde summation, which was included in [RS1].

The purpose of the present note is to obtain an elliptic extension of (2). In fact, as we will explain at the end, such an identity can be derived from the results of [RS1]. However, we prefer to give a self-contained proof, which is a straight-forward generalization of Krattenthaler's proof of (2), based on the elliptic Jackson summation

$$\sum_{l=0}^{n}\frac{\theta(aq^{2l})}{\theta(a)}\frac{(a,b,c,d,e,q^{-n})_l}{(q,aq/b,aq/c,aq/d,aq/e,aq^{n+1})_l}=\frac{(aq,aq/bc,aq/bd,aq/cd)_n}{(aq/b,aq/c,aq/d,aq/bcd)_n}, \tag{3}$$

where $a^2q^{n+1}=bcde$. The identity (3) was obtained by Date et al. [D] for special parameter values and by Frenkel and Turaev [FT] in general; see [R3] for an elementary proof.

Theorem 1. *Let a and b_j,c_j,d_j, $j=1,\dots,n$, be parameters such that the product $b_jc_jd_j$ is independent of j. Then the determinant transformation*

$$\det_{1\le j,k\le n}\left(\frac{(b_j,c_j,d_j)_{k-1}}{(a/b_j,a/c_j,a/d_j)_{k-1}}\right)=\left(\frac{a}{e}\right)^{\binom{n}{2}}\prod_{j=2}^{n}\frac{(aq^{j-2})_{j-1}}{(eq^{j-2})_{j-1}}\det_{1\le j,k\le n}\left(\frac{(a/b_jc_j,a/b_jd_j,a/c_jd_j)_{k-1}}{(a/b_j,a/c_j,a/d_j)_{k-1}}\right) \tag{4}$$

holds, where

$$e=a^2/b_jc_jd_j. \tag{5}$$

Note that if d_j is independent of j, we may write $b_j=bx_j$, $c_j=c/x_j$ and evaluate both determinants using (1). Thus, in this special case, Theorem 1 follows from Warnaar's determinant evaluation.

As far as we know, Theorem 1 is new even in the case $p=0$. In that case, letting $a\to 0$, $c_j\to 0$ and $d_j\to\infty$, keeping b_j and a/c_j fixed, so that $e\to 0$, one recovers (2) after a change of variables.

In view of its close relation to Warnaar's determinant, one may hope that Theorem 1 will also find applications to multiple elliptic hypergeometric series. However, so far we have not found any interesting results in that direction. Perhaps Theorem 1 serves to indicate that when encountering a determinant that cannot be evaluated in closed form, one should keep in mind that it may still satisfy some, potentially useful, transformation.

2. Proof of Theorem 1

Let $X=(X_{jk})_{j,k=1}^n$ and $Y=(Y_{jk})_{j,k=1}^n$ be the matrices

$$X_{jk}=\frac{(b_j,c_j,d_j)_{k-1}}{(a/b_j,a/c_j,a/d_j)_{k-1}}, \tag{6}$$

$$Y_{jk}=\frac{\theta(aq^{2j-3})}{\theta(aq^{-1})}\frac{(aq^{-1},q^{1-k},eq^{k-2})_{j-1}}{(q,aq^{k-1},aq^{2-k}/e)_{j-1}}\,q^{j-1}.$$

Note that Y is triangular, with determinant

$$\begin{aligned}\det(Y)&=\prod_{j=1}^n Y_{jj}=\prod_{j=2}^n\frac{\theta(aq^{2j-3})}{\theta(aq^{-1})}\frac{(aq^{-1},q^{1-j},eq^{j-2})_{j-1}}{(q,aq^{j-1},aq^{2-j}/e)_{j-1}}\,q^{j-1}\\&=\left(\frac{e}{a}\right)^{\binom{n}{2}}\prod_{j=2}^n\frac{(a,eq^{j-2})_{j-1}}{(aq^{j-2},e/a)_{j-1}},\end{aligned} \tag{7}$$

where we used the elementary identity

$$\frac{(x)_{j-1}}{(y)_{j-1}}=\left(\frac{x}{y}\right)^{j-1}\frac{(q^{2-j}/x)_{j-1}}{(q^{2-j}/y)_{j-1}} \tag{8}$$

in the last step. Moreover,

$$(XY)_{jk}=\sum_{l=0}^{n-1}X_{j,l+1}Y_{l+1,k}=\sum_{l=0}^{k-1}\frac{\theta(aq^{2l-1})}{\theta(aq^{-1})}\frac{(aq^{-1},q^{1-k},b_j,c_j,d_j,eq^{k-2})_l}{(q,aq^{k-1},a/b_j,a/c_j,a/d_j,aq^{2-k}/e)_l}\,q^l.$$

Assuming (5), the elliptic Jackson summation (3) gives

$$(XY)_{jk}=\frac{(a,a/b_jc_j,a/b_jd_j,a/c_jd_j)_{k-1}}{(e/a,a/b_j,a/c_j,a/d_j)_{k-1}}. \tag{9}$$

Writing out the equation $\det(X)=\det(XY)/\det(Y)$ using (6), (7) and (9) we arrive at (4).

3. An S_3 symmetry

Besides the non-trivial symmetry of Theorem 1, determinants of the form

$$\det_{1\le j,k\le n}\left(\frac{(b_j,c_j,d_j)_{k-1}}{(a/b_j,a/c_j,a/d_j)_{k-1}}\right),\qquad b_jc_jd_j\text{ independent of }j,$$

also have a trivial symmetry. Namely, reversing the order of the columns, we have

$$\begin{aligned}\det_{1\le j,k\le n}\left(\frac{(b_j,c_j,d_j)_{k-1}}{(a/b_j,a/c_j,a/d_j)_{k-1}}\right)&=(-1)^{\binom{n}{2}}\det_{1\le j,k\le n}\left(\frac{(b_j,c_j,d_j)_{n-k}}{(a/b_j,a/c_j,a/d_j)_{n-k}}\right)\\&=(-1)^{\binom{n}{2}}\prod_{j=1}^n\frac{(b_j,c_j,d_j)_{n-1}}{(a/b_j,a/c_j,a/d_j)_{n-1}}\\&\qquad\times\det_{1\le j,k\le n}\left(\frac{(q^{n-k}a/b_j,q^{n-k}a/c_j,q^{n-k}a/d_j)_{k-1}}{(q^{n-k}b_j,q^{n-k}c_j,q^{n-k}d_j)_{k-1}}\right).\end{aligned}$$

Using (8) and introducing the parameter e as in (5) gives

$$\det_{1\le j,k\le n}\left(\frac{(b_j,c_j,d_j)_{k-1}}{(a/b_j,a/c_j,a/d_j)_{k-1}}\right)=\left(-\frac{e^2}{a}\right)^{\binom{n}{2}}\prod_{j=1}^{n}\frac{(b_j,c_j,d_j)_{n-1}}{(a/b_j,a/c_j,a/d_j)_{n-1}}$$
$$\times\det_{1\le j,k\le n}\left(\frac{(q^{2-n}b_j/a,q^{2-n}c_j/a,q^{2-n}d_j/a)_{k-1}}{(q^{2-n}/b_j,q^{2-n}/c_j,q^{2-n}/d_j)_{k-1}}\right).\tag{10}$$

Denoting by σ and τ the transformation from the left-hand to the right-hand side of (4) and (10), respectively, one may check that $\sigma^2=\tau^2=(\sigma\tau)^3=\mathrm{id}$, that is, σ and τ generate an S_3 symmetry. Thus, there are three additional expressions, corresponding to $\sigma\tau$, $\tau\sigma$ and $\sigma\tau\sigma=\tau\sigma\tau$. These may be written, respectively, as

$$\begin{aligned}\det_{1\le j,k\le n}&\left(\frac{(b_j,c_j,d_j)_{k-1}}{(a/b_j,a/c_j,a/d_j)_{k-1}}\right)\\
&=q^{-6\binom{n}{3}}\left(\frac{e}{a^2}\right)^{\binom{n}{2}}\prod_{j=1}^{n}\frac{(b_j,c_j,d_j)_{n-1}}{(a/b_j,a/c_j,a/d_j)_{n-1}}\prod_{j=2}^{n}\frac{(aq^{j-2})_{j-1}}{(q^{j-n}e/a)_{j-1}}\\
&\quad\times\det_{1\le j,k\le n}\left(\frac{(a/b_jc_j,a/b_jd_j,a/c_jd_j)_{k-1}}{(q^{2-n}/b_j,q^{2-n}/c_j,q^{2-n}/d_j)_{k-1}}\right)\\
&=\left(-\frac{a^3}{e^2}\right)^{\binom{n}{2}}\prod_{j=1}^{n}\frac{(a/b_jc_j,a/b_jd_j,a/c_jd_j)_{n-1}}{(a/b_j,a/c_j,a/d_j)_{n-1}}\prod_{j=2}^{n}\frac{(aq^{j-2})_{j-1}}{(eq^{j-2})_{j-1}}\\
&\quad\times\det_{1\le j,k\le n}\left(\frac{(q^{2-n}b_j/a,q^{2-n}c_j/a,q^{2-n}d_j/a)_{k-1}}{(q^{2-n}b_jc_j/a,q^{2-n}b_jd_j/a,q^{2-n}c_jd_j/a)_{k-1}}\right)\\
&=q^{-6\binom{n}{3}}\left(\frac{a^2}{e^3}\right)^{\binom{n}{2}}\prod_{j=1}^{n}\frac{(a/b_jc_j,a/b_jd_j,a/c_jd_j)_{n-1}}{(a/b_j,a/c_j,a/d_j)_{n-1}}\prod_{j=2}^{n}\frac{(aq^{j-2})_{j-1}}{(q^{j-n}a/e)_{j-1}}\\
&\quad\times\det_{1\le j,k\le n}\left(\frac{(b_j,c_j,d_j)_{k-1}}{(q^{2-n}b_jc_j/a,q^{2-n}b_jd_j/a,q^{2-n}c_jd_j/a)_{k-1}}\right).\end{aligned}\tag{11}$$

To verify this, the elementary identities

$$\prod_{j=2}^{n}(aq^{j-2})_{j-1}=\prod_{j=2}^{n}(aq^{2n-2j})_{j-1}=(-a)^{\binom{n}{2}}q^{3\binom{n}{3}}\prod_{j=2}^{n}(q^{2-2n+j}/a)_{j-1}$$

are useful. Although each of the three transformations in (11) are equivalent to Theorem 1, it may be worthwhile to state them explicitly.

Finally, we explain how to obtain Theorem 1 from the results of [RS1]. In that paper (the elliptic extension of Corollary 7.3 contained in Theorem 8.1), we obtained the identity

$$\begin{aligned}
&\sum_{k_1,\dots,k_n=0}^{m_1,\dots,m_n} \prod_{1\le i<j\le n} q^{k_i}\theta(q^{k_j-k_i})\theta(aq^{k_i+k_j}) \\
&\qquad\times \prod_{j=1}^n \frac{\theta(aq^{2k_j})}{\theta(a)} \frac{(a,b,c_j,d_j,e_j,q^{-m_j})_{k_j}}{(q,aq/b,aq/c_j,aq/d_j,aq/e_j,aq^{1+m_j})_{k_j}}\, q^{k_j} \\
&= b^{-\binom{n}{2}} q^{-2\binom{n}{3}} \prod_{j=1}^n \frac{(aq^{2-n}/b)_{n-1}(b)_{j-1}}{(aq^{2+n-2j}/b)_{n-1}} \frac{(aq,aq/c_jd_j,aq/c_je_j,aq/d_je_j)_{m_j}}{(aq/c_j,aq/d_j,aq/e_j,aq/c_jd_je_j)_{m_j}} \\
&\qquad\times \det_{1\le j,k\le n}\left(\frac{(c_j,d_j,e_j,q^{-m_j})_{k-1}}{(aq^{2-n}/bc_j,aq^{2-n}/bd_j,aq^{2-n}/be_j,aq^{2-n+m_j}/b)_{k-1}} \right), \qquad (12)
\end{aligned}$$

where $bc_jd_je_j = a^2q^{2-n+m_j}$ for $j=1,\dots,n$. Note that the case $n=1$ is (3). Consider the special case when $m_i = n-1$ for each i. Then, the terms in the sum vanish unless $(k_1,\dots,k_n)$ is a permutation of $(0,1,\dots,n-1)$. Moreover, since

$$\prod_{1\le i<j\le n} q^{k_i}\theta(q^{k_j-k_i}) = \operatorname{sgn}(k) \prod_{0\le i<j\le n-1} q^i\theta(q^{j-i}),$$

the sum can be written as a determinant. After some elementary computation and a change of variables, one is reduced to the equality between the first and last member of (11). Thus, Theorem 1 can also be obtained as a special case of (12).

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