

Tyurin parameters of commuting pairs and infinite dimensional Grassmann manifold

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Abstract

Commuting pairs of ordinary differential operators are classified by a set of algebro-geometric data called “algebraic spectral data”. These data consist of an algebraic curve (“spectral curve”) Γ with a marked point γ_∞ , a holomorphic vector bundle E on Γ and some additional data related to the local structure of Γ and E in a neighborhood of γ_∞ . If the rank r of E is greater than 1, one can use the so called “Tyurin parameters” in place of E itself. The Tyurin parameters specify the pole structure of a basis of joint eigenfunctions of the commuting pair. These data can be translated to the language of an infinite dimensional Grassmann manifold. This leads to a dynamical system of the standard exponential flows on the Grassmann manifold, in which the role of Tyurin parameters and some other parameters is made clear.

1 Introduction

My lecture at the workshop was focussed on the Landau-Lifshitz (LL) equation. This equation is a typical soliton equation whose Lax formalism is based on an elliptic spectral parameter. My main concern is to understand this kind of equations in Sato’s (and Segal and Wilson’s) Grassmannian perspectives of soliton equations [16, 18]. Although a huge number of soliton equations have been shown to fall into this universal picture, most of them are equations with a rational spectral parameter. As regards the LL equation, such a Grassmannian approach has been achieved by Carey et al. [2]. In my lecture, I reviewed a slightly different approach of myself [19]. Since the contents of the lecture overlaps with my contribution to the proceedings of a Faro workshop [21], I will not repeat it here.

Another interesting class of material in this context can be found in Krichever’s recent work [9] on the construction of Lax (and zero-curvature) equations defined on an arbitrary compact Riemann surface. These equations have the so called “Tyurin parameters” among dynamical variables. Tyurin parameters are known in algebraic geometry as parameters of deformations of (generic) holomorphic vector bundles on an algebraic curve [22]. Krichever’s Lax equations are thus related to deformations of holomorphic vector bundles; in contrast, the LL equation is associated with a rigid (though nontrivial) bundle. I examined a very simple example of Krichever’s construction, and found that the Grassmannian perspective is valid for this case as well [20]. This result, too, is reviewed in the contribution to the Faro workshop (*loc. cit.*).

In the present article, I consider a more classical case, namely, commuting pairs (or commutative rings) of ordinary differential operators and the associated special solutions of the KP hierarchy. As elucidated in the studies in the 1970’s [3, 4, 5, 6, 7, 8, 13, 23], such commuting pairs are classified by a set of algebro-geometric data (“algebraic spectral data”). These data consist of an algebraic curve (“spectral curve”) Γ , a holomorphic vector bundle E on Γ , and some other additional data. Sect. 2–5 of this article are devoted to

a review of this subject. The nature of problem drastically changes as the rank r of E exceeds 1. The case of $r = 1$ reduces to Jacobi's inversion problem, and can be solved explicitly by the classical theory of theta functions and Abelian integrals (powered by the use of Baker-Akhiezer functions) [5]. Lacking a similar theory for vector bundles, the case of $r > 1$ gets much harder. To formulate a vector version of the inversion problem, Krichever and Novikov [6, 7, 8] employed the notion of Tyurin parameters (also referred to as "matrix divisors"; see Sect. 4 and 5). Previato and Wilson [14] translated the work of Krichever and Novikov to the language of an infinite dimensional Grassmann manifold (see Sect. 6). As I pointed out in the previous work [20], their usage of the Grassmann manifold is slightly different from the usual interpretation of soliton equations [16, 18]. The goal of this article (see Sect. 7) is to show how to interpret this case in the usual Grassmannian perspective.

2 Spectral curve

The study of commuting pairs dates back to the beginning of the twentieth century. Of particular importance are the pioneering works of Schur [17] and Burchnell and Chaundy [1]; see Mulase's review [12] for a rather detailed historical account as well as a modern, scheme-theoretical interpretation of this issue.

Although the work of Burchnell and Chaundy was done some twenty years after Schur's work, let us first recall their work. They pointed out that any commuting (i.e., $[P, Q] = 0$) pair (Q, P) of ordinary differential operators

$$\begin{aligned} Q &= \partial_x^m + u_2(x)\partial_x^{m-2} + \cdots + u_m(x), \\ P &= \partial_x^n + v_2(x)\partial_x^{n-2} + \cdots + v_n(x), \end{aligned}$$

satisfy a polynomial relation

$$F(Q, P) = 0 \tag{1}$$

with constant coefficients — a fact that had been known for a few special cases. This implies that the eigenvalues of the joint eigenvalue problem

$$Q\psi = z\psi, \quad P\psi = w\psi$$

satisfy the algebraic relation

$$F(z, w) = 0. \tag{2}$$

Roughly speaking, this equation defines the spectral curve.

Schur's standpoint is more abstract and, in a sense, closer to the modern approach to this subject. He considered the subring

$$\mathcal{A}_Q = \{A \in \mathcal{D} \mid [A, Q] = 0\} \tag{3}$$

of commutants of Q in the noncommutative ring \mathcal{D} of ordinary differential operators, and observed that \mathcal{A}_Q is a commutative ring. From this point of view, the commuting pair

is nothing but generators of \mathcal{A}_Q . The commutative subring $\mathcal{A}_Q \subset \mathcal{D}$ is a more intrinsic notion than the commuting pair (Q, P) . In the language of modern algebraic geometry, the spectral curve is nothing but the spectrum $\text{Spec}\mathcal{A}_P$ — an amusing coincidence of the usage of the word “spectrum”. It is quite easy to define the rank r in terms of \mathcal{A}_Q : r is the greatest common divisor of the orders of all operators in \mathcal{A}_Q .

Krichever [6] defined the rank in terms of commuting pairs. Let us review his definition and its implications. The definition consist of several steps.

1. The first step is to consider the action of P on the space of solutions of the ordinary differential equation $Q\psi = z\psi$. This equation has an m -tuple $\varphi_k = \varphi_k(x, x_0, z)$, $k = 0, \dots, m - 1$, of linearly independent solutions that are normalized by the initial conditions $\partial_x^j \varphi_k|_{x=x_0} = \delta_{jk}$, $k = 0, \dots, m - 1$, at a reference point x_0 . If $[P, Q] = 0$, the space of solutions of $Q\psi = z\psi$ is invariant under the action of P , so that there is an $m \times m$ matrix $M(x_0, z)$ such that

$$(P\varphi_0, \dots, P\varphi_{m-1}) = (\varphi_0, \dots, \varphi_{m-1})M(x_0, z). \tag{4}$$

More explicitly,

$$M(x_0, z) = (\partial_x^j P\varphi_k|_{x=x_0})_{j,k=0,\dots,m-1}. \tag{5}$$

The matrix elements of $M(x_0, z)$ are entire functions of z . The aforementioned polynomial $F(z, w)$ is given by the characteristic polynomial

$$F(z, w) = \det(wI - M(x_0, z)). \tag{6}$$

On the other hand, there is another $m \times m$ matrix $V(x, z) = (v_{jk}(x, z))$ such that

$$\partial_x^j P\psi = \sum_{k=0}^{m-1} v_{jk}(x, z)\partial_x^k \psi \tag{7}$$

holds for any solution of $Q\psi = z\psi$. The coefficients $v_{jk}(x, z)$ can be determined by division of differential operators, which shows that they are polynomials in z . If one applies the defining relation of $v_{jk}(x, z)$ to φ_j 's and set $x = x_0$, one readily finds that

$$V(x_0, z) = M(x_0, z). \tag{8}$$

Thus $F(z, w)$ turns out to be a polynomial in both z and w .

2. The second step is to introduce a power series solution (the so called “formal Baker-Akhiezer function”)

$$\hat{\psi}(x, x_0, \lambda) = (1 + \sum_{\ell=1}^{\infty} \phi_{\ell}(x, x_0)\lambda^{-\ell})e^{(x-x_0)\lambda} \tag{9}$$

of $Q\psi = z\psi$ under the initial condition $\hat{\psi}(x_0, x_0, \lambda) = 1$. The parameter λ is related to z as

$$z = \lambda^m. \tag{10}$$

The action of P on $\hat{\psi}(x, x_0, \lambda)$ defines a Laurent series $p(\lambda) = \lambda^n + \dots$ with constant coefficients as

$$P\hat{\psi}(x, x_0, \lambda) = p(\lambda)\hat{\psi}(x, x_0, \lambda). \tag{11}$$

Replacing $\lambda \rightarrow e^{2\pi ik/m}\lambda$ yields

$$P\hat{\psi}(x, x_0, e^{2\pi ik/m}\lambda) = p(e^{2\pi ik/m}\lambda)\hat{\psi}(x, x_0, e^{2\pi ik/m}\lambda).$$

Thus we have an m -tuple of solutions $\hat{\psi}(x, x_0, e^{2\pi ik/m}\lambda)$, $k = 0, \dots, m - 1$, to the equation $Q\psi = z\psi$ on which the action of P diagonalizes. Consequently, the characteristic polynomial $F(z, w)$ of $M(x_0, z)$ factorizes as

$$F(z, w) = \prod_{j=0}^{m-1} (w - p(e^{2\pi ij/m}\lambda)). \tag{12}$$

Note that this is an equality that holds in a neighborhood of $z = \infty$.

3. Let \tilde{m} be the smallest positive integer for which $p(e^{2\pi ik/m}\lambda)$, $k = 1, 2, \dots$, returns to $p(\lambda)$, i.e., $p(e^{2\pi ik/m}\lambda) \neq p(\lambda)$ for $k = 1, \dots, \tilde{m} - 1$ and $p(e^{2\pi i\tilde{m}/m}\lambda) = p(\lambda)$. The rank r is now defined by

$$r = m/\tilde{m}. \tag{13}$$

The Laurent series $p(\lambda)$ can be written as

$$p(\lambda) = \tilde{p}(\lambda^r) \tag{14}$$

with another Laurent series $\tilde{p}(\kappa) = \kappa^{\tilde{n}} + \dots$ ($\tilde{n} = n/r$) of

$$\kappa = \lambda^r. \tag{15}$$

Since $p(e^{2\pi ik/m}\lambda)$ is \tilde{m} -periodic with respect to k , the foregoing (local) expression of $F(z, w)$ factorizes as

$$F(z, w) = f(z, w)^r, \tag{16}$$

where

$$f(z, w) = \prod_{j=0}^{\tilde{m}-1} (w - \tilde{p}(e^{2\pi ij/\tilde{m}}\lambda^r)). \tag{17}$$

By construction, $f(z, w)$ is single-valued in a neighborhood of $z = \infty$, hence becomes a polynomial in z as well.

The equation $F(z, w) = 0$ thus turns out to be reducible. We define the (affine) spectral curve by the reduced equation

$$f(z, w) = 0. \tag{18}$$

Since the branches of the solutions of $f(z, w) = 0$ in a neighborhood of $z = \infty$ are parameterized as

$$(z, w) = (\kappa^{\tilde{m}}, \tilde{p}(e^{2\pi ik/\tilde{m}}\kappa)), \quad k = 0, \dots, \tilde{m} - 1,$$

this curve can be compactified by adding a point γ_∞ at infinity, κ^{-1} being a local coordinate in a neighborhood of γ_∞ . Let Γ denote the compactified spectral curve, and Γ_0 the affine part $\Gamma \setminus \{\gamma_\infty\}$.

3 Holomorphic vector bundle

Let $\varphi = \varphi(x, x_0, z)$ denote the row vector

$$\varphi = (\varphi_0, \dots, \varphi_{m-1})$$

of the aforementioned fundamental solutions of $Q\psi = z\psi$. If \mathbf{c} is an eigenvector of $M(x_0, z)$ with eigenvalue w , $\psi = \varphi\mathbf{c}$ gives a joint eigenfunction with spectrum (z, w) . The fact that $F(z, w)$ factorizes to $f(z, w)^r$ means that each eigenvalue of $M(x_0, z)$ is r -fold degenerate and that the eigenspace is r -fold degenerate. If we choose a basis $\mathbf{c}_0, \dots, \mathbf{c}_{r-1}$ of the eigenspace of $M(x_0, z)$, the associated joint eigenfunctions $\psi_k = \varphi\mathbf{c}_k$, $k = 0, \dots, r - 1$, form a basis of the space of joint eigenfunctions

$$E_{(z,w)} = \{\psi \mid Q\psi = z\psi, P\psi = w\psi\}. \tag{19}$$

Putting this vector space at each point (z, w) , we obtain a holomorphic vector bundle E_0 of rank r on the affine part Γ_0 of the spectral curve.¹

This bundle E_0 can be extended to a holomorphic vector bundle E on the compactified spectral curve Γ . This is the place where we find a final piece of geometric data, namely, the choice of local trivialization of E in a neighborhood of γ_∞ [14]. In the case of $r = 1$, this part of the geometric data is less important (even negligible). In the case of $r > 1$, in contrast, the choice of local trivialization of E plays a substantial role, as one can see in the work of Li and Mulase [10, 11, 12].

Actually, Krichever and Novikov [6, 7, 8] introduce an alternative set of data here. This data consist of $r - 1$ functions $w_2(x), \dots, w_r(x)$ (hence absent if $r = 1$), and related to the asymptotic behavior of the joint eigenfunctions ψ_k as $z \rightarrow \infty$. Let us normalize these joint eigenfunctions by the initial conditions

$$\partial_x^j \psi_k|_{x=x_0} = \delta_{jk}, \quad k = 0, \dots, r - 1. \tag{20}$$

As $z \rightarrow \infty$, these joint eigenfunctions behave as

$$(\psi_0, \dots, \psi_{r-1}) = (\xi_0, \dots, \xi_{r-1})\Psi_0, \tag{21}$$

where ξ_0, \dots, ξ_{r-1} are Laurent series of κ of the form

$$\xi_k = \delta_{k,0} + \sum_{\ell=1}^{\infty} \xi_{k\ell} \kappa^{-\ell}, \tag{22}$$

and $\Psi_0 = \Psi_0(x, x_0, \kappa)$ is a solution of the matrix differential equation

$$\partial_x \Psi_0 = A_0 \Psi_0 \tag{23}$$

¹Actually, the situation gets complicated when the equation $f(z, w) = 0$ for w has a multiple root, namely, when (z, w) is a branch point of the covering of $\pi : \Gamma_0 \rightarrow \mathbf{CP}^1$, $\pi(z, w) = z$. A careful analysis shows that the joint eigenfunctions persists to be holomorphic functions in a neighborhood of those branch points, so that E_0 is indeed a holomorphic vector bundle over the whole Γ_0 .

with the coefficient matrix

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ \kappa - w_r(x) & -w_{r-1}(x) & \cdots & -w_2(x) & 0 \end{pmatrix}$$

normalized by the initial condition

$$\Psi_0|_{x=x_0} = I. \tag{24}$$

Of course this matrix system is equivalent to the scalar equation

$$(\partial_x^r + w_2(x)\partial_x^{r-2} + \cdots + w_r(x))\psi = 0. \tag{25}$$

The entries of the first row of Ψ_0 are a set of fundamental solutions of this equation. The $r - 1$ functions $w_2(x), \dots, w_r(x)$ are the final data that Krichever and Novikov use in their approach to commuting pairs of differential operators.

We can now extend the bundle E_0 over Γ_0 to a bundle E over Γ using Ψ_0 as the transition function. ξ_k 's are interpreted as a basis of holomorphic sections of E in a neighborhood of γ_∞ , hence determines local trivialization of E therein.

4 Tyurin parameters

The normalized joint eigenvectors ψ_k are expressed as $\psi_k = \boldsymbol{\varphi} \mathbf{c}_k$, where $\mathbf{c}_k = (c_{jk})_{j=0, \dots, m-1}$ are eigenvectors of $M(x_0, z)$ normalized as

$$c_{jk} = \delta_{jk}, \quad j, k = 0, \dots, r - 1. \tag{26}$$

These normalization conditions uniquely determine the eigenvectors \mathbf{c}_k , which thereby become meromorphic functions $\mathbf{c}_k(x_0, \gamma)$ of $\gamma = (z, w)$ on Γ_0 . Let $\gamma_s, s = 1, \dots, N$, denote the poles of \mathbf{c}_k 's. Since the components of $\boldsymbol{\varphi}$ are entire functions of z , the joint eigenfunctions ψ_k , too, are meromorphic functions $\psi_k(x, x_0, \gamma)$ of γ on Γ_0 with poles at $\gamma_s, s = 1, \dots, N$.

According to Krichever [6], the multiplicity m_s of these poles γ_s satisfy the equality

$$\sum_{s=1}^N m_s = rg. \tag{27}$$

In a generic situation, these poles are all simple (i.e., $m_s = 1$) so that ψ_k 's have rg simple poles $\gamma_1, \dots, \gamma_{rg}$ in addition to an essential singularity at γ_∞ . In the following, we assume this generic situation.

In the case of $r = 1$ where E is a line bundle, the joint eigenfunction is nothing but the usual Baker-Akhiezer function [5]. Such a scalar Baker-Akhezer function is uniquely determined by the asymptotic behavior in a neighborhood of γ_∞ and the position of the

g poles $\gamma_1, \dots, \gamma_g$ or, rather, by the divisor $\gamma_1 + \dots + \gamma_g$. This divisor, in turn, determines the line bundle E .

If $r > 1$, the divisor $\gamma_1 + \dots + \gamma_{rg}$ is not enough to specify the vector bundle E . To overcome this difficulty, Krichever and Novikov employ the notion of “matrix divisors” (i.e., Tyurin parameters) in the sense of Tyurin [22]. Let $\boldsymbol{\psi}$ denote the row vector

$$\boldsymbol{\psi}(x) = (\psi_0, \dots, \psi_{r-1})$$

of the joint eigenfunctions. This vector-valued meromorphic function on Γ_0 (“vector Baker-Akhiezer function” in the terminology of Krichever and Novikov [7, 8]) have simple poles at $\gamma_1, \dots, \gamma_{rg}$. The Tyurin parameters $(\gamma_s, \boldsymbol{\alpha}_s)$, $s = 1, \dots, rg$, consist of the position of these poles γ_s and the complex ray $\boldsymbol{\alpha}_s \in \mathbf{P}^{r-1}$ determined by the residue of $\boldsymbol{\psi}$ at γ_s .² One can normalize the directional vectors $\boldsymbol{\alpha}_s$ as

$$\boldsymbol{\alpha}_s = (\alpha_{s,0}, \dots, \alpha_{s,r-2}, 1).$$

$\boldsymbol{\alpha}_s$ arises in the local expression of $\boldsymbol{\psi}$ in a neighborhood of γ_s as

$$\boldsymbol{\psi} = \frac{\beta_s \boldsymbol{\alpha}_s}{z - z(\gamma_s)} + O(1), \tag{28}$$

where $z(\gamma_s)$ denotes the z -coordinate of γ_s , and β_s is a scalar factor.

We have thus obtained the algebraic spectral data

$$\Sigma = (\Gamma, \gamma_\infty, \kappa, (\gamma_s, \boldsymbol{\alpha}_s)_{s=1}^g, (w_j(x))_{j=2}^r)$$

of the commuting pair (Q, P) . These are an analogue of the “scattering data” in the inverse scattering problem. The inverse problem reduces to a kind of Riemann-Hilbert problem, namely, to find from Σ a vector-valued analytic function $\boldsymbol{\psi}$ on Γ that has simple poles at $\gamma_1, \dots, \gamma_{rg}$ and essential singularity at γ_∞ , and behaves as (21) and (28) in a neighborhood of these singular points. Krichever [6] solved this problem by the standard method of integral equation with a Cauchy kernel on Γ . Previato and Wilson [14] reformulated Krichever’s method in the language of an infinite dimensional Grassmann manifold.

5 Another set of Tyurin parameters

Alongside the foregoing Tyurin parameters, there is another set of Tyurin parameters that amounts to the divisor of *zeros* of the usual Baker-Akhiezer function. To distinguish between these two sets of parameters, let $(\gamma_s(x_0), \boldsymbol{\alpha}_s(x_0))$, $s = 1, \dots, rg$, denote the previous Tyurin parameters (because they depend on the reference point x_0), and $(\gamma_s(x), \boldsymbol{\alpha}_s(x))$, $s = 1, \dots, rg$, the second set of Tyurin parameters. As the notation implies, they do depend on x and coalesce to $(\gamma_s(x_0), \boldsymbol{\alpha}_s(x_0))$ as $x \rightarrow x_0$.

²Precisely speaking, this is slightly different from the usage of Tyurin parameters in algebraic geometry. [22]. Namely, $(\gamma_s, \boldsymbol{\alpha}_s)$ ’s are Tyurin parameters of the dual bundle E^* of E [14].

To introduce the second set of Tyurin parameters, let us consider the Wronskian matrix

$$\Psi = \begin{pmatrix} \boldsymbol{\psi} \\ \partial_x \boldsymbol{\psi} \\ \vdots \\ \partial_x^{r-1} \boldsymbol{\psi} \end{pmatrix} = \begin{pmatrix} \psi_0 & \cdots & \psi_{r-1} \\ \partial_x \psi_0 & \cdots & \partial_x \psi_{r-1} \\ \vdots & \cdots & \vdots \\ \partial_x^{r-1} \psi_0 & \cdots & \partial_x^{r-1} \psi_{r-1} \end{pmatrix} \tag{29}$$

of ψ_k 's. By (21), this matrix-valued function can be expressed as

$$\Psi = \Xi \Psi_0(x, x_0, \kappa), \quad \Xi = \sum_{\ell=0}^{\infty} \Xi_\ell \kappa^{-\ell}, \tag{30}$$

in a neighborhood of γ_∞ . The leading coefficient Ξ_0 is a lower triangular matrix whose diagonal elements are all equal to 1. Moreover, Ψ_0 is unimodular (i.e., $\det \Psi_0 = 1$) because the coefficient matrix of the differential equation for Ψ_0 is trace-free. Consequently, $\det \Psi$ is no longer singular at γ_∞ but behaves as $\det \Psi = 1 + O(\kappa^{-1})$. Thus $\det \Psi$ turns out to be a meromorphic function on the whole spectral curve Γ .

$\det \Psi$ has poles at $\gamma_1(x_0), \dots, \gamma_{rg}(x_0)$ and is holomorphic elsewhere. Since $\boldsymbol{\psi}$ behaves as

$$\boldsymbol{\psi} = \frac{\boldsymbol{\beta}_s(x) \boldsymbol{\alpha}_s(x_0)}{z - z(\gamma_s(x_0))} + O(1)$$

in a neighborhood of $\gamma_s(x_0)$, the residue of the Wronskian matrix Ψ turns out to be a rank-one matrix:

$$\Psi = \frac{{}^t \boldsymbol{\beta}_s(x) \boldsymbol{\alpha}_s(x_0)}{z - z(\gamma_s(x_0))} + O(1), \tag{31}$$

where $\boldsymbol{\beta}_s(x)$ is a vector-valued function. This implies that $\det \Psi$ has at most a simple pole at $\gamma_s(x_0)$. Since $\det \Psi \rightarrow 1$ as $x \rightarrow x_0$, the zeroes of $\det \Psi$ coalesce to the poles $\gamma_1(x_0), \dots, \gamma_{rg}(x_0)$. Thus $\det \Psi$ turns out to have, generically, rg simple zeroes $\gamma_s(x)$ that tends to $\gamma_s(x_0)$ as $x \rightarrow x_0$.

The second set of Tyurin parameters consist of the pairs $(\gamma_s(x), \boldsymbol{\alpha}_s(x))$, $s = 1, \dots, rg$. Since $\det \Psi$ has a zero at $\gamma_s(x)$, Ψ^{-1} has a pole there. If the rank of the residue matrix is greater than 1, $\det \Psi^{-1}$ has a multiple pole there — a contradiction. Thus the residue of Ψ^{-1} , too, turns out to be a rank-one matrix:

$$\Psi^{-1} = \frac{{}^t \tilde{\boldsymbol{\beta}}_s(x) \boldsymbol{\alpha}_s(x)}{z - z(\gamma_s(x))} + O(1). \tag{32}$$

When Krichever and Novikov [7, 8] introduced these x -dependent Tyurin parameters, they derived these parameters from the coefficient matrix $A = A(x, x_0, \gamma)$ of the matrix differential equation

$$\partial_x \Psi = A \Psi \tag{33}$$

satisfied by Ψ . Because of the Wronskian structure, A becomes a matrix of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_r & -a_{r-1} & \cdots & -a_2 & -a_1 \end{pmatrix},$$

where a_j are meromorphic functions $a_j(x, x_0, \gamma)$ of $\gamma \in \Gamma$ with poles at $\gamma_1(x), \dots, \gamma_{rg}(x)$ and γ_∞ . Note that the matrix differential equation is equivalent to the scalar equations

$$(\partial_x^r + a_1 \partial_x^{r-1} + \cdots + a_r) \psi_k = 0 \tag{34}$$

for the joint eigenfunctions. As Previato and Wilson pointed out [15], the scalar differential operator $G = \partial_x^r + a_1 \partial_x^{r-1} + \cdots + a_r$ is nothing but the (noncommutative) greatest common divisor of $Q - z$ and $P - w$, which can be calculated by Euclidean division of ordinary differential operators.

These x -dependent Tyurin parameters $(\gamma_s(x), \alpha_s(x))$, $s = 1, \dots, rg$, play the role of dynamical variables. If Q and P obey the time evolutions

$$\frac{\partial Q}{\partial t_k} = [B_k, Q], \quad \frac{\partial P}{\partial t_k} = [B_k, P], \quad B_k = (Q^{k/m})_+, \tag{35}$$

of the KP hierarchy, the Tyurin parameters also depend on the time variables $t = (t_1, t_2, \dots)$; let $(\gamma_s(x, t), \alpha_s(x, t))$, $s = 1, \dots, rg$, denote those time-dependent Tyurin parameters. Following Krichever and Novikov [7, 8], one can reformulate the KP hierarchy to zero-curvature equations

$$\begin{aligned} [\partial_{t_k} - A_k(x, t, \gamma), \partial_x - A(x, t, \gamma)] &= 0, \\ [\partial_{t_j} - A_j(x, t, \gamma), \partial_{t_k} - A_k(x, t, \gamma)] &= 0 \end{aligned} \tag{36}$$

for $r \times r$ matrices $A(x, t, \gamma)$ and $A_k(x, t, \gamma)$ of meromorphic functions on Γ . These matrices have poles at $\gamma_1(x, t), \dots, \gamma_{rg}(x, t)$ and γ_∞ , and are holomorphic elsewhere. The poles of $A_k(x, t, \gamma)$ at $\gamma_s(x, t)$'s are simple, and exhibit the same rank-one structure as Ψ^{-1} . The zero-curvature equations are accompanied by a set of linear differential equations

$$\partial_x \Psi = A(x, t, \gamma) \Psi, \quad \partial_{t_k} \Psi = A_k(x, t, \gamma) \Psi - \Psi M_k(t, \gamma), \tag{37}$$

where $M_k(t, \gamma)$ are matrices that do not depend on x (but can depend on t). One can eliminate these extra terms by the right gauge transformation $\Psi \rightarrow \Psi C(t, \gamma)$ with a suitable matrix $C(t, \gamma)$ (details are omitted here).

6 Infinite dimensional Grassmann manifold

Although one can formulate the subsequent results in the functional analytic framework of Segal and Wilson [18] as well, let us use the algebraic or complex analytic language of Sato and Sato [16].

An algebraic model the relevant Grassmann manifold is based on the vector space

$$V^{\text{alg}} = \mathbf{C}((\kappa))^{\oplus r} \simeq \mathbf{C}((\lambda))$$

of r -component row vectors $\mathbf{f} = (f_0(\kappa), \dots, f_{r-1}(\kappa))$ of formal Laurent series of κ . The isomorphism to $\mathbf{C}((\lambda))$ is given by the mapping

$$(f_0(\kappa), \dots, f_{r-1}(\kappa)) \longmapsto \sum_{j=0}^{r-1} f_j(\lambda^r) \lambda^j.$$

Let V_-^{alg} denote the vector subspace

$$V_-^{\text{alg}} = \kappa^{-1} \mathbf{C}[[\kappa^{-1}]]^{\oplus r} \simeq \lambda^{-1} \mathbf{C}[[\lambda^{-1}]].$$

The Grassmann manifold $\text{Gr}(V^{\text{alg}})$ consists of closed (with respect to a topology of V^{alg}) vector subspaces $W \subset V^{\text{alg}}$ such that

$$\dim \text{Ker}(W \rightarrow V^{\text{alg}}/V_-^{\text{alg}}) = \dim \text{Coker}(W \rightarrow V^{\text{alg}}/V_-^{\text{alg}}) < \infty,$$

where $W \rightarrow V^{\text{alg}}/V_-^{\text{alg}}$ denote the composition of the inclusion $W \hookrightarrow V^{\text{alg}}$ and the projection $V^{\text{alg}} \rightarrow V^{\text{alg}}/V_-^{\text{alg}}$.

Actually, we need an *analytic* version of this model. The analytic model is based on the vector space V of vector-valued Laurent series

$$\mathbf{f} = (f_0, \dots, f_{r-1}) = \sum_{\ell=-\infty}^{\infty} \mathbf{f}_\ell \kappa^\ell \tag{38}$$

that converge in a neighborhood (which may depend on \mathbf{f}) of $\kappa = \infty$ except at the center. The Grassmann manifold $\text{Gr}(V)$ is defined in the same manner as the algebraic model except that V^{alg} is replaced by $V_- = V^{\text{alg}} \cap V$. Namely, a point of $\text{Gr}(V)$ is represented by a closed vector subspace $W \subset V$ such that

$$\dim \text{Ker}(W \rightarrow V/V_-) = \dim \text{Coker}(W \rightarrow V/V_-) < \infty. \tag{39}$$

Let $\text{Gr}^\circ(V)$ denote the so called “big cell”, namely, the subset that consist of W ’s for which

$$W \simeq V/V_-. \tag{40}$$

Such a subspace has a basis $\{\mathbf{w}_{n,j} \mid n = 0, 1, 2, \dots, j = 0, \dots, r - 1\}$ of the form

$$\mathbf{w}_{n,j} = \mathbf{e}_j \kappa^n + O(\kappa^{-1}), \tag{41}$$

where $\mathbf{e}_j, j = 0, \dots, r - 1$, denote the standard basis of \mathbf{C}^r : $\mathbf{e}_j = (\delta_{jk})_{k=0, \dots, r-1}$.

7 Dressed vacua in Grassmann manifold

Following Previato and Wilson [14], we now consider a special point $W_0(\boldsymbol{\gamma}, \boldsymbol{\alpha})$ of $\text{Gr}^\circ(V)$ for a given set of (constant) Tyurin parameters $(\boldsymbol{\gamma}, \boldsymbol{\alpha}) = (\gamma_s, \boldsymbol{\alpha}_s)_{s=1}^{rg}$. $W_0(\boldsymbol{\gamma}, \boldsymbol{\alpha})$ consists of r -tuples $\mathbf{f} = (f_0, \dots, f_{r-1})$ as follows:

1. $f_s = f_s(\gamma)$, $s = 0, \dots, r - 1$, are meromorphic functions on the affine part Γ_0 of the spectral curve with possible poles at $\gamma_1, \dots, \gamma_{rg}$. \mathbf{f} is identified with an elements of V by Laurent expansion at γ_∞ . It is here that the local geometric data (γ_∞, κ) play a role.
2. All poles at γ_s 's are simple. As $\gamma \rightarrow \gamma_s$, \mathbf{f} behaves as

$$\mathbf{f} = \frac{\beta_s \boldsymbol{\alpha}_s}{z - z(\gamma_s)} + O(1), \tag{42}$$

where β_s is a constant scalar.

One can show, by the Riemann-Roch theorem, that $W_0(\boldsymbol{\gamma}, \boldsymbol{\alpha})$ has a basis $\{\mathbf{w}_{n,j} \mid n = 0, 1, 2, \dots, j = 0, \dots, r - 1\}$ of the form mentioned above.

Previato and Wilson use this special point of $\text{Gr}^\circ(V)$ to reformulate the inverse construction of Krichever and Novikov. Of course, $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$ are identified with the Tyurin parameters in the algebraic spectral data. The functional data w_2, \dots, w_r are encoded into the matrix-valued function Ψ_0 . To combine these data, Previato and Wilson consider the subspace

$$W = W_0(\boldsymbol{\gamma}, \boldsymbol{\alpha})\Psi_0^{-1} \tag{43}$$

of V . This is a Grassmannian version of “dressing” that is commonly used in the theory of integrable systems; $W_0(\boldsymbol{\gamma}, \boldsymbol{\alpha})$ plays the role of “vacuum”. A clue here is the fact that $\text{Gr}^\circ(V)$ is an open subset of $\text{Gr}(V)$. Because of this fact, W remains in $\text{Gr}^\circ(V)$ as far as Ψ_0 is sufficiently close to the unit matrix (this is indeed the case if x is close to x_0 , because Ψ_0 satisfies the initial condition $\Psi_0|_{x=x_0} = I$). If W indeed belongs to $\text{Gr}^\circ(V)$, then one obtains an element $\boldsymbol{\psi}$ of W as the inverse image of \mathbf{e}_0 by the isomorphism $W \rightarrow V/V_-$. This $\boldsymbol{\psi}$ gives a solution to the Riemann-Hilbert problem in Krichever and Novikov’s inverse construction.

Our usage of $W_0(\boldsymbol{\gamma}, \boldsymbol{\alpha})$ is similar, but conceptually different. Namely, in place of Ψ_0 , we take a matrix Φ of Laurent series of the form

$$\Phi = \sum_{\ell=0}^{\infty} \Phi_\ell \kappa^{-\ell}, \tag{44}$$

where Φ_0 is a lower triangular matrix whose diagonal elements are equal to 1, and “dress” $W_0(\boldsymbol{\gamma}, \boldsymbol{\alpha})$ with Φ as

$$W = W_0(\boldsymbol{\gamma}, \boldsymbol{\alpha})\Phi. \tag{45}$$

We use this matrix Φ to encode the data of local trivialization of E in a neighborhood of γ_∞ .

The information of local trivialization of E is carried by the formal Baker-Akhiezer function $\hat{\psi} = \hat{\psi}(x, x_0, \lambda)$ as well as $w_2(x), \dots, w_r(x)$. To extract it, we split $\hat{\psi}$ into an r -tuple $\hat{\psi}_j = \hat{\psi}_j(x, x_0, \kappa)$, $j = 0, \dots, r-1$, of power series of κ as

$$\hat{\psi} = \sum_{j=0}^{r-1} \hat{\psi}_j(x, x_0, \kappa) \lambda^j \quad (46)$$

and construct the Wronskian matrix

$$\hat{\Psi} = \begin{pmatrix} \hat{\psi}_0 & \cdots & \hat{\psi}_{r-1} \\ \partial_x \hat{\psi}_0 & \cdots & \partial_x \hat{\psi}_{r-1} \\ \vdots & \cdots & \vdots \\ \partial_x^{r-1} \hat{\psi}_0 & \cdots & \partial_x^{r-1} \hat{\psi}_{r-1} \end{pmatrix}. \quad (47)$$

By construction, $\hat{\Psi}$ is a matrix-valued Laurent series of the form

$$\hat{\Psi} = \Phi \exp((x - x_0)\Lambda(\kappa)), \quad (48)$$

where

$$\Lambda(\kappa) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \kappa & & & 0 \end{pmatrix}.$$

The first factor $\Phi = \Phi(x, x_0, \kappa)$ on the right hand side is a matrix-valued Laurent series of the form mentioned above. We use this Φ to “dress” the “vacuum” $W_0(\gamma, \alpha)$.

8 Dynamical system on Grassmann manifold

We are now in a position to formulate our Grassmannian perspective of rank- r commuting pairs (and the associated solutions of the KP hierarchy).

To this end, we start from the fact that $\Psi = \Psi(x, x_0, \gamma)$ and $\hat{\Psi} = \hat{\Psi}(x, x_0, \kappa)$ satisfy the same differential equation

$$\partial_x \Psi = A\Psi, \quad \partial_x \hat{\Psi} = A\hat{\Psi}.$$

(This is a consequence of the fact that both ψ_j and $\hat{\psi}$ are joint eigenfunctions of Q and P .) This implies that the “matrix ratio” $\Psi^{-1}\hat{\Psi}$ is independent of x , which is equal to the initial value $\hat{\Psi}(x_0, x_0, \kappa)$. (Note that the initial value of $\hat{\Psi}$ can differ from the unit matrix, though the initial value of $\hat{\psi}$ is equal to 1.) Thus we have the fundamental relation

$$\Psi(x, x_0, \gamma)^{-1} \hat{\Psi}(x, x_0, \kappa) = \hat{\Psi}(x_0, x_0, \kappa) \quad (49)$$

or, equivalently,

$$\Psi(x, x_0, \gamma)^{-1} \Phi(x, x_0, \kappa) \exp((x - x_0)\Lambda(\kappa)) = \Phi(x_0, x_0, \kappa) \quad (50)$$

in terms of the matrix $\Phi = \Phi(x, x_0, \kappa)$ introduced in the end of the last section.

Ψ and its inverse Ψ^{-1} have the analytic properties mentioned in Sect. 3 and 4. Namely, Ψ have poles at $\gamma_s(x_0)$ and γ_∞ with the rank-one structure of the residue matrices specified by $\alpha_s(x_0)$; Ψ^{-1} have similar properties with $\gamma_s(x_0)$ and $\alpha_s(x_0)$ replaced by $\gamma_s(x)$ and $\alpha_s(x)$. Having these analytic properties of Ψ , we can follow a reasoning in the previous paper [20] to deduce that Ψ intertwines two “vacua” as

$$W_0(\gamma(x_0), \alpha(x_0)) = W_0(\gamma(x), \alpha(x))\Psi(x, x_0, \gamma). \quad (51)$$

Let us show an outline of the proof. A clue is the fact that $\alpha_s(x)$ is a left null vector of Ψ at the degeneration point $\gamma_s(x)$:

$$\alpha_s(x)\Psi(x, x_0, \gamma_s(x)) = \mathbf{0}. \quad (52)$$

Because of this, the poles of each element \mathbf{f} of $W_0(\gamma(x), \alpha(x))$ at $\gamma_s(x)$ disappear when multiplied by Ψ . Since Ψ itself has simple poles at $\gamma_s(x_0)$, $\mathbf{f}\Psi$ in turn develops simple poles therein. The rank-one structure of the residue matrices of Ψ is inherited by $\mathbf{f}\Psi$.

(50) and (51) imply that

$$\begin{aligned} & W_0(\gamma(x), \alpha(x))\Phi(x, x_0, \kappa) \\ &= W_0(\gamma(x_0), \alpha(x_0))\Phi(x_0, x_0, \kappa) \exp(-(x - x_0)\Lambda(\kappa)). \end{aligned}$$

It will be more impressive to rewrite $\Phi(x, x_0, \kappa)$ and $\Phi(x_0, x_0, \kappa)$ as $\Phi(x, \kappa)$ and $\Phi(x_0, \kappa)$. The foregoing relation thereby reads

$$\begin{aligned} & W_0(\gamma(x), \alpha(x))\Phi(x, \kappa) \\ &= W_0(\gamma(x_0), \alpha(x_0))\Phi(x_0, \kappa) \exp(-(x - x_0)\Lambda(\kappa)), \end{aligned}$$

which means that the “dressed vacuum”

$$W(x) = W_0(\gamma(x), \alpha(x))\Phi(x, \kappa)$$

obeys the exponential law

$$W(x) = W(x_0) \exp(-(x - x_0)\Lambda(\kappa)) \quad (53)$$

just like the usual Grassmannian perspective of soliton equations.

A similar result holds for the time evolutions

$$\begin{aligned} \gamma_s(x) &\rightarrow \gamma_s(x, t), & \alpha_s(x) &\rightarrow \alpha_s(x, t), \\ \Psi(x, x_0, \gamma) &\rightarrow \Psi(x, t, x_0, \gamma), & \hat{\Psi}(x, x_0, \kappa) &\rightarrow \hat{\Psi}(x, t, x_0, \kappa) \end{aligned}$$

induced by the KP hierarchy. Note that this includes the foregoing result, because the t_1 -flow can be identified with the x -flow. We omit details and show the final result.

Theorem 1 i) $\Psi = \Psi(x, t, x_0, \gamma)$ intertwines two “vacua” as

$$W_0(\gamma(x_0, 0), \alpha(x_0, 0)) = W_0(\gamma(x, t), \alpha(x, t))\Psi(x, t, x_0, \gamma). \quad (54)$$

ii) *The dressed vacuum*

$$W(x, t) = W_0(\boldsymbol{\gamma}(x, t), \boldsymbol{\alpha}(x, t))\Phi(x, t, x_0, \kappa)$$

obeys the exponential law

$$W(x, t) = W(x_0, 0) \exp(-(x - x_0)\Lambda(\kappa) - \sum_{\ell=1}^{\infty} t_{\ell}\Lambda(\kappa)^{\ell}). \quad (55)$$

Let us compare this result with the approach of Li and Mulase to commutative rings of ordinary differential operators [10, 11, 12]. $W_0(\boldsymbol{\gamma}, \boldsymbol{\alpha})$ is invariant under the action of the affine coordinate ring \mathcal{A} (which is isomorphic to \mathcal{A}_Q) of the spectral curve, i.e.,

$$\mathcal{A}W_0(\boldsymbol{\gamma}, \boldsymbol{\alpha}) \subseteq W_0(\boldsymbol{\gamma}, \boldsymbol{\alpha}). \quad (56)$$

The same holds for $W(x)$ and $W(x, t)$. In other words, the pair (\mathcal{A}, W) of \mathcal{A} and these subspaces W of V is a ‘‘Schur pair’’ in the terminology of Li and Mulase. In our case, the subspace W is parametrized more explicitly as $W = W_0(\boldsymbol{\gamma}, \boldsymbol{\alpha})\Phi$. Our result shows what the KP flows look like in terms of the parameters γ_s, α_s , etc. in this expression.

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