HEUN EQUATION AND PAINLEVÉ EQUATION

KOUICHI TAKEMURA

Abstract. We relate two parameter solutions of the sixth Painlevé equation and finite-gap solutions of the Heun equation by considering monodromy on a certain class of Fuchsian differential equations. In the appendix, we present formulae on differentials of elliptic modular functions, and obtain the ellitic form of the sixth Painlevé equation directly.

1. Introduction

In this paper we make a study on two differential equations. One is the Heun equation, and the other is the sixth Painlevé equation.

Heun's differential equation (or the Heun equation) is a differential equation given by

$$\left(\left(\frac{d}{dw}\right)^2+\left(\frac{\gamma}{w}+\frac{\delta}{w-1}+\frac{\epsilon}{w-t}\right)\frac{d}{dw}+\frac{\alpha\beta w-q}{w(w-1)(w-t)}\right)\tilde{f}(w)=0 \tag{1.1}$$

with the condition

$$\gamma+\delta+\epsilon=\alpha+\beta+1. \tag{1.2}$$

The Heun equation is the standard canonical form of a Fuchsian equation with four singularities. It is well known that the Fuchsian equation with three singularities is the hypergeometric differential equation.

In the 1980's, Treibich and Verdier [16] found that the Heun equation is related with the theory of the finite-gap potential, and several others have produced more precise statements and concerned results on this subject. Namely, integral representations of solutions, global monodromy in terms of hyperelliptic integrals and the Hermite-Krichever Ansatz for the case $\gamma, \delta, \epsilon, \alpha-\beta \in \mathbb{Z}+\frac{1}{2}$ are investigated (see [1, 2, 9, 11–14] etc.).

The sixth Painlevé equation is a non-linear ordinary differential equation written as

$$\begin{aligned}\frac{d^2\lambda}{dt^2}=&\frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}\\&+\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left\{\frac{\kappa_\infty^2}{2}-\frac{\kappa_0^2}{2}\frac{t}{\lambda^2}+\frac{\kappa_1^2}{2}\frac{(t-1)}{(\lambda-1)^2}+\frac{(1-\kappa_t^2)}{2}\frac{t(t-1)}{(\lambda-t)^2}\right\}.\end{aligned} \tag{1.3}$$

A remarkable property of this differential equation is that its solutions do not have movable singularities other than poles. Although generic solutions of the sixth Painlevé equation are trancedental, it may have classical solutions for special cases. If $\kappa_0=\kappa_1=\kappa_t=\kappa_\infty=0$, then Eq.(1.3) has two parameter solutions called Picard's solution [7], and if $\kappa_0=\kappa_1=\kappa_t=\kappa_\infty=1/2$, then Eq.(1.3) has two parameter solutions called Hitchin's solution [3].

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In this paper we investigate a family of solutions to the sixth Painlevé equation including Hitchin's solutions by applying Hermite-Krichever Ansatz which is used to study the Heun equation in [14]. More precisely, we develop the Hermite-Krichever Ansatz for a certain class of Fuchsian differential equations which include the linear differential equation that produce the sixth Painlevé equation by monodromy preserving deformation. By considering monodromy preserving deformation for the solutions to the linear differential equation, we obtain solutions to the sixth Painlevé equation including Hitchin's solutions.

This paper is organized as follows. In section 2, we obtain integral representations of solutions to a certain class of Fuchsian differential equations and rewrite them to the form of the Hermite-Krichever Ansatz. In section 3, we apply the results in section 2 for the Heun equation. In section 4, we show that the solutions to the linear differential equations considered in section 2 produce two parameter solutions to the sixth Painlevé equation by monodromy preserving deformation. Some explicit solutions that include Hitchin's solution are displayed. In section 5, we give concluding remarks and present an open problem. In the appendix, we present formulae on differentials of elliptic modular functions, and obtain the ellitic form of the sixth Painlevé equation directly.

2. Fuchsian differential equation and Hermite-Krichever Ansatz

In this section, we consider differential equations which have additional apparent singularities to the Heun equation. More precisely, we consider the equation

(2.1)
$$\left\{ \frac{d^2}{dw^2} + \left(\frac{\frac{1}{2}-l_1}{w} + \frac{\frac{1}{2}-l_2}{w-1} + \frac{\frac{1}{2}-l_3}{w-t} + \sum_{i'=1}^{M} \frac{-r_{i'}}{w-\tilde{b}_{i'}} \right) \frac{d}{dw} + \frac{(\sum_{i=0}^{3} l_i + \sum_{i'=1}^{M} r_{i'})(-1-l_0+\sum_{i=1}^{3} l_i + \sum_{i'=1}^{M} r_{i'})w + \tilde{p} + \sum_{i'=1}^{M} \frac{\tilde{o}_{i'}}{w-\tilde{b}_{i'}}}{4w(w-1)(w-t)} \right\} \tilde{f}(w) = 0,$$

for the case $l_i \in \mathbb{Z}_{\geq 0}$ $(0 \leq i \leq 3)$, $r_{i'} \in \mathbb{Z}_{>0}$ $(1 \leq i' \leq M)$ and the regular singular points $\tilde{b}_{i'}$ $(1 \leq i' \leq M)$ are apparent. Here, a regular singular point $x = a$ of a linear differential equation of order two is said to be apparent, if and only if the differential equation does not have a logarithmic solution at $x = a$ and the exponents at $x = a$ are integers.

Let $\wp(x)$ be the Weierstrass $\wp$-function with periods $(2\omega_1, 2\omega_3)$. We set $\omega_0 = 0$, $\omega_1 = 1/2$ $\omega_3 = \tau/2$, $\omega_2 = -\omega_1 - \omega_3$ and $e_i = \wp(\omega_i)$ $(i = 1, 2, 3)$. It is known that, if $t \neq 0, 1, \infty$, then there exists a value $\tau \in \mathbb{R} + \sqrt{-1}\mathbb{R}_{>0}$ such that $t = (e_3 - e_1)/(e_2 - e_1)$. By a certain transformation, Eq.(2.1) is rewritten in terms of elliptic functions such as

$$(H_g - \tilde{E}) f_g(x) = 0, \tag{2.2}$$

where

$$(2.3)\qquad H_g = -\frac{d^2}{dx^2} + \sum_{i'=1}^{M} \frac{r_{i'}\wp'(x)}{\wp(x)-\wp(\delta_{i'})}\frac{d}{dx} + \left(l_0 + \sum_{i'=1}^{M} r_{i'}\right)\left(l_0 + 1 - \sum_{i'=1}^{M} r_{i'}\right)\wp(x) + \sum_{i=1}^{3} l_i(l_i+1)\wp(x+\omega_i) + \sum_{i'=1}^{M} \frac{\tilde{s}_{i'}}{\wp(x)-\wp(\delta_{i'})},$$

$$(2.4)\qquad \wp(\delta_{i'}) = b_{i'}, \quad (i' = 1,\dots,M).$$

The parameter $\tilde{s}_{i'}$ $(i' = 1,\dots,M)$ corresponds to the parameter $\tilde{o}_{i'}$, and the parameter $\tilde{p}$ corresponds to $\tilde{E}$. Apparency of the singularity at $w = \pm\delta_{i'}$ on Eq.(2.2) inherits from apparency of the singularity at $w = b_{i'}$ on Eq.(2.1).

We now review the propositions on solutions to Eq.(2.2) obtained in [15]. The first one is an integral representation of solutions in terms of elliptic functions. We set

$$\Psi_g(x) = \prod_{i'=1}^{M} (\wp(x) - \wp(\delta_{i'}))^{r_{i'}/2}.$$

Proposition 2.1. *[15] Assume that $l_0,\dots,l_3 \in \mathbb{Z}_{\geq 0}$, $r_1,\dots,r_k \in \mathbb{Z}_{\geq 1}$, and the regular singular points $\{b_1,\dots,b_k\}$ are apparent. Then there exists an even doubly-periodic function $\Xi(x)$ and a value Q such that*

$$(2.5)\qquad \Lambda_g(x) = \Psi_g(x)\sqrt{\Xi(x)}\exp\int \frac{\sqrt{-Q}dx}{\Xi(x)}$$

is a solution to the differential equation (2.2).

For the constructions of $\Xi(x)$ and Q, see [15]

We now show that a solution to Eq.(2.2) can be expressed in the form of the Hermite-Krichever Ansatz. In our situation, the Hermite-Krichever Ansatz asserts that the differential equation has solutions that are expressed as a finite series in the derivatives of an elliptic Baker-Akhiezer function, multiplied by an exponential function. We set

$$(2.6)\qquad \Phi_i(x,\alpha) = \frac{\sigma(x+\omega_i-\alpha)}{\sigma(x+\omega_i)}\exp(\zeta(\alpha)x), \qquad (i = 0,1,2,3),$$

where $\sigma(x)$ (resp. $\zeta(x)$) is the Weierstrass sigma (resp. zeta) function. Then we have

$$(2.7)\qquad \left(\frac{d}{dx}\right)^j \Phi_i(x+2\omega_k,\alpha) = \exp(-2\eta_k\alpha + 2\omega_k\zeta(\alpha))\left(\frac{d}{dx}\right)^j \Phi_i(x,\alpha)$$

for $i = 0,1,2,3$, $j \in \mathbb{Z}_{\geq 0}$ and $k = 1,3$, where $\eta_k = \zeta(\omega_k)$ $(k = 1,3)$. The following proposition asserts that a solution to Eq.(2.2) is written in the form of the Hermite-Krichever Ansatz.

Proposition 2.2. *[15] (i) Set $\tilde{l}_0 = l_0 + \sum_{i'=1}^{M} r_{i'}$ and $\tilde{l}_i = l_i$ $(i = 1,2,3)$. The function $\Lambda_g(x)$ in Eq.(2.5) is expressed as*

$$(2.8)\qquad \Lambda_g(x) = \exp(\kappa x)\left(\sum_{i=0}^{3}\sum_{j=0}^{\tilde{l}_i-1} \tilde{b}_j^{(i)}\left(\frac{d}{dx}\right)^j \Phi_i(x,\alpha)\right)$$

for some values α, κ *and* $\tilde{b}_j^{(i)}$ $(i=0,\dots,3,\ j=0,\dots,\tilde{l}_i-1)$, *or*

$$\Lambda_g(x)=\exp(\bar{\kappa}x)p(x) \tag{2.9}$$

for some value $\bar{\kappa}$ *and doubly-periodic function* $p(x)$. *(For a detailed expression of* $p(x)$, *see [15].)*

(ii) If $l_0,\dots,l_3\in\mathbb{Z}_{\geq 0}$, $r_1,\dots,r_k\in\mathbb{Z}_{\geq 1}$, *and the regular singular points* $\{b_1,\dots,b_k\}$ *are apparent, then there exists a non-zero solution to Eq.(2.2) that is expressed as Eq.(2.8) or Eq.(2.9).*

The monodromy of the function $\Lambda_g(x)$ is expressed in terms of α and κ. In fact, if the function $\Lambda_g(x)$ is written as Eq.(2.8), then

$$\Lambda_g(x+2\omega_k)=\exp(-2\eta_k\alpha+2\omega_k\zeta(\alpha)+2\kappa\omega_k)\Lambda_g(x),\quad (k=1,3). \tag{2.10}$$

3. Heun equation

For the case $M=0$, Eq.(2.1) is regarded as the Heun equation (see Eq.(1.1)), and it is transformed to the equation

$$\left(-\frac{d^2}{dx^2}+\sum_{i=0}^{3}l_i(l_i+1)\wp(x+\omega_i)\right)f(x)=Ef(x), \tag{3.1}$$

If $l_0,l_1,l_2,l_3\in\mathbb{Z}_{\geq 0}$, then the function $\sum_{i=0}^{3}l_i(l_i+1)\wp(x+\omega_i)$ is called the Treibich-Verdier potential, and is an example of algebro-geometric finite-gap potential (see [2, 9, 13, 16]). For the case $M=0$ and $l_0,l_1,l_2,l_3\in\mathbb{Z}_{\geq 0}$, there is no constraint relation for the apparency of additional regular singularity. Hence Propositions 2.1 and 2.2 holds true. The function $\Xi(x)$ in Proposition 2.1 is written as

$$\Xi(x)=c_0(E)+\sum_{i=0}^{3}\sum_{j=0}^{l_i-1}b_j^{(i)}(E)\wp(x+\omega_i)^{l_i-j}, \tag{3.2}$$

where the coefficients $c_0(E)$ and $b_j^{(i)}(E)$ are polynomials in E, they do not have common divisors and the polynomial $c_0(E)$ is monic. The value Q is expressed as

$$Q=\Xi(x)^2\left(E-\sum_{i=0}^{3}l_i(l_i+1)\wp(x+\omega_i)\right)+\frac{1}{2}\Xi(x)\frac{d^2\Xi(x)}{dx^2}-\frac{1}{4}\left(\frac{d\Xi(x)}{dx}\right)^2. \tag{3.3}$$

It follows from Eq.(3.1) that Q is independent of x, and it is a monic polynomial in E (see [11]). A solution to Eq.(3.1) is expressed by an integral (see Eq.(2.5)), and it is also expressed in a form of the Hermite-Krichever Ansatz (see Proposition 2.2). It is shown in [14] that the values $\wp(\alpha)$, $\wp'(\alpha)/\sqrt{-Q}$ and $\kappa/\sqrt{-Q}$ are expressed as rational functions in E, and it follows that global monodromy of the Heun equation for the case $l_0,l_1,l_2,l_3\in\mathbb{Z}_{\geq 0}$ is written as an elliptic integral. On the other hand, it is known that global monodromy is also expressed by a hyperelliptic integral (see [13]). By comparing the two expressions, we obtain a hyperelliptic-to-elliptic integral reduction formula (see [14]).

We expressed the functions appeared in this section for the case $l_0=2$, $l_1=l_2=l_3=0$. Note that Eq.(3.1) for the case $l_1=l_2=l_3=0$ is called the Lamé equation.

3.1. **The case $M = 0$, $l_0 = 2$, $l_1 = l_2 = l_3 = 0$.** The differential equation (see Eq.(3.1)) is written as

$$\left(-\frac{d^2}{dx^2} + 6\wp(x)\right) f(x) = Ef(x). \tag{3.4}$$

Set

$$\Xi(x) = 9\wp(x)^2 + 3E\wp(x) + E^2 - 9g_2/4, \quad Q = (E^2 - 3g_2)\prod_{i=1}^{3}(E - 3e_i), \tag{3.5}$$

where $g_2 = -4(e_1e_2 + e_2e_3 + e_3e_1)$. Then the function

$$\Lambda_g(x) = \sqrt{\Xi(x)} \exp \int \frac{\sqrt{-Q}dx}{\Xi(x)}, \tag{3.6}$$

is a solution to Eq.(3.4). The monodromy formula in terms of hyperelliptic integral is written as

$$\Lambda_g(x + 2\omega_k) = \Lambda_g(x) \exp\left(-\frac{1}{2}\int_{\sqrt{3g_2}}^{E} \frac{-6\eta_k\tilde{E} + 2\omega_k(\tilde{E}^2 - 3g_2/2)}{\sqrt{-(\tilde{E}^2 - 3g_2)\prod_{i=1}^{3}(\tilde{E} - 3e_i)}}d\tilde{E}\right) \tag{3.7}$$

for $k = 1, 3$.

The function $\Lambda_g(x)$ is also expressed in the form of the Hermite-Krichever Ansatz as

$$\Lambda_g(x) = \exp(\kappa x)\left\{\bar{b}_0^{(0)}\Phi_0(x,\alpha) + \bar{b}_1^{(0)}\frac{d}{dx}\Phi_0(x,\alpha)\right\} \tag{3.8}$$

for the case $E^2 \neq 3g_2$, and the values α and κ are determined as

$$\wp(\alpha) = -\frac{E^3 - 27g_3}{9(E^2 - 3g_2)}, \quad \kappa = \frac{2}{3}\sqrt{\frac{-(E - 3e_1)(E - 3e_2)(E - 3e_3)}{(E^2 - 3g_2)}}, \tag{3.9}$$

where $g_3 = 4e_1e_2e_3$. The monodromy is written by using the values α and κ (see Eq.(2.10)). By comparing two expressions of monodromy, we obtain that

$$\int_{\infty}^{\xi} \frac{d\tilde{\xi}}{\sqrt{4\tilde{\xi}^3 - g_2\tilde{\xi} - g_3}} = -\frac{3}{2}\int_{\infty}^{E} \frac{\tilde{E}d\tilde{E}}{\sqrt{-(\tilde{E}^2 - 3g_2)\prod_{i=1}^{3}(\tilde{E} - 3e_i)}}, \tag{3.10}$$

$$\kappa = -\frac{1}{2}\int_{e_i}^{E} \frac{\tilde{E}^2 - 3g_2/2}{\sqrt{-(\tilde{E}^2 - 3g_2)\prod_{i=1}^{3}(\tilde{E} - 3e_i)}}d\tilde{E} + \int_{3e_i}^{\xi} \frac{\tilde{\xi}d\tilde{\xi}}{\sqrt{4\tilde{\xi}^3 - g_2\tilde{\xi} - g_3}}, \tag{3.11}$$

$(i = 1, 2, 3)$ for the transformation

$$\xi = -\frac{E^3 - 27g_3}{9(E^2 - 3g_2)}. \tag{3.12}$$

These formulae reduce hyperelliptic integrals of genus two to elliptic integrals.

4. Sixth Painlevé equation

We consider the Fuchsian differential equation (2.1) for the case $M = 1$, $r_1 = 1$. Then Eq.(2.1) is transformed to

(4.1)
$$\left\{-\frac{d^2}{dx^2} + \frac{\wp'(x)}{\wp(x) - \wp(\delta_1)}\frac{d}{dx} + \frac{\tilde{s}_1}{\wp(x) - \wp(\delta_1)} + \sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i) - \tilde{E}\right\} f_g(x) = 0.$$

We set

$$b_1 = \wp(\delta_1), \quad \mu_1 = \frac{-\tilde{s}_1}{4b_1^3 - g_2 b_1 - g_3} + \sum_{i=1}^{3} \frac{l_i}{2(b_1 - e_i)}, \tag{4.2}$$

$$p = \tilde{E} - 2(l_1 l_2 e_3 + l_2 l_3 e_1 + l_3 l_1 e_2) + \sum_{i=1}^{3} l_i(l_i e_i + 2(e_i + b_1)). \tag{4.3}$$

The condition that, the regular singular points $x = \pm\delta_1$ is apparent, is written as

$$\begin{aligned} p = (4b_1^3 - g_2 b_1 - g_3)&\left\{-\mu_1^2 + \sum_{i=1}^{3} \frac{l_i + \frac{1}{2}}{b_1 - e_i}\mu_1\right\} \\ &- b_1(l_1 + l_2 + l_3 - l_0)(l_1 + l_2 + l_3 + l_0 + 1). \end{aligned} \tag{4.4}$$

¿From now on we assume that $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$ and the eigenvalue $\tilde{E}$ satisfies Eqs.(4.3, 4.4). Then Propositions 2.1 and 2.2 hold true. It is known [15] that the function $\Xi(x)$ in Proposition 2.1 is written as

$$\Xi(x) = c_0 + \frac{d_0}{(\wp(x) - \wp(\delta_1))} + \sum_{i=0}^{3} \sum_{j=0}^{l_i - 1} b_j^{(i)} \wp(x+\omega_i)^{l_i - j}. \tag{4.5}$$

Ratios of the coefficients c_0/d_0 and $b_j^{(i)}/d_0$ $(i = 0, 1, 2, 3,\ j = 0, \dots, l_i - 1)$ are written as rational functions in variables b_1 and μ_1. The value Q in Proposition 2.1 is expressed as a rational function in b_1 and μ_1 multiplied by d_0^2. By Proposition 2.2, the eigenfunction $\Lambda_g(x)$ in Eq.(2.5) is also expressed in the form of the Hermite-Krichever Ansatz. Namely, it is expressed as

$$\Lambda_g(x) = \exp(\kappa x)\left(\sum_{i=0}^{3} \sum_{j=0}^{\tilde{l}_i - 1} \tilde{b}_j^{(i)} \left(\frac{d}{dx}\right)^j \Phi_i(x, \alpha)\right) \tag{4.6}$$

or

$$\Lambda_g(x) = \exp(\bar{\kappa} x)\, p(x) \tag{4.7}$$

for some doubly-periodic function $p(x)$, where $l = l_0 + l_1 + l_2 + l_3 + 1$, $\tilde{l}_0 = l_0 + 1$ and $\tilde{l}_i = l_i$ $(i = 1, 2, 3)$. For the values α and κ, we have

Proposition 4.1. *[15] Assume that $M = 1$, $r_1 = 1$, $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$ and the value p satisfies Eq.(4.4). Let α and κ be the values determined by the Hermite-Krichever Ansatz (see Eq.(4.6)). Then $\wp(\alpha)$ is expressed as a rational function in variables b_1 and μ_1, $\wp'(\alpha)$*

is expressed as a product of $\sqrt{-Q}$ *and a rational function in variables* b_1 *and* μ_1*, and* κ *is expressed as a product of* $\sqrt{-Q}$ *and a rational function in variables* b_1 *and* μ_1*.*

If $\alpha \not\equiv 0 \pmod{2\omega_1\mathbb{Z} \oplus 2\omega_3\mathbb{Z}}$, then the function $\Lambda_g(x)$ is expressed as Eq.(4.6) and we have

$$\Lambda_g(x + 2\omega_k) = \exp(-2\eta_k\alpha + 2\omega_j\zeta(\alpha) + 2\kappa\omega_k)\Lambda_g(x), \quad (k = 1, 3). \tag{4.8}$$

We now discuss the relationship between the monodromy preserving deformation of Fuchsian equations and the sixth Painlevé equation. For this purpose we recall other expressions of the sixth Painlevé equation. The sixth Painlevé equation given as Eq.(1.3) is also written in terms of a Hamiltonian system by adding the variable μ, which is called the sixth Painlevé system:

$$\frac{d\lambda}{dt} = \frac{\partial H_{VI}}{\partial \mu}, \qquad \frac{d\mu}{dt} = -\frac{\partial H_{VI}}{\partial \lambda} \tag{4.9}$$

with the Hamiltonian

$$\begin{aligned} H_{VI} =& \frac{1}{t(t-1)} \left\{ \lambda(\lambda - 1)(\lambda - t)\mu^2 \right. \\ & \left. - \left\{ \kappa_0(\lambda - 1)(\lambda - t) + \kappa_1\lambda(\lambda - t) + (\kappa_t - 1)\lambda(\lambda - 1) \right\} \mu + \kappa(\lambda - t) \right\}, \end{aligned} \tag{4.10}$$

where $\kappa = ((\kappa_0 + \kappa_1 + \kappa_t - 1)^2 - \kappa_\infty^2)/4$. The sixth Painlevé equation for λ is obtained by eliminating μ in Eq.(4.9). Set $\omega_1 = 1/2$, $\omega_3 = \tau/2$ and write

$$t = \frac{e_3 - e_1}{e_2 - e_1}, \quad \lambda = \frac{\wp(\delta) - e_1}{e_2 - e_1}. \tag{4.11}$$

Then the sixth Painlevé equation is equivalent to the following equation (see [6, 10]):

$$\frac{d^2\delta}{d\tau^2} = -\frac{1}{4\pi^2} \left\{ \frac{\kappa_\infty^2}{2}\wp'(\delta) + \frac{\kappa_0^2}{2}\wp'\left(\delta + \frac{1}{2}\right) + \frac{\kappa_1^2}{2}\wp'\left(\delta + \frac{\tau + 1}{2}\right) + \frac{\kappa_t^2}{2}\wp'\left(\delta + \frac{\tau}{2}\right) \right\}, \tag{4.12}$$

where $\wp'(z) = (\partial/\partial z)\wp(z)$. In the appendix, we obtain the elliptic form of the sixth Painlevé equation (i.e., Eq.(4.12)) from the original sixth Painlevé equation (i.e., Eq.(1.3)).

It is widely known that the sixth Painlevé equation is obtained by the monodnomy preserving deformation of a certain linear differential equation. Let us introduce the following Fuchsian differential equation:

$$\frac{d^2y}{dw^2} + p_1(w)\frac{dy}{dw} + p_2(w)y = 0, \tag{4.13}$$

where

$$p_1(w) = \frac{1 - \kappa_0}{w} + \frac{1 - \kappa_1}{w - 1} + \frac{1 - \kappa_t}{w - t} - \frac{1}{w - \lambda}, \tag{4.14}$$

$$p_2(w) = \frac{\kappa}{w(w-1)} - \frac{t(t-1)H_{VI}}{w(w-1)(w-t)} + \frac{\lambda(\lambda - 1)\mu}{w(w-1)(w-\lambda)}. \tag{4.15}$$

This equation has five regular singular points $\{0, 1, t, \infty, \lambda\}$ and the exponents at $w = \lambda$ are 0 and 2. It follows from Eq.(4.10) that the regular singular point $w = \lambda$ is apparent. Then the sixth Painlevé equation is obtained by the monodromy preserving deformation of Eq.(4.9), i.e., the condition that the monodromy of Eq.(4.13) is preserved as deforming the variable t is equivalent to that μ and λ satisfy the Painlevé system (see Eq.(4.9)), provided $\kappa_0, \kappa_1, \kappa_t, \kappa_\infty \notin \mathbb{Z}$. For details, see [5].

We transform Eq.(4.13) into the form of Eq.(4.1). We set

$$w = \frac{\wp(x) - e_1}{e_2 - e_1}, \quad y = f_g(x) \prod_{i=1}^{3} (\wp(x) - e_i)^{l_i/2}, \tag{4.16}$$

$$t = \frac{e_3 - e_1}{e_2 - e_1}, \quad \lambda = \frac{b_1 - e_1}{e_2 - e_1}, \quad \wp(\delta_1) = b_1. \tag{4.17}$$

Then we obtain Eq.(4.1) by setting

$$\kappa_0 = l_1 + 1/2, \quad \kappa_1 = l_2 + 1/2, \quad \kappa_t = l_3 + 1/2, \quad \kappa_\infty = l_0 + 1/2, \tag{4.18}$$

$$\mu = (e_2 - e_1)\mu_1, \quad \kappa = (l_1 + l_2 + l_3 + l_0 + 1)(l_1 + l_2 + l_3 - l_0), \tag{4.19}$$

$$H_{VI} = \frac{1}{t(1-t)} \left\{ \frac{p + \kappa e_3}{e_2 - e_1} + \lambda(1-\lambda)\mu \right\}, \tag{4.20}$$

(see Eqs.(4.2–4.3)), and Eq.(4.10) is equivalent to Eq.(4.4), that means that the apparency of regular singularity is inheritted. Note that the monodromy preserving deformation of Eq.(4.13) in t corresponds to the monodromy preserving deformation of Eq.(4.1) in τ.

Now we consider the monodromy preserving deformation in the variable τ ($\omega_1 = 1/2, \omega_3 = \tau/2$) by applying solutions obtained by the Hermite-Krichever Ansatz for the case $l_i \in \mathbb{Z}_{\geq 0}$ ($i = 0, 1, 2, 3$). Let α and κ be values determined by the Hermite-Krichever Ansats (see Eq.(4.6)). We consider the case $Q \neq 0$. Then a basis for solutions to Eq.(2.2) is given by $\Lambda_g(x)$ and $\Lambda_g(-x)$, and the monodromy matrix with respect to the cycle $x \to x + 2\omega_k$ ($k = 1, 3$) is diagonal with the eigenvalues $\exp(\pm(-2\eta_k\alpha + 2\omega_k\zeta(\alpha) + 2\kappa\omega_k))$ (see Eq.(4.8)). The values $-2\eta_k\alpha + 2\omega_k\zeta(\alpha) + 2\kappa\omega_k$ ($k = 1, 3$) are preserved by the monodromy preserving deformation. We set

$$-2\eta_1\alpha + 2\omega_1\zeta(\alpha) + 2\kappa\omega_1 = \pi\sqrt{-1}C_1, \tag{4.21}$$

$$-2\eta_3\alpha + 2\omega_3\zeta(\alpha) + 2\kappa\omega_3 = \pi\sqrt{-1}C_3, \tag{4.22}$$

for contants C_1 and C_3. By Legendre's relation $\eta_1\omega_3 - \eta_3\omega_1 = \pi\sqrt{-1}/2$, we have

$$\alpha = C_3\omega_1 - C_1\omega_3, \tag{4.23}$$

$$\kappa = \zeta(C_1\omega_3 - C_3\omega_1) + C_3\eta_1 - C_1\eta_3, \tag{4.24}$$

¿From Proposition 4.1, the values $\wp(C_3\omega_1 - C_1\omega_3)(= \wp(\alpha))$, $\wp'(C_3\omega_1 - C_1\omega_3)/\sqrt{-Q}$ and $(\zeta(C_1\omega_3 - C_3\omega_1) + C_3\eta_1 - C_1\eta_3)/\sqrt{-Q}$ are expressed as rational functions in variables b_1 and μ_1. By solving these equations for b_1 and μ_1 and evaluating them into Eq.(4.1), the monodromy of the solutions to the differential equation (4.1) on the cycles $x \to x + 2\omega_k$ ($k = 1, 3$) are preserved for the fixed values C_1 and C_3. Thus we obtain the following proposition which was established in [15].

Proposition 4.2. *[15] We set $\omega_1 = 1/2$, $\omega_3 = \tau/2$ and assume that $l_i \in \mathbb{Z}_{\geq 0}$ ($i = 0, 1, 2, 3$) and $Q \neq 0$. By solving the equations in Proposition 4.1 in variable $b_1 = \wp(\delta_1)$ and μ_1, we express $\wp(\delta_1)$ and μ_1 in terms of $\wp(\alpha)$, $\wp'(\alpha)$ and κ, and we replace $\wp(\alpha)$, $\wp'(\alpha)$ and*

κ with $\wp(C_3\omega_1 - C_1\omega_3)$, $\wp'(C_3\omega_1 - C_1\omega_3)$ and $\zeta(C_1\omega_3 - C_3\omega_1) + C_3\eta_1 - C_1\eta_3$. Then δ_1 satisfies the sixth Painlevé equation in the elliptic form

$$\frac{d^2\delta_1}{d\tau^2} = -\frac{1}{8\pi^2}\left\{\sum_{i=0}^{3}(l_i + 1/2)^2\wp'(\delta_1 + \omega_i)\right\}. \tag{4.25}$$

We observe the expressions of b_1 and μ_1 in detail for the cases $l_0 = l_1 = l_2 = l_3 = 0$ and $l_0 = 1$, $l_1 = l_2 = l_3 = 0$.

4.1. **The case $M = 1$, $r_1 = 1$, $l_0 = l_1 = l_2 = l_3 = 0$.** We investigate the case $M = 1$, $r_1 = 1$, $l_0 = l_1 = l_2 = l_3 = 0$ in detail. The differential equation (4.1) is written as

$$\left\{-\frac{d^2}{dx^2} + \frac{\wp'(x)}{\wp(x) - b_1}\frac{d}{dx} - \frac{\mu_1(4b_1^3 - g_2 b_1 - g_3)}{\wp(x) - b_1} - p\right\} f_g(x) = 0, \tag{4.26}$$

We assume that $b_1 \neq e_1, e_2, e_3$. The condition that the regular singular points $x = \pm\delta_1$ ($\wp(\delta_1) = b_1$) are apparent is written as

$$p = -(4b_1^3 - g_2 b_1 - g_3)\mu_1^2 + (6b_1^2 - g_2/2)\mu_1 \tag{4.27}$$

(see Eq.(4.4)). The doubly-periodic function $\Xi(x)$ (see Eq.(4.5)) is calculated as

$$\Xi(x) = 2\mu_1 + \frac{1}{\wp(x) - b_1}. \tag{4.28}$$

The value Q is calculated as

$$Q = 2\mu_1(2\mu_1(e_1 - b_1) + 1)(2(e_2 - b_1)\mu_1 + 1)(2\mu_1(e_3 - b_1) + 1). \tag{4.29}$$

We set

$$\Lambda_g(x) = \sqrt{\Xi(x)(\wp(x) - b_1)}\exp\int\frac{\sqrt{-Q}dx}{\Xi(x)}. \tag{4.30}$$

Then a solution to Eq.(4.26) is written as $\Lambda_g(x)$, and is expressed in the form of the Hermite-Krichever Ansatz as

$$\Lambda_g(x) = \bar{b}_0^{(0)}\exp(\kappa x)\Phi_0(x, \alpha) \tag{4.31}$$

for generic (μ_1, b_1). The values α and κ are determined as

$$\wp(\alpha) = b_1 - \frac{1}{2\mu_1}, \quad \wp'(\alpha) = -\frac{\sqrt{-Q}}{2\mu_1^2}, \quad \kappa = \frac{\sqrt{-Q}}{2\mu_1}. \tag{4.32}$$

Hence we have

$$\mu_1 = -\frac{\kappa}{\wp'(\alpha)}, \quad b_1 = \wp(\alpha) - \frac{\wp'(\alpha)}{2\kappa}. \tag{4.33}$$

¿From Proposition 4.2, the function δ_1 determined by

$$\wp(\delta_1) = b_1 = \wp(C_1\omega_3 - C_3\omega_1) + \frac{\wp'(C_1\omega_3 - C_3\omega_1)}{2(\zeta(C_1\omega_3 - C_3\omega_1) - (C_1\eta_3 - C_3\eta_1))} \tag{4.34}$$

is a solution to the sixth Painlevé equation in the elliptic form (see Eq.(4.25)). This solution coincides with the one found by Hitchin [3].

Now we consider the case $Q = 0$. If $Q = 0$, then $\mu_1 = 0$ or $\mu_1 = 1/(2(b_1 - e_i))$ for some $i \in \{1, 2, 3\}$. For the case $\mu_1 = 0$, the function δ_1, which is determined by

$$\wp(\delta_1) = b_1 = -\frac{D_1\eta_3 - D_3\eta_1}{D_1\omega_3 - D_3\omega_1}, \tag{4.35}$$

is a solution to the sixth Painlevé equation for constants D_1 and D_3. For the case $\mu_1 = 1/(2(b_1 - e_i))$ ($i \in \{1, 2, 3\}$), the function δ_1 determined by

$$\wp(\delta_1) = b_1 = \frac{(g_2/4 - 2e_i^2)(D_1\omega_3 - D_3\omega_1) + e_i(D_1\eta_3 - D_3\eta_1)}{e_i(D_1\omega_3 - D_3\omega_1) + (D_1\eta_3 - D_3\eta_1)} \tag{4.36}$$

is a solution to the sixth Painlevé equation.

Eqs.(4.35 ,4.36) are also obtained by suitable limits from Eq.(4.34) (see [15]), and the space of the parameters of the solutions to the sixth Painlevé equation (i.e. the space of initial conditions) for the case $l_0 = l_1 = l_2 = l_3 = 0$ is obtained by blowing up four points on the surface $\mathbb{C}/2\mathbb{Z} \times \mathbb{C}/2\mathbb{Z}$. This reflects the $A_1 \times A_1 \times A_1 \times A_1$ structure of Riccati solutions by Saito and Terajima [8].

4.2. **The case $M = 1$, $r_1 = 1$, $l_0 = 1$, $l_1 = l_2 = l_3 = 0$.** The differential equation (4.1) for this case is written as

$$\left\{-\frac{d^2}{dx^2} + \frac{\wp'(x)}{\wp(x) - b_1}\frac{d}{dx} - \frac{\mu_1(4b_1^3 - g_2b_1 - g_3)}{\wp(x) - b_1} + 2\wp(x) - p\right\} f_g(x) = 0, \tag{4.37}$$

We assume that $b_1 \neq e_1, e_2, e_3$. The condition that the regular singular points $x = \pm\delta_1$ ($\wp(\delta_1) = b_1$) are apparent is written as

$$p = -(4b_1^3 - g_2b_1 - g_3)\mu_1^2 + (6b_1^2 - g_2/2)\mu_1 + 2b_1. \tag{4.38}$$

The doubly-periodic function $\Xi(x)$ is calculated as

$$\begin{aligned}\Xi(x) =& \wp(x) + ((-4b_1^3 + b_1g_2 + g_3)\mu_1^2 + (6b_1^2 - g_2/2)\mu_1 - b_1) \\ &+ ((-4b_1^3 + b_1g_2 + g_3)\mu_1/2 + 3b_1^2 - g_2/4)/(\wp(x) - b_1),\end{aligned} \tag{4.39}$$

and the value Q is calculated as

$$\begin{aligned}Q = &-((2(4b_1^3 - b_1g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4)(2(b_1^2 + e_1b_1 + e_2e_3)\mu_1 - 2b_1 - e_1) \\ &(2(b_1^2 + e_2b_1 + e_1e_2)\mu_1 - 2b_1 - e_2)(2(b_1^2 + e_3b_1 + e_1e_3)\mu_1 - 2b_1 - e_3).\end{aligned} \tag{4.40}$$

We set

$$\Lambda_g(x) = \sqrt{\Xi(x)(\wp(x) - b_1)}\exp\int\frac{\sqrt{-Q}dx}{\Xi(x)}. \tag{4.41}$$

Then a solution to Eq.(4.37) is written as $\Lambda_g(x)$, and it is expressed in the form of the Hermite-Krichever Ansatz as

$$\Lambda_g(x) = \exp(\kappa x)\left\{\bar{b}_0^{(0)}\Phi_0(x, \alpha) + \bar{b}_1^{(0)}\frac{d}{dx}\Phi_0(x, \alpha)\right\} \tag{4.42}$$

for generic (μ_1, b_1). The values α and κ are determined as

(4.43)
$$\begin{aligned}
\wp(\alpha) &= \frac{2(4b_1^3 - b_1 g_2 - g_3)b_1\mu_1^3 + (-24b_1^3 + 4g_2 b_1 + 3g_3)\mu_1^2 + (24b_1^2 - 2g_2)\mu_1 - 8b_1}{2(4b_1^3 - b_1 g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4}, \\
\wp'(\alpha) &= \frac{-4((4b_1^3 - b_1 g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 12b_1\mu_1 - 4)}{(2(4b_1^3 - b_1 g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4)^2}\sqrt{-Q}, \\
\kappa &= \frac{2\mu_1}{2(4b_1^3 - b_1 g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4}\sqrt{-Q}.
\end{aligned}$$

Hence we have

$$b_1 = \frac{2\wp(\alpha)\kappa^3 - 3\wp'(\alpha)\kappa^2 + (6\wp(\alpha)^2 - g_2)\kappa - \wp(\alpha)\wp'(\alpha)}{2(\kappa^3 - 3\wp(\alpha)\kappa + \wp'(\alpha))}, \tag{4.44}$$

$$\mu_1 = \frac{2(\kappa^3 - 3\wp(\alpha)\kappa + \wp'(\alpha))\kappa}{-2\wp'(\alpha)\kappa^3 + (12\wp(\alpha)^2 - g_2)\kappa^2 - 6\wp(\alpha)\wp'(\alpha)\kappa + \wp'(\alpha)^2}. \tag{4.45}$$

¿From Proposition 4.2, the function δ_1 determined by

(4.46) $\wp(\delta_1) = b_1 =$
$$\frac{2\wp(\omega)(\zeta(\omega) - \eta)^3 + 3\wp'(\omega)(\zeta(\omega) - \eta)^2 + (6\wp(\omega)^2 - g_2)(\zeta(\omega) - \eta) + \wp(\omega)\wp'(\omega)}{2((\zeta(\omega) - \eta)^3 - 3\wp(\omega)(\zeta(\omega) - \eta) - \wp'(\omega))},$$
$$(\omega = C_1\omega_3 - C_3\omega_1, \quad \eta = C_1\eta_3 - C_3\eta_1),$$

is a solution to the sixth Painlevé equation in the elliptic form (see Eq.(4.25)). In the sixth Painlevé equation, it is known that the case $(\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (1/2, 1/2, 1/2, 3/2)$ is linked to the case $(\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (1/2, 1/2, 1/2, 1/2)$ by Bäcklund transformation (see [17]). By transformating the solution in Eq.(4.34) of the case $(\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (1/2, 1/2, 1/2, 1/2)$ to the one of the case $(\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (1/2, 1/2, 1/2, 3/2)$, we recover the solution in Eq.(4.46).

Now we consider the case $Q = 0$. If $Q = 0$, then μ_1 is a solution to the equation $2(4b_1^3 - b_1 g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4 = 0$ or $\mu_1 = (2b_1 + e_i)/(2(b_1^2 + e_i b_1 + e_i^2 - g_2/4))$ for some $i \in \{1, 2, 3\}$. We set $\omega = D_1\omega_3 - D_3\omega_1$ and $\eta = D_1\eta_3 - D_3\eta_1$, where D_1 and D_3 are constants. For the case that μ_1 is a solution to the equation $2(4b_1^3 - b_1 g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4 = 0$, the corresponding solutions to the sixth Painlevé equation are written as the function δ_1, where

$$\wp(\delta_1) = b_1 = \frac{4\eta^3 + g_2\omega^2\eta - 2g_3\omega^3}{\omega(g_2\omega^2 - 12\eta^2)}. \tag{4.47}$$

For the case $\mu_1 = (2b_1 + e_i)/(2(b_1^2 + e_i b_1 + e_i^2 - g_2/4))$ $(i \in \{1, 2, 3\})$, we have

$$\wp(\delta_1) = b_1 = \frac{-g_2 e_i\omega/2 + (6e_i^2 - g_2)\eta}{(6e_i^2 - g_2)\omega - 6e_i\eta}. \tag{4.48}$$

5. Summary and concluding remarks

The Heun equation (see Eq.(1.1)) is the standard canonical form of a Fuchsian equation with four singularities. By transforming the Heun equation to the form of elliptic functions, we find that solving the Heun equation is equivalent to investigating spectral and eigenstates of quantum BC_1 Inozemtsev model (see Eq.(3.1)). Note that the Hamiltonian of the quantum BC_1 Inozemtsev model [4] is given by

$$H = -\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i). \tag{5.1}$$

On the other hand, by adding an apparent singularity to the Heun equation, we obtain Fuchsian differential equations that produce the sixth Painlevé equation by monodromy preserving deformation (see section 4).

The sixth Painlevé equation is rewritten as Eq.(4.12), and it is equivalent to the Hamiltonian system (see [10])

$$2\pi\sqrt{-1}\frac{d\delta}{d\tau} = \frac{\partial \mathcal{H}}{\partial \gamma}, \qquad 2\pi\sqrt{-1}\frac{d\gamma}{d\tau} = -\frac{\partial \mathcal{H}}{\partial \delta}, \tag{5.2}$$

$$\mathcal{H} = \frac{1}{2}\left(\gamma^2 - \sum_{i=0}^{3}(l_i+1/2)^2\wp(\delta+\omega_i)\right). \tag{5.3}$$

If we replace $2\pi\sqrt{-1}\frac{d}{d\tau}$ by $\frac{d}{ds}$ (s: time variable, independent of τ) formally, we obtain the classical BC_1 Inozemtsev system [4]. In other words, the sixth Painlevé equation is a non-autonomous version of the classical BC_1 Inozemtsev system. To summarize, we present the following diagram.

$$\begin{tikzcd}
\begin{array}{c}\text{Heun equation}\\ \text{(quamtum } BC_1 \text{ Inozemtsev model)}\end{array} \arrow[r, "\text{adding apparent singularity}"] \arrow[d, "\text{classical limit}"] & \begin{array}{c}\text{Fuchsian equation with}\\ \text{an apparent singularity}\end{array} \arrow[d, "\text{monodromy preserving deformation}"] \\
\begin{array}{c}\text{Classical } BC_1\\ \text{Inozemtsec model}\end{array} \arrow[r, "\text{non-autonomous version}"] & \text{Sixth Painlevé equation}
\end{tikzcd}$$

Before starting this work, the author noticed that the parameters, that the monodromy of solutions to the Heun equation have expressions in terms of elliptic or hyperelliptic integrals, resemble the ones in the sixth Painlevé equation that has explicit two-parameter solutions. Typical two-parameter solutions are Picard's and Hitchin's solutions. In this paper, we partially obtain an explanation of this phenomena by intermediating Fuchsian differential equations with an apparent singularity, though the corresponding parameters on the sixth Painlevé equation are a little off as O_1 and $O_1 \cup O_2$, where

$$O_1 = \left\{ (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) | \kappa_0, \kappa_1, \kappa_t, \kappa_\infty \in \mathbb{Z} + \frac{1}{2} \right\}, \tag{5.4}$$

$$O_2 = \left\{ (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \middle| \begin{array}{l} \kappa_0, \kappa_1, \kappa_t, \kappa_\infty \in \mathbb{Z} \\ \kappa_0 + \kappa_1 + \kappa_t + \kappa_\infty \in 2\mathbb{Z} \end{array} \right\}. \tag{5.5}$$

For the case $(\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \in O_1$, solutions of the linear differential equation are investigated by our method, and solutions of the sixth Painlevé equation follow from them (see Proposition 4.2). By Bäcklund transformation of the sixth Painlevé equation (see [17] etc.), Hitchin's solution (i.e., solutions for the case $(\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$) is transformed to the solutions for the case $(\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \in O_1 \cup O_2$. But we cannot obtain results on integral representation and the Hermite-Krichever Ansatz by our method for the case $(\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \in O_2$. Note that the condition $(\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \in O_2$ corresponds to the condition $l_0, \dots, l_3 \in \mathbb{Z} + \frac{1}{2}$, $l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z}$. How can we investigate solutions and their monodromy of the linear differential equation for the cases $l_0, \dots, l_3 \in \mathbb{Z} + \frac{1}{2}$, $l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z}$?

Appendix A. Elliptic form of sixth Painvé equation

We calculate the differentiation of modular functions, which will be used to rewrite the sixth Painlevé equation.

Proposition A.1. *(c.f. [6]) Set $\omega_1 = 1/2$, $\omega_3 = \tau/2$, $t = (e_3 - e_1)/(e_2 - e_1)$. Then we have*

$$\frac{dt}{d\tau} = \frac{(e_2 - e_1)t(t-1)}{\pi\sqrt{-1}}, \tag{A.1}$$

$$\frac{d}{d\tau}\left(\frac{1}{(e_2 - e_1)^{1/2}}\right) = \frac{\eta_1 + e_3/2}{\pi\sqrt{-1}(e_2 - e_1)^{1/2}}, \tag{A.2}$$

$$\frac{d}{d\tau}\left((e_2 - e_1)^\alpha\right) = -\frac{\alpha(2\eta_1 + e_3)(e_2 - e_1)^\alpha}{\pi\sqrt{-1}}. \tag{A.3}$$

Proof. Set $u = (z - e_1)/(e_2 - e_1)$. Then

$$\begin{aligned} \frac{\tau}{2} &= \int_{1/2}^{(1+\tau)/2} dx = \int_{e_1}^{e_2} \frac{dz}{\wp'(x)} \\ &= \int_{e_1}^{e_2} \frac{dz}{2\sqrt{(z-e_1)(z-e_2)(z-e_3)}} = \frac{1}{2(e_2-e_1)^{1/2}} \int_0^1 \frac{du}{\sqrt{u(u-1)(u-t)}}, \end{aligned} \tag{A.4}$$

$$-\frac{1}{2} = \int_{e_1}^{\infty} \frac{dz}{2\sqrt{(z-e_1)(z-e_2)(z-e_3)}} = \frac{1}{2(e_2-e_1)^{1/2}} \int_0^{\infty} \frac{du}{\sqrt{u(u-1)(u-t)}}, \tag{A.5}$$

By differentiating Eq.(A.4) in variable τ, we have

$$\text{(A.6)} \qquad \begin{aligned} \frac{1}{2} =& \frac{d}{d\tau}\left(\frac{1}{2(e_2-e_1)^{1/2}}\right)\int_0^1 \frac{du}{\sqrt{u(u-1)(u-t)}} \\ &+ \frac{1}{2}\left(\frac{dt}{d\tau}\right)\frac{1}{2(e_2-e_1)^{1/2}}\int_0^1 \frac{du}{(u-t)\sqrt{u(u-1)(u-t)}}, \end{aligned}$$

and it follows from Eq.(A.4) that

$$\text{(A.7)} \qquad \begin{aligned} &\frac{1}{2(e_2-e_1)^{1/2}}\int_0^1 \frac{du}{(u-t)\sqrt{u(u-1)(u-t)}} = (e_2-e_1)\int_{e_1}^{e_2}\frac{dz}{(\wp(x)-e_3)\wp'(x)} \\ &= \frac{e_2-e_1}{(e_3-e_2)(e_3-e_1)}\int_{1/2}^{(1+\tau)/2}(\wp(x+\tau/2)-e_3)dx \\ &= \frac{e_2-e_1}{(e_3-e_2)(e_3-e_1)}(-\eta(1/2+\tau)+\eta(1/2+\tau/2)-e_3\tau/2) \\ &= \frac{e_2-e_1}{(e_3-e_2)(e_3-e_1)}(-\eta_3-e_3\tau/2). \end{aligned}$$

Hence

$$\text{(A.8)} \qquad \frac{1}{2} = (e_2-e_1)^{1/2}\tau\frac{d}{d\tau}\left(\frac{1}{2(e_2-e_1)^{1/2}}\right)+\frac{1}{2}\left(\frac{dt}{d\tau}\right)\frac{(e_2-e_1)(-\eta_3-e_3\tau/2)}{(e_3-e_2)(e_3-e_1)}.$$

Similarly it follows from differentiating Eq.(A.5) that

$$\text{(A.9)} \qquad 0 = -(e_2-e_1)^{1/2}\frac{d}{d\tau}\left(\frac{1}{2(e_2-e_1)^{1/2}}\right)+\frac{1}{2}\left(\frac{dt}{d\tau}\right)\frac{(e_2-e_1)(\eta_1+e_3/2)}{(e_3-e_2)(e_3-e_1)}.$$

¿From these equalities we have

$$\text{(A.10)} \qquad \left(\frac{dt}{d\tau}\right)\frac{(e_2-e_1)(\tau\eta_1-\eta_3)}{(e_3-e_2)(e_3-e_1)} = 1.$$

By Legendre's relation $\eta_1\tau-\eta_3 = \pi\sqrt{-1}$ and definition of t, it follows that

$$\text{(A.11)} \qquad \frac{dt}{d\tau} = \frac{(e_2-e_1)t(t-1)}{\pi\sqrt{-1}}.$$

Combining with Eq.(A.9), we obtain that

$$\text{(A.12)} \qquad \frac{d}{d\tau}\left(\frac{1}{(e_2-e_1)^{1/2}}\right) = \frac{\eta_1+e_3/2}{\pi\sqrt{-1}(e_2-e_1)^{1/2}}.$$

The derivation of the function $(e_2-e_1)^{\alpha}$ is calculated as

$$\text{(A.13)} \quad \frac{d}{d\tau}(e_2-e_1)^{\alpha} = -2\alpha(e_2-e_1)^{\alpha+1/2}\frac{d}{d\tau}\left(\frac{1}{(e_2-e_1)^{1/2}}\right) = -\frac{\alpha(2\eta_1+e_3)(e_2-e_1)^{\alpha}}{\pi\sqrt{-1}}.$$

□

Proposition A.2. *Set* $g_2 = -4(e_1e_2 + e_2e_3 + e_3e_1)$. *We have*

$$\frac{de_i}{d\tau} = \frac{-2\eta_1 e_i + e_i^2 - g_2/6}{\pi\sqrt{-1}}, \tag{A.14}$$

for $i = 1, 2, 3$.

Proof. It follows from Proposition A.1 that

$$\frac{d}{d\tau}(e_2 - e_1)^{\pm 1} = \mp \frac{(2\eta_1 + e_3)(e_2 - e_1)^{\pm 1}}{\pi\sqrt{-1}}. \tag{A.15}$$

Combining with Eq.(A.1), we obtain that

$$\frac{d}{d\tau}(e_3 - e_1) = -\frac{(2\eta_1 + e_2)(e_3 - e_1)}{\pi\sqrt{-1}}. \tag{A.16}$$

By adding two functions, we have

$$\frac{d}{d\tau}(-3e_1) = -\frac{-6e_1\eta_1 + 2e_2e_3 - e_1^2}{\pi\sqrt{-1}}. \tag{A.17}$$

Hence, we obtain Eq.(A.14) for the case $i = 1$. Eq.(A.14) for the case $i = 2$ (resp. $i = 3$) follows from Eqs.(A.17, A.15) (resp. Eqs.(A.17, A.16)). □

Proposition A.3.

$$\frac{d\eta_1}{d\tau} = \frac{-\eta_1^2 + g_2/48}{\pi\sqrt{-1}}. \tag{A.18}$$

Proof. It follows similarly to Eq.(A.7) that

$$\frac{1}{2}\int_0^\infty \frac{du}{(u-t)\sqrt{u(u-1)(u-t)}} = \frac{(e_2 - e_1)^{3/2}}{(e_3 - e_2)(e_3 - e_1)}(\eta_1 + e_3/2). \tag{A.19}$$

We differentiate Eq.(A.19) in t. From the l.h.s, we have

$$\frac{3}{2}\int_0^\infty \frac{du}{2(u-t)^2\sqrt{u(u-1)(u-t)}} = \frac{3}{2}\frac{(e_2 - e_1)^{5/2}}{(e_3 - e_2)^2(e_3 - e_1)^2}\int_{1/2}^0 (\wp(x + \tau/2) - e_3)^2 dx,$$

$$\int_{1/2}^0 (\wp(x+\tau/2) - e_3)^2 dx = \int_{1/2}^0 \left(\frac{\wp''(x+\tau/2)^2}{6} - 2e_3\wp(x+\tau/2) + e_3^2 + \frac{g_2}{12}\right) dx$$
$$= -2e_3\eta_1 - \frac{1}{2}\left(e_3^2 + \frac{g_2}{12}\right).$$

¿From the r.h.s, we have

$$\frac{(e_2 - e_1)^{3/2}}{(e_3 - e_2)^2(e_3 - e_1)^2}\left\{\left(\eta_1 + \frac{e_3}{2}\right)\left(\eta_1 - \frac{5e_3}{2}\right) + \pi\sqrt{-1}\frac{d}{d\tau}\left(\eta_1 + \frac{e_3}{2}\right)\right\}.$$

Hence, we obtain

$$\frac{d}{d\tau}\left(\eta_1 + \frac{e_3}{2}\right) = \frac{1}{\pi\sqrt{-1}}\left\{-\eta^2 - \eta_1 e_3 + \frac{e_3^2}{2} - \frac{g_2}{16}\right\}, \tag{A.20}$$

and Eq.(A.18). □

We now we show that the sixth Painlevé equation (see Eq.(1.3)) can be rewritten to an elliptic form (see Eq.(4.12)).

Proposition A.4. *[6] Set*

$$\omega_1 = 1/2, \quad \omega_3 = \tau/2, \quad t = \frac{e_3 - e_1}{e_2 - e_1}, \quad \lambda = \frac{\wp(\delta) - e_1}{e_2 - e_1}. \tag{A.21}$$

Then the sixth Painlevé equation

$$\begin{aligned} \frac{d^2\lambda}{dt^2} =& \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda - 1} + \frac{1}{\lambda - t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda - t}\right)\frac{d\lambda}{dt} \\ &+ \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left\{\frac{\kappa_\infty^2}{2} - \frac{\kappa_0^2}{2}\frac{t}{\lambda^2} + \frac{\kappa_1^2}{2}\frac{(t-1)}{(\lambda-1)^2} + \frac{(1-\kappa_t^2)}{2}\frac{t(t-1)}{(\lambda-t)^2}\right\} \end{aligned} \tag{A.22}$$

is equivalent to the equation

$$\frac{d^2\delta}{d\tau^2} = -\frac{1}{4\pi^2}\left\{\frac{\kappa_\infty^2}{2}\wp'(\delta) + \frac{\kappa_0^2}{2}\wp'\left(\delta + \frac{1}{2}\right) + \frac{\kappa_1^2}{2}\wp'\left(\delta + \frac{\tau+1}{2}\right) + \frac{\kappa_t^2}{2}\wp'\left(\delta + \frac{\tau}{2}\right)\right\}. \tag{A.23}$$

Proof. It follows from the relation $\lambda = (\wp(\delta) - e_1)/(e_2 - e_1)$ that

$$\delta = \int_0^\delta dx = \int_\infty^\lambda \frac{e_2 - e_1}{\wp'(x)}du = \frac{1}{2(e_2 - e_1)^{1/2}}\int_\infty^\lambda \frac{du}{\sqrt{u(u-1)(u-t)}}. \tag{A.24}$$

We differentiate Eq.(A.24) by the variable τ. Then we have

$$\begin{aligned} \frac{d\delta}{d\tau} = \frac{\eta_1 + e_3/2}{\pi\sqrt{-1}}\delta + \frac{(e_2 - e_1)^{1/2}t(t-1)}{2\pi\sqrt{-1}}&\left\{\frac{d\lambda}{dt}\frac{1}{\sqrt{\lambda(\lambda-1)(\lambda-t)}}\right. \\ &\left. + \frac{1}{2}\int_\infty^\lambda \frac{du}{(u-t)\sqrt{u(u-1)(u-t)}}\right\}. \end{aligned} \tag{A.25}$$

Note that we used Proposition A.1. We differentiate Eq.(A.25) once more. By applying formulae on the differentiation of modular functions, we obtain that

$$\begin{aligned} \frac{d^2\delta}{d\tau^2} = \frac{t^2(t-1)^2(e_1 - e_3)^{3/2}}{-2\pi^2}&\left[\frac{1}{\sqrt{\lambda(\lambda-1)(\lambda-t)}}\left\{\frac{d^2\lambda}{dt^2}\right.\right. \\ &\left. - \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 + \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}\right\} \\ &\left. + \frac{1}{4t(t-1)}\int_\infty^\lambda \frac{(u^2 + 2tu - 2u - t)du}{(u-t)^2\sqrt{u(u-1)(u-t)}}\right], \end{aligned} \tag{A.26}$$

and we have

$$\int_\infty^\lambda \frac{(u^2 + 2tu - 2u - t)du}{(u-t)^2\sqrt{u(u-1)(u-t)}} = -2\sqrt{\frac{\lambda(\lambda-1)}{(\lambda-t)^3}}. \tag{A.27}$$

It follows from $\lambda = (\wp(\delta) - e_1)/(e_2 - e_1)$ that

$$\wp'(\delta) = 2(e_1 - e_2)^{3/2}\sqrt{\lambda(\lambda-1)(\lambda-t)}, \quad \frac{\wp'(\delta+1/2)}{\wp'(\delta)} = -\frac{t}{\lambda^2}, \tag{A.28}$$

$$\frac{\wp'(\delta+(\tau+1)/2)}{\wp'(\delta)} = \frac{t-1}{(\lambda-1)^2}, \quad \frac{\wp'(\delta+\tau/2)}{\wp'(\delta)} = \frac{t(1-t)}{(\lambda-t)^2}.$$

By combining Eqs.(A.26-A.28), we obtain the equivalence of Eq.(A.22) and Eq.(A.23). □

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Kouichi Takemura
Department of Mathematical Sciences,
Yokohama City University,
22-2 Seto, Kanazawa-ku, Yokohama 236-0027, Japan.
takemura@yokohama-cu.ac.jp