q-difference shift for van Diejen's BC_n type Jackson integral arising from 'elementary' symmetric polynomials *

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Abstract

We study a q-difference equation of a BC_n type Jackson integral, which is a multiple q-series generalized from a q-analogue of Selberg's integral. The equation is characterized by some new symmetric polynomials defined via the symplectic Schur functions. As an application of it, we give another proof of a product formula for the BC_n type Jackson integral, which is equivalent to the so-called q-Macdonald-Morris identity for the root system BC_n first obtained by Gustafson and van Diejen.

1 Introduction

As it is known, the beta integral

$$B(\alpha,\beta) := \int_0^1 z^{\alpha-1}(1-z)^{\beta-1}dz \tag{1}$$

is written as the following product of the gamma functions $\Gamma(\alpha)$:

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{2}$$

Since the gamma function satisfies $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$, we easily see the following recurrence relations:

$$B(\alpha+1,\beta) = \frac{\alpha}{\alpha+\beta}B(\alpha,\beta), \quad B(\alpha,\beta+1) = \frac{\beta}{\alpha+\beta}B(\alpha,\beta). \tag{3}$$

We regard these relations as difference equations with respect to parameters. Among the solutions of (3), the beta function $B(\alpha,\beta)$ is characterized by the following asymptotic behavior:

$$B(\alpha+N,\beta+N) \sim 2^{-\alpha-\beta+1-2N}\sqrt{\pi/N} \qquad (N\to+\infty). \tag{4}$$

Conversely, we can recover the formula (2) from (3) and (4). Since

$$B(\alpha+1,\beta) = \int_0^1 z\Phi(z)dz \quad \text{or} \quad B(\alpha,\beta+1) = \int_0^1 \Phi(z)dz - \int_0^1 z\Phi(z)dz,$$

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where $\Phi(z)$ denotes the integrand $z^{\alpha-1}(1-z)^{\beta-1}$, if we want to have the recurrence relations (3) without using (2), we usually prove the following relation from the definition (1) using integration by part:

$$\int_0^1 z\Phi(z)dz = \frac{\alpha}{\alpha+\beta}\int_0^1 \Phi(z)dz,$$

which provides the equation between the integral (1) and that multiplied by a monomial z to the integrand $\Phi(z)$.

Next we consider the following *q-Selberg integral* [4, 6, 7, 10, 18, 20] defined by using the Jackson integral which is a sum over the lattice $\mathbf{Z}^n$ (For the definition of the Jackson integral, see Section 3):

$$S_q(\alpha,\beta,\tau;\xi) := \int_0^{\xi\infty} \Phi_{\mathcal{S}_n}(z)\Delta_{\mathcal{S}_n}(z)\varpi_q, \quad \varpi_q = \frac{d_qz_1}{z_1}\cdots\frac{d_qz_n}{z_n}$$

where the integrand is defined by

$$\begin{aligned}\Phi_{\mathcal{S}_n}(z) &:= \prod_{i=1}^n z_i^\alpha \frac{(qz_i)_\infty}{(bz_i)_\infty} \prod_{1\le j<k\le n} z_k^{2\tau}\frac{(qt^{-1}z_j/z_k)_\infty}{(tz_j/z_k)_\infty}\\ \Delta_{\mathcal{S}_n}(z) &:= \prod_{1\le j<k\le n}(z_k-z_j)\end{aligned}$$

and $q^\alpha=a, q^\beta=b, q^\tau=t$. Let $\eta\in(\mathbf{C}^*)^n$ be the point defined by

$$\eta := (t^{n-1}, t^{n-2}, ..., t, 1).$$

For the Jackson integral $S_q(\alpha,\beta,\tau;\xi)$, if we put $\xi=\eta$ and take the limit $q\to 1$, then the sum $S_q(\alpha,\beta,\tau;\eta)$ becomes the following so-called Selberg integral:

$$S(\alpha,\beta,\tau) = \int_{0\le z_1\le\ldots\le z_n\le 1} \prod_{i=1}^n z_i^{\alpha-1}(1-z_i)^{\beta-1}\Delta_{\mathcal{S}_n}(z)^{2\tau}dz_1...dz_n \tag{5}$$

which can be expressed as a product of gamma functions as

$$S(\alpha,\beta,\tau) = \prod_{i=1}^n \frac{\Gamma(i\tau)\Gamma(\alpha+(n-i)\tau)\Gamma(\beta+(n-i)\tau)}{\Gamma(\tau)\Gamma(\alpha+\beta+(2n-i-1)\tau)}.$$

The Selberg integral (5) is nothing but the beta function if $n=1$. For the q-Selberg integral $S_q(\alpha,\beta,\tau;\xi)$, it is also possible to express it as a product of q-gamma functions by using its q-difference equation and its asymptotic behavior. To carry it out we need the q-difference equation first. According to Aomoto [3], the following formula is known:

Proposition 1.1 *Let $e_i(z)$, $0\le i\le n$, be the ith elementary symmetric polynomial, i.e., $e_i(z)=\sum_{1\le j_1<\ldots<j_i\le n} z_{j_1}z_{j_2}...z_{j_i}$. Then*

$$\begin{aligned}&\int_0^{\xi\infty} e_i(z)\Phi_{\mathcal{S}_n}(z)\Delta_{\mathcal{S}_n}(z)\varpi_q\\ &= t^{i-1}\frac{(1-t^{n-i+1})(1-at^{n-i})}{(1-t^i)(1-abt^{2n-i-1})}\int_0^{\xi\infty} e_{i-1}(z)\Phi_{\mathcal{S}_n}(z)\Delta_{\mathcal{S}_n}(z)\varpi_q.\end{aligned}$$

Since $S_q(\alpha+1,\beta,\tau;\xi) = \int_0^{\xi\infty} z_1 z_2 ... z_n \, \Phi_{\mathcal{S}_n}(z)\Delta_{\mathcal{S}_n}(z)\varpi_q$, we easily have the following q-difference equation by repeated use of Proposition 1.1:

$$S_q(\alpha+1,\beta,\tau;\xi) = \prod_{i=1}^{n} \frac{t^{i-1}(1-at^{n-i})}{(1-abt^{2n-i-1})} S_q(\alpha,\beta,\tau;\xi), \tag{6}$$

which can be found in [20] and in [1, 2] for the case $\xi=\eta$ and $q\to 1$. For the q-Selberg integral $S_q(\alpha,\beta,\tau;\xi)$, we can construct its product expression of q-gamma functions from its q-difference equation (6) and asymptotic behavior of $S_q(\alpha+N,\beta,\tau;\eta)$ at $N\to+\infty$. (For explicit form of it, see [4, 20].)

In this paper we discuss a structure of product expression of a multiple sum generalized from the q-Selberg integral. We call it the *BC_n type Jackson integral* (See Section 3 for its definition). Like the q-Selberg integral case, for the BC_n type Jackson integral there also exist symmetric polynomials $e'_i(z)$ of middle degree i, $0\le i\le n$, such that they interpolate a q-difference equation with respect to the parameter shift $a_1\to qa_1$ as follows:

Theorem 1.2 *There exist symmetric Laurent polynomials $e'_i(z)$ of degree i, $0\le i\le n$, such that*

$$\begin{aligned}&\int_0^{\xi\infty} e'_i(z)\Phi_{B_n}(z)\Delta_{C_n}(z)\varpi_q \\ &\quad = -\frac{t^{i-1}(1-t^{n-i+1})\prod_{k=2}^{4}(1-a_k a_1 t^{n-i})}{t^{n-i}(1-t^i)a_1(1-a_1a_2a_3a_4t^{2n-i-1})}\int_0^{\xi\infty} e'_{i-1}(z)\Phi_{B_n}(z)\Delta_{C_n}(z)\varpi_q.\end{aligned}$$

Considering an analogy to Proposition 1.1, we call the polynomials $e'_i(z)$ the *'elementary' symmetric polynomials*, which are different from the symplectic Schur functions $\chi_{(1^i)}(z)$ though the polynomials $\chi_{(1^i)}(z)$ are sometimes called the *elementary symmetric polynomials.* (See Section 2 for the definition of $\chi_{(1^i)}(z)$). Moreover, the explicit forms of them are the following (Note the number of variables in the RHS):

$$\begin{aligned}
e'_0(z) &= 1,\\
e'_1(z) &= \chi_{(1)}(z_1,z_2,...,z_n) - \chi_{(1)}(a_1,a_1t,...,a_1t^{n-1}),\\
e'_2(z) &= \chi_{(1^2)}(z_1,z_2,...,z_n)\\
&\quad - \chi_{(1)}(z_1,z_2,...,z_n)\chi_{(1)}(a_1,a_1t,...,a_1t^{n-2})\\
&\quad + \chi_{(2)}(a_1,a_1t,...,a_1t^{n-2}),\\
&\ \ \vdots\\
e'_i(z) &= \sum_{j=0}^{i}(-1)^j\chi_{(1^{i-j})}(\underbrace{z_1,z_2,...,z_n}_{n})\chi_{(j)}(\underbrace{a_1,a_1t,...,a_1t^{n-i}}_{n-i+1}),\\
&\ \ \vdots\\
e'_n(z) &= \sum_{j=0}^{n}(-1)^j\chi_{(1^{n-j})}(z_1,z_2,...,z_n)\chi_{(j)}(a_1).
\end{aligned}$$

This paper is organized as follows. In Section 2, we first state a relation between the symplectic Schur functions $\chi_{(1^i)}(z)$ and $\chi_{(j)}(z)$. The relation is used in Section 4 for proving a property of the 'elementary' symmetric function. In Section 3, we introduce the BC_n type Jackson integral and its *truncated* case. In Section 4, we define the 'elementary' symmetric function for the BC_n type Jackson integral. Section 5 is devoted to the proof of Theorem 1.2, which is a main result of this paper. In Section 6, as a corollary of Theorem 1.2, we construct a product formula for the BC_n type Jackson integral as if we recover the product expression (2) of the beta function from q-difference equations (3) and asymptotic behavior (4). This is to be another proof of the product formula, which is equivalent to the so-called q-Macdonald-Morris identity [21, 25] for the root system BC_n first obtained by Gustafson [9] and van Diejen [26]. (See also [16, 23] for relations between q-Macdonald-Morris identities and the Jackson integrals associated with root systems.)

Throughout this paper we use the notations $(x)_\infty := \prod_{i=0}^{\infty}(1-q^ix)$ and $(x)_N := (x)_\infty/(q^Nx)_\infty$ where $0<q<1$.

2 Symplectic Schur functions $\chi_\lambda(z)$

Before introducing the BC_n type Jackson integral, we prove Proposition 2.4, which will be used technically when we state a property of the 'elementary' symmetric polynomials in Section 4. The formula in Proposition 2.4 indicates some relation among the symplectic Schur functions $\chi_\lambda(z)$. The relation is very similar to those between the elementary symmetric functions and the complete symmetric functions (see [8, 22] for instance).

2.1 Definition of the symplectic Schur functions $\chi_\lambda(z)$

Let $\mathcal{A}_{(i_1,i_2,\dots,i_n)}(z)$ be the function of $z\in(\mathbf{C}^*)^n$ defined in the form of the following determinant:

$$\mathcal{A}_{(i_1,i_2,\dots,i_n)}(z) := \det\left(z_j^{i_k}-z_j^{-i_k}\right)_{1\le j,k\le n}$$

for $(i_1,i_2,\dots,i_n)\in\mathbf{Z}^n$. For example,

$$\mathcal{A}_{(3,2,1)}(z_1,z_2,z_3)=\begin{vmatrix} z_1^3-z_1^{-3} & z_2^3-z_2^{-3} & z_3^3-z_3^{-3}\\ z_1^2-z_1^{-2} & z_2^2-z_2^{-2} & z_3^2-z_3^{-2}\\ z_1-z_1^{-1} & z_2-z_2^{-1} & z_3-z_3^{-1}\end{vmatrix},$$

$$\mathcal{A}_{(5,2)}(z_1,z_2)=\begin{vmatrix} z_1^5-z_1^{-5} & z_2^5-z_2^{-5}\\ z_1^2-z_1^{-2} & z_2^2-z_2^{-2}\end{vmatrix} \quad \text{and so on.}$$

Let W_{C_n} be the Weyl group of type C_n, which is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^n\rtimes\mathcal{S}_n$ where $\mathcal{S}_n$ is the symmetric group of nth order. W_{C_n} is generated by the following transformations of the coordinates $(z_1,z_2,\dots,z_n)\in(\mathbf{C}^*)^n$:

$$\begin{aligned}(z_1,z_2,\dots,z_n) &\to (z_1^{-1},z_2,\dots,z_n),\\ (z_1,z_2,\dots,z_n) &\to (z_{\sigma(1)},z_{\sigma(2)},\dots,z_{\sigma(n)}) \quad \sigma\in\mathcal{S}_n.\end{aligned}$$

For a function $f(z)$ of $z\in(\mathbf{C}^*)^n$, we denote by $\mathcal{A}f(z)$ the alternating sum over W_{C_n} defined by

$$\mathcal{A}f(z) := \sum_{w\in W_{C_n}}(\operatorname{sgn} w)\, wf(z).$$

In particular, by definition of determinant, $\mathcal{A}_{(i_1,i_2,...,i_n)}(z)$ is expanded as the following alternating sum of the monomial $z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}$:

$$\mathcal{A}_{(i_1,i_2,...,i_n)}(z) = \mathcal{A}(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n})$$

for $(i_1, i_2, ..., i_n) \in \mathbf{Z}^n$. This implies that

$$\mathcal{A}_\rho(z) = \prod_{i=1}^{n}(z_i - z_i^{-1}) \prod_{1\le j<k\le n} \frac{(z_k - z_j)(1 - z_j z_k)}{z_j z_k} \tag{7}$$

where

$$\rho := (n, n-1, ..., 2, 1) \in \mathbf{Z}^n,$$

which is the so-called Weyl denominator formula. Let P be the set of partitions defined by

$$P := \{(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbf{Z}^n\, ;\, \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0\}.$$

For $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in P$, we define the *symplectic Schur function* $\chi_\lambda(z)$ as follows:

$$\chi_\lambda(z) := \frac{\mathcal{A}_{\lambda+\rho}(z)}{\mathcal{A}_\rho(z)} = \frac{\mathcal{A}_{(\lambda_1+n,\lambda_2+n-1,...,\lambda_{n-1}+2,\lambda_n+1)}(z)}{\mathcal{A}_{(n,n-1,...,2,1)}(z)},$$

which occurs in the Weyl character formula. For $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in P$, if we denote by m_i the multiplicity of i in λ, i.e., $m_i = \#\{j\, ;\, \lambda_j = i\}$, it is convenient to use the notations

$$\lambda = (1^{m_1} 2^{m_2} \dots r^{m_r} \dots) \quad \text{and} \quad \chi_\lambda(z) = \chi_{(1^{m_1}2^{m_2}\dots r^{m_r}\dots)}(z).$$

For example, we use them like $\chi_{(2,1,1,0)}(z_1, z_2, z_3, z_4) = \chi_{(1^2 2)}(z_1, z_2, z_3, z_4)$.

2.2 A relation among $\chi_\lambda(z)$

For $i = 0, 1, 2, ..., n$, we define the determinant $D_i^{(n)}(z, y)$ of a matrix of degree $n + i + 1$ as follows:

$$D_i^{(n)}(z, y) := \begin{vmatrix} A_{n+1,n}(z) & A_{n+1,i+1}(y) \\ A_{i,n}(z) & -A_{i,i+1}(y) \end{vmatrix}$$

where $A_{\mu,\nu}(z)$ is the $\mu \times \nu$ matrix defined by

$$A_{\mu,\nu}(z) := \begin{pmatrix} z_1^{\mu} - z_1^{-\mu} & z_2^{\mu} - z_2^{-\mu} & \cdots & z_\nu^{\mu} - z_\nu^{-\mu} \\ z_1^{\mu-1} - z_1^{-(\mu-1)} & z_2^{\mu-1} - z_2^{-(\mu-1)} & \cdots & z_\nu^{\mu-1} - z_\nu^{-(\mu-1)} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{2} - z_1^{-2} & z_2^{2} - z_2^{-2} & \cdots & z_\nu^{2} - z_\nu^{-2} \\ z_1 - z_1^{-1} & z_2 - z_2^{-1} & \cdots & z_\nu - z_\nu^{-1} \end{pmatrix}.$$

Lemma 2.1 *The determinant $D_i^{(n)}(z, y)$ is divided out by*

$$\sum_{k=i+1}^{n+1} (-1)^{k+1} \mathcal{A}_{(n+1,n,n-1,...,k+1,k-1,...,2,1)}(z_1, ..., z_n) \times \mathcal{A}_{(k,i,i-1,i-2,...,2,1)}(y_1, ..., y_{i+1}). \tag{8}$$

Proof. Separating $(n+1)$ columns of $D_i^{(n)}(z,y)$ into two parts which are the forward part to the nth column and that backward from the $n+1$ one, we have a Laplace expansion (8) of $D_i^{(n)}(z,y)$ by minors of sizes n and $i+1$ up to constant. ∎

Corollary 2.2 *The following holds for $\chi_\lambda(z)$ and $\mathcal{A}_\rho(z)$:*

$$\frac{\mathcal{A}_{(n+1,n,\ldots,1)}(z_1,z_2,\ldots,z_n,z_{n+1})}{\mathcal{A}_{(n,n-1,\ldots,1)}(z_1,\ldots,z_n)\mathcal{A}_{(1)}(z_{n+1})}=\sum_{j=0}^{n}(-1)^j\chi_{(1^{n-j})}(z_1,\ldots,z_n)\chi_{(j)}(z_{n+1}).$$

Proof. If we put $y_1=z_{n+1}$ for $D_0^{(n)}(z,y)$, then

$$D_0^{(n)}(z,y)=\mathcal{A}_{(n+1,n,\ldots,1)}(z_1,z_2,\ldots,z_n,z_{n+1}).\tag{9}$$

On the other hand, from Lemma 2.1, we have

$$\begin{aligned}&D_0^{(n)}(z,y)\\&=\sum_{k=1}^{n+1}(-1)^{k+1}\mathcal{A}_{(n+1,n,n-1,\ldots,k+1,k-1,\ldots,2,1)}(z_1,\ldots,z_n)\mathcal{A}_{(k)}(y_1).\end{aligned}\tag{10}$$

From (9) and (10), it follows that

$$\begin{aligned}&\mathcal{A}_{(n+1,n,\ldots,1)}(z_1,z_2,\ldots,z_n,z_{n+1})\\&=\sum_{k=1}^{n+1}(-1)^{k+1}\mathcal{A}_{(n+1,n,n-1,\ldots,k+1,k-1,\ldots,2,1)}(z_1,\ldots,z_n)\mathcal{A}_{(k)}(z_{n+1}).\end{aligned}\tag{11}$$

Dividing both sides of (11) by $\mathcal{A}_{(n,n-1,\ldots,1)}(z_1,\ldots,z_n)\mathcal{A}_{(1)}(z_{n+1})$, we obtain Corollary 2.2. ∎

Lemma 2.3 *If we put $y_j=z_j$ for all $j\in\{1,2,\ldots,i+1\}$, then $D_i^{(n)}(z,z)=0$.*

Proof. Set $y_j=z_j$ $(1\le j\le i+1)$. For the determinant

$$D_i^{(n)}(z,z)=\begin{vmatrix}A_{n+1,n}(z)&A_{n+1,i+1}(z)\\A_{i,n}(z)&-A_{i,i+1}(z)\end{vmatrix},$$

if we subtract the jth column from the $(n+j)$th one for $j=1,2,\ldots,i+1$, by the elementary column operations, we have

$$\begin{aligned}D_i^{(n)}(z,z)&=\begin{vmatrix}A_{n+1,n}(z)&O\\A_{i,n}(z)&-2A_{i,i+1}(z)\end{vmatrix}\\&=(-2)^{i+1}\begin{vmatrix}A_{n+1,n}(z)&O\\A_{i,n}(z)&A_{i,i+1}(z)\end{vmatrix}.\end{aligned}$$

Moreover, since the rank of the $i\times(i+1)$ matrix $A_{i,i+1}(z)$ is less than or equal to i, after the processes of the elementary column operations the matrix $A_{i,i+1}(z)$ can be deformed into an $i \times (i+1)$ matrix B which has at least one column consisting of zeros only. Thus, the determinant

$$\begin{vmatrix} A_{n+1,n}(z) & O \\ A_{i,n}(z) & A_{i,i+1}(z) \end{vmatrix}$$

is divided out by

$$\begin{vmatrix} A_{n+1,n}(z) & O \\ A_{i,n}(z) & B \end{vmatrix},$$

which has the column consisting of zeros and is equal to zero. This implies $D_i^{(n)}(z,z) = 0$, which completes the proof. □

Proposition 2.4 *The following holds for $i = 0, 1, 2, ..., n$:*

$$\sum_{j=0}^{i} (-1)^j \chi_{(1^{i-j})}(z_1, z_2, ..., z_n)\chi_{(j)}(z_1, z_2, ..., z_{n-i+1}) = \begin{cases} 0 & (i \neq 0), \\ 1 & (i = 0). \end{cases}$$

Proof. From Lemma 2.3 and 2.1, it follows that

$$\sum_{k=i+1}^{n+1} (-1)^{k+1} \mathcal{A}_{(n+1,n,n-1,...,k+1,k-1,...,2,1)}(z_1, ..., z_n) \times \mathcal{A}_{(k,i,i-1,i-2,...,2,1)}(z_1, ..., z_{i+1}) = 0. \tag{12}$$

Divide both sides of (12) by $\mathcal{A}_{(n,n-1,...,1)}(z_1, ..., z_n)\mathcal{A}_{(i+1,i,...,1)}(z_1, ..., z_{i+1})$. Exchanging i with $n - i$, we obtain Proposition 2.4. □

3 Definition of BC_n type Jackson integral

For $z = (z_1, z_2, ..., z_n) \in (\mathbf{C}^*)^n$, we set

$$\begin{aligned} \Phi_{B_n}(z) &:= \prod_{i=1}^{n} \prod_{m=1}^{4} z_i^{1/2-\alpha_m} \frac{(qa_m^{-1}z_i)_\infty}{(a_m z_i)_\infty} \\ &\quad \times \prod_{1\le j<k\le n} z_j^{1-2\tau} \frac{(qt^{-1}z_j/z_k)_\infty}{(tz_j/z_k)_\infty} \frac{(qt^{-1}z_j z_k)_\infty}{(tz_j z_k)_\infty}, \\ \Delta_{C_n}(z) &:= \prod_{i=1}^{n} \frac{1-z_i^2}{z_i} \prod_{1\le j<k\le n} \frac{(1-z_j/z_k)(1-z_j z_k)}{z_j} \end{aligned}$$

where $q^{\alpha_m} = a_m, q^\tau = t$. We abbreviate $\Phi_{B_n}(z)$ and $\Delta_{C_n}(z)$ to $\Phi(z)$ and $\Delta(z)$ respectively. Weyl's denominator formula (7) says

$$\Delta(z) = (-1)^n \mathcal{A}_{(n,n-1,...,1)}(z). \tag{13}$$

For an arbitrary $\xi = (\xi_1, \xi_2, ..., \xi_n) \in (\mathbf{C}^*)^n$, we define the q-shift $\xi \to q^\nu \xi$ by a lattice point $\nu = (\nu_1, \nu_2 ..., \nu_n) \in \mathbf{Z}^n$, where

$$q^\nu \xi := (q^{\nu_1}\xi_1, q^{\nu_2}\xi_2, ..., q^{\nu_n}\xi_n) \in (\mathbf{C}^*)^n.$$

For $\xi=(\xi_1,\xi_2,...,\xi_n)\in(\mathbf{C}^*)^n$ and a function $h(z)$ of $z\in(\mathbf{C}^*)^n$, we define the sum over the lattice $\mathbf{Z}^n$ by

$$\int_0^{\xi_1\infty}\!\!\cdots\!\int_0^{\xi_n\infty} h(z)\,\frac{d_qz_1}{z_1}\cdots\frac{d_qz_n}{z_n}:=(1-q)^n\sum_{\nu\in\mathbf{Z}^n}h(q^\nu\xi), \tag{14}$$

which we call the *Jackson integral* if it converges. We abbreviate the LHS of (14) to $\displaystyle\int_0^{\xi\infty}h(z)\,\varpi_q$. We now define the Jackson integral whose integrand is $\Phi(z)\Delta(z)$ as follows:

$$J(\xi):=\int_0^{\xi\infty}\Phi(z)\Delta(z)\varpi_q, \tag{15}$$

which converges if

$$|a_1a_2a_3a_4t^{n+i-2}|>q \quad \text{for} \quad i=1,2,...,n$$

and

$$\begin{cases} t\xi_j/\xi_k,t\xi_j\xi_k\notin\{q^l\,;\,l\in\mathbf{Z}\} & \text{for} \quad 1\le j<k\le n,\\ a_m\xi_i\notin\{q^l\,;\,l\in\mathbf{Z}\} & \text{for} \quad 1\le m\le 4,\ 1\le i\le n.\end{cases}$$

We call the sum $J(\xi)$ the *BC_n type Jackson integral*. The sum $J(\xi)$ is invariant under the shifts $\xi\to q^\nu\xi$ for $\nu\in\mathbf{Z}^n$.

Since $(q^{1+m})_\infty=0$ if m is a negative integer, for the special point

$$\zeta:=(t^{n-1}a_1,t^{n-2}a_1,...,ta_1,a_1)\in(\mathbf{C}^*)^n,$$

it follows that

$$\Phi(q^\nu\zeta)=0 \quad \text{if} \quad \nu\notin D$$

where D forms the cone in the lattice $\mathbf{Z}^n$ defined by

$$D:=\{\nu\in\mathbf{Z}^n\,;\,\nu_1-\nu_2\ge0,\ \nu_2-\nu_3\ge0,...,\nu_{n-1}-\nu_n\ge0 \text{ and } \nu_n\ge0\}.$$

This implies that $J(\zeta)$ is written as a sum over the cone D as follows:

$$J(\zeta)=(1-q)^n\sum_{\nu\in D}\Phi(q^\nu\zeta)\Delta(q^\nu\zeta) \tag{16}$$

We call its Jackson integral summed over D *truncated*. We just write

$$J(\zeta)=\int_0^{\zeta}\Phi(z)\Delta(z)\varpi_q$$

omitting the notation ∞ in its region only if $\xi=\zeta$.

Let $\Theta(\xi)$ be the function defined by

$$\Theta(\xi):=\prod_{i=1}^n\frac{\xi_i\,\theta(\xi_i^2)}{\prod_{m=1}^4\xi_i^{\alpha_m}\theta(a_m\xi_i)}\prod_{1\le j<k\le n}\frac{\theta(\xi_j/\xi_k)\theta(\xi_j\xi_k)}{\xi_j^{2\tau}\theta(t\xi_j/\xi_k)\theta(t\xi_j\xi_k)} \tag{17}$$

where $\theta(x):=(x)_\infty(q/x)_\infty$. We state a lemma for the subsequent section.

Lemma 3.1 *The Jackson integral $J(\xi)$ is expressed as*

$$J(\xi) = C\,\Theta(\xi) \tag{18}$$

where C is a constant not depending on $\xi \in (\mathbf{C}^*)^n$

Proof. See [14]. $\square$

We will discuss the constant C later in Section 6.

4 'Elementary' symmetric polynomials $e'_i(z)$

For $i = 0, 1, 2, 3, ..., n$, we define the following symmetric polynomials in terms of $\chi_\lambda(z)$:

$$e'_i(z) := \sum_{j=0}^{i} (-1)^j \chi_{(1^{i-j})}(\underbrace{z_1, z_2, ..., z_n}_{n})\, \chi_{(j)}(\underbrace{a_1, a_1 t, ..., a_1 t^{n-i}}_{n-i+1}), \tag{19}$$

which we call the *ith 'elementary' symmetric polynomials* as we mentioned in Introduction. In particular,

Lemma 4.1 *The product expression of the nth 'elementary' symmetric polynomial $e'_n(z)$ is the following*:

$$e'_n(z) = \prod_{i=1}^{n} \frac{(a_1 - z_i)(1 - a_1 z_i)}{a_1 z_i}. \tag{20}$$

Proof. By using Weyl's denominator formula (7), we have

$$\prod_{i=1}^{n} \frac{(a_1 - z_i)(1 - a_1 z_i)}{a_1 z_i} = \frac{\mathcal{A}_{(n+1,n,...,1)}(z_1, z_2, ..., z_n, a_1)}{\mathcal{A}_{(n,n-1,...,1)}(z_1, ..., z_n)\mathcal{A}_{(1)}(a_1)} \tag{21}$$

Taking $z_{n+1} = a_1$ at Corollary 2.2, we have

$$\begin{aligned} \frac{\mathcal{A}_{(n+1,n,...,1)}(z_1, z_2, ..., z_n, a_1)}{\mathcal{A}_{(n,n-1,...,1)}(z_1, ..., z_n)\mathcal{A}_{(1)}(a_1)} &= \sum_{j=0}^{n} (-1)^j \chi_{(1^{n-j})}(z_1, ..., z_n)\chi_{(j)}(a_1) \\ &= e'_n(z). \end{aligned} \tag{22}$$

From (21) and (22), we have (20). $\square$

Let x be a real number satisfying $x > 0$. For $i = 1, 2, 3, ..., n+1$, we set

$$\zeta_i = (\zeta_{i1}, \zeta_{i2}, \ldots, \zeta_{in}) \in (\mathbf{C}^*)^n, \tag{23}$$

where

$$\zeta_{ij} := \begin{cases} x^{i-j} & \text{if} \quad 1 \le j < i, \\ t^{n-j} a_1 & \text{if} \quad i \le j \le n. \end{cases}$$

The explicit expression of ζ_i is the following:

$$\begin{aligned}
\zeta_1 &= (t^{n-1}a_1, t^{n-2}a_1, ..., ta_1, a_1),\\
\zeta_2 &= (x, t^{n-2}a_1, t^{n-3}a_1, ..., ta_1, a_1),\\
\zeta_3 &= (x^2, x, t^{n-3}a_1, t^{n-4}a_1, ..., ta_1, a_1),\\
&\vdots\\
\zeta_n &= (x^{n-1}, ..., x^2, x, a_1),\\
\zeta_{n+1} &= (x^n, x^{n-1}, ..., x^2, x).
\end{aligned}$$

In particular, the point $\zeta_1 \in (\mathbf{C}^*)^n$ is nothing but $\zeta \in (\mathbf{C}^*)^n$ which is defined in Section 3.

Lemma 4.2 *If $1 \le j \le i \le n$, then*

$$e_i'(z_1, z_2, ..., z_{j-1}, a_1t^{n-j}, a_1t^{n-j-1}, ..., a_1t, a_1) = 0.$$

Proof. Since $\chi_\lambda(z)$ is symmetric, by definition (19), we have

$$\begin{aligned}
&e_i'(z_1, z_2, ..., z_{j-1}, a_1t^{n-j}, a_1t^{n-j-1}, ..., a_1t, a_1)\\
&= \sum_{k=0}^{i}(-1)^k\chi_{(1^{i-k})}(z_1, z_2, ..., z_{j-1}, a_1t^{n-j}, a_1t^{n-j-1}, ..., a_1t, a_1)\\
&\qquad \times \chi_{(k)}(a_1, a_1t, ..., a_1t^{n-i})\\
&= \sum_{k=0}^{i}(-1)^k\chi_{(1^{i-k})}(a_1, a_1t, ..., a_1t^{n-i}, a_1t^{n-i+1}, ..., a_1t^{n-j}, z_1, z_2, ..., z_{j-1})\\
&\qquad \times \chi_{(k)}(a_1, a_1t, ..., a_1t^{n-i}).
\end{aligned}$$

Applying Proposition 2.4, the RHS of the above equation is equal to 0. This completes the proof. □

The explicit expression of Lemma 4.2 is the following:

$$\begin{aligned}
e_1'(a_1t^{n-1}, a_1t^{n-2}, ..., a_1t, a_1) &= 0,\\
e_2'(z_1, a_1t^{n-2}, ..., a_1t, a_1) &= 0,\\
&\vdots\\
e_n'(z_1, z_2, ..., z_{n-1}, a_1) &= 0.
\end{aligned}$$

In particular,

Corollary 4.3 *If $1 \le j \le i \le n$, then $e_i'(\zeta_j) = 0$.*

Proof. It is straightforward from definition (23) of ζ_i and Lemma 4.2. □

5 Main theorem

In this section, to specify the number of variables n, we simply use the notations $e_i^{(n)}(z)$ and $\mathcal{A}^{(n)}(z)$ instead of the 'elementary' symmetric polynomials $e_i'(z)$ and Weyl's denominator $\mathcal{A}_\rho(z)$ respectively. The notation (n) on the right shoulder of e_i or $\mathcal{A}$ indicates the number of variables of $z=(z_1,z_2,...,z_n)$ for $e_i'(z)$ or $\mathcal{A}_\rho(z)$.

Let T_{z_1} be the q-shift of variable z_1 such that $T_{z_1}: z_1 \to qz_1$. Set

$$\nabla\varphi(z) := \varphi(z) - \frac{T_{z_1}\Phi(z)}{\Phi(z)}\, T_{z_1}\varphi(z), \tag{24}$$

where $T_{z_1}\Phi(z)/\Phi(z)$ is written as follows by definition:

$$\frac{T_{z_1}\Phi(z)}{\Phi(z)} = q^{n+1}\prod_{k=1}^{4}\frac{(1-a_kz_1)}{(a_k-qz_1)}\prod_{j=2}^{n}\frac{(1-tz_1/z_j)(1-tz_1z_j)}{(t-qz_1/z_j)(t-qz_1z_j)}.$$

Lemma 5.1 *Let $\varphi(z)$ be an arbitrary function such that $\int_0^{\xi\infty}\varphi(z)\,\Phi(z)\,\varpi_q$ converges. The following holds for $\varphi(z)$:*

$$\int_0^{\xi\infty}\Phi(z)\,\nabla\varphi(z)\,\varpi_q = 0.$$

In particular,

$$\int_0^{\xi\infty}\Phi(z)\,\mathcal{A}\nabla\varphi(z)\,\varpi_q = 0. \tag{25}$$

Proof. See [17, Lemma 5.1]. □

Let τ_1 and σ_i be the reflections of the coordinates $z=(z_1,z_2,...,z_n)$ defined as follows:

$$\begin{aligned}\tau_1 &: z_1 \longleftrightarrow z_1^{-1},\\ \sigma_i &: z_1 \longleftrightarrow z_i \quad \text{for} \quad i=2,3,...,n.\end{aligned}$$

Since the Weyl group W_{C_n} of type C_n is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^n \rtimes \mathcal{S}_n$, we may write

$$W_{C_n} = \langle\tau_1,\sigma_2,\sigma_3,...,\sigma_n\rangle, \tag{26}$$

which means W_{C_n} is generated by τ_1 and σ_i, $i=2,3,...,n$.

Let $f(z)$ and $g(z)$ be the functions defined as follows:

$$\begin{aligned} f(z) &:= \prod_{m=1}^{4}(a_m - z_1)\prod_{j=2}^{n}(t-z_1/z_j)(t-z_1z_j),\\ g(z) &:= \prod_{m=1}^{4}(1-a_mz_1)\prod_{j=2}^{n}(1-tz_1/z_j)(1-tz_1z_j).\end{aligned}$$

For $i=2,3,...,n$, we set

$$f_i(z) := \sigma_i f(z), \quad g_i(z) := \sigma_i g(z) \tag{27}$$

and simply $f_1(z) := f(z)$, $g_1(z) := g(z)$. For $i = 1, 2, ..., n$, the explicit forms of $f_i(z)$ and $g_i(z)$ are the following:

$$\begin{aligned} f_i(z) &= \prod_{m=1}^{4}(a_m - z_i)\prod_{j\in I_i}(t - z_i/z_j)(t - z_i z_j), \qquad (28)\\ g_i(z) &= \prod_{m=1}^{4}(1 - a_m z_i)\prod_{j\in I_i}(1 - t z_i/z_j)(1 - t z_i z_j), \qquad (29)\end{aligned}$$

where $I_i := \{1, 2, ..., i-1, i+1, ...n\}$. By definition, we have

$$\tau_1\left(\frac{f_1(z)}{z_1^{n+1}}\right) = \frac{g_1(z)}{z_1^{n+1}}. \qquad (30)$$

Let $\overline{\varphi}_i(z)$, $1 \le i \le n$, be the function defined by

$$\overline{\varphi}_i(z) := \frac{\mathcal{A}\nabla\varphi_i(z)}{2}$$

where

$$\varphi_i(z) := \frac{f(z)}{z_1^{n+1}}\, z_2^{n-1} z_3^{n-2} \dots z_n\, e_{i-1}^{(n-1)}(z_2, z_3, ..., z_n). \qquad (31)$$

Lemma 5.2 *The functions $\overline{\varphi}_i(z)$ are expressed as*

$$\overline{\varphi}_i(z) = \sum_{k=1}^{n}(-1)^{k+1}\frac{f_k(z) - g_k(z)}{z_k^{n+1}} e_{i-1}^{(n-1)}(\widehat{z}_k)\mathcal{A}^{(n-1)}(\widehat{z}_k) \qquad (32)$$

where $(\widehat{z}_k) := (z_1, ..., z_{k-1}, z_{k+1}, ..., z_n)$. On the other hand, $\overline{\varphi}_i(z)$ are expanded by the functions $e_j^{(n)}(z)\mathcal{A}^{(n)}(z)$, $0 \le j \le i$, as follows:

$$\overline{\varphi}_i(z) = \sum_{j=0}^{i} c_{ij} e_j^{(n)}(z)\mathcal{A}^{(n)}(z). \qquad (33)$$

Proof. By definition (24) of ∇ and (31), we have

$$\nabla\varphi_i(z) = \frac{f(z) - g(z)}{z_1^{n+1}} z_2^{n-1} z_3^{n-2} \dots z_n\, e_{i-1}^{(n-1)}(\widehat{z}_1).$$

Then, from (26) and (30), it follows that

$$\begin{aligned} \overline{\varphi}_i(z) &= \mathcal{A}\nabla\varphi_i(z)/2\\ &= \frac{f_1(z) - g_1(z)}{z_1^{n+1}} e_{i-1}^{(n-1)}(\widehat{z}_1)\mathcal{A}^{(n-1)}(\widehat{z}_1)\\ &\quad + \sum_{k=2}^{n}(\operatorname{sgn}\sigma_k)\ \sigma_k\Big[\frac{f_1(z) - g_1(z)}{z_1^{n+1}} e_{i-1}^{(n-1)}(\widehat{z}_1)\,\mathcal{A}^{(n-1)}(\widehat{z}_1)\Big]. \qquad (34)\end{aligned}$$

Thus, we obtain the expression (32) by substituting (27) and the following for (34):

$$\operatorname{sgn}\sigma_k = -1,\ \sigma_k e^{(n-1)}_{i-1}(\widehat{z}_1) = e^{(n-1)}_{i-1}(\widehat{z}_k),\ \sigma_k \mathcal{A}^{(n-1)}(\widehat{z}_1) = (-1)^k \mathcal{A}^{(n-1)}(\widehat{z}_k).$$

Next, from the degrees of the monomials in the expansion of (31), we can obtain the expression (33). This completes the proof. □

Lemma 5.3 *The following hold for $f_k(z)$, $g_k(z)$ and $\zeta_j \in (\mathbf{C}^*)^n$:*

$$\begin{array}{rclll} f_k(\zeta_j) & = & 0 & \text{if} & j \le k \le n, \\ g_k(\zeta_j) & = & 0 & \text{if} & j < k \le n. \end{array}$$

Moreover,

$$\begin{aligned} &\lim_{x\to 0}\left[z_1 z_2 \ldots z_{i-1}\frac{g_i(z)}{z_i^{n+1}}\right]_{z=\zeta_i} \\ &= (-t)^{i-1}\frac{\prod_{k=2}^4(1-a_k a_1 t^{n-i})}{(1-t)(t^{n-i}a_1)^{n-i+2}}\prod_{j=0}^{n-i}(1-t^{j+1})(1-t^{n-i+j}a_1^2). \end{aligned} \tag{35}$$

Proof. From (28), $f_k(z)$ has the factor $(t - z_k/z_{k+1})$ if $1 \le k \le n-1$, and has the factor $(a_1 - z_n)$ if $k = n$. When $z = \zeta_j$, from definition (23) of ζ_j, it follows that $t - z_k/z_{k+1} = 0$ if $j \le k \le n-1$ and $a_1 - z_n = 0$ if $j \le n$. Thus $f_k(\zeta_j) = 0$ if $j \le k \le n$. From (29), it follows that $g_k(z)$ has the factor $(1 - tz_k/z_{k-1})$, so that $g_k(\zeta_j) = 0$ if $j+1 \le k \le n$.

Next, we prove the latter part of Lemma 5.3. From (29), it follows that

$$\begin{aligned} z_1 z_2 \ldots z_{i-1}\frac{g_i(z)}{z_i^{n+1}} &= \frac{\prod_{k=1}^4(1-a_k z_i)}{z_i^{n+1}} \\ &\quad\times(z_1 - tz_i)(z_2 - tz_i)\ldots(z_{i-1} - tz_i) \\ &\quad\times(1 - tz_1 z_i)(1 - tz_2 z_i)\ldots(1 - tz_{i-1}z_i) \\ &\quad\times(1 - tz_i/z_{i+1})(1 - tz_i/z_{i+2})\ldots(1 - tz_i/z_n) \\ &\quad\times(1 - tz_i z_{i+1})(1 - tz_i z_{i+2})\ldots(1 - tz_i z_n). \end{aligned}$$

Put

$$z = \zeta_i = (\underbrace{x^{i-1}, x^{i-2}, \ldots, x}_{i-1}, \underbrace{t^{n-i}a_1, t^{n-i-1}a_1, \ldots, a_1}_{n-i+1}). \tag{36}$$

Then we have

$$\begin{aligned} &\left[z_1 z_2 \ldots z_{i-1}\frac{g_i(z)}{z_i^{n+1}}\right]_{z=\zeta_i} \\ &= \frac{(1 - a_1^2 t^{n-i})\prod_{k=2}^4(1 - a_k a_1 t^{n-i})}{(t^{n-i}a_1)^{n+1}} \\ &\quad\times(x^{i-1} - t^{n-i+1}a_1)(x^{i-2} - t^{n-i+1}a_1)\ldots(x - t^{n-i+1}a_1) \\ &\quad\times(1 - x^{i-1}t^{n-i+1}a_1)(1 - x^{i-2}t^{n-i+1}a_1)\ldots(1 - xt^{n-i+1}a_1) \\ &\quad\times(1-t^2)(1-t^3)\ldots(1-t^{n-i+1}) \\ &\quad\times(1 - t^{2(n-i)}a_1^2)(1 - t^{2(n-i)-1}a_1^2)\ldots(1 - t^{n-i+1}a_1^2), \end{aligned}$$

so that

$$\begin{aligned}
&\lim_{x\to 0}\left[z_1z_2...z_{i-1}\frac{g_i(z)}{z_i^{n+1}}\right]_{z=\zeta_i}\\
&= \frac{(1-a_1^2t^{n-i})\prod_{k=2}^4(1-a_ka_1t^{n-i})}{(t^{n-i}a_1)^{n+1}}\\
&\quad\times(-t^{n-i+1}a_1)^{i-1}\prod_{j=1}^{n-i}(1-t^{j+1})(1-t^{n-i+j}a_1^2)\\
&= (-t)^{i-1}\frac{\prod_{k=2}^4(1-a_ka_1t^{n-i})}{(1-t)(t^{n-i}a_1)^{n-i+2}}\prod_{j=0}^{n-i}(1-t^{j+1})(1-t^{n-i+j}a_1^2),
\end{aligned}$$

which completes the proof. □

Lemma 5.4 *If $i\geq k$, then*

$$\lim_{x\to 0}\left[\frac{z_1}{z_k}\frac{z_2}{z_k}\cdots\frac{z_{k-1}}{z_k}\Big(f_k(z)-g_k(z)\Big)\right]_{z=\zeta_{i+1}} = (-1)^k(t^{k-1}-t^{2n-k-1}a_1a_2a_3a_4).$$

Proof. From (28) and (29), it follows that

$$\begin{aligned}
\frac{z_1}{z_k}\frac{z_2}{z_k}\cdots\frac{z_{k-1}}{z_k}f_k(z) &= \prod_{m=1}^4(a_m-z_k)\\
&\quad\times(tz_1/z_k-1)(tz_2/z_k-1)\dots(tz_{k-1}/z_k-1)\\
&\quad\times(t-z_1z_k)(t-z_2z_k)\dots(t-z_{k-1}z_k)\\
&\quad\times(t-z_k/z_{k+1})(t-z_k/z_{k+2})\dots(t-z_k/z_n)\\
&\quad\times(t-z_kz_{k+1})(t-z_kz_{k+2})\dots(t-z_kz_n),\\
\frac{z_1}{z_k}\frac{z_2}{z_k}\cdots\frac{z_{k-1}}{z_k}g_k(z) &= \prod_{m=1}^4(1-a_mz_k)\\
&\quad\times(z_1/z_k-t)(z_2/z_k-t)\dots(z_{k-1}/z_k-t)\\
&\quad\times(1-tz_1z_k)(1-tz_2z_k)\dots(1-tz_{k-1}z_k)\\
&\quad\times(1-tz_k/z_{k+1})(1-tz_k/z_{k+2})\dots(1-tz_k/z_n)\\
&\quad\times(1-tz_kz_{k+1})(1-tz_kz_{k+2})\dots(1-tz_kz_n).
\end{aligned}$$

From the above equations, if we put

$$z=\zeta_{i+1}=(\underbrace{x^i,x^{i-1},...,x}_{i},\underbrace{t^{n-i-1}a_1,t^{n-i-2}a_1,...,a_1}_{n-i}) \tag{37}$$

and suppose $k\leq i$, then we have the following:

$$\begin{aligned}
\lim_{x\to 0}\left[\frac{z_1}{z_k}\frac{z_2}{z_k}\cdots\frac{z_{k-1}}{z_k}f_k(z)\right]_{z=\zeta_{i+1}} &= (-1)^{k-1}t^{2n-k-1}a_1a_2a_3a_4,\\
\lim_{x\to 0}\left[\frac{z_1}{z_k}\frac{z_2}{z_k}\cdots\frac{z_{k-1}}{z_k}g_k(z)\right]_{z=\zeta_{i+1}} &= (-t)^{k-1}.
\end{aligned}$$

This completes Lemma 5.4. □

Lemma 5.5 *The following holds for* $1 \le j \le i+1$:

$$\lim_{x\to 0}\Big[\Big(\prod_{l=1}^{j-1} z_l^{n-l+2}\Big)\overline{\varphi}_i(z)\Big]_{z=\zeta_j} = (-1)^{j-1} c_{i,j-1}\mathcal{A}^{(n-j+1)}(t^{n-j}a_1, ..., a_1). \tag{38}$$

Proof. From (33), it follows that

$$\Big(\prod_{l=1}^{j-1} z_l^{n-l+2}\Big)\overline{\varphi}_i(z) = \sum_{k=0}^{i} c_{ik}\Big(z_1 z_2 ... z_{j-1}\, e_k^{(n)}(z)\Big)\Big(z_1^n z_2^{n-1} ... z_{j-1}^{n-j+2}\,\mathcal{A}^{(n)}(z)\Big)$$

Put

$$z = \zeta_j = (\underbrace{x^{j-1}, x^{j-2}, ..., x}_{j-1}, \underbrace{t^{n-j}a_1, t^{n-j-1}a_1, ..., a_1}_{n-j+1}) \tag{39}$$

Since $e_k^{(n)}(\zeta_j) = 0$ if $j \le k$ by Corollary 4.3, we have

$$\begin{aligned}&\Big[\Big(\prod_{l=1}^{j-1} z_l^{n-l+2}\Big)\overline{\varphi}_i(z)\Big]_{z=\zeta_j} \\ &= \sum_{k=0}^{j-1} c_{ik}\Big[\Big(z_1 z_2 ... z_{j-1}\, e_k^{(n)}(z)\Big)\Big(z_1^n z_2^{n-1} ... z_{j-1}^{n-j+2}\,\mathcal{A}^{(n)}(z)\Big)\Big]_{z=\zeta_j}.\end{aligned} \tag{40}$$

By definition (19) of $e_k^{(n)}(z)$ and the explicit expression (39) of ζ_j, we have

$$\lim_{x\to 0}\Big[\Big(z_1 z_2 ... z_{j-1}\, e_k^{(n)}(z)\Big)\Big]_{z=\zeta_j} = \begin{cases} 0 & \text{if } \ k < j-1, \\ 1 & \text{if } \ k = j-1. \end{cases} \tag{41}$$

From Weyl's denominator formula (7) and the expression (39) of ζ_j, it follows

$$\lim_{x\to 0}\Big[\Big(z_1^n z_2^{n-1} ... z_{j-1}^{n-j+2}\,\mathcal{A}^{(n)}(z)\Big)\Big]_{z=\zeta_j} = (-1)^{j-1}\mathcal{A}^{(n-j+1)}(t^{n-j}a_1, ..., a_1). \tag{42}$$

Taking the limit $x \to 0$ in both sides of (40) and using (41) and (42), we obtain (38). This completes the proof. ▯

Lemma 5.6 *Set* $(\widehat{\zeta_j}_k) := (\zeta_{j1}, ..., \zeta_{j,k-1}, \zeta_{j,k+1}, ..., \zeta_{jn}) \in (\mathbf{C}^*)^{n-1}$ *for* $\zeta_j \in (\mathbf{C}^*)^n$. *Then*

$$e_{i-1}^{(n-1)}(\widehat{\zeta_j}_k) = 0 \quad \textit{if} \ \ 1 \le k \le j < i.$$

Moreover,

$$e_{i-1}^{(n-1)}(\widehat{\zeta}_{ik}) = 0 \quad \textit{if} \ \ 1 \le k < i.$$

Proof. It is straightforward from (23) and Lemma 4.2. ▯

Lemma 5.7 *The coefficient* c_{ij} *in* (33) *vanishes if* $0 \le j < i-1$. *In particular,* $\overline{\varphi}_i(z)$ *is expanded as*

$$\overline{\varphi}_i(z) = \Big(c_{ii}e_i^{(n)}(z) + c_{i,i-1}e_{i-1}^{(n)}(z)\Big)\mathcal{A}^{(n)}(z).$$

Proof. From (38), in order to prove $c_{ij}=0$ for $0\le j<i-1$, it is sufficient to show that

$$\lim_{x\to 0}\Big[\Big(\prod_{l=1}^{j-1} z_l^{n-l+2}\Big)\overline{\varphi}_i(z)\Big]_{z=\zeta_j}=0 \tag{43}$$

if $1\le j<i$.

We now suppose $1\le j<i$. By Lemma 5.3, if $j<k\le n$, then $f_k(\zeta_j)=g_k(\zeta_j)=0$. Moreover, by Lemma 5.6, if $k\le j<i$, then $e_{i-1}^{(n-1)}(\widehat{\zeta_j}_k)=0$. Since the summand of $\overline{\varphi}_i(z)$ in form (32) has the factors $f_k(z)-g_k(z)$ and $e_{i-1}^{(n-1)}(\widehat{z}_k)$, if we put $z=\zeta_j$, then $\overline{\varphi}_i(\zeta_j)=0$. In particular, we conclude (43). $\square$

Lemma 5.8 *The coefficient $c_{i,i-1}$ in (33) is evaluated as*

$$c_{i,i-1}=\frac{1-t^{n-i+1}}{(1-t)t^{n+1-2i}}\frac{\prod_{k=2}^{4}(1-t^{n-i}a_1a_k)}{a_1}. \tag{44}$$

Proof. By Lemma 5.3, $f_k(\zeta_i)=g_k(\zeta_i)=0$ if $i<k\le n$, and $f_i(\zeta_i)=0$. Moreover, by Lemma 5.6, $e_{i-1}^{(n-1)}(\widehat{\zeta}_{ik})=0$ if $k<i$. Since the summand of $\overline{\varphi}_i(z)$ in form (32) has the factors $f_k(z)-g_k(z)$ and $e_{i-1}^{(n-1)}(\widehat{z}_k)$, if we put $z=\zeta_i$, then

$$\overline{\varphi}_i(\zeta_i)=\Big[(-1)^i\frac{g_i(z)}{z_i^{n+1}}e_{i-1}^{(n-1)}(\widehat{z}_i)\mathcal{A}^{(n-1)}(\widehat{z}_i)\Big]_{z=\zeta_i}. \tag{45}$$

Thus we have

$$\begin{aligned}
&\Big[\Big(\prod_{l=1}^{i-1} z_l^{n-l+2}\Big)\overline{\varphi}_i(z)\Big]_{z=\zeta_i} \qquad (46)\\
&= (-1)^i\Big[\Big(z_1z_2\ldots z_{i-1}\frac{g_i(z)}{z_i^{n+1}}\Big)\Big(z_1z_2\ldots z_{i-1}e_{i-1}^{(n-1)}(\widehat{z}_i)\Big)\\
&\qquad\times\Big(z_1^{n-1}z_2^{n-2}\ldots z_{i-1}^{n-i+1}\mathcal{A}^{(n-1)}(\widehat{z}_i)\Big)\Big]_{z=\zeta_i}
\end{aligned}$$

From the explicit form (36) of ζ_i and definition (19) of $e_i^{(n)}(z)$, we have

$$\lim_{x\to 0}\Big[z_1z_2\ldots z_{i-1}e_{i-1}^{(n-1)}(\widehat{z}_i)\Big]_{z=\zeta_i}=1. \tag{47}$$

Using (36) and Weyl's denominator formula (7), we also have

$$\lim_{x\to 0}\Big[z_1^{n-1}z_2^{n-2}\ldots z_{i-1}^{n-i+1}\mathcal{A}^{(n-1)}(\widehat{z}_i)\Big]_{z=\zeta_i}=(-1)^{i-1}\mathcal{A}^{(n-i)}(t^{n-i-1}a_1,\ldots,a_1). \tag{48}$$

From (46), (47) and (48), it follows that

$$\begin{aligned}
&\lim_{x\to 0}\Big[\Big(\prod_{l=1}^{i-1} z_l^{n-l+2}\Big)\overline{\varphi}_i(z)\Big]_{z=\zeta_i}\\
&= -\lim_{x\to 0}\Big[z_1z_2\ldots z_{i-1}\frac{g_i(z)}{z_i^{n+1}}\Big]_{z=\zeta_i}\mathcal{A}^{(n-i)}(t^{n-i-1}a_1,\ldots,a_1). \qquad (49)
\end{aligned}$$

Comparing (49) with (38), we have

$$c_{i,i-1}=(-1)^i\lim_{x\to 0}\left[z_1z_2...z_{i-1}\frac{g_i(z)}{z_i^{n+1}}\right]_{z=\zeta_i}\frac{\mathcal{A}^{(n-i)}(t^{n-i-1}a_1,...,a_1)}{\mathcal{A}^{(n-i+1)}(t^{n-i}a_1,...,a_1)}. \tag{50}$$

From Weyl's denominator formula (7), it follows that

$$\frac{\mathcal{A}^{(j+1)}(z_1,z_2,...,z_{j+1})}{\mathcal{A}^{(j)}(z_2,...,z_{j+1})}=-\frac{1-z_1^2}{z_1}\prod_{k=2}^{j+1}\frac{(1-z_1/z_k)(1-z_1z_k)}{z_1},$$

so that

$$\frac{\mathcal{A}^{(n-i+1)}(t^{n-i}a_1,...,a_1)}{\mathcal{A}^{(n-i)}(t^{n-i-1}a_1,...,a_1)}=\frac{-1}{(1-t^{n-i+1})}\prod_{j=0}^{n-i}(1-t^{j+1})\frac{(1-t^{n-i+j}a_1^2)}{t^{n-i}a_1}. \tag{51}$$

From (35), (50) and (51), we obtain (44). This completes the proof. □

Lemma 5.9 *The coefficient c_{ii} in* (33) *is evaluated as*

$$c_{ii}=\frac{1-t^i}{1-t}(1-t^{2n-i-1}a_1a_2a_3a_4).$$

Proof. Using Lemma 5.3, $f_k(\zeta_{i+1})=g_k(\zeta_{i+1})=0$ if $i+2\le k\le n$. Since the summand of $\overline{\varphi}_i(z)$ in form (32) has the factors $f_k(z)-g_k(z)$, if we put $z=\zeta_{i+1}$, then

$$\overline{\varphi}_i(\zeta_{i+1})=\left[\sum_{k=1}^{i+1}(-1)^{k+1}\frac{f_k(z)-g_k(z)}{z_k^{n+1}}e_{i-1}^{(n-1)}(\widehat{z}_k)\mathcal{A}^{(n-1)}(\widehat{z}_k)\right]_{z=\zeta_{i+1}}$$

where k in the sum runs from 1 to $i+1$. Thus, it follows that

$$\left[\Big(\prod_{l=1}^{i}z_l^{n-l+2}\Big)\overline{\varphi}_i(z)\right]_{z=\zeta_{i+1}}=S_1(\zeta_{i+1})+S_2(\zeta_{i+1})$$

where $S_1(z)$ and $S_2(z)$ are functions defined by the following:

$$\begin{aligned}
S_1(z) &:= \sum_{k=1}^{i}(-1)^{k+1}\frac{z_1}{z_k}\frac{z_2}{z_k}\cdots\frac{z_{k-1}}{z_k}\Big(f_k(z)-g_k(z)\Big)\\
&\qquad\times\Big(z_1z_2...z_{k-1}\,z_{k+1}...z_i\,e_{i-1}^{(n-1)}(\widehat{z}_k)\Big)\\
&\qquad\times\Big(z_1^{n-1}z_2^{n-2}...z_{k-1}^{n-k+1}z_{k+1}^{n-k}...z_i^{n-i+1}\,\mathcal{A}^{(n-1)}(\widehat{z}_k)\Big),\\
S_2(z) &:= (-1)^i z_i^{n-i+1}\Big(z_1z_2...z_i\frac{f_{i+1}(z)-g_{i+1}(z)}{z_{i+1}^{n+1}}\Big)\\
&\qquad\times\Big(z_1z_2...z_{i-1}\,e_{i-1}^{(n-1)}(\widehat{z}_{i+1})\Big)\\
&\qquad\times\Big(z_1^{n-1}z_2^{n-2}...z_{i-1}^{n-i+1}\,\mathcal{A}^{(n-1)}(\widehat{z}_{i+1})\Big).
\end{aligned}$$

Since $f_{i+1}(\zeta_{i+1})=0$ by Lemma 5.3, it follows that

$$\Big[z_i^{n-i+1}\Big(z_1z_2...z_i\frac{f_{i+1}(z)-g_{i+1}(z)}{z_{i+1}^{n+1}}\Big)\Big]_{z=\zeta_{i+1}}=-\,x^{n-i+1}\Big[z_1z_2...z_i\frac{g_{i+1}(z)}{z_{i+1}^{n+1}}\Big]_{z=\zeta_{i+1}}.$$

From (35) in Lemma 5.3, the RHS of the above equation vanishes if we take the limit $x\to 0$. Since the LHS of that is a factor of $S_2(\zeta_{i+1})$, we have $\lim\limits_{x\to 0}S_2(\zeta_{i+1})=0$.

If $k\le i$, from the explicit form (37) of ζ_{i+1} and definition (19) of $e_i^{(n)}(z)$, we have

$$\lim_{x\to 0}\Big[z_1z_2...z_{k-1}\,z_{k+1}...z_i\,e_{i-1}^{(n-1)}(\widehat{z}_k)\Big]_{z=\zeta_{i+1}}=1.$$

If $k\le i$, we also have

$$\begin{aligned}&\lim_{x\to 0}\Big[z_1^{n-1}z_2^{n-2}...z_{k-1}^{n-k+1}z_{k+1}^{n-k}...z_i^{n-i+1}\,\mathcal{A}^{(n-1)}(\widehat{z}_k)\Big]_{z=\zeta_{i+1}}\\ &=\ (-1)^{i-1}\mathcal{A}^{(n-i)}(t^{n-i-1}a_1,...,a_1)\end{aligned}$$

by using (37) and Weyl's denominator formula (7). Thus, we have

$$\begin{aligned}\lim_{x\to 0}\Big[\Big(\prod_{l=1}^{i}z_l^{n-l+2}\Big)\overline{\varphi}_i(z)\Big]_{z=\zeta_{i+1}}\ &=\ \lim_{x\to 0}S_1(\zeta_{i+1})\\ =\ &(-1)^{i-1}\mathcal{A}^{(n-i)}(t^{n-i-1}a_1,...,a_1)\\ &\times\sum_{k=1}^{i}(-1)^{k+1}\lim_{x\to 0}\Big[\frac{z_1}{z_k}\frac{z_2}{z_k}\cdots\frac{z_{k-1}}{z_k}\Big(f_k(z)-g_k(z)\Big)\Big]_{z=\zeta_{i+1}}.\end{aligned}\tag{52}$$

Comparing (38) with (52), and using Lemma 5.4, we obtain

$$\begin{aligned}c_{ii}\ &=\ -\sum_{k=1}^{i}(-1)^{k+1}\lim_{x\to 0}\Big[\frac{z_1}{z_k}\frac{z_2}{z_k}\cdots\frac{z_{k-1}}{z_k}\Big(f_k(z)-g_k(z)\Big)\Big]_{z=\zeta_{i+1}}\\ &=\ \sum_{k=1}^{i}(t^{k-1}-t^{2n-k-1}a_1a_2a_3a_4)\\ &=\ \frac{1-t^i}{1-t}(1-t^{2n-i-1}a_1a_2a_3a_4),\end{aligned}$$

which completes the proof. □

Theorem 5.10 *The following relation holds between* $e_i^{(n)}(z)$ *and* $e_{i-1}^{(n)}(z)$:

$$\int_0^{\xi\infty}e_i^{(n)}(z)\Phi(z)\mathcal{A}^{(n)}(z)\varpi_q=-\frac{c_{i,i-1}}{c_{ii}}\int_0^{\xi\infty}e_{i-1}^{(n)}(z)\Phi(z)\mathcal{A}^{(n)}(z)\varpi_q,\tag{53}$$

where the coefficient is evaluated as

$$-\frac{c_{i,i-1}}{c_{ii}}=-\frac{(1-t^{n+1-i})}{(1-t^i)t^{n+1-2i}}\frac{\prod_{k=2}^{4}(1-t^{n-i}a_1a_k)}{a_1(1-t^{2n-i-1}a_1a_2a_3a_4)}.$$

Remark. In other words, by definition (13), Theorem 5.10 is nothing but Theorem 1.2.

Proof. Since $\int_0^{\xi\infty} \Phi(z)\,\overline{\varphi}_i(z)\,\varpi_q = 0$ by (25) in Lemma 5.1, from Lemma 5.7, it follows that

$$\int_0^{\xi\infty} \Phi(z)\Big(c_{ii}e_i^{(n)}(z) + c_{i,i-1}e_{i-1}^{(n)}(z)\Big)\mathcal{A}^{(n)}(z)\varpi_q = 0.$$

We therefore obtain (53). The evaluation of the coefficient $-c_{i,i-1}/c_{ii}$ is given by Lemma 5.8 and 5.9. The proof is now complete. ∎

6 Product formula

The aim of this section is to deduce a product formula for the BC_n type Jackson integral as if reconstructing the product expression (2) of the beta function from q-difference equations (3) and asymptotic behavior (4). The following formula has been proved by van Diejen [26]. He has done it to calculate a certain multiple Jackson integral in two ways by using Fubini's theorem, following Gustafson's method [9]. We give here another proof of it as a consequence of Theorem 1.2.

Theorem 6.1 (van Diejen) *The constant C in the expression* (18) *is the following:*

$$C = (1-q)^n (q)_\infty^n \prod_{i=1}^n \frac{(qt^{-i})_\infty}{(qt^{-1})_\infty} \frac{\prod_{1\le\mu<\nu\le 4}(qt^{-(n-i)}a_\mu^{-1}a_\nu^{-1})_\infty}{(qt^{-(n+i-2)}a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1})_\infty}.$$

Before proving Theorem 6.1, we have to establish q-difference equations and asymptotic behavior for the BC_n type Jackson integral.

6.1 q-difference equations

First we deduce a recurrent relation which $J(\xi)$ satisfies, using Theorem 1.2.

Corollary 6.2 *Let T_{a_1} be the q-shift of parameter a_1 such that T_{a_1}: $a_1 \to qa_1$. Then*

$$T_{a_1}J(\xi) = (-a_1)^{-n} \prod_{i=1}^n \frac{\prod_{k=2}^4 (1-t^{n-i}a_1a_k)}{1-t^{n+i-2}a_1a_2a_3a_4} J(\xi).$$

Remark. The parameters a_1, a_2, a_3 and a_4 can be replaced symmetrically in the above equation.

Proof. The function $T_{a_1}J(\xi)$ is written

$$T_{a_1}J(\xi) = \int_0^{\xi\infty} \frac{T_{a_1}\Phi(z)}{\Phi(z)}\Phi(z)\Delta(z)\varpi_q = \int_0^{\xi\infty} e_n'(z)\Phi(z)\Delta(z)\varpi_q$$

because the following holds for $\Phi(z)$ by Lemma 4.1:

$$\frac{T_{a_1}\Phi(z)}{\Phi(z)} = \prod_{i=1}^n \frac{(a_1-z_i)(1-a_1z_i)}{a_1z_i} = e_n'(z).$$

From repeated use of Theorem 1.2, we have

$$\int_0^{\xi\infty} e_n'(z)\Phi(z)\Delta(z)\varpi_q = (-a_1)^{-n}\prod_{j=1}^{n}\frac{\prod_{k=2}^4(1-t^{n-j}a_1a_k)}{1-t^{n+j-2}a_1a_2a_3a_4}J(\xi).$$

This completes the proof. □

Let T^N be the shift of parameters for the special direction defined by

$$T^N:\left\{\begin{array}{lll} a_1 & \to & a_1q^{2N},\\ a_2 & \to & a_2q^{-N},\\ a_3 & \to & a_3q^{-N},\\ a_4 & \to & a_4q^{-N}.\end{array}\right.$$

Lemma 6.3 *The following holds for the shift T^N:*

$$\begin{aligned} J(\xi) \; = \; & \prod_{i=1}^{n}\frac{(a_1a_2^2a_3^2a_4^2t^{3(n-i)})^N\prod_{2\le\mu<\nu\le4}(qt^{-(n-i)}a_\mu^{-1}a_\nu^{-1})_{2N}}{q^{2N(N+1)}(qt^{-(n+i-2)}a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1})_N\prod_{k=2}^4(t^{n-i}a_1a_k)_N}\\ & \times T^NJ(\xi).\end{aligned}$$

Proof. Applying Corollary 6.2 to $J(\xi)$ repeatedly, we obtain the above relation between $J(\xi)$ and $T^NJ(\xi)$. □

6.2 Asymptotic behavior of truncated Jackson integral

Next we consider an asymptotic behavior of $J(\zeta)$.

Lemma 6.4 *The asymptotic behavior of the truncated Jackson integral $T^NJ(\zeta)$ at $N\to+\infty$ is the following*:

$$T^NJ(\zeta)\sim(1-q)^n\prod_{i=1}^{n}\frac{q^{2N(N+1)}(t^{n-i}a_1)^{1-\alpha_1-\cdots-\alpha_4-2(n-i)\tau}}{(a_1a_2^2a_3^2a_4^2t^{3(n-i)})^N}\frac{(q)_\infty(t)_\infty}{(t^i)_\infty}.$$

Proof. We divide $\Phi(z)\Delta(z)$ into the following three parts:

$$\Phi(z)\Delta(z) = I_1(z)I_2(z)I_3(z)$$

where

$$\begin{aligned} I_1(z) &= \prod_{i=1}^{n} z_i^{1-\alpha_1-\cdots-\alpha_4-2(n-i)\tau},\\ I_2(z) &= \prod_{i=1}^{n}(qa_1^{-1}z_i)_\infty\prod_{1\le j<k\le n}(1-z_j/z_k)\frac{(qt^{-1}z_j/z_k)_\infty}{(tz_j/z_k)_\infty},\\ I_3(z) &= \prod_{i=1}^{n}\frac{(1-z_i^2)}{(a_1z_i)_\infty}\prod_{m=2}^{4}\frac{(qa_m^{-1}z_i)_\infty}{(a_mz_i)_\infty}\\ &\quad\times\prod_{1\le j<k\le n}(1-z_jz_k)\frac{(qt^{-1}z_jz_k)_\infty}{(tz_jz_k)_\infty}.\end{aligned}$$

Thus, $T^N J(\zeta)$ is expressed as

$$\begin{aligned} T^N J(\zeta) &= (1-q)^n \sum_{\nu\in D} T^N\Big(\Phi(q^\nu\zeta)\Delta(q^\nu\zeta)\Big) \\ &= (1-q)^n \sum_{\nu\in D} T^N I_1(q^\nu\zeta)\, T^N I_2(q^\nu\zeta)\, T^N I_3(q^\nu\zeta) \end{aligned} \tag{54}$$

where

$$\begin{aligned} T^N I_1(q^\nu\zeta) &= \prod_{i=1}^n (t^{n-i} a_1 q^{\nu_i+2N})^{1-\alpha_1-\cdots-\alpha_4-2(n-i)\tau+N}, \\ T^N I_2(q^\nu\zeta) &= \prod_{i=1}^n (q t^{n-i} q^{\nu_i})_\infty \\ &\quad\times \prod_{1\le j<k\le n} (1-t^{k-j} q^{\nu_j-\nu_k}) \frac{(t^{k-j-1} q^{\nu_j-\nu_k+1})_\infty}{(t^{k-j+1} q^{\nu_j-\nu_k})_\infty}, \\ T^N I_3(q^\nu\zeta) &= \prod_{i=1}^n \frac{(1-t^{2(n-i)} a_1^2 q^{\nu_i+4N})}{(t^{n-i} a_1^2 q^{\nu_i+4N})_\infty} \prod_{m=2}^4 \frac{(t^{n-i} a_1 a_m^{-1} q^{1+\nu_i+3N})_\infty}{(t^{n-i} a_1 a_m q^{\nu_i+N})_\infty} \\ &\quad\times \Bigg[\prod_{1\le j<k\le n} (1-t^{2n-j-k} a_1^2 q^{\nu_j+\nu_k+4N}) \\ &\qquad\qquad \times \frac{(t^{2n-j-k-1} a_1^2 q^{1+\nu_j+\nu_k+4N})_\infty}{(t^{2n-j-k+1} a_1^2 q^{\nu_j+\nu_k+4N})_\infty} \Bigg]. \end{aligned}$$

Equation (54) indicates that the summand $T^N\Big(\Phi(q^\nu\zeta)\Delta(q^\nu\zeta)\Big)$ of $T^N J(\zeta)$ corresponding to $\nu = (0,0,...,0) \in D$ gives the principal term of asymptotic behavior of $T^N J(\zeta)$ at $N \to +\infty$ because the point $(0,0,...,0) \in D$ is the vertex of the cone D. Hence we have

$$T^N J(\zeta) \sim (1-q)^n T^N I_1(\zeta)\, T^N I_2(\zeta)\, T^N I_3(\zeta). \tag{55}$$

Moreover the asymptotic behavior of each $T^N I_i(\zeta)$ at $N \to +\infty$ is the following:

$$\begin{aligned} T^N I_1(\zeta) &= \prod_{i=1}^n (t^{n-i} a_1 q^{2N})^{1-\alpha_1-\cdots-\alpha_4-2(n-i)\tau+N} \\ &= \prod_{i=1}^n \frac{(t^{n-i} a_1)^{1-\alpha_1-\cdots-\alpha_4-2(n-i)\tau} q^{2N(N+1)}}{(a_1 a_2^2 a_3^2 a_4^2 t^{3(n-i)})^N}, \end{aligned} \tag{56}$$

$$\begin{aligned} T^N I_2(\zeta) &= \prod_{i=1}^n (q t^{n-i})_\infty \prod_{1\le j<k\le n} (1-t^{k-j}) \frac{(q t^{k-j-1})_\infty}{(t^{k-j+1})_\infty} \\ &= \prod_{i=1}^n \frac{(q)_\infty (t)_\infty}{(t^i)_\infty}, \end{aligned} \tag{57}$$

$$\begin{aligned} T^N I_3(\zeta) &= \prod_{i=1}^{n} \frac{(1-t^{2(n-i)}a_1^2q^{4N})}{(t^{n-i}a_1^2q^{4N})_\infty} \prod_{m=2}^{4} \frac{(t^{n-i}a_1a_m^{-1}q^{1+3N})_\infty}{(t^{n-i}a_1a_mq^N)_\infty} \\ &\quad \times \prod_{1\le j<k\le n} (1-t^{2n-j-k}a_1^2q^{4N}) \frac{(t^{2n-j-k-1}a_1^2q^{1+4N})_\infty}{(t^{2n-j-k+1}a_1^2q^{4N})_\infty} \\ &\sim 1 \quad (N\to+\infty). \end{aligned} \tag{58}$$

Combining (55), (56), (57) and (58), we obtain Lemma 6.4. □

6.3 Proof of Theorem 6.1

Theorem 6.5 *The truncated Jackson integral $J(\zeta)$ is evaluated as*

$$\begin{aligned} &J(\zeta) = (1-q)^n(q)_\infty^n \\ &\quad\times \prod_{i=1}^{n} \frac{(t)_\infty}{(t^i)_\infty} \frac{(t^{n-i}a_1)^{1-\alpha_1-\cdots-\alpha_4-2(n-i)\tau} \prod_{2\le\mu<\nu\le 4}(qt^{-(n-i)}a_\mu^{-1}a_\nu^{-1})_\infty}{(qt^{-(n+i-2)}a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1})_\infty \prod_{k=2}^{4}(t^{n-i}a_1a_k)_\infty}. \end{aligned}$$

Proof. It is straightforward from Lemma 6.3 and Lemma 6.4 □

As a consequence of Theorem 6.5, we deduce Theorem 6.1.

Proof of Theorem 6.1. The constant C is written $C = J(\xi)/\Theta(\xi)$ by virtue of Lemma 3.1. In particular, putting $\xi=\zeta$, from Theorem 6.5, we obtain

$$C = \frac{J(\zeta)}{\Theta(\zeta)} = (1-q)^n(q)_\infty^n \prod_{i=1}^{n} \frac{(qt^{-i})_\infty}{(qt^{-1})_\infty} \frac{\prod_{1\le\mu<\nu\le 4}(qt^{-(n-i)}a_\mu^{-1}a_\nu^{-1})_\infty}{(qt^{-(n+i-2)}a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1})_\infty},$$

because $\Theta(\xi)$ in (17) is evaluated at $\xi=\zeta$ as

$$\Theta(\zeta) = \prod_{i=1}^{n} \frac{\theta(t)}{\theta(t^i)} \frac{(t^{n-i}a_1)^{1-\alpha_1-\cdots-\alpha_4-2(n-i)\tau}}{\prod_{k=2}^{4}\theta(t^{n-i}a_1a_k)}.$$

The proof of Theorem 6.1 is now complete. □

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