PADÉ INTERPOLATION TABLE AND BIORTHOGONAL RATIONAL FUNCTIONS

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Abstract. We study recurrence relations and biorthogonality properties for polynomials and rational functions in the problem of the Padé interpolation in the usual scheme and in the scheme with prescribed poles and zeros. The main result is deriving explicit orthogonality and biorthogonality relations for polynomials and rational functions in both schemes. We show that the simplest linear restrictions in the Padé table (so-called, diagonal, anti-diagonal and vertical strings) lead to different explicit types of biorthogonality relations. Finally, we apply our general theory to a concrete example of the Padé interpolation with prescribed poles and zeros on the elliptic grid. This leads to two types of biorthogonality for elliptic hypergeometric functions ${}_{12}E_{11}$. The first type arises from the Kronecker (anti-diagonal) string and coincides with previously known elliptic BRF. The second type arises from the vertical string. It generates a biorthogonality relation in an infinite set of orthogonality points. This biorthogonality relation is assumed to be new.

Contents

1. Introduction

The Cauchy-Jacobi interpolation problem (CJIP) for the sequence Y_j of (nonzero) complex numbers can be formulated as follows [2], [28]. Given two nonnegative integers n, m, choose a system of (distinct) points $x_j, j = 0, 1, \dots, n+m$ on the complex plane. We are seeking polynomials $Q_m(x;n), P_n(x;m)$ of degrees m and n correspondingly such that

$$Y_j = \frac{Q_m(x_j;n)}{P_n(x_j;m)}, \quad j = 0, 1, \dots n+m \tag{1.0.1}$$

(in our notation we stress, e.g. that polynomial $Q_m(x;n)$, being degree m in x, depends on n as a parameter).

It can happens that solution of the CJIP doesn't exist. In this case it is reasonable to consider a *modified* CJIP:

$$Y_j P_n(x_j;m) - Q_m(x_j;n) = 0, \ j = 0, 1, \dots, n+m, \tag{1.0.2}$$

where polynomials $P_n(x;m), Q_m(x;n)$ can be now unrestricted. The problem (1.0.2) always has a nontrivial solution. In exceptional case, if the system (1.0.1) has no solutions, some zeroes of polynomials $P_n(z;m)$ and $Q_m(z;n)$ coincide with interpolated points x_s. Such points, in this case, are called unattainable [28].

The CJIP is called *normal* if polynomials $Q_m(x;n), P_n(x;m)$ exist for all values of $m, n = 0, 1, \dots$ and polynomials $Q_m(x;n), P_n(x;m)$ have no common zeroes. This means, in particulary, that polynomials $Q_m(x;n), P_n(x;m)$ have no roots, coinciding with interpolation points, i.e.

$$Q_m(x_j;n) \neq 0, \ P_n(x_j;m) \neq 0, \quad j = 0, 1, \dots n+m \tag{1.0.3}$$

In a special case when there exists an analytic function $f(z)$ of complex variable such that $f(x_j) = Y_j$ the corresponding CJIP is called multipoint Padé approximation problem [2].

There is a modification of CJIP with prescribed poles and zeros [39]. Let a_i and b_i be two given sequences. We will assume that $a_i \neq a_j, \ b_i \neq b_j$ if $i \neq j$ and moreover $a_i \neq b_j$ for all i, j. For all $n = 0, 1, \dots$ introduce n-th degree polynomials $A_n(x) = (x-a_1)\dots(x-a_n)$ and $B_n(x) = (x-b_1)\dots(x-b_n)$ (of course it is assumed that $A_0 = B_0 = 1$). Then we are seeking again polynomials $Q_m(x;n), P_n(x;m)$ such that

$$Y_j = \frac{A_n(x)}{B_m(x)} \frac{Q_m(x_j;n)}{P_n(x_j;m)}, \quad j = 0, 1, \dots n+m. \tag{1.0.4}$$

Note that in this case we extract explicitly the part $A_n(x)$ with prescribed zeros $a_1, a_2, \dots, n$ and the part $B_m(x)$ with prescribed poles $b_1, b_2, \dots, b_m$. Equivalently, conditions (1.0.4) can be rewritten in the form

$$Y_j = \frac{V_m(x_j;n)}{U_n(x_j;m)}, \quad j = 0, 1, \dots n+m, \tag{1.0.5}$$

where

$$V_m(x) = Q_m(x;n)/B_m(x), \ U_n(x) = P_n(x;m)/A_n(x) \tag{1.0.6}$$

are rational functions with prescribed poles. Thus, the scheme with prescribed poles and zeros is obtained from the ordinary CJIP by replacing polynomials $Q_m(x;n), P_n(x;m)$ with rational functions $V_m(x;n), U_n(x;m)$.

Recall that the standard *Padé approximation* [2] problem consists in finding polynomials $Q_m(x;n), \ P_n(x;m)$ such that for given function $f(x)$ (which is assumed to be analytical near $x = 0$) we have the condition

$$f(x) - \frac{Q_m(x;n)}{P_n(x;m)} = O(x^{n+m+1}). \tag{1.0.7}$$

It is clear that the ordinary Padé problem can be obtained by the limiting process $x_j \to 0$ for all j. In this respect, CJIP can be considered as a generalization of the ordinary Padé problem.

It is well known [2] that diagonal strings in the usual Padé table, (i.e. $n - m = N$ with fixed N) correspond to orthogonal polynomials $S_n(x) = \gamma_n x^n \, P_n(1/x; n - N) = x^n + O(x^{n-1})$, $n = 0, 1, \dots$ with some constants γ_n. This means that there exists a linear functional σ defined on the space of polynomials such that

$$\langle \sigma, S_n(x) S_m(x) \rangle = h_n \delta_{nm} \tag{1.0.8}$$

with some normalization constants h_n. Under some natural restrictions we have $h_n \neq 0$. In this (general) case the scheme is nondegenerated. Equivalently, polynomials $S_n(x)$ satisfy three-term recurrence relation

$$S_{n+1}(x) + b_n S_n(x) + u_n S_{n-1}(x) = x S_n(x) \tag{1.0.9}$$

which together with initial conditions $S_0 = 1, S_1 = x - b_0$ completely determines (for $u_n \neq 0$) the linear functional σ [8].

Thus general orthogonal polynomials can be interpreted as denominator polynomials in diagonal Padé approximations. Note that historically orthogonal polynomials first appeared in works by Chebyshev and Stieltjes just in this way [2].

Rational functions can form orthogonal systems as well as polynomials [6]. However, in contrast to the case of polynomials, the rational functions admits *biorthogonality property.* Many special examples of biorthogonal rational functions (BRF) were constructed [45], [31]. In these examples a remarkable *duality* property was observed: BRF satisfy both 3-term recurrence relation and second-order difference equation on some grid with respect to argument. Moreover, in all these examples the orthogonality grid x_s coincides with the grid for the Askey-Wilson polynomials (i.e. "quadratic", or "q-quadratic" grid using terminology of [30]).

In [49] it was established that theory of BRF is equivalent to generalized eigenvalue problem (GEVP) for two arbitrary tri-diagonal matrices J_1, J_2 with the eigenvalue z:

$$J_1 \vec{R} = z J_2 \vec{R}, \tag{1.0.10}$$

where $\vec{R} = \{R_0(z), R_1(z), \dots\}$ is a vector constructed from BRF $R_n(z)$. In [35], [36] a new explicit family of BRF was constructed. These BRF appeared to be biorthogonal on so-called *elliptic grid.* Corresponding BRF are closely related with the so-called elliptic $6j$-symbols expressed in terms of modular hypergeometric functions introduced by Frenkel and Turaev [15]. These BRF satisfy dual property and today it is assumed that they are the most general BRF with such property.

The main goal of this paper is to analyze how orthogonality and biorthogonality relations arise from different strings of the Padé interpolation table. Our main result is that theory of BRF naturally arises from the Padé interpolation table. This can be considered as a generalization of the well known results concerning relations between the ordinary orthogonal polynomials and the Padé approximation table. Some preliminary and special results were published earlier in our papers [37, 38, 50, 51, 52].

2. The ordinary Padé interpolation table. Strings and orthogonality

In this section we consider the ordinary Padé interpolation scheme. We will assume that the the interpolation scheme is normal.

In what follows we will use basic important relations for the Padé interpolants. The first one is so-called generalized orthogonality property. In order to get it we rewrite (1.0.1) in the form

$$Y_s P_n(x_s;m) = Q_m(x_s;n) \tag{2.0.1}$$

This form is equivalent to (1.0.1) in case of nondegenerate Padé interpolation problem (i.e. none of interpolated points x_s coincide with zeroes of $P_n(z;m)$ or $Q_m(z;N)$).

Introduce the so-called divided-difference operator [17], [29] $[z_0, z_1, \dots, z_N]$ which is defined on a finite set of $N+1$ points $\{z_0, z_1, \dots, z_N\}$ by the formula

$$[z_0, z_1, \dots, z_N]f(z) = \sum_{k=0}^{N} \frac{f(z_k)}{\Omega'_{N+1}(z_k)}, \tag{2.0.2}$$

where

$$\Omega_N(z) = (z - z_0)(z - z_1)\dots(z - z_N) \tag{2.0.3}$$

There is an equivalent (Hermite) form of this operator which sometimes is much more convenient for analysis:

$$[z_0, z_1, \dots, z_N] = (2\pi i)^{-1} \int_\Gamma \frac{f(\zeta)d\zeta}{\Omega_{N+1}(\zeta)}, \tag{2.0.4}$$

where the contour Γ in complex plane is chosen such that points $z_0, z_1, \dots, z_N$ lie inside the contour whereas all singularities of the function $f(z)$ lie outside the contour. The divided-difference operator has many properties similar to those for the ordinary derivative operator. In particular,

$$[z_0, z_1, \dots, z_N]f(z) \equiv 0 \tag{2.0.5}$$

if $f(z)$ is any polynomial of degree $< N$. For every polynomial $f(z) = r_N z^N + O(z^{N-1})$ of degree N with leading coefficient r_N we have also

$$[z_0, z_1, \dots, z_N]f(z) = r_N \tag{2.0.6}$$

Apply property (2.0.2) to (2.0.1) to get

$$[x_0, x_1, \dots, x_{n+m}]\{q_j(z)f(z)P_n(z;m)\} = 0, \ j = 0, 1, \dots, n-1 \tag{2.0.7}$$

where $q_j(z)$ is any polynomial of degree j and and we assume that the function $f(z)$ exists such that $f(x_s) = Y_s$. Relation can be rewritten in the Hermite form

$$\int_\Gamma \frac{q_j(\zeta)f(\zeta)P_n(\zeta;m)d\zeta}{\omega_{m+n+1}(\zeta)} = 0, \ j = 0, 1, \dots, n-1, \tag{2.0.8}$$

where $\omega_n(x)$ is defined in (2.0.3) with $z_i = x_i$.

In general case, if the function $f(z)$ doesn't exists, this relation can be presented in the form

$$\sum_{s=0}^{m+n} \frac{Y_s q_j(x_s) P_n(x_s;m)}{\omega'_{m+n+1}(x_s)} = 0, \ j = 0, 1, \ldots, n-1 \tag{2.0.9}$$

The property (2.0.9) is a *generalized orthogonality property* of the Padé interpolants [28]. In what follows we will use this property to derive orthogonality and biorthogonality properties for polynomials and rational functions.

We will use also the fundamental Frobenius-type relations for the Padé interpolants [3]. We will assume that denominator polynomials $P_n(z;m)$ are monic $P_n(z;m) = z^n + O(z^{n-1})$, whereas numerator polynomials $Q_m(z;n)$ have the leading term α_{nm}: $Q_m(z;n) = \alpha_{nm} z^m + O(z^{m-1})$. As the scheme is assumed to be normal, we have necessarily $\alpha_{nm} \neq 0$. Then the Frobenius-type relations for denominator polynomials $P_n(z;m)$ are [3]

$$\begin{aligned} &P_{n+1}(z;m) - (z - x_{n+m+1}) P_n(z;m) + \frac{\alpha_{nm}}{\alpha_{n,m+1}} P_n(z;m+1) = 0 \\ &P_{n+1}(z;m) - P_{n+1}(z;m+1) + \frac{\alpha_{n+1,m+1}}{\alpha_{n,m+1}} P_n(z;m+1) = 0 \end{aligned} \tag{2.0.10}$$

Similar relations can be written for numerator polynomials:

$$\begin{aligned} &Q_m(z;n+1) - (z - x_{n+m+1}) Q_m(z;n) + \frac{\alpha_{nm}}{\alpha_{n,m+1}} Q_{m+1}(z;n) = 0 \\ &Q_m(z;n+1) - Q_{m+1}(z;n+1) + \frac{\alpha_{n+1,m+1}}{\alpha_{n,m+1}} Q_{m+1}(z;n) = 0 \end{aligned} \tag{2.0.11}$$

These relations are compatible if condition

$$\frac{\alpha_{n+1,m}}{\alpha_{nm}} + \frac{\alpha_{nm}}{\alpha_{n,m+1}} - \frac{\alpha_{n,m-1}}{\alpha_{nm}} - \frac{\alpha_{nm}}{\alpha_{n-1,m}} + x_{n+m+1} - x_{n+m} = 0 \tag{2.0.12}$$

is fulfilled [3].

In what follows we will consider some one-dimensional *strings* in the two-dimensional table m, n. These strings will appear by imposing simple linear relations $\phi(m,n) = 0$ upon variables m, n. Any such string will generate corresponding one-dimensional set of polynomials, say $\{P_n(z;m), \ \phi(m,n) = 0\}$. As we will see, such one-dimensional sets possess remarkable orthogonality or biorthogonality properties. Note that recurrence relations for different kinds of strings in the Padé interpolation problem were considered e.g. in [19].

Finally, we give a simplest explicit example of the Padé interpolation table for the exponential function on the uniform grid [20], [50]. This example will be used in next subsections to illustrate all obtained formulas.

Assume that the interpolated grid coincides with nonnegative integers $x_s = s = 0, 1, \ldots$. For interpolated sequence Y_s we take simple exponential grid $Y_s = q^s$. Clearly, this is equivalent to interpolation of the exponential function $f(z) = \exp(\omega z)$ with $q = \exp(\omega)$. The parameter q is an arbitrary nonzero complex number with the only restriction $q^N \neq 1$ for $N = 0, 1, \ldots$.

Explicit solutions for the numerator and denominator polynomials are [50]

$$Q_m(z;n) = (-1)^n(1-1/q)^{-n}\,(1+m)_n\,{}_2F_1\left({-m,-z \atop -m-n};1-q\right),$$

$$P_n(z;m) = (-1)^n(1-1/q)^{-n}\,(1+m)_n\,{}_2F_1\left({-n,-z \atop -m-n};1-1/q\right) \qquad (2.0.13)$$

Polynomials $P_n(z;m)$ are monic and the leading term of the polynomials $Q_m(z;n)$ is

$$\alpha_{nm} = (-1)^m q^n (1-q)^{m-n}\,\frac{(1+m)_n}{(1+n)_m} \qquad (2.0.14)$$

It is easily verified that relation (2.0.12) holds identically for coefficients (2.0.14). The Frobenius-type relations (2.0.10) and (2.0.11) can be verified using standard transformation formulas for the Gauss hypergeometric function [13].

2.1. **The Kronecker strings and the ordinary orthogonal polynomials.** By the *Kronecker string* we mean an *antidiagonal* $n+m=N$ in the Padé table with fixed N. The term is justified by the Kronecker method in numerical interpolation [2], where just antidiagonals of the Padé interpolation tables are exploited. Clearly, for every $N>0$ we have exactly $N+1$ different denominator polynomials $P_0(z;N), P_1(z;N-1), \dots P_N(z;0)$ (and the same number of numerator polynomials $Q_{N-n}(z;n)$).

It is convenient to denote $S_n(z;N) = P_n(z;N-n),\ n=0,1,\dots,N$ and $\alpha_{n,N-n} = \alpha_n(N)$ From Frobenius-type relations (2.0.10) we have

$$S_n(z;N+1) = S_n(z;N) + \frac{\alpha_n(N+1)}{\alpha_{n-1}(N)} S_{n-1}(z;N),$$

$$(z-x_N)S_n(z;N-1) = S_{n+1}(z;N) + \frac{\alpha_n(N-1)}{\alpha_n(N)} S_n(z;N) \qquad (2.1.1)$$

From these relations we immediately derive three-term recurrence relation for polynomial $S_n(z;N)$ with fixed N:

$$S_{n+1}(z;N) + b_n(N)S_n(z;N) + u_n(N)S_{n-1}(z;N) = zS_n(z;N),\ n=0,1,\dots,N-1 \quad (2.1.2)$$

where

$$b_n(N) = \frac{\alpha_{n+1}(N+1)}{\alpha_n(N)} + \frac{\alpha_n(N)}{\alpha_n(N+1)} + x_{N+1},\ u_n = \frac{\alpha_n(N)}{\alpha_{n-1}(N)} \qquad (2.1.3)$$

We see that polynomials $S_n(z;N)$ satisfy standard recurrence relation (1.0.9) defining *orthogonal polynomials*. These polynomials are nondegenerate because $u_n \neq 0,\ n = 1,2,\dots,N-1$. From generalized orthogonality relation (2.0.9) we derive that these polynomials satisfy discrete orthogonality relation on the interpolation grid x_s:

$$\sum_{s=0}^{N} w_s(N) S_n(x_s;N) S_{n'}(x_s;N) = h_n(N)\delta_{nn'}, \qquad (2.1.4)$$

where the weights are

$$w_s(N) = \frac{Y_s}{\omega'_{N+1}(x_s)}. \qquad (2.1.5)$$

The normalization constants

$$h_n(N) = u_1(N)u_2(N)\dots u_n(N) \neq 0$$

are nonzero due to nondegeneracy of the Padé interpolation problem.

The fact that anti-diagonal string of the Padé interpolation table gives rise to finite-orthogonal polynomials was noticed earlier [12] (see also [11]). It is important to note that relations (2.1.1) coincide with the so-called Geronimus and Christoffel transforms for orthogonal polynomials [48], [34]. On the other hand, condition (2.0.12) in this case is equivalent to so-called shifted qd-algorithm. Recall, that the shifted qd-algorithm is a powerful computational tool in linear algebra [14] which generalizes famous qd-algorithm by Rutishauser.

Our next result will be identification of numerator polynomials $Q_{N-n}(z;n)$ in the Kronecker string with so-called "dual" orthogonal polynomials introduced in [4], [5]. Let $S_n(z),\ n = 0, 1, \dots, N$ be a finite system of nondegenerate orthogonal polynomials satisfying recurrence relation (1.0.9) for $n = 0, 1, \dots, N-1$. The "dual" polynomials are finite OP $\tilde{S}_n(z),\ n = 0, 1, \dots, N$ satisfying three-term recurrence relation

$$\tilde{S}_{n+1}(x) + b_{N-n}\tilde{S}_n(x) + u_{N+1-n}\tilde{S}_{n-1}(x) = x\tilde{S}_n(x),\ n = 0, 1, \dots N-1 \tag{2.1.6}$$

and the initial conditions

$$\tilde{S}_0(x) = 1, \quad \tilde{S}_1(x) = x - b_N \tag{2.1.7}$$

i.e. the Jacobi matrix $\tilde{J}$ for polynomials $\tilde{S}_n(z)$ is obtained from Jacobi matrix J for OP $S_n(z)$ by reflection from its main antidiagonal. This means, in particular, that transformation $S_n(z) \to \tilde{S}_n(z)$ is an involution, i.e. the dual polynomials with respect to $\tilde{S}_n(z)$ coincide with initial polynomials $S_n(z)$.

Assume that w_s be discrete weights for polynomials $S_n(z)$ on a set of (distinct) orthogonality points x_s:

$$\sum_{s=0}^{N} w_s S_n(x_s) S_{n'}(x_s) = h_n \delta_{nn'}, \tag{2.1.8}$$

Analogously let $\tilde{w}_s$ be discrete weights for polynomials $\tilde{S}_n(z)$ on a set y_s:

$$\sum_{s=0}^{N} \tilde{w}_s \tilde{S}_n(y_s) \tilde{S}_{n'}(y_s) = \tilde{h}_n \delta_{nn'}, \tag{2.1.9}$$

Normalization factors are $h_n = u_1 u_2 \dots u_n$ and $\tilde{h}_n = u_N u_{N-1} \dots u_{N+1-n}$.

Note that orthogonality points x_s and y_s are zeroes of the polynomial $S_{N+1}(z)$ and $\tilde{S}_{N+1}(z)$:

$$S_{N+1}(z) = (z - x_0)(z - x_1)\dots(z - x_N),\ \tilde{S}_{N+1}(z) = (z - y_0)(z - y_1)\dots(z - y_N) \tag{2.1.10}$$

In [4], [5] it was shown that

(i) the orthogonality points for polynomials $S_n(z)$ and $\tilde{S}_n(z)$ coincide, i.e. $y_s = x_s,\ s = 0, 1, \dots, N$;

(ii) the weights are related as

$$w_s w_s^* = \frac{h_N}{(S'_{N+1}(x_s))^2}, \quad s = 0, 1, \dots N. \tag{2.1.11}$$

In fact, properties (i)and (ii) characterize dual polynomials $\tilde{S}_n(z)$.

In [42] these properties were explained by a simple observation that the "dual" polynomials $\tilde{S}_n(z)$ coincide with $N+1-n$-associated polynomials $S_n(z)$: $\tilde{S}_n(z) = S_n^{(N+1-n)}(z)$.

Theorem 1. For the Kronecker string $m+n=N$ the numerator polynomials $Q_{N-n}(z;n)$ coincide with "dual" polynomials with respect to denominator polynomials $S_n(z;N) = P_n(z;N-n)$. More exactly, denote $T_n(z;N) = Q_n(z;N-n)/\alpha_{N-n,n}$. Then $T_n(z;N) = \tilde{S}_n(z;N)$, or, equivalently, $T_n(z;N)$ are monic orthogonal polynomials satisfying recurrence relation

$$T_{n+1}(x;N) + b_{N-n}(N)T_n(x;N) + u_{N+1-n}(N)T_{n-1}(x;N) = xT_n(x;N),\ n=0,1,\dots N-1 \tag{2.1.12}$$

and the initial conditions

$$T_0(x) = 1, \quad T_1(x) = x - b_N(N), \tag{2.1.13}$$

where $b_n(N), u_n(N)$ are recurrence coefficients defined by (2.1.3).

Proof. The simplest way to prove this theorem is to notice that the sequence $1/Y_s$ is interpolated sequence for the the same grid x_s and with exchanged numerator and denominator polynomials. Hence, for the Kronecker string, the polynomials $Q_n(z;N-n)/\alpha_{N-n,n}$ are monic orthogonal polynomials with the weights (see (2.1.5)):

$$\tilde{w}_s(N) = \frac{1/Y_s}{\omega'_{N+1}(x_s)}. \tag{2.1.14}$$

Note that $\omega_{N+1}(x) = S_{N+1}(x) = \tilde{S}_{N+1}(x)$. This leads to formula (2.1.11) which characterizes "dual" polynomials.

We thus see that the "dual" orthogonal polynomials $\tilde{S}_n(z)$ have a very natural interpretation as the numerator polynomials in the Kronecker string of the Padé table.

Moreover, there is an important "inverse" theorem allowing to reconstruct the whole Padé interpolation table starting from given set of finite orthogonal polynomials $S_n(x;N)$.

Let $S_n(x;N)$ be a set of finite orthogonal polynomials with the properties:

(i) for every $N>0$ polynomials $S_n(x;N)$ are orthogonal on a finite set of distinct points $x_s,\ s=0,1,\dots,N$:

$$\sum_{s=0}^{N} w_s(N)S_n(x_s;N)S_{n'}(x_s;N) = h_n(N)\delta_{nn'}, \tag{2.1.15}$$

(ii) the discrete weights $w_s(N+1)$ are related with $w_s(N)$ by the Geronimus transform [48], i.e.

$$w_s(N+1) = \frac{w_s(N)}{x_s - x_{N+1}},\ s=0,1,\dots,N \tag{2.1.16}$$

and $w_{N+1}(N+1) = A_{N+1}$ with arbitrary nonzero A_{N+1}. In more details, this means that transition $N \to N+1$ consists in division of all weights $w_s(N)$ at points $x_0, x_1, \dots, x_N$ by multipliers $(x_s - x_{N+1})$ and moreover by adding of a new arbitrary weight A_{N+1} at the point x_{N+1}. It is clear from (2.1.16) that we can express $w_s(N)$ by the explicit formula

$$w_s(N) = \frac{Y_s}{\omega'_{N+1}(x_s)}, \tag{2.1.17}$$

where $\omega_{N+1}(x) = \prod_{k=0}^{N}(x - x_k) = S_{N+1}(x)$ and $Y_s = \frac{A_s}{(x_s - x_0)(x_s - x_1)\dots(x_s - x_{s-1})}$. Let $\tilde{S}_n(x; N)$ be the monic "dual" polynomials corresponding to $S_n(x; N)$ defined by (2.1.6). There is an identity [42]

$$h_n \tilde{S}_{N-1}(x; N) = \tilde{S}_N(x; N) S_n(x; N) - \tilde{S}_{N+1}(x; N) S_{n-1}^{(1)}(x; N) \tag{2.1.18}$$

where $S_{n-1}^{(1)}(x; N)$ are associated polynomials. Putting $x = x_s$ and resembling that x_s are zeroes of $\tilde{S}_{N+1}(x)$ we get

$$h_n \tilde{S}_{N-n}(x_s; N) = \tilde{S}_N(x_s; N) S_n(x_s) \tag{2.1.19}$$

On the other hand, from general theory it follows [42]

$$w_s(N) = \frac{\tilde{S}_N(x_s)}{S'_{N+1}(x_s)} \tag{2.1.20}$$

and thus we have an important relation [4]

$$h_n \tilde{S}_{N-n}(x_s) = w_s(N) S'_{N+1}(x_s) S_n(x_s), \quad s = 0, 1, \dots N \tag{2.1.21}$$

Substituting expression $w_s(N)$ from (2.1.17) we get finally

$$Y_s = \frac{T_{N-n}(x_s; N)}{S_n(x_s; N)}, \quad s = 0, 1, \dots, N, \tag{2.1.22}$$

where $Q_{N-n}(x; N) = h_n \tilde{S}_{N-n}(x; N)$.

We thus see that starting from a sequence of finite orthogonal polynomials $S_n(x; N)$ and their duals $\tilde{S}_n(x; N)$ we can construct a whole Padé interpolation table for the interpolated sequence Y_s on the interpolated grid x_s. Sequence Y_s in general is an arbitrary, but we should exclude (exceptional) cases when the Padé scheme becomes non-normal, i.e. when some of interpolated points x_s coincide with zeroes of $S_n(x; N)$ or $\tilde{S}_n(x; N)$.

Using this approach, one can construct explicit Padé interpolation tables starting from known systems of finite orthogonal polynomials [53], [38].

We present here only the simplest example connected with interpolation (2.0.13) of the exponential function. From (2.1.5) we immediately find the discrete weights

$$w_s = \binom{N}{s} p^s (1 - p)^{N-s}, \tag{2.1.23}$$

where $p = q/(q - 1)$. The weights (2.1.23) describe the usual binomial distribution and hence corresponding polynomials $S_n(z) = P_n(z; N - n)$ are the Krawtchouk polynomials [22]:

$$S_n(z) = \kappa_{n2} F_1 \left({-n, -z \atop -N} ; 1/p \right), \tag{2.1.24}$$

where $\kappa_n = p^n (-N)_n$ is a normalization factor in order for polynomials $S_n(z)$ to be monic.

2.2. **The Kronecker algorithm revisited.** The Kronecker algorithm [2] consists in reconstructing of all entries $Q_{N-n}(z;n),\ P_n(z;N-n), n = 0,1,\dots,N$ of the Kronecker string in the Padé interpolation table. In this section we consider the Kronecker algorithm from the point of view of orthogonal polynomials and their "duals".

Assume that all interpolation values are nonzero: $Y_s \neq 0$. Fix the positive integer $N > 0$ and define the monic polynomial

$$S_{N+1}(z) \equiv P_{N+1}(z;-1) \equiv (z-x_0)(z-x_1)\dots(z-x_N) \tag{2.2.1}$$

Next, we should determine the polynomial $S_N(z) \equiv P_N(z;0)$. In order to do this, we notice that by definition of the Padé interpolation we have:

$$\frac{Q_0(x_s;N)}{P_N(x_s;0)} = \frac{\alpha_{N0}}{P_N(x_s;0)} = Y_s, \quad s = 0,1,\dots,N \tag{2.2.2}$$

because $Q_0(z;N) = \alpha_{N0} \equiv const$. Hence, the (non-monic!) polynomial $P_N(z;0)/\alpha_{N0}$ coincides with the ordinary Lagrange interpolation polynomial which interpolates the given sequence $Y_0, Y_1, \dots, Y_N$ on the given grid $x_0, x_1, \dots, x_N$. We thus have by the Lagrange formula

$$\frac{P_N(z;0)}{\alpha_{N0}} = \sum_{s=0}^{N} \frac{S_{N+1}(z)}{Y_s(z-x_s)S'_{N+1}(x_s)} \tag{2.2.3}$$

The value α_{N0} is determined directly from (2.2.3) by examining of leading term of the Lagrange interpolation polynomial $P_N(z;0)/\alpha_{N0}$. Thus we obtain the first entry $Q_0(z;N-n) = \alpha_{N0}$ and $P_N(z;0)$ of the Kronecker string. The only non-trivial procedure is determining of other entries $P_{N-n}(z;n)$ and $Q_n(z;N-n)$ for $n = 1,2,\dots,N$. This can be done using so-called backward algorithm for orthogonal polynomials (see, e.g. [24]). This algorithm works as follows. Let $S_{N+1}(z)$ and $S_N(z)$ be two given polynomials of degrees $N+1$ and N. We assume that these polynomials belong to a family of finite monic orthogonal polynomials $S_0(z), S_1(z), \dots, S_N, S_{N+1}(z)$. This means that there exists a 3-term recurrence relation

$$S_{n+1}(z) + b_n S_n(z) + u_n S_{n-1}(z) = z S_n(z) \tag{2.2.4}$$

with initial conditions $S_{-1} = 0, S_0 = 1$. But in our case we have different initial conditions: polynomials $S_{N+1}(z)$ and $S_N(z)$ are given and all polynomials $S_n,\ 0 \le n < N$ should be restored together with the recurrence coefficients u_n, b_n.

In principle, there are several ways to reconstruct polynomials $S_n(z)$ and recurrence coefficients u_n, b_n. We describe here the simplest one which is equivalent to the Kronecker algorithm.

We can present every polynomial $S_n(z)$ as

$$S_n(z) = z^n + \xi_n z^{n-1} + \eta_n z^{n-2} + O(z^{n-3}) \tag{2.2.5}$$

with some coefficients ξ_n, η_n. Then from recurrence relation (2.2.4) we find expression for the recurrence coefficients b_n, u_n in terms of the expansion coefficients ξ_n, η_n:

$$b_n = \xi_n - \xi_{n+1},\ u_n = \eta_n - \eta_{n+1} - b_n \xi_n \tag{2.2.6}$$

Start from the relation

$$S_{N+1}(z) + b_N S_N(z) + u_N S_{N-1}(z) = z S_N(z), \tag{2.2.7}$$

where the polynomial $S_{N-1}(z)$ and the coefficients u_N, b_N are unknown. As polynomials $S_N(z), S_{N+1}(z)$ are known, we know their expansion coefficients $\xi_N, \xi_{N+1}, \eta_N, \eta_{N+1}$. Hence, by (2.2.6), we can reconstruct recurrence coefficients b_N, u_N. Then we reconstruct polynomial $S_{N-1}(z)$ by

$$S_{N-1}(z) = \frac{(z - b_N)S_N(z) - S_{N+1}(z)}{u_N} \tag{2.2.8}$$

It is seen from (2.2.8) that our procedure leads to the unique polynomial $S_{N-1}(z)$ iff $u_N \neq 0$. And if this condition is fulfilled, then $S_{N-1} = z^{N-1} + O(z^{N-2})$ is indeed a monic polynomial of degree $N-1$. This procedure can be repeated for $n = N-1, N-2, \dots, 1$. And at each step we reconstruct recurrence coefficients u_{N-n}, b_{N-n} and monic polynomial $S_{N-n}(z)$. If $u_n \neq 0$ at each step, then the scheme is nondegenerated, i.e. all polynomials $S_{N-1}(z), S_{N-2}(z), \dots, S_0(z) \equiv 1$ are uniquely reconstructed from the given two polynomials $S_{N+1}(z), S_N(z)$. Thus the denominator polynomials $P_{N-n}(z;n) \equiv S_{N-n}(z)$ are reconstructed uniquely step-by-step for $n = N, N-1, \dots$. What about numerator polynomials $Q_n(z; N-n)$? We know, that, up to a constant factor $\alpha_{n,N-n}$, this polynomials coincide with the "dual" OP $T_n(z)$ with respect to $S_n(z)$. From (2.1.12) we can rewrite recurrence relation for polynomials $Q_n(z; N-n)$ in the form

$$u_{N-n}Q_{n+1}(z; N-n-1) + b_{N-n}Q_n(z; N-n) + Q_{n-1}(z; N-n+1) = zQ_n(z; N-n) \tag{2.2.9}$$

with initial conditions

$$Q_{-1}(z; N+1) = 0,\ Q_0(z; N) = \alpha_{N0} \tag{2.2.10}$$

Thus numerator polynomials $Q_n(z; N-n)$ can be determined *simultaneously* with denominator polynomials if the process is normal. Indeed, for the first step we have $u_N Q_1(z; N-1) = (z - b_N)Q_0(z; N) = (z - b_N)\alpha_{N0}$. We thus can find $Q_1(z; N-1)$ because coefficients u_N, b_N were determined already at the first step. If the scheme is nondegenerate (i.e. $u_N \neq 0$)) then $Q_1(z; N-1)$ is a first degree polynomial in z. This process can be continued: at the n-th step we reconstruct uniquely polynomial $Q_{n+1}(z; N-n)$ from recurrence relation (2.2.9) using already determined coefficients b_{N-n}, u_{N-n}. Again we should assume $u_n \neq 0$ for all $n = 1, 2, \dots, N$. Thus, if the process is nondegenerate, the backward algorithm for orthogonal polynomials leads uniquely to polynomials $Q_n(z; N-n)$ and $P_{N-n}(z, n)$ from the Kronecker string.

The Kronecker process will be nondegenerated, e.g. in the case if the sequence x_s is real and monotonic, say $x_{s+1} > x_s$ for all s and sequence Y_s is real and sign-changed: $Y_s Y_{s+1} < 0$ for all s. Indeed, in this case the weights w_s are all positive or negative as is easily sen from (2.1.5). But this means that $u_n \neq 0$ [8] and hence the Kronecker process is nondegenerated.

Note, however, that for many practically important cases we usually have monotonic sequence $Y_s = f(x_s)$, because it is naturally to assume that the function $f(z)$ is sufficiently monotonic on an interval containing many interpolation points x_s. Then we have $Y_{s+1}Y_s > 0$ for corresponding points of grid. This may, in principle, lead to a situation when the Kronecker algorithm is degenerated. But such a situation is exclusive.

Nondegeneracy of the Kronecker algorithm doesn't guarantee that unattainable points x_s are absent. This means that sometimes the roots of orthogonal polynomials $S_n(z) =$

$P_n(z; N-n)$ can coincide with one or more grid point x_s. We illustrate such possibility by an elementary example. Consider the sequence $Y_s = (-1)^s,\ s = 0, 1, \dots$ while the interpolation grid x_s is an arbitrary (all points x_s are distinct). Then for diagonal interpolants (i.e. $m = n$) we have two interpolation conditions [27]:

$$P_n(x_{2s}; n) = Q_n(x_{2s}; n),\ s = 0, 1, \dots, n \quad \text{and}$$
$$-P_n(x_{2s+1}; n) = Q_n(x_{2s+1}; n),\ s = 0, 1, \dots, n-1 \tag{2.2.11}$$

From the first of these relation we see that polynomials $P_n(z; n)$ and $Q_n(z; n)$ should coincide, then from the second relation we find

$$P_n(z; n) = Q_n(z; n) = (z - x_1)(z - x_3)\dots(z - x_{2n-1}) \tag{2.2.12}$$

Thus diagonal Padé interpolants in this case have all odd interpolation points $x_1, x_3, \dots, x_{2n-1}$ as unattainable.

Consider concrete example (2.1.24) connected with the Krawtchouk polynomials. For the positively definite case (i.e. $w_s > 0,\ s = 0, \dots, N$) it is necessary and sufficient that $0 < p < 1$. This means that $-\infty < q < 0$. Thus for positively definite case we have $Y_sY_{s+1} = q^s q^{s+1} < 0$ as expected. Hence we can expect (exclusive) cases when some points x_s become unattainable. This is equivalent to the case when one or several zeroes of the Krawtchouk polynomials are *integer*. The problem of finding of all integer zeroes of the Krawtchouk polynomials is one of the interesting and important problem which is connected with many branches in modern mathematics (see, e.g. [23]). We thus have

Theorem 2. The problem of existence of integer zeros of the Krawtchouk polynomials is equivalent to the problem of existence of unattainable points in Padé interpolation for the exponential function on uniform grid.

Using this observation we can immediately conclude that for $Y_s = (-1)^s$ the points $x_{(n+m)/2}$ are unattainable for all odd numbers n and m. Indeed, in this case we have so-called symmetric Krawtchouk polynomials $p = 1/2$ [23]. But for symmetric Krawtchouk polynomials there are "trivial" zeros $x = N/2$ when n is odd and N is even [23].

2.3. **Horizontal and vertical strings.** Consider now the case when either $m = const$ (vertical string) or $n = const$ (horizontal string). These cases are closely related with the Newton-Lagrange interpolation problem. Indeed, consider, e.g. the horizontal string $n = 0$. Then we have the problem: find polynomials $Q_m(z; 0)$ such that $Y_s = Q_m(x_s; 0),\ s = 0, 1, \dots, m)$. Thus $Q_m(z; 0)$ is the Newton-Lagrange interpolation polynomial for sequence Y_s [17]. Analogously, if $m = 0$ then $P_n(z; 0)/\alpha_{n0}$ is the Newton-Lagrange interpolation polynomial for sequence $1/Y_s$ (recall that α_{nm} is leading coefficient of $Q_m(z; n)$ and we can put $Q_0(z; n) = \alpha_{n0}$). For $m = const \neq 0$ we have Padé interpolation problem with fixed degree m of numerator, analogously for $n = const$ we have fixed degree n of denominator.

In what follows we consider only the vertical strings $m = const$ (for the horizontal string all results are similar due to obvious replacement $Y_s \to 1/Y_s$ when $m \leftrightarrow n$).

Consider the rational functions

$$R_n(z; m) = \frac{P_n(z; m+1)}{\omega_{n+m+2}(z)} \tag{2.3.1}$$

where $\omega_{N+1}(z) = (z-x_0)(z-x_1)\dots(z-x_N)$ is the characteristic polynomial of interpolation points. We have

Theorem 3. For fixed m the polynomials $P_n(z;m)$ and rational functions $R_n(z;m)$ form a biorthogonal system:

$$\sum_{s=0}^{N} Y_s P_n(x_s;m)\, Res(R_k(z;m))\,|_{z=x_s} = \alpha_{n,m+1}\,\delta_{nk}, \tag{2.3.2}$$

where N is any positive integer such that $N \geq k+m+1$ or, equivalently,

$$\frac{1}{2\pi i}\int_\Gamma f(\zeta)P_n(\zeta;m)R_k(\zeta;m)d\zeta = \alpha_{n,m+1}\delta_{nk},\ n,k = 0,1,\dots, \tag{2.3.3}$$

where the poles $x_0, x_1, \dots x_{k+m+1}$ of the rational function $R_k(z;m)$ lie inside the contour Γ.

Proof. For $n > k$ and $n < k$ the statement of the theorem follows easily from basic orthogonality relation (2.0.8). The only nontrivial part is calculation of the integral for $k = n$. In this case we have

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)P_n(\zeta;m)P_n(\zeta;m+1)}{(\zeta-x_0)(\zeta-x_1)\dots(\zeta-x_{n+m+1})}d\zeta =$$
$$\sum_{s=0}^{n+m+1}\frac{P_n(x_s;m)P_n(x_s;m+1)Y_s}{\omega'_{n+m+2}(x_s)} = \sum_{s=0}^{n+m+1}\frac{P_n(x_s;m)Q_{m+1}(x_s;n)}{\omega'_{n+m+2}(x_s)}$$

where in the last equation we used the main interpolation property (1.0.1). The last expression can be presented in the form (see (2.0.2))

$$\sum_{s=0}^{n+m+1}\frac{P_n(x_s;m)Q_{m+1}(x_s;n)}{\omega'_{n+m+2}(x_s)} = [x_0,\dots x_{n+m+1}]\{P_n(x;m)Q_{m+1}(x;n)\} \tag{2.3.4}$$

i.e. we have divided difference of order $n+m+1$ from polynomial of degree $n+m+1$. From (2.0.6) we immediately have that this expression is equal to leading coefficient of the polynomial and thus

$$\sum_{s=0}^{n+m+1}\frac{P_n(x_s;m)Q_{m+1}(x_s;n)}{\omega'_{n+m+2}(x_s)} = \alpha_{n,m+1}$$

and the theorem is proven.

Note that in [51] we obtained the special case of this theorem for $m = 0$. In this case $P_n(z;0)/\alpha_{n0}$ are the Newton-Lagrange interpolants for the sequence $1/Y_s$. Biorthogonal partners for these polynomials are rational functions $R_n(z;0) = P_n(z;1)/\omega_{n+2}(z)$. Thus, in order to construct biorthogonal partners for the Newton-Lagrange interpolants we need to know the "next vertical" (i.e. $m = 1$) polynomials $P_n(z;1)$. In [51] these polynomials were explicitly expressed in terms of $P_n(z;0)$ using Frobenius-type relations (2.0.10).

Now we derive three-term recurrence relation for polynomials $P_n(z;m)$ for fixed m. Excluding $P_n(z;m+1)$ from (2.0.10) we arrive at the recurrence relation

$$P_{n+1}(z;m) + \left(-z + x_{n+m+1} + \frac{\alpha_{nm}}{\alpha_{n,m+1}} - \frac{\alpha_{nm}}{\alpha_{n-1,m}}\right) P_n(z;m) + \frac{\alpha_{nm}}{\alpha_{n-1,m}}(z - x_{n+m})P_{n-1}(z;m) = 0 \tag{2.3.5}$$

Recurrence relation (2.3.5) belongs to the class of so-called R_I recurrence relations introduced by Ismail and Masson [21]. These relations determine monic polynomials $P(z)$ of R_I type and in general form they can be written as [21]

$$P_{n+1}(z) + (\xi_n - z)P_n(z) + \eta_n(z - \nu_n)P_{n-1}(z) = 0, n = 1, 2, \dots \tag{2.3.6}$$

with initial conditions

$$P_0 = 1, P_1(z) = z - \xi_0$$

Thus recurrence relation of R_I type have 3 arbitrary parameters ξ_n, η_n, ν_n.

Comparing (2.3.5) with (2.3.6) we have

$$\xi_n = x_{n+m+1} + \frac{\alpha_{nm}}{\alpha_{n,m+1}} - \frac{\alpha_{nm}}{\alpha_{n-1,m}},$$
$$\eta_n = \frac{\alpha_{nm}}{\alpha_{n-1,m}}, \quad \nu_n = x_{n+m} \tag{2.3.7}$$

We see that polynomials of R_I type appear naturally as denominators of the vertical strings ($m = const$) in the Padé interpolation. In the simplest case of pure Newton-Lagrange interpolation (i.e. $m = 0$) corresponding recurrence relation was derived in [51].

We illustrate results of this subsection by the simplest example of the exponential function (2.0.13). In this case $Y_s = q^s$ and biorthogonal partners $R_n(z$ are rational functions (2.3.1):

$$R_n(z;m) = \frac{P_n(z;m+1)}{z(z-1)\dots(z-n-m-1)} \tag{2.3.8}$$

Using standard transformation formulas for the Gauss hypergeometric function [13] we can rewrite (2.3.8) in the form

$$R_n(z;m) = {}_2F_1\left(\begin{matrix}-n, m+2\\ 2+m-z\end{matrix}; \frac{1}{1-q}\right) \tag{2.3.9}$$

In [51] we obtain a special case $m = 0$ of formula (2.3.9) corresponding to the Lagrange-Newton interpolation of the exponential function.

We see that for fixed m the polynomials $P_n(z;m)$ (2.0.13) and rational functions (2.3.9) form a biorthogonal system.

3. Diagonal strings and biorthogonal rational functions

In previous sections we showed that for Kronecker string the denominator polynomials $P_n(z;m)$ are finite orthogonal polynomials; for the vertical string ($m = const$) they coincide with so-called polynomials of R_I type. In this section we show that for diagonal string the denominator polynomials are of R_{II} type (in terminology of [21]). These polynomials allow one to construct a pair of biorthogonal rational functions.

Assume that $m = n + M$, where M is a fixed positive or negative integer. Every M defines a straight line parallel to the main diagonal $m = n$. We say that such straight line corresponds to *diagonal string* of the Padé interpolants.

Consider denominator polynomials $G_n(z;M) \equiv P_n(z;n+M)$. We should derive 3-term recurrence relation for polynomials $G_n(z;j)$ for fixed j.

For this goal we use Frobenius-type relations (2.0.10). It is convenient to introduce notation

$$G_n(z;M) = P_n(z,n+M),\ F_n(z;M) = P_n(z,n+M+1) \tag{3.0.1}$$

Then from (2.0.10) we get relations between polynomials $F_n(z;M)$ and $G_n(z;M)$:

$$\begin{aligned}
&G_{n+1}(z;M) - (z - x_{2n+M+1})G_n(z;M) + \frac{\alpha_{n,n+M} - \alpha_{n+1,n+M+1}}{\alpha_{n,n+M+1}} F_n(z;M) = 0\\
&F_n(z;M) + \frac{\alpha_{n,n+M+1} - \alpha_{n-1,n+M}}{\alpha_{n-1,n+M}} G_n(z;M) -\\
&\frac{\alpha_{n,n+M+1}}{\alpha_{n-1,n+M}} (z - x_{2n+M})F_{n-1}(z;M) = 0
\end{aligned} \tag{3.0.2}$$

Excluding $F_n(z;M)$ from the first relation (3.0.2) and substituting to the second relation, we obtain 3-term recurrence relation for polynomials $G_n(z;M)$:

$$\begin{aligned}
&G_{n+1}(z;M) + \eta_n(M)(z - x_{2n+M})(z - x_{2n+M-1})G_{n-1}(z;M) +\\
&\quad (\xi_n(M)z + x_{2n+M+1} + \eta_n(M)x_{2n+M} + \zeta_n(M))G_n(z;M) = 0,
\end{aligned} \tag{3.0.3}$$

where

$$\begin{aligned}
&\eta_n(M) = \frac{\alpha_{n+1,n+M+1} - \alpha_{n,n+M}}{\alpha_{n,n+M} - \alpha_{n-1,n+M-1}},\ \xi_n(M) = \frac{\alpha_{n-1,n+M-1} - \alpha_{n+1,n+M+1}}{\alpha_{n,n+M} - \alpha_{n-1,n+M-1}}\\
&\zeta_n(M) = \frac{(\alpha_{n,n+M} - \alpha_{n+1,n+M+1})(\alpha_{n-1,n+M} - \alpha_{n,n+M+1})}{\alpha_{n-1,n+M}\alpha_{n,n+M+1}}
\end{aligned} \tag{3.0.4}$$

Note that $\xi_n(M) + \eta_n(M) = -1$ as follows from comparing terms z^{n+1} in (3.0.3).

Recurrence relation (3.0.3) belongs to the class of so-called R_{II} 3-term relations [21]. As shown in [49] (see also [37], [36]) this relation is equivalent to GEVP (1.0.10) defining a pair of biorthogonal rational functions (BRF).

3.1. **Explicit biorthogonality relation on the scheme with shifted grid.** In our case this pair of BRF can be constructed directly. For this goal we first rearrange interpolation points x_s in the following order. In what follows we will assume for simplicity that $M > 0$. Let y_s be the same grid x_s but with another numeration:

$$\begin{aligned}
&y_i = x_i, \quad \text{if} \quad i = 0,1,\dots,M,\\
&y_{-i} = x_{M+2i-1},\ y_{M+i} = x_{M+2i},\ i = 1,2,\dots,n
\end{aligned} \tag{3.1.1}$$

Thus for any $n = 0,1,2,\dots$ the grid $y_{-n}, y_{-n+1},\dots,y_0,y_1,\dots,y_{n+M}$ coincide with the grid $x_0, x_1, \dots x_{2n+M}$. Hence we can rewrite orthogonality property (2.0.8) in the form

$$\int_\Gamma \frac{f(\zeta)G_n(\zeta;M)q_j(z)d\zeta}{(\zeta - y_{-n})(\zeta - y_{-n+1})\dots(\zeta - y_{n+M})} = 0, \quad j = 0,1,\dots,n-1 \tag{3.1.2}$$

or, equivalently

$$\sum_{s=-n}^{n+M} \frac{Y_s G_n(y_s;M) q_j(y_s)}{H_n'(y_s)}, \quad j = 0, 1, \dots, n-1, \tag{3.1.3}$$

where $H_n(z;M) = (z - y_{-n}) \dots (z - y_{n+M})$.

This scheme can be further generalized if one admits to start with $s = -n + j$ with arbitrary integer parameter j. In this case we have the following Padé interpolation problem

$$Y_s = \frac{Q_m(y_s;n,j)}{P_n(y_s;n,j)}, \quad s = -n+j, -n+j+1, \dots, m+j \tag{3.1.4}$$

If $j = 0$ we return to the scheme (3.1.1). For fixed j the Frobenius-type relations remain valid with the only difference that the coefficients $\alpha_{nm}(j)$ will depend on j:

$$P_{n+1}(z;m+1,j) + \mu_{nm}(j) P_n(z;m+1,j) - (z - y_{m+1+j}) P_n(z;m,j) = 0 \tag{3.1.5}$$

$$P_{n+1}(z;m+1,j) + (\nu_{nm}(j) - 1) P_{n+1}(z;m,j) - \nu_{nm}(j)(z - y_{-n-1+j}) P_n(z;m,j) = 0 \tag{3.1.6}$$

where

$$\mu_{nm}(j) = \frac{\alpha_{nm}(j) - \alpha_{n+1,m+1}(j)}{\alpha_{n,m+1}(j)},$$

$$\nu_{nm}(j) = \frac{\alpha_{n+1,m+1}(j)}{\alpha_{nm}(j)}$$

There are also relations with different j:

$$P_n(z;m+1,j) + \sigma_{nm}(j)(z - y_{-n+j}) P_n(z;m,j+1) = \sigma_{nm}(j)(z - y_{m+j+1}) P_n(z;m,j) \tag{3.1.7}$$

and

$$P_{n+1}(z;m+1,j) + \kappa_{nm}(j)(z - y_{m+j+1}) P_n(z;m,j) = (1 + \kappa_{nm}(j))(z - y_{-n+j}) P_n(z;m,j+1) \tag{3.1.8}$$

where

$$\sigma_{nm}(j) = \frac{\alpha_{n,m+1}(j)}{\alpha_{nm}(j) - \alpha_{nm}(j+1)}$$

and

$$\kappa_{nm}(j) = \frac{\alpha_{n+1,m+1}(j) - \alpha_{nm}(j+1)}{\alpha_{nm}(j+1) - \alpha_{nm}(j)}$$

Introduce the following rational functions:

$$R_n(z;M,j) = \frac{P_n(z;M,j)}{\Omega_{-n+j,-1+j}(z)}, \quad T_n;(zM,j) = \frac{P_n(z;M,j+1)}{\Omega_{M+2+j,n+M+j+1}(z)} \tag{3.1.9}$$

where now

$$\Omega_{p_1,p_2}(z) \equiv (z - y_{p_1})(z - y_{p_1+1}) \dots (z - y_{p_2}) \tag{3.1.10}$$

where p_1, p_2 are arbitrary integers such that $p_1 < p_2$. Clearly, $\deg(\Omega_{p_1,p_2}(z)) = p_2 - p_1 + 1$.

By definition, these rational functions are of type $[n/n]$ (in case of normal Padé scheme zeroes of polynomials $P_n(z; M, j)$ do not coincide with zeroes of denominators in (3.1.9)).

Orthogonality property (2.0.9) now can be rewritten as

$$S = \sum_{s=-n+j}^{m+j} \frac{Y_s P_n(y_s; m, j) q_k(y_s)}{\Omega'_{-n+j,m+j}(y_s)} = 0, \quad k = 0, 1, \ldots, n-1 \tag{3.1.11}$$

where $q_k(z)$ is an arbitrary polynomial of degree k.

Theorem 4. The functions $R_n(z; M, j), T_n(z; M, j)$ are biorthogonal with respect to the following scalar product

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta) R_n(\zeta; M, j) T_k(\zeta; M, j)}{\Omega_{j,M+j+1}(\zeta)} = h_n(M, j)\delta_{nk} \tag{3.1.12}$$

or, equivalently,

$$\sum_{s=-n+j}^{k+M+j+1} Y_s Res \left\{ \frac{R_n(z; M, j) T_k(z; M, j)}{\Omega_{j,M+j+1}(z)} \right\} \Bigg|_{z=y_s} = h_n(M, j)\delta_{nk} \tag{3.1.13}$$

where normalization coefficient is

$$h_n(M, j) = \frac{\alpha_{n,n+M+1}(j)\alpha_{n-1,n+M}(j)}{\alpha_{n-1,n+M}(j) - \alpha_{n,n+M+1}(j)} \tag{3.1.14}$$

Proof. Denote by I the integral in lhs of (3.1.12). When $n \neq k$ it is easily obtained that $I = 0$ from orthogonality relation (3.1.11). So, we need only to check relation (3.1.12) for $n = k$. From relation (3.1.6) we can express the first polynomial in (3.1.12) as

$$P_n(z; m, j) = \frac{P_n(z; m+1) - \nu_{n-1,m}(j)(z - y_{-n+j}) P_{n-1}(z; m, j)}{1 - \nu_{n-1,m}(j)}. \tag{3.1.15}$$

Substituting (3.1.15) into (3.1.12) we can express the integral in (3.1.12) as $I = I_1 + I_2$, where

$$I_1 = \frac{1}{2\pi i} (1 - \nu_{n-1,m}(j))^{-1} \int_\Gamma \frac{f(z) P_n(z, n+M+1, j) P_n(z, n+M, j+1)}{\Omega_{-n+j,n+M+j+1}(z)} \tag{3.1.16}$$

and

$$I_2 = \frac{1}{2\pi i} \frac{\nu_{n-1,m}(j)}{1 + \nu_{n-1,m}(j)} \int_\Gamma \frac{f(z) P_{n-1}(z, n+M, j) P_n(z, n+M, j+1)}{\Omega_{-n+j+1,m+j+1}(z)} \tag{3.1.17}$$

In I_1 we can replace $f(z)P_n(z; n+M+1, j)$ with $Q_{n+M+1}(z; n, j)$ using interpolation property (3.1.4). Then I_1 becomes

$$I_1 = (1 - \nu_{n-1,m}(j))^{-1}[-n+j, n+M+j+1]\{P_n(z; n+M, j+1) Q_{n+M+1}(z; n, j)\} = \frac{\alpha_{n,n+M+1}(j)\alpha_{n-1,n+M}(j)}{\alpha_{n-1,n+M}(j) - \alpha_{n,n+M+1}(j)} \tag{3.1.18}$$

(in the last equality we exploited formula (2.0.6)).

The second integral $I_2 = 0$. Indeed, we can replace $f(z)P_n(z; n+M, j+1)$ with $Q_{n+M}(z; n, j+1)$. Then we have

$$I_2 = [-n+j+1, n+M+j+1]\{P_{n-1}(z; n+M, j+1)Q_{n+M}(z; n, j)\} = 0$$

because the order of divided difference operator is $2n+M$, which is greater than degree $2n+M-1$ of the polynomial $P_n(z; n+M, j+1)Q_{n+M+1}(z; n, j)$.

Thus we have finally $I = I_1 = h_n(M, j)$, where $h_n(M, j)$ is given by (3.1.14).

We introduced the new grid y_i by rearranging points x_s. But it is possible to consider the grid y_s independently of the grid x_s. In this case all results concerning biorthogonal rational functions $R_n(z; M)$ and $T_n(z; M)$ remain valid.

We illustrate this using our example of the interpolation of the exponential function.

Take the polynomials $P_n(z; m), Q_m(z; n)$ in (2.0.13) and change the argument $z \to z+n$:

$$\begin{aligned} Q_m(z; n) &= (-1)^n (1-1/q)^{-n}\, (1+m)_n \, {}_2F_1\left(\begin{matrix} -m, -z-n \\ -m-n \end{matrix}; 1-q\right), \\ P_n(z; m) &= (-1)^n (1-1/q)^{-n}\, (1+m)_n \, {}_2F_1\left(\begin{matrix} -n, -z-n \\ -m-n \end{matrix}; 1-1/q\right) \end{aligned} \tag{3.1.19}$$

Clearly, these polynomials again satisfy the interpolation property

$$\frac{Q_m(y_s; n)}{P_n(y_s; m)} = \exp(\omega y_s) = q^s, \tag{3.1.20}$$

where $y_s = s = -n, -n+1, \dots, m$.

In what follows we will assume that $m = n+M > n$. By (3.1.9) we can construct a pair of corresponding rational functions:

$$\begin{aligned} R_n(z; M) &= \frac{(-1)^n (1-1/q)^{-n}\, (1+n+M)_n}{(z+1)_n} \, {}_2F_1\left(\begin{matrix} -n, -z-n \\ -2n-M \end{matrix}; 1-1/q\right), \\ T_n(z; M) &= \frac{(-1)^n (1-1/q)^{-n}\, (2+n+M)_n}{(z-n-M-1)_n} \, {}_2F_1\left(\begin{matrix} -n, -z-n \\ -2n-M-1 \end{matrix}; 1-1/q\right) \end{aligned}$$

For more general explicit examples of BRF connected with diagonal strings in the Padé interpolation table as well as for their algebraic treatment see [54].

4. Reduction to a special case of orthogonal rational functions

So far, we considered the case when all points x_s of the interpolated grid are distinct $x_n \neq x_m$ if $n \neq m$. Equivalently, this means $y_n \neq y_m$ in the scheme with modified grid y_s. In this case for diagonal strings $m = n+M$ with fixed M we obtained biorthogonal rational functions $R_n(z; M, j), T_n(z; M, j)$ on the shifted grid $y_{-n+j}, y_{-n+j+1}, \dots y_{n+M+j+1}$. Note that denominators of rational functions $R_n(z; M, j), T_n(z; M, j)$ are distinct as is seen from expression (3.1.9) . Hence for the scheme with distinct interpolated points y_s we obtain non-coinciding rational functions: $R_n(z; M, j) \neq T_n(z; M, j)$. Nevertheless, it is possible to obtain coinciding rational functions:

$$R_n(z; M, j) = \sigma_n\, T_n(z; M, j), \quad n = 0, 1, \dots \tag{4.0.1}$$

by some degenerating procedure. Here σ_n is a nonzero constant (we will not distinguish rational functions which differ by a common constant factor). If condition (4.0.1) holds, we then deal with the special case of orthogonal rational functions [6]. In what follows we restrict ourselves with the case $j = 0$. The case of arbitrary j can be considered in a similar manner. Consider, when condition (4.0.1) holds. First of all, denominators of the functions $R_n(z;M), T_n(z;M)$ should coincide for all $n = 1, 2, 3 \dots$. (For simplicity, we will not indicate the variable $j = 0$ in notation). From (3.1.9) we see that this is equivalent to the condition

$$y_{-n} = y_{M+n+1}, \quad n = 1, 2, \dots. \tag{4.0.2}$$

Next, numerators of the rational functions $R_n(z;M), T_n(z;M)$ should coincide up to a common constant factor. From (3.1.9) we conclude that this is equivalent to the condition that the shifted grid $y_{-n+1}, y_{-n+2}, \dots y_{n+M+1}$ should coincide with initial grid $y_{-n}, y_{-n+1}, \dots y_{n+M}$ for all $n = 1, 2, \dots$. This is possible iff $y_{-n} = y_{n+M+1}$, i.e. under the same conditions (4.0.2) as for coinciding of denominators. Thus condition (4.0.2) is necessary and sufficient in order for rational functions $R_n(z;M), T_n(z;M)$ to coincide up to a common constant factor (4.0.1).

But in this case we obtain a set of *orthogonal* rational functions $R_n(z;M)$. Indeed, biorthogonality property (3.1.12) is transformed to

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)R_n(\zeta;M)R_k(\zeta;M)}{(\zeta-y_0)(\zeta-y_1)\dots(\zeta-y_{M+1})} = h_n(M)\delta_{nk} \tag{4.0.3}$$

Some remarks should be done concerning Padé interpolating scheme corresponding to the degenerating case (4.0.2). It is seen that in this case for any n the interpolated grid consists of $M+3$ simple (i.e. pairwise distinct) points $y_0, y_1, \dots, y_{M+1}, y_{n+M+1}$ and $n-1$ double points $y_{M+2}, y_{M+3}, \dots, y_{M+n}$. This means that in $n-1$ double points we should interpolate not only values of the function $f(z)$ but its derivative $f'(z)$ as well. I.e. the interpolated scheme now can be presented as

$$f(x_s) = \frac{Q_{n+M}(x_s;M)}{P_n(x_s;M)} \tag{4.0.4}$$

for $s = 0, 1, \dots, M+n+1$ (as in the usual Padé interpolation scheme) and in addition

$$f'(x_s) = \frac{d}{dz}\left.\frac{Q_{n+M}(z;M)}{P_n(z;M)}\right|_{z=x_s} \tag{4.0.5}$$

for double nodes $s = M+2, M+3, \dots, M+n$ of the interpolation grid.

The orthogonal rational functions $R_n(z;M)$ are constructed now from the denominator polynomials $P_n(z;M)$ of the osculatory interpolation scheme by the formula

$$R_n(z;m) = \frac{P_n(z;M)}{(z-y_{M+2})(z-y_{M+4})\dots(z-y_{M+n+1})} \tag{4.0.6}$$

The Padé interpolation schemes with double (and multiple, in general) interpolation nodes is called Hermite-Padé interpolation scheme, or, sometimes, rational osculatory interpolation problem. For theory of this problem see e.g. [9], [47].

We thus showed that the theory of rational orthogonal functions developed in [6] can be considered as a special, degenerated case of general diagonal Padé interpolation scheme when some pairs of interpolated nodes merge and become double points. For such double points we should apply osculatory rational interpolation problem. In general case for non-coinciding interpolation nodes we obtain only non-coinciding (i.e. biorthogonal) pairs of rational functions $R_n(z;M), T_n(z;M)$.

5. Scheme with prescribed poles and zeroes

In this section we derive fundamental properties of the Padé interpolation with prescribed poles and zeroes (PPZ scheme for shorten notation).

In what follows we will assume that the Padé scheme (1.0.4) is normal, i.e. all zeroes of polynomials $P_n(z;m)$ do not coincide with zeroes of polynomial $Q_m(z;n)$ and with prescribed zeroes a_i. Analogously, we assume that zeroes of $Q_m(z;n)$ do not coincide with prescribed poles b_i. We will assume also that sets of prescribed zeroes and poles do not overlap. Moreover it is assumed that $Y_s \neq 0$. Clearly, polynomials $P_n(z;m)$ and $Q_m(z;n)$ have degrees n and m correspondingly. As for the case of the ordinary Padé interpolation scheme [28] we have

Theorem 5. If the Padé interpolation scheme (1.0.4) is normal, then polynomials $P_n(z;m)$ and $Q_m(z;n)$ are unique up to a common factor.

The proof of this theorem is almost the same as for the ordinary case [28] and we omit it.

The normal PPZ-scheme has an important property of invariance under linear rational substitutions:

Proposition 1. Assume that the PPZ scheme (1.0.4) is normal. Perform the Möbius transformation $x_s \to \tilde{x}_s = (\alpha_1 x_s + \alpha_2)/(\alpha_3 x_s + \alpha_4)$ with some constants α_i such that $\alpha_1\alpha_4 \neq \alpha_2\alpha_3$ Replace zeros and poles a_n, b_n by similar expressions (with the same coefficients α_i) and replace $P_n(x;m) \to \tilde{P}_n(x;m) = (\alpha_3 x + \alpha_4)^n P_n(\tilde{x};m)$ (and similar replacements for $Q_m(x;n), A_n(x), B_n(x)$). Then we obtain a new normal PPZ-scheme (1.0.4) with the same interpolated sequence Y_s and with modified grid $x_s \to \tilde{x}_s$ and interpolants $P_n(x;m) \to \tilde{P}_n(x;m), \dots$.

The proof of this proposition is almost obvious and we omit it (see e.g. our paper [49] concerning general case of BRF). Nevertheless, this property is useful because sometimes it is possible to reduce the grid x_s to as simple form as possible using an appropriate Möbius transform.

As for the ordinary case, we have generalized orthogonality property for the denominator polynomials:

$$[x_0, x_1, \dots, x_{n+m}] \left\{ \frac{q_j(z) f(z) B_m(z) P_n(z;m)}{A_n(z)} \right\} = 0, \; j = 0, 1, \dots, n-1 \qquad (5.0.1)$$

where $q_j(z)$ is any polynomial of degree j. Formula (5.0.1) is a trivial consequence of the scheme (1.0.4). If function $f(z)$ doesn't exist, one can rewrite (5.0.1) in equivalent form

$$\sum_{s=0}^{m+n} \frac{Y_s q_j(x_s) P_n(x_s;m) B_m(x_s)}{A_n(x_s)\omega'_{m+n+1}(x_s)} = 0, \ j = 0,1,\dots,n-1 \tag{5.0.2}$$

In contrast to the ordinary Padé interpolation scheme, we will not fix condition that polynomials $P_n(z;m)$ are monic. The reason is that sometimes it is more convenient not to fix the leading coefficients of polynomials $P_n(z;m), Q_m(z;n)$.

Now we present the Frobenius-type relations for the Padé interpolants:

$$\begin{aligned} &\kappa_{nm} P_{n+1}(z;m) - (z - x_{n+m+1}) P_n(z;m) - \rho_{nm}(z - b_{m+1}) P_n(z;m+1) = 0, \\ &\mu_{nm} P_{n+1}(z;m) - P_{n+1}(z;m+1) - \nu_{nm}(z - a_{n+1}) P_n(z;m+1) = 0 \end{aligned} \tag{5.0.3}$$

with some nonzero coefficients $\kappa_{nm}, \rho_{nm}, \mu_{nm}, \nu_{nm}$.

Similar relations can be written for numerator polynomials:

$$\begin{aligned} &\kappa_{nm} Q_m(z;n+1)(z - a_{n+1}) - (z - x_{n+m+1}) Q_m(z;n) - \rho_{nm} Q_{m+1}(z;n) = 0, \\ &\mu_{nm}(z - b_{m+1}) Q_m(z;n+1) - Q_{m+1}(z;n+1) - \nu_{nm} Q_{m+1}(z;n) = 0 \end{aligned} \tag{5.0.4}$$

Sometimes, it is convenient to rewrite these relations in an equivalent form using rational functions $U_n(z;m) = P_n(z;m)/A_n(z)$ and $V_m(z;n) = Q_m(z;n)/B_m(z)$ instead of polynomials $P_n(z;m), Q_m(z;n)$:

$$\begin{aligned} &\kappa_{nm}(z - a_{n+1}) U_{n+1}(z;m) - (z - x_{n+m+1}) U_n(z;m) - \\ &\rho_{nm}(z - b_{m+1}) U_n(z;m+1) = 0, \\ &\mu_{nm} U_{n+1}(z;m) - U_{n+1}(z;m+1) - \nu_{nm} U_n(z;m+1) = 0 \end{aligned} \tag{5.0.5}$$

and

$$\begin{aligned} &\kappa_{nm}(z - a_{n+1}) V_m(z;n+1) - (z - x_{n+m+1}) V_m(z;n) - \\ &\rho_{nm}(z - b_{m+1}) V_{m+1}(z;n) = 0, \\ &\mu_{nm} V_m(z;n+1) - V_{m+1}(z;n+1) - \nu_{nm} V_{m+1}(z;n) = 0 \end{aligned} \tag{5.0.6}$$

Relations (5.0.3) and (5.0.4) are exact analogues of corresponding relations (2.0.10) and (2.0.11). As far as we know, these relations were not appeared in literature yet. The proof of these relations is almost the same as for the ordinary case [3]. We provide, e.g. proof of the first of relations (5.0.5).

Consider expressions

$$\begin{aligned} &U_n^*(z;m) = \frac{\kappa_{nm}(z - a_{n+1}) U_{n+1}(z;m) - (z - x_{m+n+1}) U_n(z;m)}{z - b_{m+1}} = \\ &\frac{\kappa_{nm} P_{n+1}(z;m) - (z - x_{m+n+1}) P_n(z;m)}{A_n(z)(z - b_{m+1})} \end{aligned} \tag{5.0.7}$$

and

$$V^*_{m+1}(z;n) = \frac{\kappa_{nm}(z-a_{n+1})V_m(z;n+1)-(z-x_{m+n+1})V_m(z;n)}{z-b_{m+1}} = \frac{Q^*_{m+1}(z;n)}{B_{m+1}(z)}, \tag{5.0.8}$$

where $Q^*_{m+1}(z;n)$ is a polynomial of degree $\le m+1$.

Choose

$$\kappa_{nm} = \frac{(b_{m+1}-x_{n+m+1})P_n(b_{m+1};m)}{P_{n+1}(b_{m+1};m)}$$

Such κ_{nm} does exist because roots of $P_n(z;m)$ do not coincide with b_i. Moreover, $\kappa_{nm} \neq 0$ because interpolation points x_i do not coincide with prescribed poles and zeroes. Then the pole $z = b_{m+1}$ in (5.0.7) disappears and we have

$$U^*_n(z;m) = \frac{P^*_n(z;m+1)}{A_n(z)},$$

where $P^*_n(z;m+1)$ is a polynomial of degree $\le n$.

Now from (1.0.4) we have

$$Y_s = \frac{V^*_{m+1}(x_s;n)}{U^*_n(x_s;m+1)} = \frac{Q^*_{m+1}(x_s;n)A_n(x_s)}{B_{m+1}(x_s)P^*_n(x_s;m+1)}, \quad ,s=0,1,\dots,n+m+1 \tag{5.0.9}$$

From uniqueness of the Padé interpolation scheme we see that

$$U^*_n(z;m+1) = \rho_{nm}U_n(z;m+1), \quad V^*_{m+1}(z;n) = \rho_{nm}V_{m+1}(z;n)$$

with some nonzero coefficients ρ_{nm}. This leads to the first relations in (5.0.3) and (5.0.4). Second relations are derived analogously.

In what follows we need two statements concerning polynomials denominator $P_n(z;m)$.

Lemma 1. For the normal scheme of the PI with prescribed poles and zeroes the roots of polynomials $P_n(z;m)$ and $P_{n+1}(z;m)$ do not coincide. Similarly, roots of polynomials $P_n(z;m)$ and $P_{n+1}(z;m-1)$ do not coincide

For the proof, let us assume that there exists z_0 such that $P_n(z_0;m) = P_{n+1}(z_0;m) = 0$, i.e. z_0 is a common root. Then from Frobenius-type relations (as all coefficients $\kappa_{nm}, \rho_{nm}, \mu_{nm}, \nu_{nm}$ are nonzero) it is seen that also $P_{n-1}(z_0;m) = 0$. Repeating this procedure we arrive at $P_0(z_0;m) = 0$ which is impossible because for the normal scheme we should $P_0(z;m) = const \neq 0$. Quite analogously we can prove the statement for roots of polynomials $P_n(z;m)$ and $P_{n+1}(z;m-1)$.

Lemma 2. Assume that for fixed m one has relation

$$F_1(z;n,m)P_n(z;m) + F_2(z;n,m)P_{n+1}(z;m) \equiv 0 \tag{5.0.10}$$

where $F_{1,2}(z;n,m)$ are polynomials in z of *fixed* degrees (i.e. not depending on n,m). Then necessarily $F_1(z;n,m) \equiv F_2(z;n,m) \equiv 0$.

Proof. Assume that polynomials $F_1(z;n,m), F_2(z;n,m)$ do not vanish identically. Then $F_1(z;n,m)/F_2(z;n,m) = P_{n+1}(z;m)/P_n(z;m)$. But the lhs of this relation is a rational function of fixed degree, whereas the degree of rhs increases infinitely with n. (Indeed, by Lemma **1** we know that roots of polynomials $P_{n+1}(z;m)$ and $P_n(z;m)$ do not coincide). This is impossible, hence both polynomials $F_1(z), F_2(z)$ should vanish identically.

Using these statements it is easily to prove that for the normal scheme all the Frobenius coefficients $\mu_{nm}, \nu_{nm}, \kappa_{nm}, \rho_{nm}$ are nonzero. Indeed, assume that, say, $\mu_{nm} = 0$ for some n, m. Then from (5.0.3) it follows that either polynomials $P_{n+1}(x;m+1), P_n(x;m+1)$ have a common zero, or the polynomial $P_{n+1}(x;m+1)$ has a zero coinciding with the prescribed zero a_{n+1}. Both possibilities are forbidden by our assumptions of normality.

The Lemma **2** allows one to obtain a compatibility conditions for 4 parameters κ_{nm}, ρ_{nm}, μ_{nm}, ν_{nm}. To do this, we obtain 3-term recurrence relations for polynomials $P_n(z;m)$ with fixed parameter m. There are two possibilities:

(i) first, express $P_n(z;m+1)$ in terms of $P_n(z;m)$ from the first relation (5.0.3) and substitute it to the second relation. We get

$$\begin{aligned}&P_{n+1}(z;m) - \frac{\mu_{n-1,m}\rho_{nm}}{\kappa_{nm}}(z-b_{m+1})P_n(z;m) - \\ &\frac{z-x_{n+m+1}}{\kappa_{nm}}P_n(z;m) + \frac{\rho_{nm}\kappa_{n-1,m}\nu_{n-1,m}}{\rho_{n-1,m}\kappa_{nm}}(z-a_n)P_n(z;m) - \\ &\frac{\rho_{nm}\nu_{n-1,m}}{\rho_{n-1,m}\kappa_{nm}}(z-x_{n+m})(z-a_n)P_{n-1}(z;m) = 0 \qquad (5.0.11)\end{aligned}$$

(ii) one can express $P_n(z;m)$ in terms of $P_n(z;m+1)$ from the second relation (5.0.3), then substitute it to the first relation and then to shift $m \to m-1$. We get:

$$\begin{aligned}&P_{n+1}(z;m) - \frac{\mu_{n,m-1}\rho_{n,m-1}}{\kappa_{n,m-1}}(z-b_m)P_n(z;m) - \\ &(z-x_{n+m})\,\frac{\mu_{n,m-1}}{\mu_{n-1,m-1}\kappa_{n,m-1}}P_n(z;m) + \nu_{n,m-1}(z-a_{n+1})P_n(z;m) - \\ &\frac{\mu_{n,m-1}\nu_{n-1,m-1}}{\mu_{n-1,m-1}\kappa_{n,m-1}}(z-x_{n+m})(z-a_n)P_{n-1}(z;m) = 0 \qquad (5.0.12)\end{aligned}$$

Subtracting then (5.0.11) and (5.0.12) we obtain relation of the type

$$F_1(z;n,m)P_n(z;m) + F_2(z;n,m)P_{n-1}(z;m) = 0, \qquad (5.0.13)$$

where $F_1(z;n,m)$ is a polynomial of the first degree in z and $F_2(z;n,m)$ is a polynomial of the second degree in z. By Lemma **2** we have $F_1(z;n,m) \equiv F_2(z;n,m) = 0$ which leads to 3 conditions:

$$\frac{\rho_{nm}\nu_{n-1,m}}{\kappa_{nm}\mu_{n,m-1}} = \frac{\rho_{n-1,m}\nu_{n-1,m-1}}{\kappa_{n,m-1}\mu_{n-1,m-1}}; \tag{5.0.14}$$

$$\frac{1+\kappa_{nm}\nu_{n,m-1}+\rho_{nm}\mu_{n-1,m}}{\kappa_{nm}\mu_{n,m-1}} = \frac{1+\kappa_{n-1,m}\nu_{n-1,m-1}+\rho_{n,m-1}\mu_{n-1,m-1}}{\kappa_{n,m-1}\mu_{n-1,m-1}}; \tag{5.0.15}$$

$$\frac{x_{n+m+1}+\kappa_{nm}\nu_{n,m-1}a_{n+1}+\rho_{nm}\mu_{n-1,m}b_{m+1}}{\kappa_{nm}\mu_{n,m-1}} = \frac{x_{n+m}+\kappa_{n-1,m}\nu_{n-1,m-1}a_n+\rho_{n,m-1}\mu_{n-1,m-1}b_m}{\kappa_{n,m-1}\mu_{n-1,m-1}} \tag{5.0.16}$$

Of course, coefficients $\kappa_{nm},\dots,\mu_{nm}$ are not independent. They can be further specified under some normalization conditions for polynomials $P_n(z;m), Q_m(z;n)$. For example, if $P_n(z;m)=z^n+O(z^{n-1})$ are chosen to be monic and $Q_m(z;n)=\alpha_{nm}z^m+O(z^{m-1}$ (as for the case of the ordinary Padé interpolation) we have

$$\kappa_{nm}=\frac{\alpha_{nm}-\alpha_{n,m+1}}{\alpha_{n+1,m}-\alpha_{n,m+1}}, \quad \rho_{nm}=\kappa_{nm}-1,$$
$$\mu_{nm}=\frac{\alpha_{n+1,m+1}-\alpha_{n,m+1}}{\alpha_{n+1,m}-\alpha_{n,m+1}}, \quad \nu_{nm}=\mu_{nm}-1, \tag{5.0.17}$$

Then it is easily verified that conditions (5.0.14) and (5.0.15) hold identically, and the only remaining condition is (5.0.16) which is rewritten as

$$\frac{(-\alpha_{n-1,m}+\alpha_{n-1,m+1})(-\alpha_{n+1,m-1}+\alpha_{nm})}{(\alpha_{nm}-\alpha_{n-1,m+1})(\alpha_{nm}-\alpha_{n-1,m})}a_n+\frac{\alpha_{n+1,m-1}-\alpha_{n+1,m}}{\alpha_{nm}-\alpha_{n+1,m}}a_{n+1}+$$
$$\frac{\alpha_{n,m-1}-\alpha_{n+1,m-1}}{\alpha_{nm}-\alpha_{n,m-1}}b_m+\frac{(\alpha_{n,m+1}-\alpha_{n-1,m+1})(\alpha_{n,m}-\alpha_{n+1,m-1})}{(\alpha_{nm}-\alpha_{n-1,m+1})(\alpha_{nm}-\alpha_{n,m+1})}b_{m+1}+$$
$$\frac{(\alpha_{n,m}-\alpha_{n+1,m-1})(\alpha_{n-1,m}-\alpha_{n,m-1})}{(\alpha_{nm}-\alpha_{n-1,m})(\alpha_{nm}-\alpha_{n,m-1})}x_{m+n}+$$
$$\frac{(\alpha_{n,m}-\alpha_{n+1,m-1})(\alpha_{n+1,m}-\alpha_{n,m+1})}{(\alpha_{nm}-\alpha_{n,m+1})(\alpha_{nm}-\alpha_{n+1,m})}x_{m+n+1} \tag{5.0.18}$$

Relations similar to (5.0.14) – (5.0.16) appeared in [35] as some compatibility conditions for a set of two linear Darboux transformations for polynomials of R_{II}-type. Careful analysis of these nonlinear equations lead to some explicit solution in terms of elliptic functions (see details in [35]). In the present approach these relations arise naturally as compatibility condition for the Frobenius-type relations for PPZ-scheme.

In applications, however, we will use another representation of the functions $U_n(z;m)$, $V_m(z;n)$. Indeed, these functions are rational ones with prescribed poles $\{a_1,\dots,a_n\}$ and $\{b_1,\dots,b_m\}$ correspondingly. Hence, one can decompose these functions in terms of elementary rational functions with prescribed poles

$$\phi_k(z)=\frac{\omega_k(z)}{A_k(z)}, \quad \chi_k(z)=\frac{\omega_k(z)}{B_k(z)}, \tag{5.0.19}$$

where, recall, $\omega_k(z) = (z - x_0)\dots(z - x_{k-1})$. The function $\phi_k(z)$ has prescribed poles $a_1, \dots, a_k$ whereas the function $\chi_k(z)$ has prescribed poles $b_1, \dots, b_k$.

We have

$$U_n(z;m) = \sum_{s=0}^{n} E_s(n,m)\phi_s(z), \quad V_m(z;n) = \sum_{s=0}^{m} D_s(n,m)\chi_s(z) \tag{5.0.20}$$

with some coefficients $E_s(n,m), D_s(n,m)$ not depending on z. Decomposition (5.0.20) is especially convenient when rational functions $U_n(z;m), V_m(z;n)$ are presented in terms of hypergeometric functions (or their basic and elliptic analogues) as we will see below.

Note that from our conditions of normality and nondegeneracy it follows that $E_n(n,m) \neq 0$, $D_m(m,n) \neq 0$. Indeed, if, say, $E_n(n,m) = 0$ then the rational function $U_n(z;m)$ will be a ratio of two polynomials of degree $n-1$. But this means that at least one root of the polynomial $P_n(z;m)$ coincides with a_n which is forbidden by our conditions. Moreover, it is obvious, that the expansion coefficients $E_s(n,m), D_s(n,m)$ are defined *uniquely* for the given rational functions $U_n(z;m), V_m(z;n)$.

Substituting (5.0.20) to Frobenius-type relations (5.0.5) and (5.0.6) and equating coefficients in front of terms with highest poles we arrive at the relations

$$\mu_{nm} = \frac{E_{n+1}(n+1,m+1)}{E_{n+1}(n+1,m)}, \quad \nu_{nm} = -\frac{D_{m+1}(n+1,m+1)}{D_{m+1}(n,m+1)} \tag{5.0.21}$$

which allow to express the Frobenius coefficients μ_{nm}, ν_{nm} in terms of the expansion coefficients. Note that from (5.0.21) it follows again that coefficients μ_{nm}, ν_{nm} are well-defined and nonzero, as we already proved.

The second pair of relations

$$\begin{gathered}\kappa_{nm}(a_n - x_n)E_{n+1}(n+1,m) + \kappa_{nm}(a_n - a_{n+1})E_n(n+1,m) - \\ (a_n - x_{n+m+1})E_n(n,m) - \rho_{nm}(a_n - b_{m+1})E_n(n,m+1) = 0, \\ \\ \kappa_{nm}(b_m - a_{n+1})D_m(n+1,m) - (b_m - x_{n+m+1})D_m(n,m) - \\ \rho_{nm}(b_m - x_m)D_{m+1}(n,m+1) - \rho_{nm}(b_m - b_{m+1})D_m(n,m+1) = 0\end{gathered} \tag{5.0.22}$$

allows one to express the second pair of the Frobenius coefficients κ_{nm}, ρ_{nm} in terms of expansion coefficients.

6. Shifted PPZ scheme

In this section we consider a generalization of the PPZ scheme with two arbitrary shifts j_1, j_2, where j_1, j_2 are two arbitrary positive integers. Fix these integers and consider the interpolation problem

$$Y_s = \frac{(x_s - a_1)(x_s - a_2)\dots(x_s - a_{n-j_1})}{(x_s - b_1)(x_s - b_2)\dots(x_s - b_{m-j_2})}\,\frac{Q_m^{(j_1,j_2)}(x_s;n)}{P_n^{(j_1,j_2)}(x_s;m)}, \; s = 0,1,\dots,n+m, \quad n \geq j_1, m \geq j_2 \tag{6.0.1}$$

We see that the scheme (6.0.1) differs from the scheme (1.0.4) only by the numbers of prescribed poles and zeros taken into account: the number of prescribed zeros is $n - j_1$, the number of prescribed poles is $m - j_2$.

Note that our shifted PPZ scheme is *not defined* for $n < j_1$ and $m < j_2$. We can define shifted PPZ scheme for these values of n, m by different possible ways. In what follows we will deal only with case $j_1 = j_2 = 1$. In this case it is natural to introduce two new parameters a_0, b_0 and then $(1,1)$-shifted PPZ-scheme can be presented in the form

$$\tilde{f}(x_s) = \frac{(x_s - a_0)\dots(x_s - a_{n-1})}{(x_s - b_0)\dots(x_s - a_{m-1})} \frac{Q_m^{(1,1)}(x_s; n)}{P_n^{(1,1)}(x_s; m)}, \; s = 0, 1, \dots, n+m, \tag{6.0.2}$$

where $\tilde{f}(z) = (z - a_0)f(z)/(z - b_0)$.

Then for new interpolated function $\tilde{f}(z)$ we obtain the same (non-shifted) interpolated PPZ-scheme but with "shifted" zeros and poles: $a_n \to a_{n-1}, b_m \to b_{m-1}$. Thus all formulas including the Frobenius-type relations (5.0.3), (5.0.4) are still valid for the shifted PPZ-scheme (but of course, with modified coefficients $\kappa_{nm}, \dots \nu_{nm}$).

In many explicit cases of PPZ-scheme (see below) the shifted $(1,1)$-scheme differs from unshifted $(0,0)$-scheme only by a shift of one or more parameters of the interpolated function $f(z)$ (or, equivalently, the sequence Y_s). This means some "self-similarity" property of the PPZ-scheme. Only for schemes with such property it is possible to construct explicit examples in terms of known special functions (say, hypergeometric ones), as we will see below.

The ordinary PPZ scheme corresponds to the case $j_1 = j_2 = 0$. Of course, the values j_1, j_2 may be negative as well. Then no restrictions for n, m are imposed.

For adjacent polynomials $P_n^{(j_1,j_2)}(z; m)$ and $P_n^{(j_3,j_4)}(z; m)$ with $|j_3 - j_1| \le 1, |j_4 - j_2| \le 1$ there are algebraic relations similar to the Frobenius-type relations. We will not write down them here.

7. Simple explicit example of the PPZ scheme

In this section we construct a simple concrete example of the PPZ scheme. This example is connected with a ratio of two gamma-functions.

Choose the interpolated function

$$f(z) = \kappa \frac{\Gamma(\alpha - \beta - z)}{\Gamma(\alpha - z)} \tag{7.0.1}$$

where κ is an arbitrary non-zero constant. We choose for convenience

$$\kappa = \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta)} \tag{7.0.2}$$

In what follows we will assume that both α and β do not take integer values, and moreover, the difference $\alpha - \beta$ is not integer as well. This means in particular, that function $f(z)$ is not rational in z. Moreover, all zeros and poles of the function $f(z)$ are not integer and don't overlap .

Function $f(z)$ has zeros and poles at

$$a_n = \alpha + n - 1, \; b_n = \alpha - \beta + n - 1, \; n = 1, 2, \dots. \tag{7.0.3}$$

So it is natural to choose PPZ scheme for the function $f(z)$ with explicit prescribed zeros a_n and poles b_n.

Choose $x_s = s = 0, 1, 2, \dots$ (uniform grid).

Then obviously

$$Y_s = f(x_s) = \frac{(1-\alpha)_s}{(1-\alpha+\beta)_s}, \; A_n(x) = (-1)^n(\alpha - x)_n, \; B_n(x) = (-1)^n(\alpha-\beta-x)_n \quad (7.0.4)$$

where $(x)_n = x(x+1)\dots(x+n-1)$ is shifted factorial (Pochhammer symbol).

Then we have

$$\begin{aligned} Q_m(x;n) &= (-1)^m(\alpha-\beta)_m \, {}_3F_2\left(\begin{matrix} -m, -x, -\beta-m \\ -n-m, 1-m-\alpha+\beta \end{matrix}; 1\right) \\ P_n(x;m) &= (-1)^n(\alpha)_n \, {}_3F_2\left(\begin{matrix} -n, -x, -\beta-n \\ -n-m, 1-n-\alpha \end{matrix}; 1\right) \end{aligned} \quad (7.0.5)$$

Equivalently, we have rational functions

$$\begin{aligned} V_m(x;n) &= Q_m(x;n)/B_m(x) = {}_3F_2\left(\begin{matrix} -m, -x, -\beta-n \\ -n-m, \alpha-\beta-x \end{matrix}; 1\right) \\ U_n(x;m) &= P_n(x;m)/A_n(x) = {}_3F_2\left(\begin{matrix} -n, -x, \beta-m \\ -n-m, \alpha-x \end{matrix}; 1\right) \end{aligned} \quad (7.0.6)$$

Now it is directly verified that the main interpolation property (1.0.5) holds (it is sufficient to use standard transformation formulas for hypergeometric function ${}_3F_2(1)$).

The PPZ scheme holds for all $m, n = 0, 1, 2, \dots$.

The Frobenius-type relations (5.0.3), (5.0.4) follow from contiguous relations for hypergeometric function ${}_3F_2(1)$. We obtain

$$\begin{aligned} \kappa_{nm} = \rho_{nm} &= \frac{m+n+1}{\beta+n-m} \\ \mu_{nm} &= \frac{(m+1)(\beta-m-1)}{(m+n+2)(\beta+n-m)}, \\ \nu_{nm} &= -\frac{(n+1)(\beta+n+1)}{(m+n+2)(\beta+n-m)} \end{aligned}$$

Compatibility conditions (5.0.14) - (5.0.16), or, equivalently, (5.0.21), (5.0.22) are easily verified to hold.

The shifted PPZ scheme with $j_2 = j_1 = j$ in our case is equivalent to initial scheme with $\alpha \to \alpha - j$.

Indeed, consider the usual PPZ-scheme for our case:

$$Y_s \equiv \frac{(1-\alpha)_s}{(1-\alpha+\beta)_s} = \frac{(\alpha-s)_n}{(\alpha-\beta-s)_n} \frac{Q_m(s;n)}{P_n(s;m)}, \; s = 0, 1, \dots, n+m \quad (7.0.7)$$

For the $(1,1)$ shifted PPZ-scheme we have

$$Y_s \equiv \frac{(1-\alpha)_s}{(1-\alpha+\beta)_s} = \frac{(\alpha-s)_{n-1}}{(\alpha-\beta-s)_{m-1}} \frac{Q_m^{(1,1)}(s;n)}{P_n^{(1,1)}(s;m)}, \; s = 0, 1, \dots, n+m \quad (7.0.8)$$

After elementary transformations expression (7.0.8) can be rewritten in the form

$$\frac{(2-\alpha)_s}{(2-\alpha+\beta)_s} = \kappa \, \frac{(\alpha-1-s)_n}{(\alpha-1-\beta-s)_m} \, \frac{Q_m^{(1,1)}(s;n)}{P_n^{(1,1)}(s;m)}, \; s = 0,1,\dots,n+m \tag{7.0.9}$$

where $\kappa = (1-\alpha+\beta)/(1-\alpha)$. We thus see that (7.0.9) coincides with usual PPZ-scheme (7.0.7) with

$$P_n^{(1,1)}(z;m) = P_n(z;m)|_{\alpha\to\alpha-1}, \quad Q_m^{(1,1)}(z;n) = \kappa Q_m(z;n)|_{\alpha\to\alpha-1}$$

i.e. we should take polynomials from ordinary PPZ-scheme and just shift the parameter α. By induction, it can be easily proven that for (j,j) shifted PPZ-scheme we obtain the same polynomials $Q_m(z;n), P_n(z;m)$ with $\alpha \to \alpha - j$.

This remarkable observation allows one to construct biorthogonal functions which have both the same structure in terms of hypergeometric functions. Of course, such phenomenon can occurs only for exceptional cases of interpolated sequences Y_s, a_n, b_n, x_s.

8. Strings and biorthogonality in the scheme with prescribed poles and zeros

8.1. **Vertical string.** Consider the case of vertical string $m = const$ in the PPZ scheme. We already found 3-term recurrence relations for these polynomials (5.0.11), or, equivalently, (5.0.12). From structure of these relations it follows that for fixed m polynomials $P_n(z;m)$ in the PPZ scheme belong to the polynomials of R_{II}-type in terms of [21]. As was shown in [49] this means that corresponding rational functions $U_n(z;m)$ should satisfy a biorthogonality property with respect to another set $U_n^*(z;m)$ of rational functions. In this section we show that the biorthogonal partners $U_n^*(z;m)$ belong to some modified PPZ scheme.

Indeed, introduce the shifted $j_1 = j_2 = 1$ PPZ scheme for the same function $f(z)$:

$$f(x_s) = \frac{(x_s-a_1)\dots(x_s-a_{n-1})}{((x_s-b_1)\dots(x_s-b_{m-1}))} \, \frac{Q_m^{(1,1)}(x_s;n)}{P_n^{(1,1)}(x_s;m)}, \; s=0,1,\dots,n+m, \; n,m \ge 1 \tag{8.1.1}$$

Define rational functions

$$U_n^*(z;m) = \frac{P_n^{(1,1)}(z;m+1)}{(z-x_{m+2})(z-x_{m+3})\dots(z-x_{n+m+1})} \tag{8.1.2}$$

We then have

Theorem 6. Rational functions $U_n(z;m)$ and $U_n^*(z;m)$ are biorthogonal:

$$\frac{1}{2\pi i}\int_\Gamma w(\zeta;m)U_n(\zeta;m)U_j^*(\zeta;m)d\zeta = h_n(m)\,\delta_{nj}, \tag{8.1.3}$$

where the weight function is

$$w(z) = \frac{f(z)B_m(z)}{(z-x_0)(z-x_1)\dots(z-x_{m+1})} \tag{8.1.4}$$

and the contour Γ should contain all interpolation points $x_0, x_1, \dots, x_{n+m+1}$ inside and not contain prescribed zeros $a_i, i = 1,2,\dots,n$. The normalization coefficient $h_n(m)$ will be calculated below.

Proof. Assume first that $j < n$. Then we have

$$I_{nj} \equiv \int_\Gamma w(\zeta;m)U_n(\zeta;m)U_j^*(\zeta;m)d\zeta = \int_\Gamma \frac{f(\zeta;m)B_m(\zeta)P_n(\zeta;m)P_j^{(1,1)}(\zeta;m+1)}{A_n(\zeta)(z-x_0)\dots(z-x_{j+m+1})}d\zeta$$

Multiply numerator and denominator of the last expression by a polynomial $q_{n-j-1}(\zeta) = (\zeta - x_{j+m+2})\dots(\zeta - x_{n+m})$ of degree $n-j-1$. Then we have

$$I_{nj} = \int_\Gamma \frac{f(\zeta;m)B_m(\zeta)q_{n-j-1}(\zeta)P_n(\zeta;m)P_j^{(1,1)}(\zeta;m+1)}{A_n(\zeta)(z-x_0)\dots(z-x_{n+m})}d\zeta$$

In the last expression we can change the contour Γ extending it to encircle additional points $x_{j+m+2}, x_{j+m+3}, \dots x_{n+m}$. Such modification of the contour is admissible because it leads to adding of zero terms coming from zero residues from these added points. But now the last expression is zero $I_{nj} = 0,\ j < n$ due to main orthogonality property (5.0.1).

Assume that $j > n$. Then

$$I_{nj} = \int_\Gamma \frac{f(\zeta;m)B_m(\zeta)P_n(\zeta;m)P_j^{(1,1)}(\zeta;m+1)}{A_n(\zeta)(z-x_0)\dots(z-x_{j+m+1})}d\zeta =$$

$$\int_\Gamma \frac{f(\zeta;m)B_m(\zeta)P_n(\zeta;m)P_j^{(1,1)}(\zeta;m+1)\pi_{j-n-1}(\zeta)}{A_{j-1}(\zeta)(z-x_0)\dots(z-x_{j+m+1})}d\zeta$$

(we multiplied numerator and denominator by the polynomial $\pi_{j-n-1}(\zeta) = (\zeta - a_{n+1})\dots(\zeta - a_{j-1}))$. Again the orthogonality property (5.0.1) yields $I_{nj} = 0$ for $j > n$.

The remaining problem is calculation of the normalization coefficient $h_n(m)$. From (8.1.3) it follows that

$$h_n(m) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta;m)B_m(\zeta)U_n(\zeta;m)P_j^{(1,1)}(\zeta;m+1)}{(z-x_0)\dots(z-x_{j+m+1})}d\zeta. \tag{8.1.5}$$

Now we use the expansion (5.0.20) for $U_n(z;m)$. Substituting this expansion to (8.1.5) and using orthogonality property, we arrive at the expression

$$h_n = E_n(n,m)\,\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta;m)B_m(\zeta)\omega_n(\zeta;m)P_j^{(1,1)}(\zeta;m+1)}{A_n(\zeta)(\zeta-x_0)\dots(\zeta-x_{j+m+1})}d\zeta. \tag{8.1.6}$$

From the PPZ-scheme (1.0.4) we can replace under the integral the expression $f(z)B_m(z)P_n^{(1,1)}(z;m+1)/A_{n-1}(z)$ with $Q_{m+1}^{(1,1)}(z;n)$, hence, after simplification

$$h_n(m) = \frac{E_n(n,m)}{2\pi i}\int_\Gamma \frac{Q_{m+1}^{(1,1)}(\zeta;n)}{(\zeta-a_n)(\zeta-x_n)\dots(\zeta-x_{n+m+1})}d\zeta \tag{8.1.7}$$

Using Frobenius-type relation (5.0.4) we can replace in the integral:

$$\frac{Q_{m+1}^{(1,1)}(z;n)}{z-a_n} = \frac{\tilde\kappa_{nm}}{\tilde\rho_{nm}}Q_m^{(1,1)}(z;n+1) - \frac{(z-x_{n+m+1})Q_m^{(1,1)}(z;n)}{\tilde\rho_{nm}(z-a_n)}$$

where by $\tilde\kappa_{nm}, \tilde\rho_{nm}$ we denote for simplicity the Frobenius coefficients for polynomials $Q_m^{(1,1)}(z;n)$.

Thus

$$h_n(m) = \frac{E_n(n,m)\tilde{\kappa}_{nm}}{2\pi i \tilde{\rho}_{nm}} \int_\Gamma \frac{Q_m^{(1,1)}(\zeta; n+1)}{(\zeta - x_n)\dots(\zeta - x_{n+m+1})} d\zeta -$$

$$\frac{E_n(n,m)}{2\pi i \tilde{\rho}_{nm}} \int_\Gamma \frac{Q_m^{(1,1)}(\zeta; n)}{(\zeta - a_n)(\zeta - x_n)\dots(\zeta - x_{n+m})} d\zeta \qquad (8.1.8)$$

The first integral in (8.1.8) is zero because it is reduced to divided difference derivative of order $m+1$ from the m-th degree polynomial $Q_m^{(1,1)}(z;n)$. The second integral in (8.1.8) has the same structure as the integral (8.1.7) but with $m \to m-1$. This process can be repeated and we arrive after $m+1$ steps at the expression

$$h_n(m) = (-1)^{m+1} \frac{E_n(n,m)}{2\pi i \tilde{\rho}_{nm}\tilde{\rho}_{n,m-1}\dots\tilde{\rho}_{n0}} \int_\Gamma \frac{Q_0^{(1,1)}(\zeta; n)}{(\zeta - a_n)(\zeta - x_n)} d\zeta \qquad (8.1.9)$$

The last integral is calculated elementary: $Q_0^{(1,1)}(z;n) = \tilde{D}_0(n,0)$ is a constant and

$$\frac{1}{2\pi i} \int_\Gamma \frac{1}{(\zeta - a_n)(\zeta - x_n)} d\zeta$$

is a complex integral with contour Γ encircling the point $z = x_n$ but avoiding the point $z = a_n$. Hence, by the Cauchy theorem, we have

$$h_n(m) = (-1)^{m+1} \frac{E_n(n,m)\tilde{D}_0(n,0)}{\tilde{\rho}_{nm}\tilde{\rho}_{n,m-1}\dots\tilde{\rho}_{n0}(x_n - a_n)} \qquad (8.1.10)$$

Note that $h_n(m)$ is well-defined and nonzero because all quantities in (8.1.9) are nonzero due to normality of the PPZ-scheme. Thus, the normalization constant $h_n(m)$ can be expressed in terms of expansion and Frobenius coefficients.

The expression for the normalization coefficient $h_n(m)$ can be presented in an equivalent form. Indeed, from the first Frobenius-type relation in the pair (5.0.4) we have (for normal PPZ-scheme)

$$\tilde{\rho}_{nm} = (a_n - x_{n+m+1}) \frac{Q_m^{(1,1)}(a_n; n)}{Q_{m+1}^{(1,1)}(a_n; n)}$$

and hence

$$\tilde{\rho}_{nm}\tilde{\rho}_{n,m-1}\dots\tilde{\rho}_{n0} = \frac{Q_0^{(1,1)}(a_n; n)}{Q_{m+1}^{(1,1)}(a_n; n)} (a_n - x_{n+1})(a_n - x_{n+2})\dots(a_n - x_{n+m+1})$$

Substituting this to (8.1.10) we get

$$h_n(m) = (-1)^m \frac{E_n(n;m) Q_{m+1}^{(1,1)}(a_n; n)}{(a_n - x_n)(a_n - x_{n+1})\dots(a_n - x_{n+m+1})} \qquad (8.1.11)$$

Note that the biorthogonality property can be rewritten in equivalent form as

$$\sum_{s=0}^{\infty} Y_s B_m(x_s)\, Res \left(\frac{U_n(z) U_j^*(z)}{(z - x_0)(z - x_1)\dots(z - x_{m+1})} \right) \Bigg|_{z=x_s} = h_n\, \delta_{nj} \qquad (8.1.12)$$

The summation in (8.1.12) is really restricted by a finite number of terms, because $Res(...)|_{z=x_s} = 0$ if $s > j+m+1$. Nevertheless, biorthogonality relation (8.1.12) generates an *infinite* system of functions $U_n(z), U_n^*(z)$, $n = 0, 1, \dots$ in case if the PPZ scheme exists for all $n, m = 0, 1, \dots$.

Consider a simple explicit example of the biorthogonal functions for a vertical string. We choose solutions (7.0.5) for PPZ scheme for $f(x)$ a ratio of two gamma-functions (7.0.1). The rational functions $U_n(z; m)$ for fixed m are already known (7.0.6). We need only to construct their biorthogonal partners $U_n^*(x; m)$ using recipe (8.1.2). Polynomials $P_n^{(1,1)}(x; m+1)$ are obtained from $P_n(x; m+1)$ by shifting the parameter $\alpha \to \alpha - 1$:

$$P_n^{(1,1)}(x; m+1) = (-1)^n (\alpha-1)_n \, {}_3F_2 \left(\begin{matrix} -n, -x, -\beta-n \\ -n-m-1, 2-n-\alpha \end{matrix}; 1 \right) \tag{8.1.13}$$

Thus for corresponding rational functions we have

$$\begin{aligned} U_n^*(x; m) = \frac{(\alpha-1)_n}{(m+2-x)_n} \, {}_3F_2 \left(\begin{matrix} -n, -x, -\beta-n \\ -n-m-1, 2-n-\alpha \end{matrix}; 1 \right) = \\ \frac{(\alpha-1)_n}{(m+2)_n} \, {}_3F_2 \left(\begin{matrix} -n, -x, 2+\beta-\alpha \\ 2+m-x, 2-n-\alpha \end{matrix}; 1 \right) \end{aligned} \tag{8.1.14}$$

In this case we have an infinite system of rational functions satisfying biorthogonality property (8.1.12) with $x_s = s = 0, 1, 2, \dots$. The normalization constant h_n in this case can be easily calculated using formula (8.1.10):

$$h_n(m) = -\frac{(\beta-m)_n(-\beta-n)_{m+1}}{(\alpha-\beta-1)(m+1)_n(n+1)_{m+1}} \tag{8.1.15}$$

8.2. **The Kronecker string.** In this section we consider the Kronecker (i.e. anti-diagonal) string for the PPZ scheme.

Fix $N = n+m$ and denote $P(z; N-n) = K_n(z; N)$ and $\mu^{(N)} = \mu_{n,N-n}$ and so on. Then from Frobenius-type relations (5.0.3) we get 3-term recurrence relation for polynomials $K_n(z; N)$:

$$\begin{aligned} &\kappa_n^{(N)} \mu_n^{(N-1)} K_{n+1}(z; N) - \kappa_n^{(N)} \nu_n^{(N-1)} (z - a_{n+1}) K_n(z; N) - \\ &(z - x_{N+1}) K_n(z; N) - \rho_n^{(N)} (z - b_{N-n+1}) \mu_{n-1}^{(N-1)} K_n(z; N) - \\ &\rho_n^{(N)} \nu_{n-1}^{(N-1)} (z - a_n)(z - b_{N-n+1}) K_{n-1}(z; N) = 0 \end{aligned} \tag{8.2.1}$$

We recognize again the recurrence relation of R_{II} type for polynomials $K_n(z; N)$ with fixed N. Hence we can expect that there are pairs of corresponding biorthogonal rational functions.

Indeed, define a pair of rational functions of type $[n/n]$:

$$\begin{aligned} T_n(z; N) &= \frac{K_n(z; N)}{(z-a_1)(z-a_2)\dots(z-a_n)} \\ T_n^*(z; N) &= \frac{P_n^{(1,1)}(z; N-n)}{(z-b_{N-n})(z-b_{N-n+1})\dots.(z-b_{N-1})} \end{aligned} \tag{8.2.2}$$

where $P_n^{(1,1)}(z; m)$ denotes denominator polynomials in the shifted PPZ scheme (6.0.1) We have

Theorem 7. Rational functions $T_n(z;N)$, $T_n^*(z;N)$ are biorthogonal with respect to scalar product

$$\int_\Gamma \frac{f(\zeta)B_{N-1}(\zeta)}{\omega_{N+1}(\zeta)} T_n(\zeta;N)T_k^*(\zeta;N)d\zeta = h_n(N)\delta_{nk} \tag{8.2.3}$$

where $\omega_{n+1}(z) = (z-x_0)\dots(z-x_n)$ and h_n are some normalization constants. The contour Γ should contain all interpolation points $x_0, x_1, \dots x_N$ inside but not the points a_i, b_i. Equivalently, we have finite biorthogonality relation

$$\sum_{s=0}^{N} w_s T_n(x_s;N)T_k^*(x_s;N) = h_n\delta_{nk}, \tag{8.2.4}$$

with discrete weights

$$w_s = \frac{Y_s B_{N-1}(x_s)}{\omega'_{N+1}(x_s)} \tag{8.2.5}$$

Expression for the normalization coefficient is

$$h_n(N) = (-1)^{N-n} \frac{E_n(n;N-n)Q_0^{(1,1)}(0;n)}{(x_n - a_n)\tilde{\rho}_{n,N-n-1}\tilde{\rho}_{n,N-n-2}\cdots\tilde{\rho}_{n0}} \tag{8.2.6}$$

The proof of this proposition is almost the same as for vertical string of the PPZ scheme, i.e. we should consider separately case $k < n$ and $k > n$ and apply orthogonality property (5.0.1). Expression (8.2.6) for $h_n(N)$ is obtained if one considers the case $k = n$ by the same procedure as for the case of vertical string.

Just as in the case of the Kronecker string for the ordinary Padé interpolation problem, one can prove the "inverse" statement:

Theorem 8. Assume that there exists three finite sequences of pairwise non-coinciding points $x_0, x_i, a_i, b_i, i = 1\dots, N$ and a system of rational functions

$$T_n(x) = \frac{K_n(x)}{(x-a_1)(x-a_2)\dots(x-a_n)}, \quad T_n^*(x) = \frac{K_n^*(x)}{(x-b_{N-1})(x-b_{N-2})\dots(x-b_{N-n})}$$

where $K_n(x), K_n^*(x)$ are polynomials of n-th degree having no common zeros and no zeros coinciding with x_i, a_i, b_i. Let $T_n(x), T_n^*(x)$ be biorthogonal (8.2.4) with respect to some nonzero discrete weights w_s.

Then there exists a finite nonzero sequence Y_s and a finite set of polynomials $Q_n(x)$, $n = 0, 1, \dots, N$ such that anti-diagonal Padé interpolation property

$$Y_s = \frac{A_n(x_s)}{B_{N-n}(x_s)} \frac{Q_{N-n}(x_s)}{K_n(x_s)}, \; s = 0, 1, \dots, N \tag{8.2.7}$$

holds.

The proof of this proposition is similar to that for the ordinary Padé interpolation case (see section 2.1) and we omit it.

Thus *any* pair of rational functions with finite orthogonality property (8.2.4) (and with conditions of nondegeneracy) corresponds to some Kronecker string in the PPZ scheme.

Consider an explicit example connected with PPZ scheme for ratio of to gamma-functions. As we already seen, the shifted PPZ-scheme for $j_1 = j_2 = 1$ is equivalent to initial scheme with $\alpha \to \alpha - 1$. Hence we have

$$P_n^{(1,1)}(x;m) = (-1)^n(\alpha-1)_n\,{}_3F_2\left(\begin{matrix}-n,-x,-\beta-n\\ -n-m,2-n-\alpha\end{matrix};1\right)$$

From (8.2.2), using transformation formulas for hypergeometric function ${}_3F_2(1)$ we easily obtain expression for biorthogonal partners

$$T_n^*(x;N) = {}_3F_2\left(\begin{matrix}-n,x-N,\beta+n-N\\ -N,x+\beta+2-N-\alpha\end{matrix};1\right) \tag{8.2.8}$$

The discrete weight function (8.2.5) in this case is

$$w_s = -\frac{(\alpha-\beta)_{N-1}}{N!}\,\frac{(1-\alpha)_s(-N)_s}{s!(2+\beta-\alpha-N)_s} \tag{8.2.9}$$

i.e. w_s coincides with the finite hypergeometric distribution. For the normalization constant we get from (8.2.6)

$$h_n(N) = \frac{(\beta-N)(-\beta)_N(1+\beta)_n n!}{(1+\beta-\alpha)N!(-N)_n(\beta-N+2n)(\beta-N)_n}$$

8.3. **Biorthogonal rational functions on elliptic grid and PPZ scheme.** In this section we show how BRF on elliptic grid introduced in [35], [36], [37] are related with PPZ scheme of the Padé interpolation.

We need expression for elliptic hypergeometric functions [15], [36], [16] (in what follows we adopt notation of [33])

$${}_{r+1}E_r = \sum_{n=0}^{\infty}\frac{[u_1]_n[u_1+2n]}{[n]![u_1]}\prod_{m=1}^{r-4}\frac{[u_{3+m}]_n}{[u_1+1-u_{3+m}]_n} \tag{8.3.1}$$

where $\sum_{m=1}^{r-4} u_{3+m} = u_1(r-5)/2 + (r-7)/2$. The elliptic shifted factorials are defined as follows

$$[u]_n \equiv [u][u+1]\dots[u+n-1],\ [n]! = [1]_n.$$

and $[u] \equiv \theta_1(\pi h u)/\theta_1(\pi h)$, where $\theta_1(u)$ is the standard theta function (see, e.g. [44]) and h is an arbitrary ("quantum") parameter.

Introduce the interpolated sequence

$$Y_s = \frac{[1+\alpha-\beta]_s[1-\epsilon+\beta]_s[\delta-\alpha]_s[\epsilon+\delta]_s}{[1+\alpha]_s[1-\epsilon]_s[\beta+\delta-\alpha]_s[\epsilon+\delta-\beta]_s}, \tag{8.3.2}$$

where $\alpha,\beta,\epsilon,\delta$ are arbitrary parameters. Then we have solution of the Padé interpolation problem (1.0.6) where both rational functions $V_m(x;n), U_n(x;m)$ are expressed as elliptic hypergeometric functions ${}_{12}E_{11}$ with the parameters:

$$u_1 = \alpha-\beta,\ u_4 = -m,\ u_5 = -\beta-n,\ u_6 = \alpha-\beta+1+m+n,\ u_7 = \epsilon+\alpha-\beta,$$

$$u_8 = 1+\alpha-\epsilon-\delta,\ u_9 = -t,\ u_{10} = \delta+t$$

for the numerator functions $V_m(x(t);n)$ and

$$u_1 = \alpha,\ u_4 = -n,\ u_5 = \beta-m, u_6 = \alpha+1+m+n,\ u_7 = \epsilon+\alpha-\beta,$$

$$u_8 = 1 + \alpha - \epsilon - \delta,\ u_9 = -t,\ u_{10} = \delta + t$$

for the denominator functions $U_n(x(t), m)$.

The argument $x(t)$ is parameterized as

$$x(t) = \frac{[t + t_0 - \xi][t + t_0 + \xi]}{[t + t_0 - \eta][t + t_0 + \eta]}, \tag{8.3.3}$$

where ξ, η are arbitrary parameters ($\xi \neq \eta$) and $2t_0 = \delta$. The interpolation grid x_s is

$$x_s = x(s),\ s = 0, 1, \ldots, N \tag{8.3.4}$$

The prescribed zeros are $a_j = x(-\alpha - j)$ and prescribed poles are $b_j = x(\beta - \alpha - j)$.

The interpolation property (1.0.6) in our case can be easily verified with the help of the Bailey-type transformation for the elliptic hypergeometric functions [15], [16].

Note that expression (8.3.3) defines $x(t)$ as an arbitrary elliptic function of second order with prescribed periods $2\omega, 2\omega'$ and with two poles in the fundamental parallelogram [44]. Thus the zeros a_n and poles b_n are also elliptic functions of the second order in the argument n.

The parameters ξ, η do not enter to the hypergeometric parameters u_i of the rational functions $U_n(x(t); m), V_m(x(t); n)$ and play the role of free (gauge) parameters. Freedom in choice of the parameters ξ, η can be explained as follows. As we already saw, the Möbius transform of the grid $x_s \to (\alpha x_s + \beta)/(\gamma x_s + \delta)$ leads to similar PPZ-scheme (with appropriately changed interpolants $P_n(x; m), Q_m(x; n)$, see Proposition **1**). It is easily verified that the Möbius transform of the elliptic grid $x(t)$ (8.3.3) is equivalent (up to a non-essential common factor) to changing of parameters ξ, η. Thus freedom in the choice of the gauge parameters ξ, η means freedom in choice of the Möbius transform of the elliptic grid $x(t)$. In particular, by an appropriate choice of ξ, η it is always possible to reduce the grid x_s to one of standard forms: either the Weierstrass function $x(t) = \wp(qt)$, or elliptic sin function: $x(t) = sn(qt; k)$ with some parameters q, k.

The obtained rational functions $U_n(x(t); m), V_m(x(t); n)$ satisfy the Frobenius-type relations (5.0.5) and (5.0.6) with the coefficients

$$\kappa_{nm} = -\frac{[m+n+1][\epsilon+\delta+n][n+1+\beta-\epsilon]}{[\beta+n-m][\alpha+n+1][\alpha-\beta+\epsilon+m+n+1]} \times \\ \frac{[\alpha-\beta+n+2m+2][\alpha+\delta/2+n+1+\eta][\alpha+\delta/2+n+1-\eta]}{[\delta-\alpha+\epsilon-n-m-2][n+m+1+\delta/2+\eta][n+m+1+\delta/2-\eta]}$$

$$\rho_{nm} = \frac{[m+n+1][\epsilon-m-1][\epsilon+\delta-\beta+m]}{[\beta+n-m][\alpha+n+1][\alpha-\beta+\epsilon+m+n+1]} \times \\ \frac{[\alpha+2n+m+2][\beta-\alpha+\delta/2-m-1+\eta][\beta-\alpha+\delta/2-m-1-\eta]}{[\delta-\alpha+\epsilon-n-m-2][n+m+1+\delta/2+\eta][n+m+1+\delta/2-\eta]}$$

$$\mu_{nm} = \frac{[m+1][1+\alpha-\beta+m][\beta-m-1][\alpha+m+2n+3]}{[m+n+2][\alpha-\beta+m+n+2][\beta-m+n][\alpha+m+n+2]}$$

$$\nu_{nm} = -\frac{[n+1][\alpha+n+1][\beta+n+1][\alpha-\beta+n+2m+3]}{[m+n+2][\alpha-\beta+m+n+2][\beta-m+n][\alpha+m+n+2]}$$

These coefficients can be easily found from the system of relations (5.0.22), (5.0.21). On the other hand, the Frobenius-type relations (5.0.5) and (5.0.6) in this case are equivalent to contiguous relations for the elliptic hypergeometric functions ${}_{12}E_{11}$ found in [36].

As we have solution for PPZ scheme with arbitrary m, n, we can construct biorthogonal rational functions for the vertical and Kronecker strings.

First, we should consider the shifted PPZ scheme (6.0.1) for $j_1 = j_2 = 1$ because the polynomials $P_n^{(1,1)}(z;m)$ are building blocks for construction the biorthogonal partners in both schemes with vertical and Kronecker strings (see formulas (8.1.2) and (8.2.2)). It is easily seen that the shifted scheme with $j_1 = j_2 = 1$ is equivalent to the non-shifted scheme with $\alpha \to \alpha - 1$. More exactly, this means

$$Y_s \to Y_s|_{\alpha\to\alpha-1},\ P_n^{(1,1)}(z;m) = P_n(z;m)|_{\alpha\to\alpha-1},\ Q_m^{(1,1)}(z;n) = c\,Q_m(z;n)|_{\alpha\to\alpha-1} \quad (8.3.5)$$

where c is a constant which is inessential (this means merely, that we multiply Y_s by this constant).

Consider first the case of the Kronecker string. Fix $N = m + n$ and for biorthogonal partners $T_n(x;N), T_n^*(x;N)$ we have from (8.2.2) that $T_n(x)$ coincides with $U_n(x(t); N-n)$ and it is elliptic hypergeometric function ${}_{12}E_{11}$ with the parameters

$$\begin{gathered} u_1 = \alpha,\ u_4 = -n,\ u_5 = \beta - N + n, u_6 = \alpha + 1 + N, \\ u_7 = \epsilon + \alpha - \beta, u_8 = 1 + \alpha - \epsilon - \delta,\ u_9 = -t,\ u_{10} = \delta + t \end{gathered} \quad (8.3.6)$$

For dual biorthogonal partners we have

$$T_n^*(x;N) = \left\{ \frac{A_n(x)}{(x - b_{N-1}) \dots (x - b_{N-n+1})}\, P_n(x; N-n) \right\} \Bigg|_{\alpha\to\alpha-1} \quad (8.3.7)$$

Using the elliptic analogue of the Bailey transformation [15], one can simplify expression (8.3.7) reducing it to the function ${}_{12}E_{11}$ with the parameters

$$\begin{gathered} u_1 = \beta - \alpha - N + \delta,\ {}_4 = -n,\ u_5 = \beta - \alpha + \delta + 1,\ u_6 = \delta - \alpha + \epsilon - N \\ u_7 = \beta - \alpha - \epsilon - N + 1,\ u_8 = \beta - N + n, u_9 = -t, u_{10} = \delta + t \end{gathered} \quad (8.3.8)$$

For the corresponding discrete weight function w_s in the biorthogonality relation (8.2.4) we obtain after simple calculations

$$\begin{gathered} w_s = Y_s \frac{B_{N-1}(x_s)}{\Omega_{N+1}(x_s)} = \kappa_N\, [2s+\delta] \frac{[-N]_s[\delta]_s[\alpha - \beta + N]_s}{[s]![\alpha+1]_s[1-\epsilon]_s[\epsilon + \delta - \beta]_s} \times \\ \frac{[1+\beta-\epsilon]_s[\delta - \alpha]_s[\epsilon+\delta]_s}{[\delta + N + 1]_s[1 + \beta + \delta - \alpha - N]_s} \end{gathered} \quad (8.3.9)$$

where κ_N is a coefficient which doesn't depend on s. Expression (8.3.9) coincides with that for the weight function of the "elliptic" BRF obtained in [36]

We obtain that for the Kronecker string corresponding BRF $T_n(x;N), T_n^*(x;N)$ coincide with those found in our previous papers [35], [36], [37]. Here we see that these BRF are easily *derived* from the PPZ scheme using explicit formulas (8.2.2).

Consider the case of the vertical string $m = const$. The rational functions $U_n(x;m)$ coincide with denominator rational functions in PPZ scheme on elliptic grid, hence they are ${}_{12}E_{11}$ elliptic hypergeometric functions with the parameters

$$u_1 = \alpha,\; u_4 = -n,\; u_5 = \beta - m, u_6 = \alpha + 1 + m + n,\; u_7 = \epsilon + \alpha - \beta,$$

$$u_8 = 1 + \alpha - \epsilon - \delta,\; u_9 = -t,\; u_{10} = \delta + t.$$

For biorthogonal partners $U_n^*(x;m)$ we have formula (8.1.2). Again, after the Bailey-type transformation we reduce this expression (up to an inessential factor which doesn't depend on x) to the elliptic hypergeometric function ${}_{12}E_{11}$ with the parameters

$$u_1 = \delta + m + 1,\; u_4 = -n,\; u_5 = \delta + \beta - \alpha + 1, u_6 = \delta - \beta + \epsilon + m - 1,\; u_7 = m + 2 - \epsilon,$$

$$u_8 = 1 + \alpha + m + n,\; u_9 = -t,\; u_{10} = \delta + t$$

For the vertical string we obtain *a new* system of BRF on elliptic grids. These BRF form an infinite system ($n = 0, 1, \dots$) in contrast to the case of the Kronecker string. It is interesting to note that in this case the set of interpolating points $x_s, s = 0, 1, \dots$ may form a dense set on a finite real interval.

Indeed, the grid x_s given by expression (8.3.3), is an arbitrary elliptic function of the second degree. Using an appropriate Möbius transformation $x_s \to (\alpha_1 x_s + \alpha_2)/(\alpha_3 x_s + \alpha_4)$ we can reduce x_s to some simple standard form, e.g. for the elliptic sine function

$$x_s = sn(q(s - s_0); k) \tag{8.3.10}$$

with some parameters q, s_0, k. Assume that these parameters are real and $0 < k < 1$ (standard choice for k) and moreover $j_1 q \neq 4K(k) j_2$, where $4K(k)$ is a real period of the elliptic function $sn(x;k)$ [44] and j_1, j_2 are some integers. We then obtain that the grid x_s for all $s = 0, 1, \dots$ is a set of real numbers dense on the interval $[-1, 1]$. In biorthogonality relation (8.1.12) summation is made over all interpolated points $x_0, x_1, \dots$. We thus see that in the case of the vertical string the corresponding elliptic BRF have unusual property: they are biorthogonal on the dense set of points on an interval. In theory of orthogonal polynomials such measures are very interesting from both physical and mathematical point of view (see, e.g. [40], [25], [41]). We mention also an interesting paper by Magnus [26] where a simple explicit example of such polynomials orthogonal on a dense set of points was constructed.

Finally, note that the interpolated sequence Y_s defined by (8.3.2) has a remarkable property: if one changes the parameters

$$\alpha \to \alpha - \beta,\; \beta \to -\beta,\; \epsilon \to \epsilon - \delta,\; \delta \to \delta \tag{8.3.11}$$

then $Y_s \to 1/Y_s$, i.e. such transformation leads to to exchange of rational functions $U_n(z;m) \to V_n(z;m)$ $V_m(z;n) \to U_m(z;n)$. Note that transformation (8.3.11) is an involution, i.e. square of this transformation preserves the parameters $\alpha, \beta, \epsilon, \delta$ unchanged, as expected. Thus, both $U_n(z;m)$ and $V_n(z;m)$ functions can equally be exploited to construct pairs of corresponding BRF. This is also can be seen comparing hypergeometric parameters u_i corresponding to functions $U_n(z;m)$ and $V_n(z;m)$. These parameters are connected by transformation (8.3.11).

9. The case of finite orthogonal rational functions

Finally, consider the case when $T_n^*(x;N) = T_n(x;N)$ for some fixed N, i.e. when biorthogonal rational functions from the Kronecker string become orthogonal. Comparing expressions (8.2.2) we see that denominators of the functions $T_n(x;N)$ and $T_n^*(x;N)$ coincide iff condition

$$b_k = a_{N-k}, \ k = 1, 2, \dots N-1 \tag{9.0.1}$$

holds. But condition (9.0.1) for the fixed N means that in the Kronecker string the PPZ scheme is fully degenerated: $A_n(x) = B_{N-n}(x)$. Hence the PPZ scheme becomes the ordinary Padé interpolation scheme:

$$Y_s = \frac{Q_{N-n}(x_s;n)}{P_n(x_s;N-n)}, \quad s,n = 0,1,\dots,N \tag{9.0.2}$$

But then the shifted scheme (6.0.1) with $j_1 = j_2$ and with $m = N-n$ will coincide with (9.0.2). We thus have simple

Theorem 9. Biorthogonal rational functions $T_n^*(x;N)$, $T_n(x;N)$ in the Kronecker string of the PPZ scheme with fixed $N = 1,2,\dots$ become orthogonal if and only if condition (9.0.1) holds.

Now we can rewrite biorthogonality condition (8.2.4) as

$$\sum_{s=0}^{N} w_s T_n(x_s;N) T_k(x_s;N) = h_n \delta_{nk}, \tag{9.0.3}$$

with discrete weights

$$w_s = \frac{(x_s - a_1)Y_s}{\omega'_{N+1}(x_s)}, \tag{9.0.4}$$

where we used property $B_{N-1}(x) = A_1(x) = (x - a_1)$ which holds for the case of pure orthogonality.

Remark. Coincidence of zeros and poles (9.0.1) for one fixed Kronecker string doesn't mean such coincidence for other Kronecker string (i.e. with other value of N).

In particular, for the case of ${}_3F_2$ BRF we see from (7.0.3) that condition (9.0.1) is impossible for any N. Thus in this case there are no purely orthogonal functions.

For the case of elliptic BRF, nevertheless, there is a special case of pure orthogonality. Indeed, condition (9.0.1) implies

$$x(-\alpha - n) = x(\beta - \alpha - N + n), \ n = 1, 2, \dots, N-1 \tag{9.0.5}$$

where $x(t)$ is an elliptic function of the second order given by formula (8.3.3). Using well known Riemann identity for theta functions [44] one can transform the difference $x(t) - x(r)$ for arbitrary t and r as

$$x(t) - x(r) = \frac{[t-r][t+r+2t_0][\xi-\eta][\xi+\eta]}{[t+t_0+\xi][t+t_0-\xi][t+t_0+\eta][t+t_0-\eta]} \tag{9.0.6}$$

Thus equation $x(t) = x(r)$ has solutions of two types:

(i) $t - r = 2m_1\omega_1 + 2m_2\omega_2$;

(ii) $t + r + 2t_0 = 2m_1\omega_1 + 2m_2\omega_2$

where $2\omega_{1,2}$ are two minimal independent periods of the elliptic function $x(t)$ and m_1, m_2 are arbitrary integers.

In our case $t = -\alpha - n$, $r = \beta - \alpha - N + n$. We see that condition (i) is impossible, while condition (ii) is possible if (we are restricted with the simplest choice $m_1 = m_2 = 0$)

$$\beta - 2\alpha + \delta = N \tag{9.0.7}$$

Thus under condition (9.0.7) the elliptic BRF $T_n(x;N), T_n^*(x;N)$ should be *orthogonal*: $T_n(x;N) = T_n^*(x;N)$. Indeed, this can be easily verified comparing hypergeometric parameters (8.3.7) and (8.3.8) under condition (9.0.7). Note that condition (9.0.7) can be valid only for the only fixed (i.e. for the only N) Kronecker string.

10. Conclusions

To our opinion the following results obtained in the present paper are assumed to be new:

(i) new relations between the Kronecker algorithm for rational interpolation, the finite orthogonal polynomials and their duals;

(ii) relations between horizontal and diagonal strings in the ordinary Padé interpolation table and corresponding systems of BRF;

(iii) explicit formulas for biorthogonality relations (3.1.12), (2.3.2);

(iv) appearance of rational orthogonal functions (in spirit of [6]) as a degenerated case of diagonal strings in the ordinary Padé interpolation scheme.

(v) Frobenius-type relations (5.0.3), (5.0.4) for the scheme with prescribed poles and zeros;

(vi) appearance of BRF in the PPZ scheme for vertical and Kronecker strings and corresponding explicit formulas for biorthogonality.

We have showed that biorthogonal rational functions appear from the Padé interpolation scheme as naturally as orthogonal polynomials from the Padé approximation scheme. In contrast to the case of orthogonal polynomials, the Padé interpolation scheme leads to non-coinciding, biorthogonal rational functions. Purely orthogonal rational functions appear only as a special degenerated case of the Padé interpolation scheme when some pairs of interpolated nodes x_s merge leading to double points. It is interesting to note that history of theory of biorthogonal functions and their relation to approximation and interpolation schemes goes back to works of armenian mathematicians (see, e.g. review by Djrbashian [10]). Biorthogonality was also considered, but without connections to Padé interpolation scheme.

Theory of rational *orthogonal* functions was intensively developed by international group of mathematicians during last 20 years, see monograph [6] and recent survey [7]. In this direction many general results concerning different types of orthogonality measures, asymptotic properties, behavior of roots etc were obtained. However, concrete examples of such orthogonal functions connected with special functions and (or) with problems of modern mathematical physics were not considered.

On the other hand, many concrete examples of *biorthogonal* rational functions connected with generalized hypergeometric functions were constructed by experts in special functions and orthogonal polynomials (see, e.g. [18], [21], [31], [45] and reference therein). In particular, some explicit systems of BRF connected with hypergeometric function ${}_9F_8$ (or basic hypergeometric function ${}_{10}\Phi_9$) were constructed. In [21] it was an attempt to understand theory of BRF starting from three-term recurrence relations of special type (R_I and R_{II} types of recurrence relations) and their connection with new types of continued fractions. In [49] the general theory of BRF was shown to be equivalent to generalized spectral problem for two arbitrary Jacobi matrices. This observation lead to a construction of new examples of BRF on elliptic grids in [35], [36], [37]. On the other hand, these elliptic BRF are related with important objects in modern mathematical physics - so-called elliptic 6j-symbols. Namely, Frenkel and Turaev [15] were able to express these objects in terms of new special functions - modular hypergeometric functions. Recently, Rosengren [32] showed how elliptic BRF are connected with elliptic 6j-symbols. He also proposed an elegant algebraic interpretation of the elliptic BRF connected with the Sklyanin algebra.

We hope that the present paper can be considered as a bridge between approaches of authors dealing with orthogonal functions [6] and those dealing with special systems of BRF. Now we see that Padé interpolation problem is a natural source of these objects. Starting from simple examples of interpolation functions $f(x)$ and corresponding interpolation grids x_s one can obtain all known explicit systems of OP and BRF [53].

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