ELLIPTIC RUIJSENAARS OPERATORS AND ELLIPTIC HYPERGEOMETRIC INTEGRALS

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Abstract. We study a family of mutually commutative difference operators associated with root systems and discuss their simultaneous eigenvectors in a special case. For root systems with rank n, we construct $3n$ commutative difference operators, which are a generalization of elliptic Ruijsenaars operators. In particular, for the BC_1 root system, we construct an explicit simultaneous eigenvector of these operators described in terms of elliptic hypergeometric integrals.

1. Introduction

In [14] Ruijsenaars introduced the operators acting on the space of meromorphic functions which are defined by

$$Y_n = \sum_{\substack{I\subset\{1,\ldots,l\}\\ |I|=n}} \left(\prod_{\substack{j\in I\\ k\in I^c}} \frac{\sigma(x_j - x_k + \mu; \omega_1, \omega_2)}{\sigma(x_j - x_k; \omega_1, \omega_2)} \right) \prod_{j\in I} \tau_j(\omega_3), \tag{1.1}$$

where $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ are arbitrary such that $\omega_1/\omega_2 \notin \mathbb{R}$, and $\omega_3, \mu \in \mathbb{C} \setminus \{0\}$ and the action of $\tau_j(\omega)$ is defined by $(\tau_j(\omega)f)(x_1, \ldots, x_j, \ldots, x_n) = f(x_1, \ldots, x_j - \omega, \ldots, x_n)$. It is shown that these operators are mutually commutative. The first result of this article is a generalization of the elliptic Ruijsenaars operators. We define

$$Y_n^{(p)} = \sum_{\substack{I\subset\{1,\ldots,l\}\\ |I|=n}} \left(\prod_{\substack{j\in I\\ k\in I^c}} e^{\nu_p(x_j - x_k)} \frac{\sigma(x_j - x_k + \mu_p; \omega_q, \omega_r)}{\sigma(x_j - x_k; \omega_q, \omega_r)} \right) \prod_{j\in I} \tau_j(\omega_p), \tag{1.2}$$

where $p, q, r \in \mathbb{Z}/3\mathbb{Z}$ are distinct, and for $p \in \mathbb{Z}/3\mathbb{Z}$, $\omega_p \in \mathbb{C}\setminus\{0\}$ are such that $\omega_p/\omega_q \notin \mathbb{R}$ if $p \neq q$. Put $\eta_{pq} = 2\zeta(\omega_p/2; \omega_p, \omega_q)$ with Weierstrass' zeta function ζ and $a_r = \eta_{pq}\omega_q - \eta_{qp}\omega_p$, then $a_r = \pm 2\pi i$. If ν_p, μ_p satisfy three equations $\nu_p\omega_q + \mu_p\eta_{qr} = \nu_q\omega_p + \mu_q\eta_{pr}$ for distinct $p, q, r \in \mathbb{Z}/3\mathbb{Z}$, then all $Y_n^{(p)}$ are shown to be mutually commutative. For instance, these equations are solved by

$$\nu_1 = \frac{a_1(\nu_3\omega_1 + \mu_3\eta_{12}) - (a_2\mu_2 + a_3\mu_3)\eta_{32}}{a_1\omega_3}, \tag{1.3}$$

$$\nu_2 = \frac{\nu_3\omega_2 + \mu_3\eta_{21} - \mu_2\eta_{31}}{\omega_3}, \tag{1.4}$$

$$\mu_1 = \frac{a_2\mu_2 + a_3\mu_3}{a_1}, \tag{1.5}$$

where ν_3, μ_2, μ_3 are regarded as free parameters. Although the discussion above is for A-type root system, the construction can be applied to arbitrary root systems.

The second result is a construction of a simultaneous eigenvector of the elliptic Ruijsenaars operators of type BC_1. We obtain an explicit meromorphic eigenvector described in terms of the elliptic hypergeometric integral. Note that for the A-type root system, some classes of eigenvectors are discussed in [1, 4, 6, 12, 15–18].

2. Affine Root Systems

We summarize some facts about affine root systems and affine Weyl groups [2, 3, 7, 8]. In this article, we will omit $A_{2l}^{(2)}$-type root system because of simplicity, though it is straightforward. The notation and symbols are a little different from those in the previous papers [9, 10] in order to generalize the results.

Let Δ be the irreducible reduced finite root system of type X_l in a complex vector space V with $\dim V = l$ and the inner product $\langle\cdot,\cdot\rangle$, $I = \{1, \dots, l\}$ a set of indices, $\Pi = \{\alpha_i \mid i \in I\} \subset V$ the set of simple roots, $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\} \subset V$ the set of simple coroots, Q and $Q^\vee$ the root and coroot lattices, P and $P^\vee$ the weight and coweight lattices, $\{\Lambda_i \mid i \in I\}$ and $\{\Lambda_i^\vee \mid i \in I\}$ the fundamental weights and fundamental coweights such that $\langle\alpha_i, \Lambda_j^\vee\rangle = \langle\Lambda_i, \alpha_j^\vee\rangle = \delta_{ij}$. Then we have

$$Q = \bigoplus_{i\in I} \mathbb{Z}\,\alpha_i \subset P = \bigoplus_{i\in I} \mathbb{Z}\,\Lambda_i \subset V, \tag{2.1}$$

$$Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\,\alpha_i^\vee \subset P^\vee = \bigoplus_{i\in I} \mathbb{Z}\,\Lambda_i^\vee \subset V. \tag{2.2}$$

The inner product $\langle\cdot,\cdot\rangle$ is normalized such that $\langle\alpha,\alpha\rangle = 2$ for the longer roots α. Let Δ_+ and Δ_- be the set of positive roots and negative roots respectively.

Let $\Delta_s \subset \Delta$ be the set of shorter roots and $\Delta_l \subset \Delta$ the set of longer roots. Let r be the ratio of the square lengths of longer roots and shorter roots. Fix parameters γ_α for $\alpha \in \Delta$, such that in nontwisted case all $\gamma_\alpha = 1$, and in twisted case, $\gamma_\alpha = r$ if $\alpha \in \Delta_l$, $\gamma_\alpha = 1$ otherwise. Let $\widehat{V} = V \oplus \mathbb{C}\delta$ with $\langle\alpha_i, \delta\rangle = \langle\delta,\delta\rangle = 0$ and its linear extension. Then the associated affine root system $\widehat{\Delta} \subset \widehat{V}$ is written as

$$\widehat{\Delta} = \{\alpha + n\gamma_\alpha\delta \mid \alpha \in \Delta, n \in \mathbb{Z}\}. \tag{2.3}$$

Let $\widehat{\Delta}_+$ and $\widehat{\Delta}_-$ be the set of positive affine roots and negative affine roots respectively. We denote by $\overline{v}$ for $v \in \widehat{V}$ the natural projection on V.

For $\alpha \in \widehat{\Delta}$, let s_α be a reflection defined by

$$s_\alpha(v) := v - \langle \alpha, v \rangle \alpha^\vee, \qquad v \in V. \tag{2.4}$$

The Weyl group W is generated by the fundamental reflections $\{s_i := s_{\alpha_i} \mid i \in I\}$ on V and the affine Weyl group $\widehat{W}$ is generated by $\{s_i \mid i \in \widehat{I}\}$, where $\widehat{I} = I \cup \{0\}$ and $\alpha_0 = \delta - \theta$ with θ the highest root in nontwisted case and the highest short root in twisted case.

The defining relations are given by $s_i^2 = id$ and the Coxeter relations:

$$(s_i\, s_j)^{m_{ij}} = id, \qquad \text{for } i \neq j \in \widehat{I}, \tag{2.5}$$

where $m_{ij} = 2$ if α_i and α_j are disconnected in the Dynkin diagram and $m_{ij} = 3, 4, 6$ if 1,2,3 lines respectively connect α_i and α_j. For $\mu \in V$, we define endomorphisms τ_μ of the vector space V by

$$\tau_\mu(\lambda) := \lambda - \langle \lambda, \mu \rangle \delta. \tag{2.6}$$

Let $M := \mathbb{Z}(W \cdot \theta^\vee) \subset V$. For an arbitrary lattice L, we denote by T_L the corresponding group of translations of L. Then one sees that $\widehat{W}$ is the semidirect product $\widehat{W} = W \ltimes T_M$. Let $\widetilde{M} := \{\lambda \in V \mid \alpha \in \Delta, \langle \alpha, \lambda \rangle \in \gamma_\alpha \mathbb{Z}\}$. The extended affine Weyl group $\widetilde{W}$ is defined by the semidirect product $\widetilde{W} := W \ltimes T_{\widetilde{M}}$. Let Ω be the subgroup of $\widetilde{W}$ which stabilizes the affine Weyl chamber C. Then one sees that $\widetilde{W}$ is isomorphic to the semidirect product $\widehat{W} \rtimes \Omega$. Here are the explicit description of $\widetilde{M}$ and its canonical basis $\{\lambda_i \mid i \in I\}$:

$$\widetilde{M} = \begin{cases} P^\vee, & \text{nontwisted case,} \\ P, & \text{twisted case,} \end{cases} \qquad \lambda_i = \begin{cases} \Lambda_i^\vee, & \text{nontwisted case,} \\ \Lambda_i, & \text{twisted case.} \end{cases} \tag{2.7}$$

We also use $\widetilde{M}_- := \oplus_{i \in I} \mathbb{Z}_{\leq 0} \lambda_i$.

The length $\ell(w)$ of $w \in \widetilde{W}$ is defined by the length ℓ of a reduced decomposition:

$$w = s_{i_1} \dots s_{i_\ell} \omega, \qquad i_k \in \widehat{I}, \omega \in \Omega. \tag{2.8}$$

It is equivalent to the number of the negative roots made positive by $\hat{w}$:

$$\ell(\hat{w}) := |\Delta_{\hat{w}}|, \qquad \Delta_{\hat{w}} := \widehat{\Delta}_+ \cap \hat{w} \widehat{\Delta}_-. \tag{2.9}$$

The set $\Delta_{\hat{w}}$ is explicitly described as $\Delta_{\hat{w}} = \{\alpha^{(1)} = \alpha_{i_1}, \alpha^{(2)} = s_{i_1}(\alpha_{i_2}), \dots, \alpha^{(\ell)} = w s_{i_\ell}(\alpha_{i_\ell})\}$. By definition, $\Delta_{\hat{w}}$ is independent of reduced expressions. One sees that $\Omega = \{\omega \in \widetilde{W} \mid \ell(\omega) = 0\}$. A weight $\lambda \in \widetilde{M}$ is said to be minuscule if $\Delta_{\tau_{-\lambda}} \subset \Delta_+$.

In the following, we use the constants

$$\rho_x := \sum_{i \in I} x_{\alpha_i} \Lambda_i = \frac{1}{2} \sum_{\alpha \in \Delta_+} x_\alpha \alpha, \tag{2.10}$$

where $x_\alpha \in \mathbb{C}$ which depends only on the length of roots,

We shall define the root algebras after Cherednik [3, 10].

Definition 2.1. *Root algebra $\mathcal{R}$ is generated by independent variables $\{R_\alpha \mid \alpha \in \widehat{\Delta}\}$ and $\{\tau_\lambda \mid \lambda \in \widetilde{M}\}$ with the following defining relations:*

$$\underbrace{R_{w\alpha_i} R_{ws_i\alpha_j} R_{ws_is_j\alpha_i} \cdots}_{m_{ij}\ factors} = \underbrace{R_{w\alpha_j} R_{ws_j\alpha_i} R_{ws_js_i\alpha_j} \cdots}_{m_{ij}\ factors}, \quad \text{for } w \in \widetilde{W} \tag{2.11}$$

$$\tau_\lambda R_\alpha = R_{\tau_\lambda \alpha} \tau_\lambda, \tag{2.12}$$

$$\tau_\lambda \tau_{\lambda'} = \tau_{\lambda+\lambda'}. \tag{2.13}$$

Theorem 2.2 (Cherednik). 1. *There exists a unique set $\{R_w \mid w \in \widetilde{W}\} \subset \mathcal{R}$ satisfying the relations:*

$$R_{v\,w} = R_v\, {}^v\!R_w, \qquad R_{s_i} = R_{\alpha_i} \quad (i \in \widehat{I}), \qquad R_\omega = 1, \tag{2.14}$$

where $\omega \in \Omega$, $v, w \in \widetilde{W}$ and $\ell(v\,w) = \ell(v) + \ell(w)$, and ${}^v(R_{\alpha_1} \dots R_{\alpha_i}) = R_{v\alpha_1} \dots R_{v\alpha_i}$

2. *We have the R-matrix for $w \in \widetilde{W}$ and its arbitrary reduced decomposition $w = s_{i_1} \dots s_{i_\ell} \omega$ as*

$$\begin{aligned} &R_w = R_{\alpha^{(1)}} \dots R_{\alpha^{(\ell)}}, \\ &\alpha^{(1)} = \alpha_{i_1}, \quad \alpha^{(2)} = s_{i_1}(\alpha_{i_2}), \quad \dots, \quad \alpha^{(\ell)} = ws_{i_\ell}(\alpha_{i_\ell}) \in \Delta_w. \end{aligned} \tag{2.15}$$

Theorem 2.3. *The subalgebra $\mathcal{S} \subset \mathcal{R}$ generated by $\{Y^\lambda := R_{\tau_\lambda} \tau_\lambda \mid \lambda \in \widetilde{M}_-\}$ forms a commutative algebra and is generated by $\{Y^{-\lambda_i} \mid i \in I\}$.*

3. Representation and Difference Operators

Let $\gamma_\alpha^{(1)} = \gamma_\alpha$ and $\gamma_\alpha^{(2)}$, $\gamma_\alpha^{(3)}$ be taken similarly as $\gamma_\alpha^{(1)}$. Accordingly let $\widetilde{M}^{(1)} = \widetilde{M}$ and $\widetilde{M}^{(2)}$, $\widetilde{M}^{(3)}$ be taken similarly. Here $\gamma_\alpha^{(i)}$ and $\gamma_\alpha^{(j)}$ may differ if $i \neq j$. Let $\mathcal{M}$ be the set of meromorphic functions on V. To define the action of $\widetilde{W}$ on $\mathcal{M}$, it is sufficient to specify the action of s_i for $i \in I$ and τ_λ for $\lambda \in \widetilde{M}^{(1)}$. For $f \in \mathcal{M}$, we define

$$s_i(f)(v) = f(s_i(v)), \qquad \tau_\lambda(f)(v) = \tau_\lambda^{(1)}(f)(v) = f(v - \omega_1 \lambda). \tag{3.1}$$

Fix $\xi^{(1)}$, $\zeta^{(1)} \in V$ and let $\mu_\alpha^{(1)}$, $\nu_\alpha^{(1)}$ be constants depending only on the length of roots. Then one can check that (3.1) and

$$(R_\alpha f)(v) := H_\alpha(\mu_\alpha^{(1)}, \nu_\alpha^{(1)}) f(v) - H_\alpha(\langle \xi^{(1)}, \alpha^\vee \rangle, \langle \zeta^{(1)}, \alpha^\vee \rangle) f(s_\alpha v), \tag{3.2}$$

satisfy the defining relations of the root algebra [11], where $H_\alpha(\eta, \kappa)$ is a meromorphic function defined by

$$H_\alpha(\eta, \kappa)(v) := e^{\kappa\alpha(v)} \frac{\sigma(\mu_\alpha^{(1)}; \gamma_\alpha^{(2)}\omega_2, \gamma_\alpha^{(3)}\omega_3)}{\sigma(\eta; \gamma_\alpha^{(2)}\omega_2, \gamma_\alpha^{(3)}\omega_3)} \frac{\sigma(\alpha(v) + \eta; \gamma_\alpha^{(2)}\omega_2, \gamma_\alpha^{(3)}\omega_3)}{\sigma(\alpha(v); \gamma_\alpha^{(2)}\omega_2, \gamma_\alpha^{(3)}\omega_3)}, \tag{3.3}$$

and a root α acts on V as an affine linear functional $\alpha(v) = \langle \alpha, v\rangle + n\omega_1$ for $v \in V$ and $\alpha = \alpha' + n\delta$, $\alpha' \in \Delta$. Then we have the following theorems [9, 10]

Theorem 3.1. *Let* $\mathcal{V} := \mathcal{M}^W$*, the* W*-invariant subspace of* $\mathcal{M}$ *and let* $\xi^{(1)} = -\rho_{\mu^{(1)}}$, $\zeta^{(1)} = -\rho_{\nu^{(1)}}$*. Then* $Y^{(1)}_\lambda := Y^\lambda \in \mathrm{End}_{\mathbb{C}}\,\mathcal{V}$.

Theorem 3.2. *Let* $(-\lambda) \in \widetilde{M}^{(1)}$ *be minuscule. Then we have*

$$Y^\lambda|_{\mathcal{V}} = \frac{1}{|W_\lambda|}\sum_{w\in W} w\Bigg(\prod_{\substack{\alpha\in\Delta_+\\ \langle\lambda,\alpha\rangle=-\gamma_\alpha}} H_\alpha(\mu^{(1)}_\alpha, \nu^{(1)}_\alpha)\,\tau^{(1)}_\lambda\Bigg)\Bigg|_{\mathcal{V}}, \tag{3.4}$$

where W_λ *is the stabilizer of* λ *in* W.

Let $Y^{(2)}_\lambda$ be the operator obtained by changing the role of the indices 1 and 2 in the construction of $Y^{(1)}_\lambda$ and $Y^{(3)}_\lambda$ be obtained in the same manner. In the following, we assume that $\mu^{(j)}_\alpha \eta_{kl} + \nu^{(j)}_\alpha \omega_k = \mu^{(k)}_\alpha \eta_{jl} + \nu^{(k)}_\alpha \omega_j$ where $j \neq k \neq l \neq j$. Then one sees that $Y^{(j)}_\lambda$ for $\lambda \in \widetilde{M}^{(j)}$ are commutative by Theorem 2.3 for a fixed j. Let $\xi^{(j)} = -\rho_{\mu^{(j)}}$ and $\zeta^{(j)} = -\rho_{\nu^{(j)}}$ for $j = 1, 2, 3$.

Theorem 3.3. *Let* $j, k \in \{1, 2, 3\}$ *and* $-\lambda \in \widetilde{M}^{(j)}, -\nu \in \widetilde{M}^{(k)}$*. Then* $Y^{(j)}_\lambda$ *and* $Y^{(k)}_\nu$ *are commutative.*

For minuscule weights $-\lambda$, the periodicity of the coefficients is easily obtained since the explicit forms of the operators Y^λ are calculated [10]. Hence the commutativity of the operators for A-type root system follows because all the fundamental weights are minuscule in this root system. However Y^λ for general λ is complicated and the proof of the commutativity requires a further investigation.

4. BC_1-type Operators and Eigenvector

Generally, it is very difficult to construct explicit eigenvectors of the elliptic Ruijsenaars operators. However we can construct simultaneous eigenvectors in the BC_1 root system. In this root system, there are three mutually commutative operators.

Let $\omega_1, \omega_2, \mu_0, \dots, \mu_6 \in \mathbb{C}$ such that $\Im\omega_1, \omega_2 > 0$ and $2\mu_0 + \mu_1 + \dots + \mu_6 + \omega_1 + \omega_2 = 0$. Then the operators are given by

$$Y^{(1)} = e^{2\pi i x} \frac{\vartheta(x - \mu_0 + \omega_1; \omega_2) \prod_{j=0}^{6} \vartheta(x - \mu_j; \omega_2)}{\vartheta(2x; \omega_2)\vartheta(2x + \omega_1; \omega_2)} (\tau(\omega_1) - 1), + (x \leftrightarrow -x) \tag{4.1}$$

$$Y^{(2)} = e^{2\pi i x} \frac{\vartheta(x - \mu_0 + \omega_2; \omega_1) \prod_{j=0}^{6} \vartheta(x - \mu_j; \omega_1)}{\vartheta(2x; \omega_1)\vartheta(2x + \omega_2; \omega_1)} (\tau(\omega_2) - 1), + (x \leftrightarrow -x) \tag{4.2}$$

$$Y^{(3)} = \tau(1) + \tau(-1) - 2, \tag{4.3}$$

where ϑ denotes the Jacobi odd theta function. For these commutative difference operators we can construct an explicit simultaneous eigenvector:

Theorem 4.1. *Let*

$$\begin{aligned} \phi(c) = &\frac{\prod_{i=1}^{7} \Gamma(pqc_0/c_i; p, q)}{\Gamma(pqc_0; p, q) \prod_{1 \le i < j \le 7} \Gamma(pqc_0/c_i c_j; p, q)} \\ &\times \int_C \frac{\Gamma(pq/(pqc_0)^{1/2} z^{\pm 1}; p, q) \prod_{j=1}^{7} \Gamma((pqc_0)^{1/2}/c_j z^{\pm 1}; p, q)}{\Gamma(z^2, z^{-2}; p, q)} \frac{dz}{z}, \end{aligned} \tag{4.4}$$

with

$$c_0 = e^{2\pi i(\mu_0 - \mu_1 - \omega_1 - \omega_2)}, \quad c_1 = e^{2\pi i(\mu_0 + \mu_2)}, \quad c_2 = e^{2\pi i(\mu_0 + \mu_3)},$$
$$c_3 = e^{2\pi i(\mu_0 + \mu_4)}, \quad c_4 = e^{2\pi i(\mu_0 + \mu_5)}, \quad c_5 = e^{2\pi i(\mu_0 + \mu_6)},$$
$$c_6 = e^{2\pi i(-x - \mu_1)}, \quad c_7 = e^{2\pi i(x - \mu_1)},$$
$$p = e^{2\pi i \omega_1}, \quad q = e^{2\pi i \omega_2},$$

and C a closed circle taken appropriately, and $\Gamma(z; p, q)$ elliptic Gamma function defined by

$$\Gamma(z; p, q) = \frac{(pqz^{-1}; p, q)_\infty}{(z; p, q)_\infty}.$$

Then $\phi(c)$ is a simultaneous eigenvector of $Y^{(j)}$ with the eigenvalues

$$E^{(1)} = e^{-2\pi i\mu_0}\prod_{i=1}^{6}\vartheta(\mu_0+\mu_i;\omega_2), \quad E^{(3)} = 0,$$
$$E^{(2)} = e^{-2\pi i\mu_0}\prod_{i=1}^{6}\vartheta(\mu_0+\mu_i;\omega_1).$$

An integral analogue of Bailey transformation and the other properties of the integral factor of (4.4) are investigated in [13].

5. Special Cases

In this section, we will clarify the relation between $\phi(c)$ and the elliptic hypergeometric series ${}_{12}V_{11}$ introduced by Frenkel and Turaev [5].

Theorem 5.1. *If $c_j = p^{-M}q^{-N}$ for some j, then $\phi(c)$ (4.4) splits into two elliptic hypergeometric series,*

$$\phi(c) = \tilde{\phi}(c_0,\dots,q^{-N},\dots,c_7;p)\times\tilde{\phi}(c_0,\dots,p^{-M},\dots,c_7;q), \tag{5.1}$$

where

$$\tilde{\phi}(c_0,\dots,c_7;r) = \sum_{k=0}^{\infty}\frac{\langle (pq)^{2k}c_0;r\rangle}{\langle c_0;r\rangle}\frac{\langle\!\langle c_0;r\rangle\!\rangle_k}{\langle\!\langle pq;r\rangle\!\rangle_k}\prod_{i=1}^{7}\frac{\langle\!\langle c_i;r\rangle\!\rangle_k}{\langle\!\langle pqc_0/c_i;r\rangle\!\rangle_k}, \tag{5.2}$$

and $\langle\!\langle u;r\rangle\!\rangle_k = \langle u;r\rangle\cdots\langle u(pq)^{k-1};r\rangle$ with $\langle u,r\rangle = u^{-1/2}(u;r)_\infty(u^{-1}r;r)_\infty$.

Now we state the relation between $\phi(c)$ and ${}_{12}V_{11}$ which is defined by

$${}_{12}V_{11}(b_0;b_1,\dots,b_7;r) = \sum_{k=0}^{\infty}\frac{\langle q^{2k}b_0;r\rangle}{\langle b_0;r\rangle}\frac{\langle b_0;r\rangle_k}{\langle r;r\rangle_k}\prod_{i=1}^{7}\frac{\langle b_i;r\rangle_k}{\langle rb_0/b_i;r\rangle_k},$$

where one of b_i should be r^{-N} for the series to terminate and $\langle u;r\rangle_k = \langle u;r\rangle\cdots\langle ur^{k-1};r\rangle$. The following is shown by a direct calculation.

Theorem 5.2.

$$\tilde{\phi}(c_0,\dots,c_7;p) = {}_{12}V_{11}(pc_0;c_1,\dots,pc_j,\dots,c_7;p),$$

which is independent of the choice of j.

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