Generalised Elliptic $6j$-Symbols in Terms of the Vertex-Face Intertwining Vectors

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Abstract

We review a recent result on a formula of the generalized elliptic $6j$-symbols expressed in terms of the fusion of the vertex-face intertwining vectors. The formula is derived by identifying the k fusion intertwining vectors with the change of base matrix elements from Sklyanin's standard base to Rosengren's natural base in the space of even theta functions of order $2k$. We also give a list of explicit expressions of the elliptic $6j$-symbols for $k = 1, 2$.

1 Introduction

Recently, Spiridonov and Zhedanov succeeded to give a proper elliptic analogue ${}_{r+3}V_{r+2}$ of the very-well-poised basic hypergeometric series ${}_{r+1}\varphi_r$ based on the theory of biorthogonal rational functions[1, 2]. This yielded a generalization of the elliptic $6j$-symbols introduced by Frenkel and Turaev[3]. At the same time, they reached a relevant scheme for dealing with a family of biorthogonal rational functions as the generalized eigenvalue problem (GEVP) associated with two Jacobi matrices.

Concerning a generalisation of the elliptic $6j$-symbols, Rosengren found a relevant GEVP recently[4, 5]. It is deeply related with the representation theory of the Sklyanin algebra on the space of theta functions Θ_k[6]. He found a *natural basis* of the space as a set of solutions of the GEVP and identified a change of base matrix elements between the two natural bases depending on a different parameters with the generalized elliptic $6j$-symbols. He then succeeded to derive an expression of the generalised elliptic $6j$-symbols in terms of ${}_{12}V_{11}$.

His natural basis turned out to be equivalent to a fusion of the vertex-face intertwining vectors realised as the vectors in the space of theta functions Θ_k[7]. However in a context of the solvable lattice models, it is standard to take intertwining vectors as the elements of the vector space V, on which tensor product $V \otimes V$ the fusion of the R-matrix acts. They map a fusion of the elliptic R-matrix to a fusion of the face weight of the SOS model[8, 9].

In this paper, according to [10], we investigate a connection between these two realisations of the intertwining vectors, one in Θ_k and the other in V. We then derive a new formula which makes an exact relation between the generalised elliptic $6j$-symbols and the fusion of the standard intertwining vectors. The new formula allows a simple derivation of various properties of the elliptic $6j$-symbols such as addition formula, biorthogonality property, fusion formula and Yang-Baxter relation. We give a summary of them.

1.1 Notations

Let $p = e^{-\frac{\pi K'}{K}}$, $q = -e^{-\frac{\pi\lambda}{2K}}$ and $\zeta = e^{-\frac{\pi\lambda u}{2K}}$. We introduce x, τ and r by $x = -q$, $\tau = \frac{2iK}{K'}$ and $r = \frac{K'}{\lambda}$. Then $p = e^{-\frac{2\pi i}{\tau}} = x^{2r}$. The parameter r plays a role of restriction height in the restricted SOS models. Through this paper, we assume $\mathrm{Im}\tau > 0$. Let $\tilde{p} = e^{2\pi i\tau}$. We use the theta functions

$$\begin{aligned}
\vartheta_1(u|\tau) &= 2\tilde{p}^{1/8}(\tilde{p};\tilde{p})_\infty \sin \pi u \prod_{n=1}^{\infty}(1 - 2\tilde{p}^n \cos 2\pi u + \tilde{p}^{2n}),\\
\vartheta_0(u|\tau) &= -ie^{\pi i(u+\tau/4)}\vartheta_1\left(u + \frac{\tau}{2}\middle|\tau\right),\\
\vartheta_2(u|\tau) &= \vartheta_1\left(u + \frac{1}{2}\middle|\tau\right),\\
\vartheta_3(u|\tau) &= e^{\pi i(u+\tau/4)}\vartheta_1\left(u + \frac{\tau+1}{2}\middle|\tau\right).
\end{aligned}$$

We also use the symbol $[u]$ defined by

$$[u] = C\vartheta_1\left(\frac{u}{r}\middle|\tau\right), \quad C = x^{-\frac{r}{4}}e^{-\frac{\pi i}{4}}\tau^{\frac{1}{2}}.$$

The elliptic shifted factorials are defined by

$$[u]_n = \prod_{j=0}^{n-1}[u+j]$$

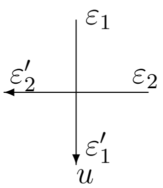
with the convention

$$[u_1, u_2, \cdots, u_k]_n = \prod_{i=1}^{k}[u_i]_n.$$

2 Vertex-Face Correspondence and Fusion

2.1 R-matrix, face weight and intertwining vectors

The vertex-face correspondence is a relationship between Baxter's R-matrix $R(u)$ and the eight-vertex SOS face weight $W\left(\begin{matrix} a_1 & a_2 \\ a_4 & a_3 \end{matrix}\middle| u\right)$. These are given as follows.

$R(u-v)^{\varepsilon_1\varepsilon_2}_{\varepsilon'_1\varepsilon'_2} = v$ 

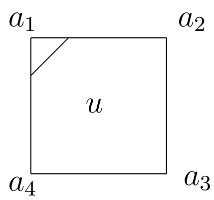
$W\left(\begin{matrix} a_1 & a_2 \\ a_4 & a_3 \end{matrix}\middle| u\right) =$ 

Fig.1: The vertex model weight

Fig.2: The SOS model face weight

Let $V = \mathbb{C}v_{\varepsilon_1} \oplus \mathbb{C}v_{\varepsilon_2}$. We define $R(u) \in \mathrm{End}(V \otimes V)$ as follows[8].

$$R(u)v_{\varepsilon_1} \otimes v_{\varepsilon_2} = \sum_{\varepsilon'_1,\varepsilon'_2} R(u)^{\varepsilon_1\varepsilon_2}_{\varepsilon'_1\varepsilon'_2} v_{\varepsilon'_1} \otimes v_{\varepsilon'_2}$$

with

$$R(u)=R_0(u)\begin{pmatrix} a(u) & & & d(u)\\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ d(u) & & & a(u)\end{pmatrix}, \tag{2.1}$$

where $z=\zeta^2=x^{2u}$ and

$$\begin{aligned}
R_0(u) &= z^{-\frac{r-1}{2r}}\frac{(px^2z;x^4,p)_\infty(x^2z;x^4,p)_\infty(p/z;x^4,p)_\infty(x^4/z;x^4,p)_\infty}{(px^2/z;x^4,p)_\infty(x^2/z;x^4,p)_\infty(pz;x^4,p)_\infty(x^4z;x^4,p)_\infty},\\
a(u) &= \frac{\vartheta_2\left(\frac{1}{2r}\middle|\frac{\tau}{2}\right)\vartheta_2\left(\frac{u}{2r}\middle|\frac{\tau}{2}\right)}{\vartheta_2\left(0\middle|\frac{\tau}{2}\right)\vartheta_2\left(\frac{1+u}{2r}\middle|\frac{\tau}{2}\right)}, \qquad b(u)=\frac{\vartheta_2\left(\frac{1}{2r}\middle|\frac{\tau}{2}\right)\vartheta_1\left(\frac{u}{2r}\middle|\frac{\tau}{2}\right)}{\vartheta_2\left(0\middle|\frac{\tau}{2}\right)\vartheta_1\left(\frac{1+u}{2r}\middle|\frac{\tau}{2}\right)},\\
c(u) &= \frac{\vartheta_1\left(\frac{1}{2r}\middle|\frac{\tau}{2}\right)\vartheta_2\left(\frac{u}{2r}\middle|\frac{\tau}{2}\right)}{\vartheta_2\left(0\middle|\frac{\tau}{2}\right)\vartheta_1\left(\frac{1+u}{2r}\middle|\frac{\tau}{2}\right)}, \qquad d(u)=-\frac{\vartheta_1\left(\frac{1}{2r}\middle|\frac{\tau}{2}\right)\vartheta_1\left(\frac{u}{2r}\middle|\frac{\tau}{2}\right)}{\vartheta_2\left(0\middle|\frac{\tau}{2}\right)\vartheta_2\left(\frac{1+u}{2r}\middle|\frac{\tau}{2}\right)}.
\end{aligned}$$

On the other hand, the face weight $W\left(\begin{matrix} a_1 & a_2\\ a_4 & a_3\end{matrix}\middle| u\right)$ is given by

$$\begin{aligned}
W\left(\begin{matrix} a & a\pm1\\ a\pm1 & a\pm2\end{matrix}\middle| u\right) &= R_0(u),\\
W\left(\begin{matrix} a & a\pm1\\ a\pm1 & a\end{matrix}\middle| u\right) &= R_0(u)\frac{[a\mp u][1]}{[a][1+u]},\\
W\left(\begin{matrix} a & a\pm1\\ a\mp1 & a\end{matrix}\middle| u\right) &= R_0(u)\frac{[a\pm 1][u]}{[a][1+u]}.
\end{aligned} \tag{2.2}$$

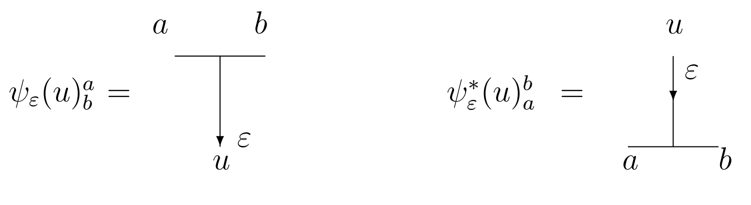
Here we allow only the configurations satisfying the so-called the admissibility condition $|a_j-a_k|=1$ for any two adjacent local heights a_j and a_k.

Then the vertex-face correspondence is stated as follows[8]. Let us define the intertwining vectors $\psi(u)^a_b$ ($|a-b|=1$) by

$$\begin{aligned}
&\psi(u)^a_b=\psi_+(u)^a_b\ v_+ + \psi_-(u)^a_b\ v_- \ \in V, \\
&\psi_+(u)^a_b=\vartheta_0\left(\frac{(a-b)u+a}{2r}\middle|\frac{\tau}{2}\right), \qquad \psi_-(u)^a_b=\vartheta_3\left(\frac{(a-b)u+a}{2r}\middle|\frac{\tau}{2}\right).
\end{aligned} \tag{2.3}$$

Then we have the following identity.

$$\sum_{\varepsilon_1',\varepsilon_2'} R(u-v)^{\varepsilon_1'\varepsilon_2'}_{\varepsilon_1\varepsilon_2}\ \psi_{\varepsilon_1'}(u)^a_b\psi_{\varepsilon_2'}(v)^b_c=\sum_{b'\in\mathbb{Z}}\psi_{\varepsilon_2}(v)^a_{b'}\psi_{\varepsilon_1}(u)^{b'}_c W\left(\begin{matrix} a & b\\ b' & c\end{matrix}\middle| u-v\right). \tag{2.4}$$



(a) (b)

Fig.3: (a) The intertwining vector ; (b) the dual intertwining vector

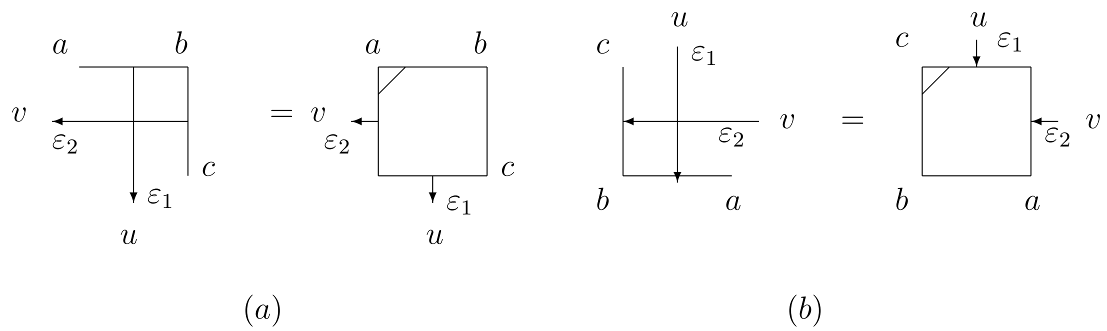


Fig.4 The vertex-face correspondence: (a) via the intertwining vector ; (b) via the dual intertwining vector

In addition, due to the crossing symmetry properties of R and W, we have the following relation.

$$\sum_{\varepsilon_1',\varepsilon_2'} R(u-v)_{\varepsilon_1'\varepsilon_2'}^{\varepsilon_1\varepsilon_2}\ \psi^*_{\varepsilon_1'}(u)_b^a \psi^*_{\varepsilon_2'}(v)_c^b = \sum_{s\in\mathbb{Z}} \psi^*_{\varepsilon_2}(v)_{b'}^a \psi^*_{\varepsilon_1}(u)_c^{b'} W\left(\begin{array}{cc} c & b' \\ b & a \end{array}\middle| u-v\right), \quad (2.5)$$

where we defined the dual intertwining vectors $\psi^*(u)_b^a \in V^*$ by

$$\begin{aligned} \psi^*(u)_b^a\ v_\varepsilon &= \psi^*_\varepsilon(u)_b^a, \quad v_\varepsilon \in V, \\ \psi^*_\varepsilon(u)_b^a &= -\varepsilon\frac{a-b}{2[b][u]}C^2\ \psi_{-\varepsilon}(u-1)_b^a. \end{aligned} \quad (2.6)$$

By a direct calculation, we have the following inversion relations.

$$\sum_{\varepsilon=\pm} \psi^*_\varepsilon(u)_b^a \psi_\varepsilon(u)_c^b = \delta_{a,c}, \quad (2.7)$$

$$\sum_{a=b\pm1} \psi^*_{\varepsilon'}(u)_b^a \psi_\varepsilon(u)_a^b = \delta_{\varepsilon',\varepsilon}. \quad (2.8)$$

2.2 Fusion

Fusion of the vertex-face correspondence relationship was considered systematically in [9] (see also [12] for the 2×2 fusion case). Let $V_j, V_{\bar{j}}$ be copies of V. Let us define the operator $\Pi_{1\cdots k} \in \mathrm{End}(V^{\otimes k}\otimes V^{\otimes k})$ by

$$\Pi_{1\cdots k} = \frac{1}{k!}(P_{1k}+\cdots+P_{k-1k}+I)\cdots(P_{13}+P_{23}+I)(P_{12}+I),$$

where $P_{ij}(v_{\varepsilon_1}\otimes v_{\varepsilon_2}) = v_{\varepsilon_2}\otimes v_{\varepsilon_1}$ is the transposition between the vectors in the ith and jth vector space. This yields the projection on the space $V^{(k)}$ of the symmetric tensors in $V^{\otimes k}$. We define

$$R^{(k,1)}_{1\cdots k,\bar{j}}(u) = \Pi_{1\cdots k} R_{1\bar{j}}(u+k-1)\cdots R_{k-1\bar{j}}(u+1)R_{k\bar{j}}(u)\ \in \mathrm{End}(V^{(k)}\otimes V_{\bar{j}}).$$

Then we obtain the $k\times l$ fusion R-matrix $R^{(k,l)}(u)$ as follows.

$$R^{(k,l)}(u) = \Pi_{\bar{1}\cdots\bar{l}} R^{(k,1)}_{1\cdots k,\bar{l}}(u) R^{(k,1)}_{1\cdots k,\overline{l-1}}(u-1)\cdots R^{(k,1)}_{1\cdots k,\bar{1}}(u-l+1). \quad (2.9)$$

This is an operator in $\mathrm{End}(V^{(k)} \otimes V^{(l)})$. It satisfies the Yang-Baxter equation (YBE) on $V^{(k)} \otimes V^{(l)} \otimes V^{(m)}$.

$$R^{(k,l)}(u-v)R^{(k,m)}(u)R^{(l,m)}(v) = R^{(l,m)}(v)R^{(k,m)}(u)R^{(k,l)}(u-v).$$

The $k \times l$ fusion of the face weight $W^{(k,l)}$ is obtained similarly. We first define

$$\begin{aligned}
&W^{(k,1)}\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| u\right) \\
&= \sum_{d_1,..,d_{k-1}} W\left(\begin{matrix} a & a_1 \\ d & d_1 \end{matrix}\middle| u+k-1\right) W\left(\begin{matrix} a_1 & b \\ d_1 & c \end{matrix}\middle| u+k-2\right) \cdots W\left(\begin{matrix} a_{k-1} & b \\ d_{k-1} & c \end{matrix}\middle| u\right).
\end{aligned}$$

Then the RHS is independent of the choice of $a_1,..,a_{k-1}$ provided $|a-a_1| = |a_1-a_2| = \cdots = |a_{k-1}-b| = 1$. Then we have

$$\begin{aligned}
&W^{(k,l)}\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| u\right) \\
&= \sum_{a_1,..,a_{l-1}} W^{(k,1)}\left(\begin{matrix} a & b \\ a_1 & b_1 \end{matrix}\middle| u-l+1\right) W^{(k,1)}\left(\begin{matrix} a & b \\ a_1 & b_1 \end{matrix}\middle| u-l+2\right) \cdots W^{(k,1)}\left(\begin{matrix} a_{l-1} & b_{l-1} \\ d & c \end{matrix}\middle| u\right).
\end{aligned} \tag{2.10}$$

The RHS is independent of the choice of $b_1,..,b_{l-1}$ provided $|b-b_1| = |b_1-b_2| = \cdots = |b_{l-1}-c| = 1$. In $W^{(k,l)}$, the dynamical variables satisfy the extended admissible condition $a-b \ \in \{-k,-k+2,..,k\}$ for any two horizontally adjacent local heights a,b, while $a-d \ \in \{-l,-l+2,..,l\}$ for any two vertically adjacent local heights a,d. The $k \times l$ fusion face weight $W^{(k,l)}$ satisfies the face type YBE.

$$\begin{aligned}
&\sum_g W^{(k,l)}\left(\begin{matrix} a & b \\ f & g \end{matrix}\middle| u\right) W^{(k,m)}\left(\begin{matrix} f & g \\ e & d \end{matrix}\middle| v\right) W^{(m,l)}\left(\begin{matrix} b & c \\ g & d \end{matrix}\middle| u-v\right) \\
&\quad = \sum_g W^{(m,l)}\left(\begin{matrix} a & g \\ f & e \end{matrix}\middle| u-v\right) W^{(k,m)}\left(\begin{matrix} a & b \\ g & c \end{matrix}\middle| v\right) W^{(k,l)}\left(\begin{matrix} g & c \\ e & d \end{matrix}\middle| u\right).
\end{aligned} \tag{2.11}$$

Next let us consider the fusion of the vertex-face relationships (2.4), (2.5). We define the k fusion of the intertwining vectors by[9]

$$\psi^{(k)}(u)^a_b = \Pi_{1\cdots k}\ \psi(u+k-1)^a_{c_1} \otimes \psi(u+k-2)^{c_1}_{c_2} \otimes \cdots \otimes \psi(u)^{c_{k-1}}_b. \tag{2.12}$$

Here the RHS is independent of the choice of $c_1,..,c_{k-1}$ provided $|a-c_1| = |c_1-c_2| = \cdots = |c_{k-1}-b| = 1$. The local heights a and b now satisfy the extended admissible condition $a-b \in \{-k,-k+2,..,k\}$. For $k>1$, the basis $\{v^{(k)}_\mu\}_{\mu=-k,-k+2,..,k}$ of $V^{(k)}$ is given by a fusion of the basis vectors v_{ε_i} $(\varepsilon_i = \pm)$ of V.

$$v^{(k)}_\mu \;=\; \frac{1}{k!}\sum_{\sigma\in S_k} v_{\varepsilon_{\sigma(1)}} \otimes v_{\varepsilon_{\sigma(2)}} \otimes \cdots \otimes v_{\varepsilon_{\sigma(k)}}, \tag{2.13}$$

where S_k being the symmetric group and we set $\mu=\sum_{j=1}^k \varepsilon_j$. Substituting (2.3) to (2.12), we obtain

$$\begin{aligned}\psi^{(k)}(u)^a_b &= \sum_{\mu\in\{-k,-k+2,..,k\}} v^{(k)}_\mu \psi^{(k)}_\mu(u)^a_b,\\ \psi^{(k)}_\mu(u)^a_b &= \sum_{\substack{\varepsilon_1,..,\varepsilon_k=+,-\\ \mu=\sum_{j=1}^k\varepsilon_j}} \psi_{\varepsilon_1}(u+k-1)^a_{c_1}\psi_{\varepsilon_2}(u+k-2)^{c_1}_{c_2}\cdots\psi_{\varepsilon_k}(u)^{c_{k-1}}_b. \qquad (2.14)\end{aligned}$$

From (2.4), (2.9), and (2.10), it follows that the fused intertwining vectors satisfy the $k\times l$ fusion vertex-face correspondence relations.

$$\sum_{\mu_1',\mu_2'} R^{(k,l)}(u-v)^{\mu_1'\mu_2'}_{\mu_1\mu_2}\ \psi^{(k)}_{\mu_1'}(u)^a_b\psi^{(l)}_{\mu_2'}(v)^b_c=\sum_{b'\in\mathbb{Z}}\psi^{(k)}_{\mu_2}(v)^a_{b'}\psi^{(l)}_{\mu_1}(u)^{b'}_c W^{(k,l)}\left(\begin{array}{cc|c} a & b & \\ b' & c & u-v\end{array}\right). \qquad (2.15)$$

Similarly, the dual intertwining vectors can be fused k times in the following way[11].

$$\psi^{*(k)}(u)^b_a=\sum_{c_1,..,c_{k-1}}\psi^*(u+k-1)^{c_1}_a\otimes\psi^*(u+k-2)^{c_2}_{c_1}\otimes\cdots\otimes\psi^*(u)^b_{c_{k-1}} \qquad (2.16)$$

with the property

$$\Pi_{1\cdots k}\ \psi^{*(k)}(u)^b_a=\psi^{*(k)}(u)^b_a\ \Pi_{1\cdots k}.$$

As above, it follows immediately from (2.5), (2.9) and (2.10), that we have

$$\sum_{\mu_1',\mu_2'} R^{(k,l)}(u-v)^{\mu_1'\mu_2'}_{\mu_1\mu_2}\ \psi^{*(k)}_{\mu_1'}(u)^a_b\psi^{*(l)}_{\mu_2'}(v)^b_c=\sum_{b'\in\mathbb{Z}}\psi^{*(k)}_{\mu_2}(v)^a_{b'}\psi^{*(l)}_{\mu_1}(u)^{b'}_c W^{(k,l)}\left(\begin{array}{cc|c} c & b' & \\ b & a & u-v\end{array}\right). \qquad (2.17)$$

Finally, using (2.7) and (2.8), the following inversion relations hold.

$$\begin{aligned}\sum_{\mu\in\{-k,-k+2,\cdots,k\}}\psi^{*(k)}_\mu(u)^a_b\psi^{(k)}_\mu(u)^b_c &= \delta_{a,c}, \qquad (2.18)\\ \sum_{a\in\{b-k,b-k+2,..,b+k\}}\psi^{*(k)}_{\mu'}(u)^a_b\psi^{(k)}_\mu(u)^b_a &= \delta_{\mu',\mu}. \qquad (2.19)\end{aligned}$$

3 The Elliptic $6j$-symbols

Through this section, we use the abbreviation

$$\begin{aligned}&[u\pm z]=[u+z][u-z],\\ &\vartheta_\alpha(u\pm z|\tau)=\vartheta_\alpha(u+z|\tau)\vartheta_\alpha(u-z|\tau) \qquad \alpha=0,1,2,3.\end{aligned}$$

3.1 The natural basis

Let Θ_k be the space of even theta functions of order $2k$ with quasi-period $(1,\tau)$ and zero characteristics.

$$\Theta_k = \left\{ f(z) : \text{entire} \mid f(z+1)=f(z), f(z+\tau)=e^{-2\pi i k(2z+\tau)}f(z), f(-z)=f(z) \right\}.$$

This space forms a $k+1$ dimensional vector space.

Let us consider the vectors of Θ_k given by

$$e_n^k(z;\alpha,\beta) = [\alpha \pm rz]_n[\beta \pm rz]_{k-n} \quad (n=0,1,..,k). \tag{3.1}$$

These vectors are linearly independent, if α,β satisfy the following conditions[4].

$$\begin{aligned} &\frac{\alpha-\beta+j}{r} \notin \mathbb{Z}+\tau\mathbb{Z},\ j=1-k,2-k,...,k-1, \\ &\frac{\alpha+\beta+j}{r} \notin \mathbb{Z}+\tau\mathbb{Z},\ j=0,1,...,k-1. \end{aligned}$$

Hence a system of vectors $\{e_n^k(z;\alpha,\beta)\}_{n=0}^k$ forms a basis of Θ_k, and is called the natural basis.

Rosengren showed that the change of base coefficients $R_n^m(\alpha,\beta,\gamma,\delta;k;q,p)$ in

$$e_n^k(z;\alpha,\beta) = \sum_{m=0}^k R_n^m(\alpha,\beta,\gamma,\delta;k;q,p)e_m^k(z;\gamma,\delta) \tag{3.2}$$

can be regarded as a generalisation of the elliptic $6j$-symbols. In fact, the vectors $e_n^k(z;\alpha,\beta)$ are natural elliptic analogue of the product $h_n(x;\alpha)h_{k-n}(x;\beta)$ of the Askey-Wilson monomials $h_n(x;\alpha) = (\alpha\xi;q)_n(\alpha\xi^{-1};q)_n$. Here $x=\xi+\xi^{-1}$ and $(z;q)_n = (1-z)(1-zq)\cdots(1-zq^{n-1})$. For generic α,β, the set $\{h_n(x;\alpha)h_{k-n}(x;\beta)\}_{n=0}^k$ forms a basis of the space of polynomials of degree $\leq k$. Then the trigonometric coefficients $R_n^m(\alpha,\beta,\gamma,\delta;k;q)$ in

$$h_n(x;\alpha)h_{k-n}(x;\beta) = \sum_{m=0}^k R_n^m(\alpha,\beta,\gamma,\delta;k;q)h_m(x;\gamma)h_{k-m}(x;\delta)$$

gives a biorthogonal function generalization of q-Racah polynomials[4].

Furthermore he found an expression of the coefficients $R_n^m(\alpha,\beta,\gamma,\delta;k;q,p)$ in terms of the elliptic analogue of the very-well-poised balanced basic hypergeometric series, ${}_{12}V_{11}$.

Theorem 3.1 *[4]*

$$\begin{aligned} &R_n^m(\alpha,\beta,\gamma,\delta;k;q,p) \\ &= \frac{[1]_k}{[1]_m[1]_{k-m}}\frac{[\beta-\delta,\beta+\delta-1+k]_n[\alpha-\gamma,\alpha+\gamma]_n[\beta-\gamma,\beta+\gamma]_{k-n}[\beta-\gamma]_{k-m}}{[\gamma-\delta+m-k,\beta+\gamma]_m[\delta-\gamma-m]_{k-m}[\delta+\gamma,\beta-\gamma]_k} \\ &\times {}_{12}V_{11}(\gamma-\beta-k;-n,-m,\alpha-\beta+n-k,\gamma-\delta+m-k,\gamma+\delta,\alpha-\beta+1-k,\gamma-\beta+1). \end{aligned} \tag{3.3}$$

Here ${}_{s+1}V_s$ is defined by[2]

$$ {}_{s+1}V_s(u_0;u_1,\cdots,u_{s-4}) = \sum_{j=0}^{\infty} \frac{[u_0+2j]}{[u_0]} \prod_{i=0}^{s-4} \frac{[u_i]_j}{[u_0+1-u_i]_j} $$

with the balancing condition

$$ \sum_{i=1}^{s-4} u_i = \frac{s-7}{2} + \frac{s-5}{2} u_0. $$

3.2 Relation with the intertwining vectors

We next consider the standard basis of Θ_k introduced by Sklyanin[6] and make a connection between $R_n^m(\alpha,\beta,\gamma,\delta;k;q,p)$ and the vertex-face intertwining vectors. For $k=1$, the following two vectors form a basis of Θ_1.

$$ \begin{aligned} v_+(z) &= \vartheta_3\left(2z|\,2\tau\right) - \vartheta_2\left(2z|\,2\tau\right), \\ v_-(z) &= \vartheta_3\left(2z|\,2\tau\right) + \vartheta_2\left(2z|\,2\tau\right). \end{aligned} $$

For $k>1$, we obtain the basis $\{v_\mu^{(k)}(z)\}_{\mu=-k,-k+2,..,k}$ of Θ_k by fusing the basis vectors $v_\varepsilon(z)$ $(\varepsilon=\pm)$ of Θ_1.

$$ v_\mu^{(k)}(z) \;=\; \frac{1}{k!}\sum_{\sigma\in S_k} v_{\varepsilon_{\sigma(1)}}(z) v_{\varepsilon_{\sigma(2)}}(z)\cdots v_{\varepsilon_{\sigma(k)}}(z), \tag{3.4} $$

with $\mu=\sum_{j=1}^k \varepsilon_j$.

Now let us consider the intertwining vectors in the standard basis.

$$ \psi(u;z)_b^a \;=\; \sum_{\varepsilon=\pm} v_\varepsilon(z)\psi_\varepsilon(u)_b^a = \vartheta_3\left(z \pm \frac{(a-b)u+a}{2r}\middle|\,\tau\right). $$

Fusion of the intertwining vectors $\psi(u;z)_b^a$ are then given by

$$ \psi^{(k)}(u;z)_b^a = \Pi_{1,2,..,k}\psi(u+k-1;z)_{c_1}^a \otimes \psi(u+k-2;z)_{c_2}^{c_1} \otimes \cdots \otimes \psi(u;z)_b^{c_{k-1}}. $$

$\psi^{(k)}(u)_b^a$ is independent of the choice of $c_1,..,c_{k-1}$ and a,b satisfy the admissible condition $a-b\in\{-k,-k+2,..,k\}$. Let us set $a-b=k-2n$ $(n=0,1,2,..,k)$. Evaluating $\psi^{(k)}(u;z)_b^a$ in two ways, we found the following formula[10].

$$ \begin{aligned} \psi^{(k)}(u;z)_b^a \;&=\; \sum_{\mu\in\{-k,-k+2,..,k\}} v_\mu^{(k)}(z)\psi_\mu^{(k)}(u)_b^a \\ &=\; (-)^k e^{-\pi i k(\frac{\tau}{2}+2z)} C^{-k} e_n^k\left(z+\frac{\tau+1}{2}; \frac{-u+a-k+1}{2}, \frac{-u-a-k+1}{2}\right), \end{aligned} \tag{3.5} $$

where $\psi_\mu^{(k)}(u)_b^a$ is given in (2.14). In the derivation of the second line we took a choice $c_{j+1}=c_j+1, c_0=a$ for $j=0,1,2,..,n$ and $c_{j+1}=c_j-1, c_k=b$ for $j=n,n+1,..,k-1$[7].

C is a constant given in §1.1. This formula indicates that the components $\psi^{(k)}_\mu(u)^a_b$ of the vertex-face intertwining vector play the role of the change of base matrix elements from $\{v^{(k)}_\mu(z)\}$ to $\{e^k_n(z; \frac{-u+a-k+1}{2}, \frac{-u-a-k+1}{2})\}$ in Θ_k. This role is similar to the one of the generalized group elements (= Babelon's vertex-IRF transformations[14]) in the theory of q-$6j$ symbols studied by Rosengren[15].

Using (3.5), we obtain from (3.2)

$$\psi^{(k)}_\mu(u)^a_b = \sum_{m=0}^{k} R^m_n(\alpha,\beta,\gamma,\delta;k;q,p)\psi^{(k)}_\mu(u)^c_d,$$

for $a-b=k-2n$, $c-d=k-2m$ and $\alpha=\frac{-u+a-k+1}{2}, \beta=\frac{-u-a-k+1}{2}, \gamma=\frac{-u+c-k+1}{2}, \delta=\frac{-u-c-k+1}{2}$. Then the inversion relation (2.18) yields the following formula for R^m_n.

Theorem 3.2 *[10]*

$$R^m_n(\alpha,\beta,\gamma,\delta;k;q,p) = \sum_{\mu\in\{-k,-k+2,..,k\}} \psi^{*(k)}_\mu(u)^d_c\psi^{(k)}_\mu(u)^a_b. \tag{3.6}$$

Note that Rosengren derived a similar scalar product expression for R^m_n ((11.2) in [5]), where the scaler product is defined by Sklyanin's invariant metric on Θ_k.

It turns out that the expression appeared in the RHS of (3.6) is nothing but a matrix $L^{(k)}$ introduced by Lashkevich and Pugai[13] for $k=1$ and extended to higher k by Kojima, Weston and the present author [11]. Namely,

$$L^{(k)}\left(\begin{array}{cc|c} a & b & \\ c & d & u\end{array}\right) = \sum_{\mu\in\{-k,-k+2,..,k\}} \psi^{*(k)}_\mu(u)^d_c\psi^{(k)}_\mu(u)^a_b = R^m_n(\alpha,\beta,\gamma,\delta;k;q,p). \tag{3.7}$$

Combining (3.7) and (3.3), we obtain a full expression of $L^{(k)}\left(\begin{array}{cc|c} a & b & \\ c & d & u\end{array}\right)$ for arbitrary $k\in\mathbb{Z}_{>0}$.

Corollary 3.3 *For $a-b=k-2n, c-d=k-2m$,*

$$\begin{aligned}
&L^{(k)}\left(\begin{array}{cc|c} a & b & \\ c & d & u\end{array}\right)\\
&=\frac{[1]_k[-\frac{a-c}{2},-u-\frac{a+c}{2}]_m[-\frac{a+c}{2}]_{k-m}[\frac{a-c}{2},-u+\frac{a+c}{2}-k+1]_n[-\frac{a+c}{2},-u-\frac{a-c}{2}-k+1]_{k-n}}{[1]_m[1]_{k-m}[c+m-k,-u-\frac{a-c}{2}-k+1]_m[-c-m]_{k-m}[-u-k+1,-\frac{a+c}{2}]_k}\\
&\times\sum_{j=0}^{\min(n,m)}\frac{[\frac{a+c}{2}-k+2j]}{[\frac{a+c}{2}-k]}\frac{[\frac{a+c}{2}-k,-n,-m,a+n-k]_j}{[1,\frac{a+c}{2}+1+n-k,\frac{a+c}{2}+1+m-k,-\frac{a-c}{2}+1-n]_j}\\
&\qquad\times\frac{[c+m-k,-u-k+1,u+k-1,\frac{a+c}{2}+1]_j}{[\frac{a-c}{2}+1-m,u+\frac{a+c}{2},-u+\frac{a+c}{2}+1-k,-k]_j}.
\end{aligned}$$

In [11], some of the $L^{(k)}$-matrix elements were calculated by fusion. In the below we give a list of them. They agree with this formula. In some cases, one need to apply elliptic Jackson's summation formula (See for example [10]).

The k=1 full expressions:

$$\begin{aligned} L^{(1)}\left(\begin{array}{cc|c} a & a\pm 1 & \\ c & c\pm 1 & u \end{array}\right) &= \frac{[u\pm\frac{c-a}{2}][\frac{c+a}{2}]}{[u][c]}, \\ L^{(1)}\left(\begin{array}{cc|c} a & a\mp 1 & \\ c & c\pm 1 & u \end{array}\right) &= \frac{[u\pm\frac{c+a}{2}][\frac{c-a}{2}]}{[u][c]}. \end{aligned}$$

The k=2 full expressions:

$$\begin{aligned} L^{(2)}\left(\begin{array}{cc|c} a & a+2 & \\ c & c+2 & u \end{array}\right) &= \frac{[\frac{c+a}{2}][\frac{c+a}{2}+1][u+1+\frac{c-a}{2}][u+\frac{c-a}{2}]}{[c][c+1][u+1][u]}, \\ L^{(2)}\left(\begin{array}{cc|c} a & a & \\ c & c+2 & u \end{array}\right) &= \frac{[\frac{c+a}{2}][\frac{c-a}{2}][u+1+\frac{c+a}{2}][u+1+\frac{c-a}{2}]}{[c][c+1][u+1][u]}, \\ L^{(2)}\left(\begin{array}{cc|c} a & a-2 & \\ c & c+2 & u \end{array}\right) &= \frac{[\frac{c-a}{2}][\frac{c-a}{2}+1][u+1+\frac{c+a}{2}][u+\frac{c+a}{2}]}{[c][c+1][u+1][u]}, \\ L^{(2)}\left(\begin{array}{cc|c} a & a+2 & \\ c & c & u \end{array}\right) &= \frac{[\frac{c+a}{2}][\frac{c-a}{2}][u-\frac{c+a}{2}][u+\frac{c-a}{2}][2]}{[c-1][c+1][u+1][u][1]}, \end{aligned}$$

$$\begin{aligned} & L^{(2)}\left(\begin{array}{cc|c} a & a & \\ c & c & u \end{array}\right) \\ &= \frac{[c-1][\frac{c-a}{2}][\frac{c-a}{2}+1][u+1+\frac{c+a}{2}][u-\frac{c+a}{2}]+[c+1][\frac{c+a}{2}][\frac{c+a}{2}-1][u+1-\frac{c-a}{2}][u+\frac{c-a}{2}]}{[u+1][u][c-1][c][c+1]} \\ &= \frac{[c-1][\frac{c+a}{2}][\frac{c+a}{2}+1][u+1+\frac{c-a}{2}][u-\frac{c-a}{2}]+[c+1][\frac{c-a}{2}][\frac{c-a}{2}-1][u+1-\frac{c+a}{2}][u+\frac{c+a}{2}]}{[u+1][u][c-1][c][c+1]}, \end{aligned}$$

$$\begin{aligned} L^{(2)}\left(\begin{array}{cc|c} a & a-2 & \\ c & c & u \end{array}\right) &= \frac{[\frac{c+a}{2}][\frac{c-a}{2}][u+\frac{c+a}{2}][u-\frac{c-a}{2}][2]}{[c-1][c+1][u+1][u][1]}, \\ L^{(2)}\left(\begin{array}{cc|c} a & a+2 & \\ c & c-2 & u \end{array}\right) &= \frac{[\frac{c-a}{2}][\frac{c-a}{2}-1][u+1-\frac{c+a}{2}][u-\frac{c+a}{2}]}{[c-1][c][u+1][u]}, \\ L^{(2)}\left(\begin{array}{cc|c} a & a & \\ c & c-2 & u \end{array}\right) &= \frac{[\frac{c-a}{2}][\frac{c+a}{2}][u+1-\frac{c+a}{2}][u+1-\frac{c-a}{2}]}{[c-1][c][u+1][u]}, \\ L^{(2)}\left(\begin{array}{cc|c} a & a-2 & \\ c & c-2 & u \end{array}\right) &= \frac{[\frac{c+a}{2}][\frac{c+a}{2}-1][u+1-\frac{c-a}{2}][u-\frac{c-a}{2}]}{[c-1][c][u+1][u]}. \end{aligned}$$

The $k\in\mathbb{Z}_{>0}$ partial results:

$$L^{(k)}\left(\begin{array}{cc|c} a & a+k-2j & \\ c & c+k & u \end{array}\right) = \frac{[\frac{c+a}{2}+k-1+j]_{k-j}[\frac{c-a}{2}-1+j]_j}{[c+k-1]_k}$$

$$
\begin{aligned}
& \times \quad \frac{[-u+\frac{a-c}{2}-j]_{k-j}[-u-\frac{a+c}{2}+j-k]_j}{[-u]_k}, \\
L^{(k)}\left(\begin{array}{cc|c} a & a+k-2j & \\ c & c-k & u \end{array}\right) \quad = \quad & \frac{[\frac{c+a}{2}]_j[\frac{c-a}{2}]_{k-j}}{[c]_k} \\
& \times \quad \frac{[-u+\frac{a+c}{2}-j]_{k-j}[-u+\frac{c-a}{2}+j-k]_j}{[-u]_k}.
\end{aligned}
$$

Using the formula (3.6), we can derive various properties of the elliptic $6j$-symbols $R_n^m(\alpha,\beta,\gamma,\delta;k;q,p)$[10]. They are summarized as follows.

Addition formula :

$$
R_n^m(\alpha,\beta,\gamma,\delta;k;q,p)=\sum_{l=0}^{k} R_n^l(\alpha,\beta,\rho,\sigma;k;q,p)R_l^m(\rho,\sigma,\gamma,\delta;k;q,p).
$$

Biorthogonality property :

$$
\sum_{m=0}^{k} R_n^m(\alpha,\beta,\gamma,\delta;k;q,p)R_m^l(\gamma,\delta,\alpha,\beta;k;q,p)=\delta_{n,l}.
$$

Fusion formula (combinatorial formula [4]):

$$
\begin{aligned}
& R_n^m(\alpha,\beta,\gamma,\delta;k;q,p) \\
= & \sum_{\substack{0\le m_j\le 1 \\ \sum_{j=1}^k m_j=m}} R_{n_1}^{m_1}(\alpha,\alpha_1,\gamma,\gamma_1;1;q,p)R_{n_2}^{m_2}(\alpha_1,\alpha_2,\gamma_1,\gamma_2;1;q,p)\cdots R_{n_k}^{m_k}(\alpha_{k-1},\beta,\gamma_{k-1},\delta;1;q,p),
\end{aligned}
$$

Yang-Baxter relation :

$$
\begin{aligned}
& \sum_d W^{(k,l)}\left(\begin{array}{cc|c} a & b & \\ d & c & u-v \end{array}\right) L^{(k)}\left(\begin{array}{cc|c} d & c & \\ f & e & u \end{array}\right) L^{(l)}\left(\begin{array}{cc|c} a & d & \\ g & f & v \end{array}\right) \\
= & \sum_d L^{(k)}\left(\begin{array}{cc|c} a & b & \\ g & d & u \end{array}\right) L^{(l)}\left(\begin{array}{cc|c} b & c & \\ d & e & v \end{array}\right) W^{(k,l)}\left(\begin{array}{cc|c} g & d & \\ f & e & u-v \end{array}\right).
\end{aligned}
$$

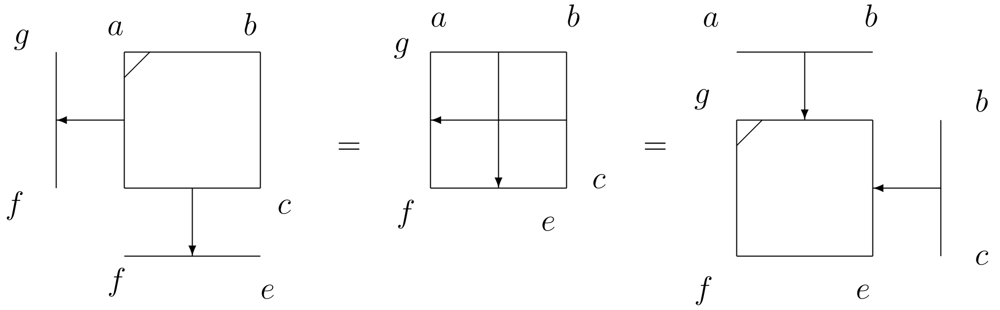


Fig.5: The Yang-Baxter equation for the elliptic $6j$-symbol.

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