CLASSICAL LEONARD TRIPLES

ALYONA KOROVNICHENKO AND ALEXEI ZHEDANOV

ABSTRACT. We study classical Leonard triples from the point of view of nonlinear Poisson algebras. It is shown that these triples essentially are equivalent to the classical Askey-Wilson algebra. Chains of involutions of these triples give rise to dynamical systems with discrete time. In special cases they can be solved explicitly in terms of elliptic functions of second order. Relations with the generalized Markov problem in number theory and the Poncelet problem in projective geometry are demonstrated.

CONTENTS

1. INTRODUCTION

Assume that $X(q,p)$ and $Y(q,p)$ are two independent dynamical variables of canonical variables q,p with the standard Poisson bracket (PB) $\{q,p\}=1$. As usual, independence of functions X,Y means that in some domain of interest of the phase space (q,p) they satisfy the condition

$$\frac{\partial(X,Y)}{\partial(q,p)} \equiv \frac{\partial X}{\partial q}\frac{\partial Y}{\partial p} - \frac{\partial X}{\partial p}\frac{\partial Y}{\partial q} = \{X,Y\} \neq 0, \tag{1.1}$$

where $\partial(X,Y)/\partial(q,p)$ is the Jacobian of a change of variables.

According [13], two independent variables X and Y are said to form a *classical Leonard pair* (CLP) if there exist two different canonical transformations $(q,p) \to (x,y)$ and $(q,p) \to (\xi,\eta)$ such that in the first case one has

$$X = \varphi(x), \quad Y = A_1(x)\, e^y + A_2(x)\, e^{-y} + A_3(x) \tag{1.2}$$

and in the second one

$$Y = \psi(\xi), \quad X = B_1(\xi)\, e^\eta + B_2(\xi)\, e^{-\eta} + B_3(\xi), \tag{1.3}$$

where (x,y) and (ξ,η) are canonical pairs (i.e. $\{x,y\}=\{\xi,\eta\}=1$) and $\varphi(x)$, $A_i(x)$, $\psi(\xi)$, $B_i(\xi)$ are some functions. Using canonical transformations $y \to \kappa y$, $x \to x/\kappa$ and taking the limit $\kappa \to 0$ one can obtain from (1.2) the limiting form $Y = a_1(x)y^2 + a_2(x)y + a_3(x)$.

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Therefore we shall assume that CLP admit such degenerate forms of Y in (1.2) (or of X in (1.3)) without further reservations.

Define the variable

$$W = \{X, Y\}. \tag{1.4}$$

We assume that there exists a region of values of X, Y where X and Y are independent variables, i.e. $Z \neq 0$. The latter means that in this domain one can invert (at least locally) the changes of variables to find $x = x(X, Y), y = y(X, Y)$ and consider W as a function of X and Y, $W = W(X, Y)$. The condition that X and Y form a CLP allows one to establish the explicit form of this function $W(X, Y)$.

Namely, one can show (see [13], [29]) that there exist 9 arbitrary constants α_{ik}, $i, k = 0, 1, 2$, such that

$$W^2 = \sum_{i,k=0}^{2} \alpha_{ik} X^i Y^k = F(X, Y). \tag{1.5}$$

Vice versa, it can be shown that starting from the condition (1.5) for arbitrary α_{ik} one arrives at a CLP (including its degenerate form mentioned above). The condition $F = 0$ determines the region of the phase space with complex values of q, p where such a consideration is broken down.

From (1.5) it follows that the dynamical variables X, Y and $W = \{X, Y\}$ form a Poisson algebra with the relations (1.4) and

$$\{W, X\} = -\frac{1}{2}\frac{\partial F(X, Y)}{\partial Y}, \qquad \{Y, W\} = -\frac{1}{2}\frac{\partial F(X, Y)}{\partial X}, \tag{1.6}$$

which are known as the classicla Askey-Wilson algebra relations [9]. It can be shown [13], [29] that the Poisson algebra (1.6) generates (1.5) and the constant α_{00} is interpreted as a value of the corresponding Casimir element. The derived algebra is a particular example of the quadratic algebras, the most popular representative of which is given by the Sklyanin algebra [23].

Suppose that X is the Hamiltonian of some physical system. Then the first canonical transformation $(q, p) \to (x, y)$ is, in fact, an action-angle transformation: it maps X into a function depending on only one canonical variable x. Similarly, canonical transformation $(q, p) \to (\xi, \eta)$ is an action-angle variables transformation for a system with the Hamiltonian Y. Existence of a CLP can be considered as some duality property of two Hamiltonians with respect to prescribed dependence on the momenta y and η of the "conjugated" Hamiltonians (i.e. Y and X, respectively). From this point of view, the CLP property is equivalent to the notion of duality discussed in the theory of integrable systems, see e.g. [22, 7].

Note that the quantum analogue of CLP property coincides with the standard Leonard's duality [14] or the bispectrality condition [5] for two 3-diagonal $N \times N$ matricies L, M. In this case the matrix M is tridiagonal in the basis formed by eigenvectors ϕ_k of the matrix L, whereas M is 3-diagonal in the basis formed by eigenvectors ψ_k of M:

$$L\phi_k = \lambda_k \phi_k, \quad M\phi_k = \alpha_{k+1}\phi_{k+1} + \beta_k \phi_k + \gamma_k \phi_{k-1}, \tag{1.7}$$

and

$$M\psi_k = \mu_k \psi_k, \quad L\psi_k = \xi_{k+1}\psi_{k+1} + \eta_k \psi_k + \zeta_k \psi_{k-1}, \tag{1.8}$$

where $k = 1, 2 \ldots, N$. It is assumed that eigenvalues λ_k, μ_k are nondegenerate, so that the vectors ϕ_k and ψ_k form two independent complete bases. In such a form the problem of classifying all ("quantum") Leonard pairs L, M was investigated by Terwilliger [26]. It is equivalent to the original Leonard problem [14] in the following sense. Let us decompose the vectors ψ_k in the basis of vectors ϕ_k

$$\psi_k = \sum_{s=1}^{N} P_{ks}\phi_s, \tag{1.9}$$

with some expansion coefficients P_{ks}. It follows from (1.7), (1.8) that the coefficients P_{ks} satisfy simultaneously two three term recurrence relations

$$\zeta_{s+1}P_{k,s+1} + \eta_s P_{ks} + \xi_{s-1}P_{k,s-1} = \mu_k P_{ks} \tag{1.10}$$

and

$$\gamma_{k+1}P_{k+1,s} + \beta_k P_{ks} + \alpha_{k-1}P_{k-1,s} = \lambda_s P_{ks}, \tag{1.11}$$

which mean that P_{ks} can be expressed in terms of some orthogonal polynomials of the argument λ_s or μ_k. It appears that these polynomials are self-dual: permutation of the discrete variables k and s is equivalent to some permutation of parameters entering the recurrence coefficients η_s, ξ_s (for details see [14, 26, 28]).

Leonard's theorem [14] states that the q-Racah polynomials, discovered by Askey and Wilson [1], are the most general self-dual orthogonal polynomials. As shown in [28], the quantum analogue of the algebra (1.6) with the generators L, M and $N = [L, M] \equiv LM - ML$ describes these polynomials through the representation theory (see also [26] for similar algebraic treatments). Relations of this algebra with the standard $sl_q(2)$ quantum algebra have been established in [10, 11].

The main purpose of the present paper is to derive relations for so-called classical Leonard triples, i.e. for 3 dynamical variables x, y, z such that all pairs $(x, y), (y, z), (z, x)$ form a CLP. In quantum case (i.e. on the level of operators acting on a finite-dimensional space) such Leonard triples were studied recently in [21]. We observe interesting and rather unexpected relations between this problem and other problems in pure mathematics (e.g. Markov cubic in number theory) and mathematical physics.

2. Classical Leonard triples and generalized Markov cubic

Let $x(q, p)$ and $y(q, p)$ be two variables which form a CLP in sense of the previous section or [29], i.e. we assume that PB $w = \{x, y\}$ satisfies the condition

$$w^2 = F(x, y) = \sum_{i,k=0}^{2} \alpha_{ik}x^i y^k \tag{2.1}$$

with some coefficients $\alpha_{i,k}$ which are not all zero. In this case variables x, y are independent as well as canonical variables q, p.

Equivalently, CLP is characterized by 3 relations for the so-called classical AW-algebra for 3 generators x, y, w [9], [29]:

$$\{x, y\} = w, \ \{y, w\} = -F_x/2, \ \{w, x\} = -F_y/2, \tag{2.2}$$

where F_x, F_y are partial derivatives of the polynomial $F(x,y)$ defined by (2.1) with arbitrary (but not all zero) coefficients α_{ik}. We see that PB $\{y,w\}$ is linear in x, whereas PB $\{w,x\}$ is linear in y. For the AW-algebra (2.2) the function $Q = w^2 - F(x,y)$ plays the role of the Casimir element, i.e. $\{x,Q\} = \{y,Q\} = \{w,Q\} = 0$. Thus $Q = const$, which is compatible with relation (2.1).

In what follows we will assume that $\alpha_{22} = \rho^2 \neq 0$. Under this assumption we can define a new generator z by the relation

$$z = \xi(w - \rho xy - A_1 y - A_2 x) + \eta, \tag{2.3}$$

where $A_1 = \alpha_{12}/(2\rho)$, $A_2 = \alpha_{21}/(2\rho)$ and ξ, η are arbitrary parameters. Then, excluding variable w from (2.1) and (2.3) we arrive at expression

$$\Phi(x,y,z) = 0, \tag{2.4}$$

where $\Phi(x,y,z)$ is a cubic polynomial

$$\Phi(x,y,z) = \rho xyz + A_1 yz + A_2 xz + A_3 xy + B_1 x^2 + B_2 y^2 + B_3 z^2 + C_1 x + C_2 y + C_3 z + E = 0 \tag{2.5}$$

with the coefficients

$$A_3 = \frac{\xi\alpha_{12}\alpha_{21} - 4\eta\rho^3 - 2\xi\alpha_{11}\rho^2}{4\rho^2},\ B_1 = \xi\frac{-4\rho^2\alpha_{20} + \alpha_{21}^2}{8\rho^2},$$
$$B_2 = \xi\frac{-4\rho^2\alpha_{02} + \alpha_{12}^2}{8\rho^2},\ B_3 = 1/(2\xi),\ C_1 = -\frac{\xi\rho\alpha_{10} + \alpha_{21}\eta}{2\rho}$$
$$C_2 = -\frac{\xi\rho\alpha_{01} + \alpha_{12}\eta}{2\rho},\ C_3 = -\eta/\xi,\ E = -\frac{-\eta^2 + \xi^2\alpha_{0,0} + \xi^2 Q}{2\xi}$$

It is easily sen that 3 generators x, y, z satisfy Poisson brackets relations

$$\{x,y\} = \Phi_z,\ \{y,z\} = \Phi_x,\ \{z,x\} = \Phi_y \tag{2.6}$$

which is a special realization of general Poisson structure with 3 dynamical variables with Casimir element $\Phi(x,y,z)$ [19].

By affine transformations of the generators $x_i \to \alpha_i x_i + \beta_i$ with appropriate constants α_i, β_i it is possible to kill all bi-quadratic terms xy, yz, zx in $\Phi(x,y,z)$. Then we have

$$\Phi(x,y,z) = \rho xyz + B_1 x^2 + B_2 y^2 + B_3 z^2 + C_1 x + C_2 y + C_3 z, \tag{2.7}$$

where $B_1 = B_2 = 1$ for nondegenerate case and $B_1 = 0, B_2 = 1$ or $B_1 = 0, B_2 = 1$ or $B_1 = B_2 = 0$. The nondegenerate case occurs if

$$D_2 = \alpha_{12}^2 - 4\alpha_{22}\alpha_{02} \neq 0,\ D_1 = \alpha_{21}^2 - 4\alpha_{22}\alpha_{20} \neq 0. \tag{2.8}$$

The degenerate cases take place when one or both discriminants $D_{1,2}$ are zero.

Explicitly, relations for corresponding Poisson algebra are as follows:

$$\begin{aligned} \{x,y\} &= \rho xy + 2B_3 z + C_3, \\ \{y,z\} &= \rho yz + 2B_1 x + C_1, \\ \{z,x\} &= \rho xz + 2B_2 y + C_2. \end{aligned} \tag{2.9}$$

As far as we know, firstly a special case of the algebra (2.9) appeared in Dubrovin's paper [6] as a Poisson algebra for the Stokes matrices in the isomonodromy problem for a matrix differential equation. In the Dubrovin case the algebra's parameters are $\rho = -1, B_1 = B_2 = B_3 = 1, C_1 = C_2 = C_3 = 0$. Recently, the Poisson algebra (2.9) with general parameters ρ, B_i, C_i appeared in Oblomkov work [18] as the non-trivial Poisson structure corresponding to some "classical limit" of the double affine Hecke algebra.

Now we start from the Poisson algebra (2.9) with generic coefficients ρ, B_i, C_i. We will assume only that $\rho \neq 0$. (We don't include terms xy, yz, zx because they always can be removed by an appropriate shifts of the generators). It is elementary verified that the polynomial $\Phi(x, y, z)$ defined by (2.11) is the Casimir element of the algebra: $\{x, \Phi\} = \{y, \Phi\} = \{z, \Phi\} = 0$. The main property of the Poisson algebra (2.9) is that all pairs of variables $(x, y), (y, z), (z, x)$ form CLP. Indeed, assume that $B_3 \neq 0$. Then we have $z = (w - \rho xy - C_3)/(2B_3)$, where $w = \{x, y\}$. Substituting z into the Casimir element $\Phi(x, y, z) = q$, where q is a fixed constant for a given realization of the algebra (2.9), we get relation (2.1) with

$$\begin{aligned} &\alpha_{22} = \rho^2,\ \alpha_{12} = \alpha_{21} = 0,\ \alpha_{11} = 2\rho C_3,\ \alpha_{20} = -4B_1B_3,\ \alpha_{02} = -4B_2B_3, \\ &\alpha_{10} = -4C_1B_3,\ \alpha_{01} = -4C_2B_3,\ \alpha_{00} = 4qB_3 + C_3^2 \end{aligned} \tag{2.10}$$

Hence the variables x, y form a CLP. If $B_3 = 0$ then from the first relation in (2.9) we have

$$w^2 = (\rho xy + C_3)^2$$

and hence again the pair (x, y) is a CLP of a special type.

Due to the obvious symmetry of the algebra (2.9) with respect to each variable x, y, z we get that two other pairs (y, z) and (z, x) are CLP for every choice of parameters C_i, B_i.

In what follows we will consider only the nondegenerate case (2.8). Then all coefficients in front of squares x^2, y^2, z^2 can be chosen to be equal to 1:

$$\Phi(x, y, z) = \rho xyz + x^2 + y^2 + z^2 + C_1x + C_2y + C_3z. \tag{2.11}$$

Corresponding Poisson algebra takes the form

$$\begin{aligned} \{x, y\} &= \rho xy + 2z + C_3, \\ \{y, z\} &= \rho yz + 2x + C_1, \\ \{z, x\} &= \rho xz + 2y + C_2. \end{aligned} \tag{2.12}$$

This algebra possesses 3 remarkable anti-automorphisms. Indeed, return to relation (2.3). We can define an alternative variable $\tilde{z}$ by just replacing $w \to -w$:

$$\tilde{z} = -\xi(w + \rho xy + A_1y + A_2x) + \eta, \tag{2.13}$$

It is easily seen that variables $x, y, \tilde{z}$ satisfy the polynomial equation $\Phi(x, y, \tilde{z}) = 0$ with the same polynomial (2.5). However corresponding Poisson algebra for variables $x, y, \tilde{z}$ is obtained from the algebra (2.9) by changing of signs of PB:

$$\{x, y\} = -\Phi_{\tilde{z}},\ \{y, \tilde{z}\} = -\Phi_x,\ \{\tilde{z}, x\} = -\Phi_y \tag{2.14}$$

Geometrical meaning of transformation $z \to \tilde{z}$ is obvious: z, and $\tilde{z}$ are two roots of the quadratic equation $\Phi(x, y, z) = const$ with x, y being fixed parameters, i.e. points $z, \tilde{z}$ are two point of intersection of the 3-dimensional surface $\Phi(x, y, z) = const$ and the straight line parallel to the axis z. Obviously variables $(x, y, \tilde{z})$ form classical Leonard triple as well as the variables (x, y, z). This means, that starting from the given CLT (x, y, z) one can construct many other triples. We will investigate this problem in the next section.

3. 9 fundamental involutions

Consider first the simplest case $C_1 = C_2 = C_3 = 0$, i.e.

$$\Phi(x, y, z) = \rho xyz + x^2 + y^2 + z^2. \tag{3.1}$$

We will call $\Phi(x, y, z)$ a *Markov cubic* because it was A.Markov who investigated this polynomial from the point of view of number theory [16]. More exactly, Markov considered the case $\rho = -3$. It appears that such cubic is connected with theory of quadratic forms. Markov investigated a problem of finding all integer (nonzero) solutions of the equation $\Phi(x, y, z) = 0$. For $\rho = -3$ one such solution is obvious: $(x_0, y_0, z_0) = (1, 1, 1)$. Markov showed that all integer solutions can be obtained from $(1, 1, 1)$ by application of 9 fundamental involutions of the cubic $\Phi(x, y, z)$. Among these involutions there are obvious 3 permutations

$$P_x : (x, y, z) \to (x, z, y),\ P_y : (x, y, z) \to (z, y, x),\ P_z : (x, y, z) \to (y, x, z) \tag{3.2}$$

and 3 reflections:

$$\begin{aligned} &R_x : (x, y, z) \to (x, -y, -z),\ R_y : (x, y, z) \to (-x, y, -z), \\ &R_z : (x, y, z) \to (-x, -y, z). \end{aligned} \tag{3.3}$$

The only non-trivial automorphisms are

$$\begin{aligned} &T_x : (x, y, z) \to (-x - \rho yz, y, z),\ T_y : (x, y, z) \to (x, -y - \rho xz, z), \\ &T_z : (x, y, z) \to (x, y, -z - \rho xy). \end{aligned} \tag{3.4}$$

The involutions T_x, T_y, T_z have the same geometrical meaning as in the end of the previous section: e.g. (x, y, z) and $(-x - \rho yz, y, z)$ are two points of intersection of the surface $\Phi(x, y, z) = 0$ with the line parallel to the axis x. Further deep analysis of this problem was done by Frobenius [8] who showed, in particular, that $\rho = -1$ and $\rho = -3$ are the only values of the parameter ρ when nonzero integer solutions of the equation $\Phi(x, y, z) = 0$ exist. In these admissible cases all nonzero integer solutions of such equation can be obtained by application of 9 fundamental involutions to the simplest ones $(1, 1, 1)$ (for $\rho = -3$) and $(3, 3, 3)$ (for $\rho = -1$). It is interesting to note that in the Dubrovin example [6] one encounters just with the second admissible case $\rho = -1$. The reason for this, as well as relations of the Dubrovin's Poisson algebra with the Diofantine equations (and theory of quadratic forms) are still unclear.

In what follows we will use the following rule for compositions of transformations $T_\alpha, R_\alpha, P_\alpha$. Let $f(x, y, z)$ be an arbitrary function of 3 variables x, y, z. Define the transformed function $\tilde{f}(x, y, z)$ as $\tilde{f}(x, y, z) = f(\tilde{x}, \tilde{y}, \tilde{z})$, where variables $\tilde{x}, \tilde{y}, \tilde{z}$ are given by above mentioned transformation formulas. Thus we can define action of the corresponding operators, e.g. $T_x f(x, y, z) = \tilde{f}(x, y, z) = f(-x - \rho yz, y, z)$. Then composition of operators is defined by standard rules, e.g. $T_y T_x f(x, y, z) = T_y \tilde{f}(x, y, z)$ etc.

Dubrovin noticed [6] that one can construct two operators

$$\sigma_1 = R_z T_y P_x,\ \sigma_2 = R_y T_x P_z \tag{3.5}$$

which are generators of the braid group, i.e. these generators satisfy the relation

$$\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1. \tag{3.6}$$

Explicitly, the braid operators act as $\sigma_1(x, y, z) = (-x, z, y + \rho xz)$, $\sigma_2(x, y, z) = (y, x + \rho yz, -z)$. Moreover, there is relation

$$(\sigma_1\sigma_2)^3 = (\sigma_2\sigma_1)^3 = 1 \tag{3.7}$$

Dubrovin found several finite orbits of the corresponding braid group. Some of these orbits are connected with perfect polyhedra (i.e. tetrahedron, cube, icosahedron), for others geometric meaning is still unknown.

Now, let us consider 9 fundamental involutions for the case of generic polynomial $\Phi(x, y, z)$ given by (2.11). Involutions T_x, T_y, T_z have almost the same expression and the same geometric meaning:

$$T_x : \ (x, y, z) \to (-x - \rho yz - C_1, y, z),\ T_y : \ (x, y, z) \to (x, -y - \rho xz - C_2, z),$$
$$T_z : \ (x, y, z) \to (x, y, -z - \rho xy - C_3). \tag{3.8}$$

Involutions P_x, P_y, P_z take now the form

$$\begin{aligned} P_x &: \ (x, y, z) \to (x, z + \frac{C_2 - C_3}{\rho x - 2}, y + \frac{C_3 - C_2}{\rho x - 2}),\\ P_y &: \ (x, y, z) \to (z + \frac{C_1 - C_3}{\rho y - 2}, y, x + \frac{C_3 - C_1}{\rho y - 2}),\\ P_z &: \ (x, y, z) \to (y + \frac{C_1 - C_2}{\rho z - 2}, x + \frac{C_2 - C_1}{\rho z - 2}, z) \end{aligned} \tag{3.9}$$

It is easily seen that $P_x^2 = P_y^2 = P_z^2 = 1$ (i.e. these are indeed involutions) and for $C_1 = C_2 = C_3$ we return to permutations (3.2).

Involutions R_x, R_y, R_z are constructed as follows:

$$\begin{aligned} R_x &: \ (x, y, z) \to (x, -y + \frac{2(2C_2 - \rho C_3 x)}{\rho^2 x^2 - 4}, -z + \frac{2(2C_3 - \rho C_2 x)}{\rho^2 x^2 - 4}),\\ R_y &: \ (x, y, z) \to (-x + \frac{2(2C_1 - \rho C_3 y)}{\rho^2 y^2 - 4}, y, -z + \frac{2(2C_3 - \rho C_1 y)}{\rho^2 y^2 - 4}),\\ R_z &: \ (x, y, z) \to (-x + \frac{2(2C_1 - \rho C_2 z)}{\rho^2 z^2 - 4}, -y + \frac{2(2C_2 - \rho C_1 z)}{\rho^2 z^2 - 4}, z) \end{aligned} \tag{3.10}$$

Again it is easily verified that $R_x^2 = R_y^2 = R_z^2 = 1$ and these involutions become simple reflections (3.3) when $C_1 = C_2 = C_3 = 0$.

There are simple relations between involutions. E.g. in the simplest case $C_1 = C_2 = C_3 = 0$ all reflections R_i commute and we have

$$R_x R_y = R_y R_x = R_z \tag{3.11}$$

From (3.11) we can obtain other relations of such type, e.g. $R_y R_z = R_x, R_x R_z = R_y$. However, in general case involutions (3.10) do not commute. Nevertheless, in general case there are simple relations between involutions, e.g. all involutions R_α commute with all involutions T_β if $\alpha \neq \beta$:

$$[R_\alpha, T_\beta] = 0, \quad \text{if } \beta \neq \alpha. \tag{3.12}$$

Moreover, there are relations

$$[R_\alpha, P_\alpha] = 0, \quad \alpha = x, y, z \tag{3.13}$$

and

$$P_\alpha T_\beta P_\alpha = T_\gamma, \quad \text{if } \alpha \neq \beta \neq \gamma \tag{3.14}$$

By direct computation it is verified the following important property of these involutions:

Proposition 1. The involutions R_α, P_α and T_α $(\alpha = x, y, z)$ lead to anti-automorphisms of the AW-algebra, i.e. $\{x_\alpha, x_\beta\} \to -\{x_\alpha, x_\beta\}$.

In general case when C_i are nonzero, the operators $\sigma_{1,2}$ constructed by (3.5) do not satisfy braid relations (3.6). The open problems are:
(i) describe all integer solutions of the equation $\Phi(x, y, z) = 0$;
(ii) describe the group generated by 9 involutions P, R, T and find its finite orbits;
(iii) find relation of classical AW-algebra with isomonodromy problem.

4. Trigonometric chain of CLT

We describe in this section an algorithm based on involutions T_x and T_y which leads to a chain of CLT.

Fix the value of the variable z and consider the operator $K = T_y T_x$. As both T_x and T_y are involutions, the operator K can be considered as a "shift". From proposition **1** we conclude that operator K is an automorphism of the AW-algebra. Denote (x_n, y_n, z) variable which are obtained as $(x_n, y_n, z) = K^n(x, y, z),\ n = 1, 2, \ldots$. Clearly, for all n the triple (x_n, y_n, z) is a CLP satisfying the same AW-relations as initial triple (x, y, z). In general $K^n \neq 1$, however there are special cases when $K^N = 1$ for some N. In these cases we have periodic orbit $(x_n, y_n, z),\ n = 0, 1, 2, \ldots, N-1$.

From (3.8) we obtain formula for action of K on the triple (x, y, z):

$$K(x, y, z) = (-x - \rho yz - C_1, \rho zx + (\rho^2 z^2 - 1)y + \rho z C_1 - C_2, z) \tag{4.1}$$

As the variable z is fixed under action of K, we can restrict ourselves with variables (x, y) only. Then we have recursively

$$x_{n+1} = -x_n - \rho y_n z - C_1, \quad y_{n+1} = \rho z x_n + (\rho^2 z^2 - 1)y_n + \rho z C_1 - C_2 \tag{4.2}$$

The system (4.2) is a linear recurrence system with constant coefficients for two discrete variables x_n, y_n. If $z \neq 0$ and $\rho z \neq \pm 2$ general solution of this system can be presented in the form

$$x_n = \mu_1 q^n + \nu_1 q^{-n} + \kappa_1, \quad y_n = \mu_2 q^n + \nu_2 q^{-n} + \kappa_2 \tag{4.3}$$

with some parameters $q, \mu_i, \nu_i, \kappa_i, i = 1, 2$ depending on z. For q we have quadratic equation

$$q^2 + (2 - \rho^2 z^2)q + 1 = 0. \tag{4.4}$$

If, however, $z = 0$ or $\rho z = \pm 2$ then instead we have solution

$$x_n = \mu_1 n^2 + \nu_1 n + \kappa_1, \quad y_n = \mu_2 n^2 + \nu_2 n + \kappa_2 \tag{4.5}$$

It is intersting to note that in both cases we have for x_n, y_n so-called Askey-Wilson grids [17],[15].

Denote $q = exp(i\omega)$. From (4.4) it is seen that

$$2\cos\omega = \rho^2 z^2 - 2 \tag{4.6}$$

Thus if z is chosen such that condition $\rho^2 z^2 = 2 + 2\cos(2\pi M/N)$ holds for some integers M, N then we have a N- *periodic orbit* of automorphisms of classical AW-algebras with the same Casimir element. This condition can be rewritten in the form

$$\rho z = \pm 2\cos(\pi M/N).$$

5. Elliptic chain of CLP and the Poncelet problem

In previous section we constructed a sequence of automorphisms of the classical AW-algebra, connected with involutions T_x, T_y of the generalized Markov cubic (2.11) $\Phi(x, y, z) = 0$.

In this section we construct another sequence of automorphisms of the AW-algebra which is connected with involutions R_x, R_y. We show that these involutions have transparent geometrical meaning and are closely related with the famous Poncelet problem in projective geometry.

Return to canonical representation of the CLP in terms of bi-quadratic polynomial $F(x, y)$ (2.1). It is possible to present it in two forms:

$$F(x, y) = U_2(y)x^2 + U_1(y)x + U_0(y) = V_2(x)y^2 + V_1(x)y + V_0(x)$$

where $U_i(y), V_i(x)$ are polynomials of degree ≤ 2. Take an arbitrary point (x_0, y_0) on the bi-quadric $F(x, y) = 0$ and construct two involutions I_1 and I_2 having obvious geometrical meaning. Involution I_1 transforms the point (x_0, y_0) to the point (x_0, y_1) where y_1 is another point of intersection between vertical line $x_0 = const$ and biquadric $F(x, y) = 0$. Explicitly

$$I_1(x_0, y_0) = (x_0,\ -y_0 - V_1(x_0)/V_2(x_0))$$

Analogously we introduce involution I_2:

$$I_2(x_0, y_0) = (-x_0 - U_1(y_0)/U_2(y_0),\ y_0)$$

where the point $-x_0 - U_1(y_0)/U_2(y_0)$ is the second point of intersection of the biquadric with horizontal line $y_0 = const$. Obviously both operations I_1, I_2 are involutions: $I_1^2 = I_2^2 = E$, where E is identical transformation.

Define the transformation $S = I_2 I_1$ as a product of two successive involutions. Clearly,

$$S(x_0, y_0) = (x_1, y_1) = (-x_0 - U_1(y_1)/U_2(y_1),\ -y_0 - V_1(x_0)/V_2(x_0))$$

We can repeat this process and obtain a sequence of points $(x_n, y_n) \equiv S^n(x_0, y_0)$ belonging to the curve $F(x, y) = 0$. We have recurrence relation

$$(x_{n+1}, y_{n+1}) = (-x_n - U_1(y_{n+1})/U_2(y_{n+1}),\ -y_n - V_1(x_n)/V_2(x_n))$$

Now we interpret x, y as dynamical variables with the Poisson bracket $w = \{x, y\}$, where $w^2 = F(x, y)$.

Proposition 2. Transformation S is an automorphism of the classical AW-algebra. i.e. $\{x_n, y_n\} = \{x_0, y_0\} = w$ for $n = 1, 2, \dots$.

For the proof of this proposition it is sufficient to verify that both involutions I_1 and I_2 are anti-automorphisms of the AW-algebra (this is obvious from their definition), hence their product $S = I_2 I_1$ is an automorphism.

From this proposition it follows that starting from a given CLP (x_0, y_0) we can construct a sequence of CLP (x_n, y_n) with the same algebraic structure, i.e. variables (x_n, y_n, w) form classical AW-algebra which commutation relations (2.2) do not depend on n.

Consider a special case of the bi-quadratic polynomial

$$F(x,y) = \rho^2 x^2 y^2 - 4(x^2 + y^2) + 2\rho C_3 xy - 4C_1 x - 4C_2 y + 4q \tag{5.1}$$

(in general situation every bi-quadratic function $F(x, y)$ can be transformed to this form by an appropriate affine transformation of variables x, y). Then we have, equivalently, AW-algebra connected with the cubic (2.11). In this case it easily verified that involution I_1 of the bi-quadric curve $F(x, y) = 0$ is equivalent to the involution R_x for the cubic surface $\Phi(x, y, z) = 0$. Analogously, involution I_2 is equivalent to the involution R_y. Thus we can present the shift operator S on the bi-quadratic curve a s a product of two involutions $S = R_y R_x$.

Explicit solution for the points $(x_n, y_n) = S^n(x_0, y_0)$ can be found in terms of elliptic functions:

$$x_n = \Xi_1(qn + \phi_1), \quad y_n = \Xi_2(qn + \phi_2), \tag{5.2}$$

where ϕ_1, ϕ_2, q are some parameters and $\Xi(z)$ is an elliptic function of second degree [27]:

$$\Xi(z) = \mu \, \frac{\sigma(z - \xi)\sigma(z + \xi)}{\sigma(z - \eta)\sigma(z + \eta)}. \tag{5.3}$$

The function $\sigma(z)$ is the standard Weierstrass zeta-function (with quasi-periods $2\omega_1, 2\omega_2$ which are not indicated in our notation). Subscripts $\Xi_{1,2}$ mean that parameters μ, ξ, η can take different values for x_n and y_n.

The proof of this proposition is based on the observation that an algorithm of construction of points (x_n, y_n) on the bi-quadratic curve $F(x, y) = 0$ is equivalent to famous Poncelet problem in projective geometry [3]. We refer to the paper [25] for details of this identification. From this proposition it follows that the set of points (x_n, y_n) is finite if the corresponding Poncelet problem has periodic solution (the so-called Poncelet porism [3]).

More precisely, the parameter q and periods $2\omega_1, 2\omega_2$ of the functions $\Xi_{1,2}(z)$ depend only on parameters of the bi-quadratic curve $F(x, y) = 0$ (or, equivalently, on parameters of the AW-algebra). Then periodicity condition $(x_N, y_N) = (x_0, y_0)$ means

$$qN = 2\omega_1 M_1 + 2\omega_2 M_2, \tag{5.4}$$

where M_1, M_2 are some integers. In general, condition (5.4) is rather complicated: there are no simple explicit formulas allowing to rewrite this condition in terms of the parameters of the bi-quadratic $F(x, y) = 0$. The only known tool is so-called Cayley approach: some determinant relations arising from corresponding Poncelet problem [3].

In special cases, trivial periodicity conditions arise. Consider. e.g. the case when the function $F(x, y)$ is an even function in y, i.e. $F(x, y) = V_2(x)y^2 + V_0(x)$. In this case it is almost obvious that the period is 2 for generic initial point: $(x_2, y_2) = (x_0, y_0)$. Such case occurs, e.g. for the Markov cubic (3.1) when $C_1 = C_2 = C_3 = 0$. In this case

$S(x_0, y_0) = (-x_0, -y_0)$ and action of the shift S is trivial. Thus, non-trivial action of this transformation occurs only for curves $F(x, y) = 0$ which are not symmetric with respect to co-ordinate axis x or y.

In another special case we have $C_1 = C_2 = 0, C_3 \neq 0$ in (5.1). Corresponding bi-quadratic curve

$$F(x, y) = \rho^2 x^2 y^2 - 4(x^2 + y^2) + 2\rho C_3 xy + 4q = 0 \tag{5.5}$$

is called the Euler-Baxter curve. It appeared first in Euler's work on addition theorem for elliptic sine function (in modern terms, of course). Equation (5.5) appeared also in Baxter's approach to solve the so-called 8-vertex model [2]. Baxter showed that the transformation S in this case admits a simple solution in terms of elliptic sine function:

$$x_n = \mu\, sn(qn + \phi_1),\ y_n = \mu\, sn(q(n + 1/2) + \phi_1), \tag{5.6}$$

with some parameters μ, q, ϕ_1. Periodicity condition in this case can be presented as

$$qN = 4K(k)M,$$

where $K(k)$ is complete elliptic integral of the first kind.

There is also an interesting relation with the so-called John algorithm in theory of two-dimensional wave equation [12]. Consider a generic closed curve Γ such that any vertical and horizontal line intersects Γ in no more than two points. Then it is possible to construct two involutions I_1, I_2 in exactly the same manner as our involutions described above. Corresponding operator $S = I_2 I_1$ generates a shift $(x_n, y_n) = S^n(x_0, y_0)$ on the curve Γ, where (x_0, y_0) is an arbitrary initial point on Γ. John showed [12] that the Dirichlet problem for the wave equation $\psi_{xy} = 0$ on Γ has unique solutions only if operator S has no finite orbits, i.e. $(x_n, y_n) \neq (x_0, y_0)$ for any n and any initial point (x_0, y_0). In our case the curve Γ is a bi-quadric curve $F(x, y) = 0$. From the Poncelet porism it follows that periodicity property $(x_N, y_N) = (x_0, y_0)$ for some N depends only on parameters of the bi-quadratic curve and does not depend on initial point (this is the famous big Poncelet theorem [3]). This means, in particular, that the periodicity property for the AW-automorphisms under transformation S is determined only by the value of the parameters of the AW-algebra.

For details concerning relations between the Dirichlet problem for the wave equation, John algorithm and the Poncelet problem see, e.g. [4].

Thus a set of automorphisms $(x_n, y_n) = S^n(x_0, y_0)$ of the classical AW-algebra is equivalent to the Poncelet problem and to the John algorithm in theory of differential equations [12].

Now we identify involutions I_1, I_2 with involutions R_x, R_y introduced in Section **3**. Indeed, involution R_x defined by (3.10) leads to anti-automorphism

$$(\tilde{x}, \tilde{y}, \tilde{z}) = (x, -y + \frac{2(2C_2 - \rho C_3 x)}{\rho^2 x^2 - 4}, -z + \frac{2(2C_3 - \rho C_2 x)}{\rho^2 x^2 - 4})$$

of the Poisson algebra (2.9). Introduce the variable

$$\tilde{w} = \{\tilde{x}, \tilde{y}\} = -\{x, y\} = -w$$

We see that variables $(\tilde{x}, \tilde{y})$ form a new CLP with new variable $w = -w$ which is equivalent to involution I_1 on the bi-quadric $F(x, y) = w^2 = const$. Similarly, involution R_y is equivalent to involution I_2 on the same bi-quadric.

We thus conclude that involutions R_x, R_y on cubic surface (2.7) are equivalent to involutions I_1, I_2 on the bi-quadric $F(x, y) = const$

It would be interesting to obtain similar geometrical meaning of involutions P_x, P_y.

6. Quantum case

In this section we briefly describe situation in the quantum case, when x, y, z are non-commuting variables satisfying AW-algebra [11]

$$xy - qyx = z + C_1, \ yz - qzy = x + C_1, \ zx - qxz = y + C_2, \tag{6.1}$$

where $q \neq \pm 1$ is a fixed real parameter. We see that the algebra (6.1) has almost the same structure as classical AW-algebra (2.12) with replacing of Poisson brackets $\{,\}$ by commutators $[,]$.

It is interesting to note that algebra of type (6.1) appeared in the list of so-called 3-dimensional skew-polynomial algebras introduced by Bell and Smith [24], [20]. Recently the algebra (6.1) was studied by Rosengren and Terwilliger [21] in connection with Leonard triples on the space of finite-dimensional matrices.

First, we establish existence of the Casimir operator for the algebra (6.1) which has the form coinciding with classical one (2.5).

Proposition 3. The Casimir operator Q commuting with all the elements x, y, z has the expression

$$Q = (q^2 - 1)xyz + x^2 + z^2 + q^2y^2 + (q + 1)(C_1x + C_3z + qC_2y) \tag{6.2}$$

Proof of this proposition is direct verification of relations $[Q, x] = [Q, y] = [Q, z] = 0$ using commutation relations (6.1).

The main difference with respect to classical case is sensibility of the expression for Q on order of elements x, y, z. There are $6 = 3!$ possibilities of choice of elements in the product xyz. However, it is a nice property of the algebra (6.1) that for all such choices the expression for Q will have similar form with the possible change of value of coefficients in front of the quadratic and linear terms. Indeed, consider, e.g. permutation of the elements x and y in cubic term in (6.2):

$$xyz = (qyx + z + C_3)z = qyxz + z^2 + C_3z$$

(where we use the first of relations (6.1)). Thus we have instead of (6.2)

$$Q = q(q^2 - 1)yxz + x^2 + q^2y^2 + q^2z^2 + (q + 1)(C_1x + qC_2y + qC_3z) \tag{6.3}$$

Similarly, for all 6 possible choices of the cubic term in Q we obtain expressions similar to (6.2) and (6.3).

It is interesting to investigate possible automorphisms of the "quantum" polynomial Q similar to classical involutions T, P, R. For generic values of parameters C_1, C_2, C_3 we have a quantum analogue of the involution T_x of the form:

$$T_x : \ (x, y, z) \rightarrow (-x + (1 - q^2)yz - (q + 1)C_1, y/q, zq) \tag{6.4}$$

It is easily verified that the polynomial $Q(x,y,z)$ remains unchanged under transformation.

We mention here two essential points concerning "quantum" transformation (6.4):

(i) this transformation changes variables y,z;

(ii) this transformation leads to a new algebra for variables $\tilde{x},\tilde{y},\tilde{z}$:

$$\tilde{y}\tilde{x} - q\tilde{x}\tilde{y} = q^{-1}\tilde{z} + C_3$$
$$\tilde{z}\tilde{y} - q\tilde{y}\tilde{z} = q^{-1}\tilde{x} + C_1$$
$$\tilde{x}\tilde{z} - q\tilde{z}\tilde{x} = q^3\tilde{y} + C_2q^2 \qquad (6.5)$$

New variables $\tilde{x},\tilde{y},\tilde{z}$ satisfy AW-relations and hence form a "quantum" Leonard triple. It would be interesting to investigate other automorphisms of the "quantum cubic".

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Alyona Korovnichenko
Department of mathematics, University of Notre-Dame,
255 Hurley Building, Notre-Dame, IN, 46556 USA

Alexei Zhedanov
Donetsk Institute for Physics and Technology,
Donetsk 83114, Ukraine