

## INTEGRABLE SYSTEMS ASSOCIATED WITH ELLIPTIC ALGEBRAS

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ABSTRACT. We construct new integrable systems (IS), both classical and quantum, associated with elliptic algebras. Our constructions are based both on a construction of commuting families in skew fields and on properties of the elliptic algebras and their representations. We give some examples showing how these IS are related to previously studied systems.

**Introduction.** This paper is an attempt to establish a direct connection between two close subjects of modern Mathematical Physics: integrable systems (IS) and elliptic algebras. More precisely, we construct IS on a large class of elliptic algebras.

We will start with a short account of the subject of the story and then we will briefly describe the IS's under consideration.

In [12], B. Enriquez and the second author proposed a construction of commuting families of elements in skew fields. They used the Poisson version of this construction to give a new proof of the integrability of the Beauville-Mukai integrable systems associated with a K3 surface  $S$  ([1]).

The Beauville-Mukai systems appear as Lagrangian fibrations of the form  $S^{[g]} \rightarrow |\mathcal{L}| = \mathbb{P}(H^0(S, \mathcal{L}))$ , where  $S^{[g]}$  is the Hilbert scheme of  $g$  points of  $S$ , equipped with the symplectic structure introduced in [19], and  $\mathcal{L}$  is a line bundle on  $S$ . Later, the authors of [8] explained that these systems are natural deformations of the “separated” (in the sense of [13]) versions of Hitchin’s integrable systems, more precisely, of their description in terms of spectral curves (already present in [15]). Beauville-Mukai systems can be generalized to surfaces with a Poisson structure (see [3]). When  $S$  is the canonical cone  $\text{Cone}(C)$  of an algebraic curve  $C$ , these systems coincide with the separated version of Hitchin’s systems.

A quantization of Hitchin’s system was proposed in [2]. The paper [12] shows that the construction of commuting families yields a quantization of the separated version of this system on the canonical cone. This construction depends on a choice of quantization of functions on  $\text{Cone}(C)$ . In [12], we also conjectured that one can determine such choices in the construction of quantized Beauville-Mukai systems that these systems become isomorphic to the Beilinson-Drinfeld systems at the birational level. A part of this program was realized in [9] for the case of  $S = T^*(\mathbb{P}^1 - P)$ , where  $P \subset \mathbb{P}^1$  is a finite subset.

Another main theme of the paper is a certain family of elliptic algebras. These algebras (with 4 generators) appeared in the works of Sklyanin [20, 21] and were later generalized (for any number of generators) and intensively studied by Feigin and one of us ([22, 27, 28]). They can be considered as deformations of certain quadratic Poisson structures on symmetric algebras.

The geometric meaning of these Poisson structures was explained in [29] (see also [41, 36]): these are natural Poisson structures of moduli spaces of holomorphic bundles on an elliptic curve. We will use the recent survey [30] as our main source of results and references in the theory of elliptic algebras.

Here is an account of the relations between elliptic algebras and IS. Elliptic algebras appeared in Sklyanin's approach to integrability of the Landau-Lifshitz model ([20, 21]) using the methods of the Faddeev school (quantum inverse scattering method and  $R$ -matrix approach). Later, Cherednik observed a relation between the elliptic algebras defined in [22] and the Baxter-Belavin  $R$ -matrix (see [32]). An interesting observation of Krichever and Zabrodin giving an interpretation of a generator in the Sklyanin elliptic algebra as a hamiltonian of the 2-point Ruijsenaars IS ([16]) was later generalized in [4] to the case of the double-elliptic 2-point classical model. However, all applications of these algebras to the IS theory had a somewhat indirect character until the last two years.

Another relation between elliptic algebras and IS was obtained in a recent paper of Sokolov-Tsyganov ([34]), who constructed (using Sklyanin's definition of quadratic Poisson structures) some classical commuting families associated with these Poisson algebras. The integrability of these families is implied by Sklyanin's method of separation of variables (SoV) and is technically based on a generalization of classical methods going back to Jacobi, Liouville and Stäckel (which are close to the classical part of theorems in [12] and [35]). However, all their results are stated in a "non-elliptic" language.

Recently, an example of an integrable system associated with linear and quadratic Poisson brackets given by the elliptic Belavin-Drinfeld classical  $r$ -matrix was proposed in [18]. This system (an elliptic rotator) appears both in finite and infinite-dimensional cases. The authors of [18] give an elliptic version of 2-dimensional ideal hydrodynamics on the symplectomorphism group of the 2-dimensional torus as well as on a non-commutative torus. The elliptic algebras also appear in the context of non-commutative geometry (see [40]). It would be interesting to relate them to the numerous modern attempts to define a non-commutative version of IS theory.

On the other hand, the IS which we construct here appear directly in the frame of elliptic algebras.

Let us describe the results of this paper. We construct two commuting families in elliptic algebras (Theorem 3.1 and Proposition 3.4).

The first commuting family is related to a quantum version of a bi-hamiltonian system which was introduced in [31]. In this paper, a family of compatible elliptic Poisson structures was introduced. This family contains three quadratic Poisson brackets such that their generic linear combination is the quasi-classical limit  $q_n(\mathcal{E})$  of an elliptic algebra  $Q_n(\mathcal{E}, \eta)$ . The famous Magri-Lenard scheme yields the existence of a classical integrable system associated with the elliptic curve. The corresponding integrable system has as its phase space a  $2m$ -dimensional component of the moduli space of parabolic rank two bundles on the given elliptic curve  $\mathcal{E}$ . More precisely, the coordinate ring of the open dense part of this component has the structure of a quadratic Poisson algebra isomorphic to  $q_{2m}(\mathcal{E})$ .

The quantization of this system was unknown because it is not known how to quantize the Magri-Lenard scheme. When  $n = 2m$ , we construct a quantum integrable system based on the approach of [12]; we conjecture that this is a quantization of the classical system from [31] and check this in the first nontrivial case ( $m = 3$ ).

Whereas Theorem 3.1 takes place for  $n = 2m$ , we are sure that there are some interesting integrable quantum systems in the case  $n = 2m + 1$ . It would be interesting

to study the bi-hamiltonian structures giving the algebra  $q_{2m+1}(\mathcal{E})$  using the results of Gelfand-Zakharevich ([33]) on the geometry of bi-hamiltonian systems in the case of odd-dimensional Poisson manifolds. The precise quantum version of these systems in the context of the elliptic algebras  $Q_{2m+1}(\mathcal{E}, \eta)$  is still obscure and should be a subject of further studies.

The second commuting family (Proposition 3.4) is associated with a special choice of the elliptic algebra. This family is obtained, on one hand, by direct application of the construction from [12] to the elliptic algebras and, on the other hand, by using the properties of the “bosonization” homomorphism, constructed in earlier works ([27, 28]). Some of these families (under the appropriate choice of numerical parameters) may be interpreted as algebraic examples of completely integrable systems. We give a geometric interpretation to some of them describing a link with the Lagrangian fibrations on symmetric powers of the cone over an elliptic curve, giving a version of the Beauville-Mukai systems (see [1, 13, 12, 39]).

The theorems in [12] may also be interpreted as an algebraic version of the SoV method (as it was argued in [35]). Hence, it is very plausible that some of our quantum commuting families arising from the generalization of the Jacobi-Liouville-Stäckel conditions are the quantum elliptic versions of the IS from [34]. We hope to return to this question in a future paper.

We give also some low-dimensional examples of our construction.

## 1. COMMUTING FAMILIES IN SOME NON-COMMUTATIVE ALGEBRAS

**1.1. Commuting families in skew fields.** Let  $A$  be an associative algebra with unit. We will assume later that  $A$  is contained in a skew field  $K$ .

Fix a natural number  $n \geq 2$ . We will assume that there are  $n$  subalgebras  $B_i \subset A$ ,  $1 \leq i \leq n$  such that for any pair of indices  $i \neq j$ ,  $1 \leq i, j \leq n$ , any pair of elements  $b_{(i)} \in B_i$  and  $b_{(j)} \in B_j$  commute with each other (while the algebras  $B_i$  are not assumed to be commutative).

Let us consider a matrix  $\mathcal{M}$  of size  $n \times (n + 1)$

$$\begin{pmatrix} b_{(1)}^0 & b_{(1)}^1 & \cdots & b_{(1)}^{n-1} & b_{(1)}^n \\ b_{(2)}^0 & b_{(2)}^1 & \cdots & b_{(2)}^{n-1} & b_{(2)}^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{(n)}^0 & b_{(n)}^1 & \cdots & b_{(n)}^{n-1} & b_{(n)}^n \end{pmatrix} \quad (1)$$

such that all the elements of  $i$ th row belong to the  $i$ th subalgebra  $B_{(i)}$ .

We will denote by  $\mathcal{M}^i$  the  $n \times n$  matrix obtained from the matrix  $\mathcal{M}$  by removing  $i$ th row. The corresponding Cartier-Foata determinant will be denoted by  $M^i$ .

Its definition repeats *verbatim* the standard one: in each matrix  $\mathcal{M}^i$  the entries lying in *different rows* commute together, so that each summand in the standard definition of the determinant is (up to sign) the product of  $n$  elements of different rows, whose product is order-independent.

The following theorem was proved in [12].

**Theorem 1.1.** *Assume that the matrix  $\mathcal{M}^0$ , is invertible. Then the elements  $H_i := (M^0)^{-1}M^i, i = 1, \dots, n$  pairwise commute.<sup>1</sup>*

The proof of theorem is achieved by some tedious but straightforward induction procedure.

Similar results were obtained in the framework of multi-parametric spectral problems in Operator Analysis ([38]) and in the framework of Seiberg-Witten integrable systems associated with a hyperelliptic spectral curves in [35].

The important step in the proof of Theorem 1.1 is the following “triangle” relations which are similar to the usual Yang -Baxter relation:

$$M^i(M^0)^{-1}M^j = M^j(M^0)^{-1}M^i \quad (2)$$

$$B^{ij}(M^0)^{-1}B^{kj} = B^{ik}(M^0)^{-1}B^{ij}, \quad (3)$$

where  $B^{ij}$  is the co-factor of the matrix element  $b_j^i, 0 \leq i, j \leq n$ .

Theorem 1.1 can be reformulated to give the following result:

**Corollary 1.1.** *Let  $A$  be an algebra,  $(f_{i,j})_{0 \leq i \leq n, 1 \leq j \leq n}$  be elements of  $A$  such that*

$$f_{i,j}f_{k,\ell} = f_{k,\ell}f_{i,j}$$

for any  $i, j, k, \ell$  such that  $j \neq \ell$ . For any  $I \subset \{0, \dots, n\}, J \subset \{1, \dots, n\}$  of the same cardinality, we set  $\Delta_{I,J} = \sum_{\sigma \in \text{Bij}(I,J)} \epsilon(\sigma) \prod_{i \in I} f_{i,\sigma(i)}$ . Here  $\text{Bij}(I, J)$  denotes the set of bijections between  $I$  and  $J$ . Assume that the  $\Delta_{I,J}$  are all invertible. Set  $\Delta_i := \Delta_{\{1, \dots, n\}, \{0, \dots, \check{i}, \dots, n\}}$ . Then the elements

$$H_i = (\Delta_0)^{-1} \Delta_i$$

all commute together.

**1.2. Poisson commuting families.** The following observation is straightforward:

**Lemma 1.1.** *If  $B$  is an integral Poisson algebra, then there is a unique Poisson structure on  $\text{Frac}(B)$  extending the Poisson structure of  $B$ .*

This structure is uniquely defined by the relations

$$\{1/f, g\} = -\{f, g\}/f^2, \{1/f, 1/g\} = \{f, g\}/(f^2g^2).$$

Theorem 1.1 has a Poisson counterpart.

**Theorem 1.2.** *Let  $A$  be a Poisson algebra. Assume that  $A$  is integral, and let  $\pi : A \hookrightarrow \text{Frac}(A)$  be its injection in its fraction field. Let  $B_1, \dots, B_n \subset A$  be Poisson subalgebras*

<sup>1</sup>In [12] is proved that the images of  $H_i$  under the embedding  $A \subset K$  pairwise commute, which obviously implies the above statement.

of  $A$ , such that  $\{B_i, B_j\} = 0$  for  $i \neq j$ . We will write the analogue of the matrix (1) as a row-vector:  $\mathcal{M} = [b^0, b^1, \dots, b^n]$ , where

$$b^i = \begin{pmatrix} b_1^i \\ b_2^i \\ \vdots \\ b_n^i \end{pmatrix}.$$

We set

$$\Delta_i^{\text{class}} = \det[b^0, \dots, \check{b}^i, \dots, b^n].$$

Here as usual we denote by  $\check{b}^i$  the  $i$ th omitted column. Then if  $\Delta_0^{\text{class}}$  is nonzero we set  $H_i^{\text{class}} = \Delta_i^{\text{class}} / \Delta_0^{\text{class}}$  and the family  $(H_i^{\text{class}})_{i=1, \dots, n}$  is Poisson-commutative:

$$\{H_i^{\text{class}}, H_j^{\text{class}}\} = 0$$

for any pair  $(i, j)$ .

*Remark 1.* The elements  $b_i^k$  and  $b_j^l$  of the matrix  $\mathcal{M}$  belong to different subalgebras  $B_i$  and  $B_j$  if  $i \neq j$  and hence Poisson commute. This condition reminds the classical constraints on the Poisson brackets between matrix elements which appeared in the 19th century papers of Stäckel on the separation of variables of Hamilton-Jacobi systems ([37]). So the conditions of our theorem can be considered as an algebraic version of the Stäckel conditions.

1.2.1. *Plücker relations.* Here we remind an important step of the second proof in [12] which shows the relations between the commuting elements and the Plücker-like equations.

We have to prove

$$\Delta_i^{\text{class}} \{\Delta_j^{\text{class}}, \Delta_k^{\text{class}}\} + \text{cyclic permutation in } (i, j, k) = 0. \quad (4)$$

We have

$$\Delta_i^{\text{class}} = \sum_{p=1}^n \sum_{\alpha=0}^n (-1)^{p+\alpha} (b^\alpha)^{(p)} (\Delta_{\alpha, i})^{(1 \dots \check{p} \dots n)},$$

where (if  $\alpha \neq i$ )

$$\Delta_{\alpha, i}^{(1 \dots \check{p} \dots n)} = (-1)^{1_{\alpha < i}} \det[b^0 \dots \check{b}^\alpha \dots \check{b}^i \dots b^n]^{(p)}$$

(which means that the  $p$ th row in the matrix  $[b^0, \dots, \check{b}^\alpha, \dots, \check{b}^i \dots b^n]$  should be omitted.)

We set  $1_{\alpha < i} = 1$  if  $\alpha < i$  and 0 otherwise. If  $\alpha = i$  we assume  $\Delta_{\alpha, i} = 0$ . Now we have

$$\{\Delta_i^{\text{class}}, \Delta_j^{\text{class}}\} = \sum_{p=1}^n \sum_{\alpha, \beta=0}^n (-1)^{\alpha+\beta} (\{b^\alpha, b^\beta\})^{(p)} (\Delta_{\alpha, i} \Delta_{\beta, j} - \Delta_{\beta, i} \Delta_{\alpha, j})^{(1 \dots \check{p} \dots n)},$$

so identity (4) is a consequence of

$$\forall (i, j, k, \alpha, \beta, \gamma), \quad \sum_{\sigma \in \text{Perm}(i, j, k)} \epsilon(\sigma) \Delta_{\alpha, \sigma(i)} \Delta_{\beta, \sigma(j)} \Delta_{\gamma, \sigma(k)} = 0. \quad (5)$$

When  $\text{card}\{\alpha, \dots, k\} = 3$ , this identity follows from the antisymmetry relation  $\Delta_{i, j} + \Delta_{j, i} = 0$ .

When  $\text{card}\{\alpha, \dots, k\} = 4$  (resp., 5, 6), it follows from the following Plücker identities (to get (5), one should set  $V = (A^{\otimes n})^{\oplus n}$  and  $\Lambda$  some partial determinant).

Let  $V$  be a vector space. Then

– if  $\Lambda \in \wedge^2(V)$ , and  $a, b, c, d \in V^*$ , then

$$\Lambda(a, b)\Lambda(c, d) - \Lambda(a, c)\Lambda(b, d) + \Lambda(a, d)\Lambda(b, c) = 0; \quad (6)$$

– if  $\Lambda \in \wedge^3(V)$  and  $a, b, c, b', c' \in V^*$ , then

$$\begin{aligned} & \Lambda(b, c, c')\Lambda(a, c, b')\Lambda(b, b', c') + \Lambda(b, c, b')\Lambda(c, b', c')\Lambda(a, b, c') \\ & - \Lambda(b, c, b')\Lambda(a, c, c')\Lambda(b, b', c') - \Lambda(b, c, c')\Lambda(c, b', c')\Lambda(a, b, b') = 0; \end{aligned}$$

– if  $\Lambda \in \wedge^4(V)$  and  $a, b, c, a', b', c' \in V^*$ , then

$$\begin{aligned} & \Lambda(b, c, b', c')\Lambda(a, c, a', c')\Lambda(a, b, a', b') + \Lambda(b, c, a', c')\Lambda(a, c, a', b')\Lambda(a, b, b', c') \\ & + \Lambda(b, c, a', b')\Lambda(a, c, b', c')\Lambda(a, b, a', c') \\ & - \Lambda(b, c, b', c')\Lambda(a, c, a', b')\Lambda(a, b, a', c') - \Lambda(b, c, a', b')\Lambda(a, c, a', c')\Lambda(a, b, b', c') \\ & - \Lambda(b, c, a', c')\Lambda(a, c, b', c')\Lambda(a, b, a', b') = 0. \end{aligned} \quad (7)$$

We refer to [12] for a proof of these identities. We will need them below in some special situation arising from the commuting elements in associative and Poisson algebras which are directly connected with elliptic curves and vector bundles over them. These Plücker relations can be interpreted as a kind of Riemann-Fay identities which are in their turn related to integrable (difference) equations in Hirota bilinear form.

## 2. ELLIPTIC ALGEBRAS

Now we describe one of the main heroes of our story – the family of elliptic algebras. We will follow the survey [30] for notation and as our a main source of results and proofs in this section.

**2.1. Definition and main properties.** The elliptic algebras are associative quadratic algebras  $Q_{n,k}(\mathcal{E}, \eta)$  which were introduced in the papers [22, 28]. Here  $\mathcal{E}$  is an elliptic curve and  $n, k$  are coprime integer numbers, such that  $1 \leq k < n$  while  $\eta$  is a complex number and  $Q_{n,k}(\mathcal{E}, 0) = \mathbb{C}[x_1, \dots, x_n]$ .

Let  $\mathcal{E} = \mathbb{C}/\Gamma$  be an elliptic curve defined by a lattice  $\Gamma = \mathbb{Z} \oplus \tau\mathbb{Z}$ ,  $\tau \in \mathbb{C}$ ,  $\Im\tau > 0$ . The algebra  $Q_{n,k}(\mathcal{E}, \eta)$  has generators  $x_i$ ,  $i \in \mathbb{Z}/n\mathbb{Z}$  subject to the relations

$$\sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+r(k-1)}(0)}{\theta_{j-i-r}(-\eta)\theta_{kr}(\eta)} x_{j-r} x_{i+r} = 0$$

and have the following properties:

1)  $Q_{n,k}(\mathcal{E}, \eta) = \mathbb{C} \oplus Q_1 \oplus Q_2 \oplus \dots$  such that  $Q_\alpha * Q_\beta = Q_{\alpha+\beta}$ , here  $*$  denotes the algebra multiplication. In other words, the algebras  $Q_{n,k}(\mathcal{E}, \eta)$  are  $\mathbb{Z}$ -graded;

2) The Hilbert function of  $Q_{n,k}(\mathcal{E}, \eta)$  is  $\sum_{\alpha \geq 0} \dim Q_\alpha t^\alpha = \frac{1}{(1-t)^n}$ .

We consider here the set of theta-functions  $\{\theta_i(z), i = 0, \dots, n-1\}$  as a basis of the space of order  $n$  theta-functions  $\Theta_n(\Gamma)$ . These functions satisfy the quasi-periodicity relations

$$\theta_i(z+1) = \theta_i(z), \theta_i(z+\tau) = (-1)^n \exp(-2\pi\sqrt{-1}nz)\theta_i(z), i = 0, \dots, n-1.$$

The order one theta-function  $\theta(z) \in \Theta_1(\Gamma)$  satisfies the conditions  $\theta(0) = 0$  and  $\theta(-z) = \theta(z+\tau) = -\exp(-2\pi\sqrt{-1}z)\theta(z)$ .

When  $\mathcal{E}$  is fixed, the algebra  $Q_{n,k}(\mathcal{E}, \eta)$  is a flat deformation of the polynomial ring  $\mathbb{C}[x_1, \dots, x_n]$ . The first order term in  $\eta$  of this deformation gives rise to a quadratic Poisson algebra  $q_{n,k}(\mathcal{E})$ .

The geometric meaning of the algebras  $Q_{n,k}$  was explained in [29, 41], where it was shown that the quadratic Poisson structure  $q_{n,k}(\mathcal{E})$  identifies with a natural Poisson structure on  $\mathbb{P}^{n-1} = \mathbb{P}\text{Ext}^1(E, \mathcal{O})$ , where  $E$  is a stable vector bundle of rank  $k$  and degree  $n$  on the elliptic curve  $\mathcal{E}$ .

In what follows we will denote the algebras  $Q_{n,1}(\mathcal{E}, \eta)$  by  $Q_n(\mathcal{E}, \eta)$ .

## 2.2. The algebra $Q_n(\mathcal{E}, \eta)$ .

2.2.1. *Construction.* For any  $n \in \mathbb{N}$ , any elliptic curve  $\mathcal{E} = \mathbb{C}/\Gamma$ , and any point  $\eta \in \mathcal{E}$ , we construct a graded associative algebra  $Q_n(\mathcal{E}, \eta) = \mathbb{C} \oplus F_1 \oplus F_2 \oplus \dots$ , where  $F_1 = \Theta_n(\Gamma)$  and  $F_\alpha = S^\alpha \Theta_n(\Gamma)$ . By construction,  $\dim F_\alpha = \frac{n(n+1)\dots(n+\alpha-1)}{\alpha!}$ . It is clear that the space  $F_\alpha$  can be realized as the space of holomorphic symmetric functions of  $\alpha$  variables  $f(z_1, \dots, z_\alpha)$  such that

$$\begin{aligned} f(z_1+1, z_2, \dots, z_\alpha) &= f(z_1, \dots, z_\alpha), \\ f(z_1+\tau, z_2, \dots, z_\alpha) &= (-1)^n e^{-2\pi i n z_1} f(z_1, \dots, z_\alpha). \end{aligned} \tag{8}$$

For  $f \in F_\alpha$  and  $g \in F_\beta$  we define the symmetric function  $f * g$  of  $\alpha + \beta$  variables by the formula

$$\begin{aligned} f * g(z_1, \dots, z_{\alpha+\beta}) &= \frac{1}{\alpha! \beta!} \sum_{\sigma \in S_{\alpha+\beta}} f(z_{\sigma_1} + \beta\eta, \dots, z_{\sigma_\alpha} + \beta\eta) g(z_{\sigma_{\alpha+1}} - \alpha\eta, \dots, z_{\sigma_{\alpha+\beta}} - \alpha\eta) \times \\ &\quad \times \prod_{\substack{1 \leq i \leq \alpha \\ \alpha+1 \leq j \leq \alpha+\beta}} \frac{\theta(z_{\sigma_i} - z_{\sigma_j} - n\eta)}{\theta(z_{\sigma_i} - z_{\sigma_j})}. \end{aligned} \tag{9}$$

In particular, for  $f, g \in F_1$  we have

$$f * g(z_1, z_2) = f(z_1 + \eta)g(z_2 - \eta) \frac{\theta(z_1 - z_2 - n\eta)}{\theta(z_1 - z_2)} + f(z_2 + \eta)g(z_1 - \eta) \frac{\theta(z_2 - z_1 - n\eta)}{\theta(z_2 - z_1)}.$$

Here  $\theta(z)$  is a theta-function of order one.

**Proposition 2.1.** *If  $f \in F_\alpha$  and  $g \in F_\beta$ , then  $f * g \in F_{\alpha+\beta}$ . The operation  $*$  defines an associative multiplication on the space  $\bigoplus_{\alpha \geq 0} F_\alpha$ .*

2.2.2. *Main properties of the algebra  $Q_n(\mathcal{E}, \eta)$ .* By construction, the dimensions of the graded components of the algebra  $Q_n(\mathcal{E}, \eta)$  coincide with those for the polynomial ring in  $n$  variables. For  $\eta = 0$  the formula for  $f * g$  becomes

$$f * g(z_1, \dots, z_{\alpha+1}) = \frac{1}{\alpha! \beta!} \sum_{\sigma \in S_{\alpha+\beta}} f(z_{\sigma_1}, \dots, z_{\sigma_\alpha}) g(z_{\sigma_{\alpha+1}}, \dots, z_{\sigma_{\alpha+\beta}}).$$

This is the formula for the ordinary product in the algebra  $S^* \Theta_n(\Gamma)$ , that is, in the polynomial ring in  $n$  variables. Therefore, for a fixed elliptic curve  $\mathcal{E}$  (that is, for a fixed modular parameter  $\tau$ ) the family of algebras  $Q_n(\mathcal{E}, \eta)$  is a deformation of this polynomial ring. In particular, it defines a Poisson structure on this ring, which we denote by  $q_n(\mathcal{E})$ . One can readily obtain the formula for the Poisson bracket on this polynomial ring from the formula for  $f * g$  by expanding the difference  $f * g - g * f$  in Taylor series with respect to  $\eta$ . It follows from semi-continuity arguments that the algebra  $Q_n(\mathcal{E}, \eta)$  with generic  $\eta$  is determined by  $n$  generators and  $\frac{n(n-1)}{2}$  quadratic relations. One can prove (see §2.6 in [30]) that this is the case if  $\eta$  is not a point of finite order on  $\mathcal{E}$ , that is,  $N\eta \notin \Gamma$  for any  $N \in \mathbb{N}$ .

The space  $\Theta_n(\Gamma)$  of the generators of the algebra  $Q_n(\mathcal{E}, \eta)$  is endowed with an action of a finite group  $\widetilde{\Gamma}_n$  which is a central extension of the group  $\Gamma/n\Gamma$  of points of order  $n$  on the curve  $\mathcal{E}$ . It can be checked that it extends to an action of this group by algebra automorphisms of  $Q_n(\mathcal{E}, \eta)$ .

2.2.3. *Bosonization of the algebra  $Q_n(\mathcal{E}, \eta)$ .* The main approach to obtain representations of the algebra  $Q_n(\mathcal{E}, \eta)$  is to construct homomorphisms from this algebra to other algebras with a simple structure (close to the Weyl algebras) which have a natural set of representations. These homomorphisms are referred to as bosonizations, by analogy with the known constructions of quantum field theory.

Let  $B_{p,n}(\eta)$  be the algebra generated by the commutative algebra of meromorphic functions  $f(u_1, \dots, u_p)$  and by the elements  $e_1, \dots, e_p$  with the defining relations

$$\begin{aligned} e_\alpha f(u_1, \dots, u_p) &= f(u_1 - 2\eta, \dots, u_\alpha + (n-2)\eta, \dots, u_p - 2\eta) e_\alpha, \\ e_\alpha e_\beta &= e_\beta e_\alpha, \quad f(u_1, \dots, u_p) g(u_1, \dots, u_p) = g(u_1, \dots, u_p) f(u_1, \dots, u_p). \end{aligned} \quad (10)$$

Then  $B_{p,n}(\eta)$  is a  $\mathbb{Z}^p$ -graded algebra whose space of degree  $(\alpha_1, \dots, \alpha_p)$  is

$$\{f(u_1, \dots, u_p) e_1^{\alpha_1} \dots e_p^{\alpha_p}\},$$

where  $f$  ranges over the meromorphic functions of  $p$  variables.

We note that the subalgebra of  $B_{p,n}(\eta)$  consisting of the elements of degree  $(0, \dots, 0)$  is the commutative algebra of all meromorphic functions of  $p$  variables with the ordinary multiplication.

**Proposition 2.2.** *Let  $\eta \in \mathcal{E}$  be a point of infinite order. For any  $p \in \mathbb{N}$  there is a homomorphism  $\phi_p: Q_n(\mathcal{E}, \eta) \rightarrow B_{p,n}(\eta)$  defined on the generators of the algebra  $Q_n(\mathcal{E}, \eta)$  by the formula :*

$$\phi_p(f) = \sum_{1 \leq \alpha \leq p} \frac{f(u_\alpha)}{\prod_{i \neq \alpha} \theta(u_\alpha - u_i)} e_\alpha. \quad (11)$$



Here  $f \in \Theta_n(\Gamma)$  is a degree 1 generator of  $Q_n(\mathcal{E}, \eta)$ .

2.2.4. *Symplectic leaves.* We recall that  $Q_n(\mathcal{E}, 0)$  is the polynomial ring  $S^*\Theta_n(\Gamma)$ . For a fixed elliptic curve  $\mathcal{E} = \mathbb{C}/\Gamma$  we obtain the family of algebras  $Q_n(\mathcal{E}, \eta)$ , which is a flat deformation of the polynomial ring. We denote the corresponding Poisson algebra by  $q_n(\mathcal{E})$ . We obtain a family of Poisson algebras, depending on  $\mathcal{E}$ , that is, on the modular parameter  $\tau$ . Let us study the symplectic leaves of this algebra. To this end, we note that, when passing to the limit  $\eta \rightarrow 0$ , the homomorphism  $\phi_p$  of associative algebras gives a homomorphism of Poisson algebras. Namely, let us denote by  $b_{p,n}$  the Poisson structure on the algebra  $\bigoplus_{\alpha_1, \dots, \alpha_p \geq 0} \{f_{\alpha_1, \dots, \alpha_p}(u_1, \dots, u_p)e_1^{\alpha_1} \dots e_p^{\alpha_p}\}$ , where  $f_{\alpha_1, \dots, \alpha_p}$  runs over all meromorphic functions, defined by

$$\{u_\alpha, u_\beta\} = \{e_\alpha, e_\beta\} = 0; \quad \{e_\alpha, u_\beta\} = -2e_\alpha; \quad \{e_\alpha, u_\alpha\} = (n - 2)e_\alpha,$$

where  $\alpha \neq \beta$ .

The following assertion results from Proposition 6 in the limit  $\eta \rightarrow 0$ .

**Proposition 2.3.** *There is a Poisson algebra homomorphism  $\psi_p: q_n(\mathcal{E}) \rightarrow b_{p,n}$  given by the following formula: if  $f \in \Theta_n(\Gamma)$ , then*

$$\psi_p(f) = \sum_{1 \leq \alpha \leq p} \frac{f(u_\alpha)}{\theta(u_\alpha - u_1) \dots \theta(u_\alpha - u_p)} e_\alpha.$$

Let  $\{\theta_i(u); i \in \mathbb{Z}/n\mathbb{Z}\}$  be a basis of the space  $\Theta_n(\Gamma)$  and let  $\{x_i; i \in \mathbb{Z}/n\mathbb{Z}\}$  be the corresponding basis in the space of elements of degree one in the algebra  $Q_n(\mathcal{E}, \eta)$  (this space is isomorphic to  $\Theta_n(\Gamma)$ ). Consider the embedding of the elliptic curve  $\mathcal{E} \subset \mathbb{P}^{n-1}$  by means of theta functions of order  $n$ , by  $z \mapsto (\theta_0(z) : \dots : \theta_{n-1}(z))$ . We denote by  $C_p\mathcal{E}$  the variety of  $p$ -chords, that is, the union of projective spaces of dimension  $p - 1$  passing through  $p$  points of  $\mathcal{E}$ . Let  $K(C_p\mathcal{E})$  be the corresponding homogeneous manifold in  $\mathbb{C}^n$ . It is clear that  $K(C_p\mathcal{E})$  consists of the points with the coordinates

$$x_i = \sum_{1 \leq \alpha \leq p} \frac{\theta_i(u_\alpha)}{\theta(u_\alpha - u_1) \dots \theta(u_\alpha - u_p)} e_\alpha,$$

where  $u_\alpha, e_\alpha \in \mathbb{C}$ .

Let  $2p < n$ . Then one can show that  $\dim K(C_p\mathcal{E}) = 2p$  and  $K(C_{p-1}\mathcal{E})$  is the manifold of singularities of  $K(C_p\mathcal{E})$ . It follows from Proposition 7 and from the fact that the Poisson bracket is non-degenerate on  $b_{p,n}$  for  $2p < n$  and  $e_\alpha \neq 0$  that the non-singular part of the manifold  $K(C_p\mathcal{E})$  is a  $2p$ -dimensional symplectic leaf of the Poisson algebra  $q_n(\mathcal{E})$ .

Let  $n$  be odd. One can show that the equation defining the manifold  $K(C_{\frac{n-1}{2}}\mathcal{E})$  is of the form  $C = 0$ , where  $C$  is a homogeneous polynomial of degree  $n$  in the variables  $x_i$ . This polynomial is a central function of the algebra  $q_n(\mathcal{E})$ .

Let  $n$  be even. The manifold  $K(C_{\frac{n-2}{2}}\mathcal{E})$  is defined by equations  $C_1 = 0$  and  $C_2 = 0$ , where  $\deg C_1 = \deg C_2 = n/2$ . The polynomials  $C_1$  and  $C_2$  are central in the algebra  $q_n(\mathcal{E})$ .

3. INTEGRABLE SYSTEMS

There are (at least two) different ways to construct commuting families and IS associated with the elliptic algebras. We will start with general statements about the commuting elements arising from the ideas and constructions of Section 1.

**3.1. Commuting elements in the algebras  $Q_n(\mathcal{E}; \eta)$ .** Let us consider the following Weyl-like algebra  $\mathcal{V}_n$  with the set of “generators”  $f_1, \dots, f_n, z_1, \dots, z_n$  subject to the relations

$$0 = [f_i, f_j] = [z_i, z_j] = [f_i, z_j] \ (i \neq j), \ f_i z_i = (z_i - n\eta) f_i.$$

$\mathcal{V}_n$  is spanned as a vector space by the elements of the form  $F(z_1, \dots, z_n) f_1^{m_1} \dots f_n^{m_n}$ , where  $F$  is a meromorphic function in  $n$  variables. So we have the following commutation relations between the functions in variables  $z_i$  and the elements  $f_j$ :

$$f_j F(z_1, \dots, z_n) = F(z_1, \dots, z_j - n\eta, \dots, z_n) f_j.$$

*Remark 2.* The algebra  $\mathcal{V}_n$  looks different from the above-mentioned Weyl-like algebras  $B_{p;n}$  but it is isomorphic to the algebra  $B_{n;n}$ . We will return below to a geometric interpretation of the algebra  $\mathcal{V}_n$ .

Now we consider the following determinant

$$D_0 = \begin{vmatrix} \theta_0(z_1) & \theta_1(z_1) & \dots & \theta_{n-2}(z_1) & \theta_{n-1}(z_1) \\ \theta_0(z_2) & \theta_1(z_2) & \dots & \theta_{n-2}(z_2) & \theta_{n-1}(z_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_0(z_n) & \theta_1(z_n) & \dots & \theta_{n-2}(z_n) & \theta_{n-1}(z_n) \end{vmatrix} =$$

$$c \exp(z_2 + 2z_3 + \dots + (n-1)z_n) \prod_{1 \leq i < j \leq n} \theta(z_i - z_j) \theta\left(\sum_{i=1}^n z_i\right),$$

where the constant  $c$  is irrelevant for us because it will be cancelled in future computations.

Then we define the partial determinants  $D_i$  deleting the  $i$ th column and adjoining a  $n$ th column of  $f_\alpha$ 's:

$$D_i = \begin{vmatrix} \theta_0(z_1) & \theta_1(z_1) & \dots & i & \theta_{n-1}(z_1) & f_1 \\ \theta_0(z_2) & \theta_1(z_2) & \dots & | & \theta_{n-1}(z_2) & f_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_0(z_n) & \theta_1(z_n) & \dots & | & \theta_{n-1}(z_n) & f_n \end{vmatrix} =$$

$$\sum_{1 \leq \alpha \leq n} (-1)^{\alpha+n} \begin{vmatrix} \theta_0(z_1) & \theta_1(z_1) & \dots & \theta_{n-1}(z_1) \\ \theta_0(z_2) & \theta_1(z_2) & \dots & \theta_{n-1}(z_2) \\ \vdots & \vdots & \vdots & \vdots \\ \theta_0(z_n) & \theta_1(z_n) & \dots & \theta_{n-1}(z_n) \end{vmatrix}_{\alpha,i} f_\alpha.$$

Here the subscript  $|_{\alpha,i}$  means that we omit the  $i$ th column and the  $\alpha$ th row.

The immediate corollary of (1.1) is the following

**Proposition 3.1.** *The determinant ratios form a commutative family:*

$$[D_0^{-1} D_i, D_0^{-1} D_j] = 0.$$

The result of the proposition can be expressed in an elegant way in the form of a commutation relation of generating functions.

Let us define a generating function  $T(u)$  of a variable  $u \in \mathbb{C}$ :

$$T(u) = D_0^{-1} \sum_{1 \leq j \leq n} (-1)^j \theta_j(u) D_j.$$

Then we can express the function  $T(u)$ , using the formulas for the determinants of the theta-functions as

$$\begin{aligned} T(u) &= D_0^{-1} \sum_{1 \leq \alpha \leq n} (-1)^\alpha \begin{vmatrix} \theta_0(z_1) & \theta_1(z_1) & \dots & \theta_{n-1}(z_1) \\ \vdots & \vdots & \vdots & \vdots \\ \theta_0(u) & \theta_1(u) & \dots & \theta_{n-1}(u) \\ \vdots & \vdots & \vdots & \vdots \\ \theta_0(z_n) & \theta_1(z_n) & \dots & \theta_{n-1}(z_n) \end{vmatrix} f_\alpha = \quad (12) \\ &= \sum_{1 \leq \alpha \leq n} \frac{\theta(u + \sum_{\beta \neq \alpha} z_\beta) \prod_{1 \leq \beta \neq \alpha \leq n} \theta(u - z_\beta)}{\prod_{\beta \neq \alpha} \theta(z_\alpha - z_\beta)} \tilde{f}_\alpha, \end{aligned}$$

where we denote by  $\tilde{f}_\alpha$  the normalization

$$\tilde{f}_\alpha = \frac{f_\alpha}{\theta(\sum_{i=1}^n z_i)}.$$

We remark that the commutation relations between the variables  $z_1, \dots, z_n, \tilde{f}_1, \dots, \tilde{f}_n$  are the same as they were between the variables  $z_1, \dots, z_n, f_1, \dots, f_n$ .

In this notation the proposition now reads:

**Proposition 3.2.** *The “transfer-like” operators  $T(u)$  commute for different values of the parameter  $u$ :*

$$[T(u), T(v)] = 0.$$

Now we will apply this result to a construction of commuting elements in the algebra  $Q_n(\mathcal{E}; \eta)$  for even  $n$ .

Let  $n = 2m$ . It is known in this case that the center  $Z(Q_{2m}(\mathcal{E}; \eta))$  for  $\eta$  of infinite order is generated by Casimir elements  $f(z_1, \dots, z_m) \in S^m(\Theta_{2m}(\Gamma))$  such that  $f(z_1, \dots, z_m) = 0$  for  $z_2 = z_1 + 2m\eta$ . A straightforward computation shows that the space of such elements is two-dimensional and has as a basis the elements of the form

$$C_\alpha = \theta_\alpha(z_1 + \dots + z_m) \prod_{1 \leq i \neq j \leq m} \theta(z_i - z_j - 2m\eta),$$

where  $\theta_\alpha \in \Theta_2(\Gamma), \alpha \in \mathbb{Z}/2\mathbb{Z}$ .

Let  $S^m(\Theta_{2m}(\Gamma))$  be the space of symmetric polynomials of degree  $m$  in theta-functions of order  $2m$ .

**Proposition 3.3.** Fix an element  $\Psi(z) \in \Theta_{m+5}(\Gamma)$  and two complex numbers  $a, b \in \mathbb{C}$ . There exists a family  $f(u)(z_1, \dots, z_m) \in S^m(\Theta_{2m}(\Gamma))$ , indexed by  $u \in \mathbb{C}$ , such that

$$\begin{aligned} & f(u)(z_1, z_1 + 2m\eta, z_2, \dots, z_{m-1}) \tag{13} \\ = & \Psi(z_1 + 4m^2\eta - \frac{1}{m+5}(a + (m-2)b + 2m(m-2)\eta))\theta(z_1 + \dots + z_{m-1} + a)\theta(z_1 + z_2 + b) \dots \\ & \theta(z_1 + z_{m-1} + b)\theta(z_2 - z_1 - 4m\eta) \dots \theta(z_{m-1} - z_1 - 4m\eta)\theta(z_2 - z_1 + 2m\eta) \dots \theta(z_{m-1} - z_1 + 2m\eta) \times \\ & \theta(u + z_2 + \dots + z_{m-1} - a - b + 2m\eta)\theta(u - z_2) \dots \theta(u - z_{m-1}) \times \\ & \exp(2\pi i(2(m-2)z_1 + z_2 + \dots + z_{m-1})) \prod_{2 \leq i \neq j \leq m-1} \theta(z_i - z_j - 2m\eta). \end{aligned}$$

*Remark 3.* The elements  $f(u)$  are well-defined up to a linear combination of the Casimirs  $C_1, C_2$ .

*Sketch of the proof.* Consider the space  $\mathcal{F}(\Delta_{m,\eta})$  of symmetric functions defined on the subvariety of all shifted diagonals  $z_i = z_j + 2m\eta, 1 \leq i \neq j \leq m$ ; we have a restriction map  $S^m(\Theta_{2m}(\Gamma)) \rightarrow \mathcal{F}(\Delta_{m,\eta})$ . Formula (13) defines a family of functions in  $\mathcal{F}(\Delta_{m,\eta})$ , indexed by  $u \in \mathbb{C}$ , whose quasi-periodicity properties are the same as those of the elements of the image of the restriction map  $S^m(\Theta_{2m}(\Gamma)) \rightarrow \mathcal{F}(\Delta_{m,\eta})$ .

We now show that the elements  $f(u)$  can be lifted to a family of degree  $m$  polynomials in theta-functions of order  $2m$ . The proof is based on the existence of a certain general filtration in symmetric polynomial rings and their deformations which was introduced in [45]. The elliptic algebra  $Q_n(\mathcal{E}, \eta)$  admits the following filtration  $F_{r_1, \dots, r_p}$  with  $r_1 \geq r_2 \geq \dots \geq r_p$ : an element  $x \in Q_n(\mathcal{E}, \eta)$  belongs to  $F_{r_1, \dots, r_p}$  if the image  $\phi_p(x) \in B_{p,n}$  under the bosonization homomorphism  $\phi_p$  (11) has the following form:

$$\phi_p(x) = f(u_1, \dots, u_p)e_1^{r_1} \dots e_p^{r_p} + \text{elements of smaller order.}$$

Here we mean the natural ordering on homogeneous components of  $B_{n,p}$ :  $(r_1, \dots, r_p) < (s_1, \dots, s_p)$  if

$$r_1 = s_1, \dots, r_t = s_t, r_{t+1} < s_{t+1},$$

for some  $t$ .

For example, the subspace  $F_{1, \dots, 1}$  is generated by two central elements  $C_1, C_2$ . The most difficult part of the proof is to identify the associated graded of this filtration with some spaces of holomorphic functions satisfying certain symmetry and quasi-periodicity properties.

For example, let us consider the highest order terms of the filtration:

$$F_{p-1, 1, 0, \dots, 0} \subset F_{p, 0, \dots, 0}$$

and take the bosonization of an element  $x \in F_{p, 0, \dots, 0} \subset Q_n(\mathcal{E}, \eta)$ :

$$\phi_p(x) = \alpha_1(u)e_1^p + \alpha_2(u)e_1^{p-1}e_2 + \dots$$

The element  $x \in Q_n(\mathcal{E}, \eta)$  belongs to  $F_{p-1, 1, 0, \dots, 0}$  if the coefficient function  $\alpha_1$  vanishes ( $\alpha_1$  is a theta-function of order  $np$ ). Hence the elements of the quotient  $F_{p, 0, \dots, 0}/F_{p-1, 1, 0, \dots, 0}$  identify with theta-functions of order  $np$ .

Using this identification we see that the condition (13) on  $f(u)$  determines the image of  $f(u)$  in the quotient  $F_{2,1,\dots,1}/F_{1,\dots,1}$ . Since the space  $F_{1,\dots,1}$  is the space generated by Casimirs  $C_{1,2}$ , the element  $f(u)$  can be lifted to an element in

$$F_{2,1,\dots,1} \subset S^m(\Theta_{2m}(\Gamma)).$$

Now we apply these observations to establish the following result:

**Theorem 3.1.** *In the elliptic algebra  $Q_{2m}(\mathcal{E}, \eta)$  the following relation holds*

$$[f(u), f(v)] = f(u) * f(v) - f(v) * f(u) = 0.$$

*Proof.* We will use the homomorphism  $Q_{2m}(\mathcal{E}, \eta) \rightarrow B_{m-1,2m}$  from Subsection 2.2.3. The element  $f(u)$  is taken by this homomorphism to the element

$$\sum_{1 \leq \alpha \leq m-1} \frac{\theta(u + \sum_{\beta \neq \alpha} z_\beta) \prod_{1 \leq \beta \neq \alpha \leq m-1} \theta(u - z_\beta)}{\prod_{\beta \neq \alpha} \theta(z_\beta - z_\alpha)} f_\alpha,$$

where we denote by  $f_\alpha$  the following expression

$$\begin{aligned} f_\alpha &= \Psi(z_\alpha + 4m^2\eta - \frac{1}{m+5}(a + (m-2)b + 2m(m-2)\eta)) \times \\ &\quad \theta(\sum_{\beta=1}^{m-1} z_\beta + a) \prod_{\beta \neq \alpha} \theta(z_\alpha + z_\beta + b) \times \\ &\quad \exp(2\pi i(2(m-2)z_\alpha + \sum_{\beta \neq \alpha} z_\beta)) e_\alpha e_1 \dots e_{m-1}. \end{aligned}$$

Then, formula (12) and Proposition 3.2 immediately imply that the images of  $f(u)$  and  $f(v)$  under the homomorphism  $\phi_{m-1}$  commute in  $B_{m-1,2m}(\eta)$ . It is known that the image of the algebra  $Q_{2m}(\mathcal{E}, \eta)$  in  $B_{m-1,2m}(\eta)$  is the quotient of  $Q_{2m}(\mathcal{E}, \eta)$  by the ideal  $\langle C_1, C_2 \rangle$  generated by  $C_1, C_2$ . Hence the commutator

$$[f(u), f(v)] = f(u) * f(v) - f(v) * f(u)$$

belongs to this ideal. To show that  $[f(u), f(v)] = 0$  we can consider the injective homomorphism into  $B_{m,2m}(\eta)$  and it is sufficient to verify that the coefficient before  $(e_1)^2 \dots (e_m)^2$  equals zero, or (which is equivalent) that

$$[f(u), f(v)](z_1, z_1 + 2m\eta, z_2, z_2 + 2m\eta, \dots, z_m, z_m + 2m\eta) = 0,$$

which is a direct verification.

*Remark 4.* We should observe that the commuting elements constructed here are parametrized by the choice of the element  $\Psi(z) \in \Theta_{m+5}(\Gamma)$ .

*Remark 5.* In [31], we constructed a family of compatible quadratic Poisson structures, a linear combination of which is isomorphic to the Poisson structure of the classical elliptic algebra  $q_n(\mathcal{E})$ . The Magri-Lenard scheme yields a family of Poisson commuting elements (“hamiltonians” in involution) in the algebra  $q_n(\mathcal{E})$ . These elements have degree  $n$  if  $n$  is odd and  $n/2$  otherwise.

Let  $n = 2m$ . We conjecture that the commuting family of  $Q_{2m}(\mathcal{E})$  constructed in 3.1 (for a proper choice of the parametrizing element  $\Psi(z)$ ) is the quantum analogue of the family of commuting “hamiltonians” in  $q_{2m}(\mathcal{E})$  generated by the Magri-Lenard scheme applied to the classical Casimirs  $C_0^{(m)}, C_1^{(m)}$  of [31]. This conjecture is true in the first non-trivial case  $n = 6$ , where it is verified by direct computation.

**3.2. Commuting elements in  $B_{2,\dots,2}(\eta)$ .** The elliptic algebra  $Q_{n,n-1}(\mathcal{E}, \eta)$  is commutative and the bosonization homomorphism yields a large class of commuting families in the corresponding Weyl-like algebra. This gives rise to a new integrable system.

**3.2.1. Bosonization of the algebra  $Q_{n,n-1}(\mathcal{E})$ .** The homomorphism  $\phi_n : Q_n(\mathcal{E}, \eta) \rightarrow B_{n;n}(\eta)$  from 2.2.3 (corresponding to  $p = n$ ) may be generalized to the case of the elliptic algebras  $Q_{n,k}(\mathcal{E}, \eta)$  ([27]). The structure of the Weyl-like algebra similar to  $B_{n;n}(\eta)$  turns out to be more complicated for  $k > 1$ . We will use this generalization in the special case  $k = n - 1$ , i.e., for the commutative algebra  $Q_{n,n-1}(\mathcal{E}, \eta)$ .

Let us describe the generalization of  $\phi_n$ . Assume that  $1 \leq k \leq n - 1$  and expand  $n/k$  in continued fractions as follows

$$\frac{n}{k} = p_1 - \frac{1}{p_2 - \dots - \frac{1}{p_q}}$$

Let  $B_{n,p_1,p_2,\dots,p_q}(\eta)$  be the associative algebra with generators

$$\begin{aligned} & z_{1,1}, \dots, z_{p_1,1}; e_{1,1}, \dots, e_{p_1,1}; \\ & z_{1,2}, \dots, z_{p_2,2}; e_{1,2}, \dots, e_{p_2,2}; \vdots \\ & z_{1,q}, \dots, z_{p_q,q}; e_{1,q}, \dots, e_{p_q,q}; \end{aligned}$$

and

$$\begin{aligned} & t_{1,2}, t_{2,3} \dots, t_{q-1,q}; \\ & f_{1,2}, f_{2,3} \dots, f_{q-1,q}. \end{aligned}$$

We will impose the following commutation relations between them:

$$\begin{aligned} e_{\alpha,\gamma} z_{\beta,\gamma} &= (z_{\beta,\gamma} - n\eta) e_{\alpha,\gamma}; \quad \alpha \neq \beta \\ f_{\alpha,\alpha+1} t_{\alpha,\alpha+1} &= (t_{\alpha,\alpha+1} - n\eta) f_{\alpha,\alpha+1}. \end{aligned}$$

The other pairs of generators commute.

There is a “bosonization” homomorphism  $\phi_{p_1,\dots,p_q} : Q_{n,k}(\mathcal{E}, \eta) \rightarrow B_{n,p_1,p_2,\dots,p_q}(\eta)$ .

A generating series for elements of the image of this morphism is “transfer-function”

$$\tilde{T}(u) =$$

$$\sum_{\substack{1 \leq \alpha_1 \leq p_1 \\ \dots \\ 1 \leq \alpha_q \leq p_q}} \frac{\theta(u - z_{\alpha_1,1})\theta(u + z_{\alpha_1,1} - z_{\alpha_2,2})\theta(u + z_{\alpha_2,2} - z_{\alpha_3,3}) \dots \theta(u + z_{\alpha_q,q})}{\prod_{\substack{\beta \neq \alpha_\gamma \\ 1 \leq \beta \leq p_\gamma \\ 1 \leq \gamma \leq q}} \theta(z_{\alpha_\gamma,\gamma} - z_{\beta,\gamma})} \quad (14)$$

$$\theta(z_{\alpha_1,1} + z_{\alpha_2,2} - t_{1,2})\theta(z_{\alpha_2,2} + z_{\alpha_3,3} - t_{2,3}) \dots \theta(z_{\alpha_{q-1,q-1}} + z_{\alpha_q,q} - t_{q-1,q})$$

$$e_{\alpha_1,1} \dots e_{\alpha_q,q} f_{1,2} \dots f_{q-1,q}.$$

(All these statements are contained in [28].)

The expansion of  $\frac{n}{n-1}$  in continued fraction is

$$\frac{n}{n-1} = 2 - \frac{1}{2 - \dots - \frac{1}{2}},$$

so  $q = n - 1$  and  $p_1 = \dots = p_q = 2$ .

**Proposition 3.4.** *The relation*

$$[\tilde{T}(u), \tilde{T}(v)] = 0$$

holds in  $B_{n,2,\dots,2}(\eta)$ .

*Proof.* This follows from the commutativity of  $Q_{n,n-1}(\mathcal{E})$ . □

#### 4. SOME EXAMPLES OF ELLIPTIC INTEGRABLE SYSTEMS

**4.1. Low-dimensional example: the algebra  $q_2(\mathcal{E}, \eta)$ .** The elliptic integrable system arising in 3.1 becomes transparent in the case  $m = 1$ . Then the commutative elliptic algebra  $Q_2(\mathcal{E}, \eta)$  has functional dimension 2 and its Poisson counterpart admits the Poisson morphism

$$\psi_2 : q_2(\mathcal{E}) \rightarrow b_2,$$

where the algebra  $b_2$  consists of the elements

$$\sum_{\alpha,\beta} f_{\alpha,\beta}(z_1, z_2) e_1^\alpha e_2^\beta,$$

where  $f_{\alpha,\beta}(z_1, z_2)$  are meromorphic functions and the Poisson structure in  $b_2$  is given by

$$\{z_i, z_j\} = \{e_i, e_j\} = \{e_i, z_i\} = 0, \quad \{e_i, z_j\} = -2e_i (i \neq j), \quad i, j = 1, 2. \quad (15)$$

The explicit formula for this mapping (for  $f \in \Theta_2(\Gamma)$  a given theta-function of order 2) is the following:

$$\psi_2(f) = \frac{f(z_1)}{\theta(z_1 - z_2)} e_1 + \frac{f(z_2)}{\theta(z_2 - z_1)} e_2. \quad (16)$$

Now, let  $\theta_1, \theta_2$  be the basic theta-functions of order 2, and let us compute the Poisson brackets between their images  $\psi_2(\theta_1)$  and  $\psi_2(\theta_2)$ :

**Proposition 4.1.** *These theta-functions commute in  $b_2$*

$$\{\psi_2(\theta_1), \psi_2(\theta_2)\} = \psi_2(\{\theta_1, \theta_2\}) = 0.$$

**4.2. SOS eight-vertex model of Date-Miwa-Jimbo-Okado.** We recall some ingredients of the SOS eight-vertex model (see [23]) and the construction of its transfer-operators. We then establish their relation with our operators  $T(u)$ .

The 8-vertex model is an IRF (interaction round a face) statistical mechanical model, more precisely, it is a version of the Baxter model related to Felder's elliptic quantum group  $E_{\tau,\eta}(sl_2)$ . This model was studied by Sklyanin's method of separation of variables (under antiperiodic boundary conditions) in [25] (see also [24] for the representation theory of  $E_{\tau,\eta}(sl_2)$ ). We will use the results of [25] in a form which we need.

The antiperiodic boundary conditions of the model are fixed by the family of transfer-matrix  $T_{\text{SOS}}(u, \lambda)$  where  $u \in \mathbb{C}$  is a parameter and the family  $T_{\text{SOS}}(u, \lambda)$  is expressed as (twisted) traces of the  $L$ -operators  $L_{\text{SOS}}(u, \lambda)$  defined using an "auxiliary" module over the elliptic quantum group  $E_{\tau,\eta}(sl_2)$ . The  $L$ -operator is built out in the tensor product of the fundamental representations of the elliptic quantum group and is twisted by the matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

The  $L$ -operator is usually represented as a  $2 \times 2$ -matrix of the form

$$L_{\text{SOS}}(u, \lambda) = \begin{pmatrix} a(u, \lambda) & b(u, \lambda) \\ c(u, \lambda) & d(u, \lambda) \end{pmatrix},$$

with matrix entries meromorphic in  $u$  and  $\lambda$  and obeys the dynamical  $RLL$ -commutation relations for the Felder elliptic  $R$ -matrix

$$R(u, \lambda) = \begin{pmatrix} \theta(u + 2\eta) & 0 & 0 & 0 \\ 0 & \frac{\theta(u)\theta(\lambda+2\eta)}{\theta(\lambda)} & \frac{\theta(u-\lambda)\theta(2\eta)}{\theta(\lambda)} & 0 \\ 0 & \frac{\theta(\lambda+u)\theta(2\eta)}{\theta(\lambda)} & \frac{\theta(u)\theta(\lambda-2\eta)}{\theta(\lambda)} & 0 \\ 0 & 0 & 0 & \theta(u + 2\eta) \end{pmatrix}.$$

We will have to deal with the functional representations of  $E_{\tau,\eta}(sl_2)$  which are described by the pairs  $(F, L)$ . Here  $F$  is a complex vector space of meromorphic functions  $f(z_1, \dots, z_n, \lambda)$  (or a subspace of functions which are holomorphic in a part of the variables) and  $L$  is the  $L$ -operator as above. The entries of the  $L$ -operator are acting by difference operators in the tensor product  $V \otimes W$  where  $\dim(V) = 2$  and  $W$  is an appropriate subspace in the functional space  $F$ .

The Bethe ansatz method works in the case of the SOS model with periodic boundary conditions, according to Felder and Varchenko ([24]). In the antiperiodic case the family of transfer matrices

$$T_{\text{SOS}}(u, \lambda) = \text{tr}(KL_{\text{SOS}}(u, \lambda)), \quad u \in \mathbb{C}$$

is commutative:  $[T_{\text{SOS}}(u, \lambda), T_{\text{SOS}}(v, \lambda)] = 0$ , as it follows from the  $RLL$ -relations by tedious computations in [26].

On the other hand, it is possible to establish the explicit one-to-one correspondence between the families of antiperiodic SOS transfer-matrices and the auxiliary transfer-matrices  $T_{\text{aux}}(u, \lambda)$  (see 4.4.3 in [26]). This isomorphism is established by a version of the separation of variables.



The explicit expression of the auxiliary transfer-matrix is

$$T_{\text{aux}}(u, \lambda) = \sum_{\alpha=1}^n \frac{\theta(u + z_\alpha - \lambda)}{\theta(\lambda)} \prod_{1 \leq \beta \neq \alpha \leq n} \frac{\theta(u + z_\beta)}{\theta(z_\beta - z_\alpha)} (\theta(z_\alpha + \eta) T_{z_\alpha}^{-2\eta} + \theta(z_\alpha - \eta) T_{z_\alpha}^{2\eta}),$$

where (to compare with our “transfer”-operators in the elliptic integrable systems) we have put in the formulas of ([26], ch.4.6)

$$x_\alpha = 0, \Lambda_\alpha = 1, \alpha = 1, \dots, n.$$

(The choice  $\Lambda_\alpha = 1$  corresponds to the case of separated variables (Proposition 4.36 in [26]).)

The operators  $T_{z_\alpha}^{\pm 2\eta}$  are acting similarly to the generators  $f_\alpha$  above:

$$T_{z_\alpha}^{\pm 2\eta} f(z_1, \dots, z_n) = f(z_1, \dots, z_\alpha \pm 2\eta, \dots, z_n) T_{z_\alpha}^{\pm 2\eta}.$$

Let us make the following change of the variables

$$f_\alpha^\pm = \theta(z_\alpha \mp \eta) T_{z_\alpha}^{\pm 2\eta}.$$

Now a simple inspection of the formulas shows that

$$T_{\text{aux}}(u, \lambda) = T_+(u, \lambda) + T_-(u, \lambda),$$

and the commutation results of Section 3.2 can be applied. So we obtain

**Proposition 4.2.** *The “transfer”-operator  $T(u)$  coincides (up to an inessential numerical factor, depending on the normalization in the definition of theta-functions and rescaling of the parameter  $\eta$ ) with the combination of the twisted traces of the auxiliary  $L$ -operator of the antiperiodic SOS-model under the following change of variables:*

$$\lambda = \sum_{\alpha=1}^n z_\alpha, \quad x_\alpha = 0, \quad f_\alpha^\pm = \theta(z_\alpha \mp \eta) T_{z_\alpha}^{\pm 2\eta}.$$

Proposition 3.2 gives a simple proof of the commutation relations for the twisted traces of the SOS-model in this special case. We should observe that the commutation relations

$$[T_+(u), T_-(v)] + [T_-(u), T_+(v)] = 0$$

follow from the same arguments as in Section 3.2.

On the other hand, this gives additional evidence to our belief that the “integrability” condition from [12] is equivalent in some sense to a *RLL*-type integrability condition.

We also believe that the role of the elliptic integrable systems in IRF models can be generalized and we hope to return to this point in future.

**4.3. Elliptic analogs of the Beauville-Mukai systems and Fay identity.** Let us describe a geometric meaning of our IS’s. We will start with the observation that the classical analogue of the Weyl-like algebra  $\mathcal{V}_n$  given by the Poisson brackets

$$0 = \{f_i, f_j\} = \{z_i, z_j\} = \{f_i, z_j\} \quad (i \neq j), \quad \{f_i, z_i\} = -nf_i$$

can be identified with the Poisson algebra of functions on the symmetric power  $S^n(\text{Cone}(\mathcal{E}))$  of a cone over an elliptic curve (more precisely, on the Hilbert scheme  $(\text{Cone}(\mathcal{E}))^{[n]}$  of  $n$  points on this surface.)

The  $n$  commuting elements  $h_i = D_0^{-1}D_i$  in the Poisson algebra obtained in Section 3.2 can be interpreted as an elliptic version of the Beauville-Mukai systems associated with this Poisson surface.

We will develop in more details the first interesting case ( $n = 3$ ) of the Poisson commuting conditions (the Plücker relations from Section 2) for this system.

The Beauville-Mukai hamiltonians have the following form in this case:

$$H_1 = \frac{\det[e, \theta_1, \theta_2]}{\det[\theta_0, \theta_1, \theta_2]}, H_2 = \frac{\det[e, \theta_0, \theta_2]}{\det[\theta_0, \theta_1, \theta_2]}, H_3 = \frac{\det[e, \theta_0, \theta_1]}{\det[\theta_0, \theta_1, \theta_2]},$$

where the vectors-columns  $e, \theta_i$ ,  $i = 0, 1, 2$  have the following entries

$$e = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \theta_i = \begin{pmatrix} \theta_i(z_1) \\ \theta_i(z_2) \\ \theta_i(z_3) \end{pmatrix}, i = 0, 1, 2$$

Now the integrability condition (6) may be expressed as a kind of 4-Riemann identity (known also as the trisecant Fay's identity):

$$\begin{aligned} \tilde{\theta}(v)\tilde{\theta}(u + \int_B^A \omega + \int_D^C \omega) &= \tilde{\theta}(u + \int_D^A \omega)\tilde{\theta}(u + \int_B^C \omega) \frac{E(A, B)E(D, C)}{E(A, C)E(D, B)} + \\ &\tilde{\theta}(u + \int_B^A \omega)\tilde{\theta}(u + \int_D^C \omega) \frac{E(A, D)E(C, B)}{E(A, C)E(D, B)}, \end{aligned} \quad (17)$$

where  $\omega$  is a chosen holomorphic differential on  $\mathcal{E}$ , and for  $X, Y \in \mathcal{E}$ ,  $\int_X^Y \omega \in \text{Jac}(\mathcal{E}) = \mathcal{E}$ ,  $E(X, Y) = -E(Y, X)$  is a prime form:

$$\frac{E(A, B)E(D, C)}{E(A, C)E(D, B)} = \frac{\tilde{\theta}(\int_A^B) \tilde{\theta}(\int_D^C)}{\tilde{\theta}(\int_A^C) \tilde{\theta}(\int_D^B)}$$

and  $\tilde{\theta}(u)$  is an odd theta-function (see [46].)

Let  $a, b, c, d$  correspond to the points  $A, B, C, D$  in (17) and set  $u = a + c$ .

Then (17) reads

$$\begin{aligned} \tilde{\theta}(a + c)\tilde{\theta}(a - c)\tilde{\theta}(b + d)\tilde{\theta}(b - d) - \tilde{\theta}(a + b)\tilde{\theta}(a - b)\tilde{\theta}(c + d)\tilde{\theta}(c - d) + \\ \tilde{\theta}(a + d)\tilde{\theta}(a - d)\tilde{\theta}(c + b)\tilde{\theta}(c - b) = 0, \end{aligned} \quad (18)$$

where we easily recognize the commutativity conditions (6) for the case  $n = 3$  and an appropriate choice of a linear relation between 4 points  $a, b, c, d$  and  $z_1, z_2, z_3, \eta/3$  (modulo an irrelevant exponential factor entering in the relations between the theta-functions in different normalizations).

*Remark 6.* 1. The appearance of the Riemann-Fay relations as commutativity or integrability conditions in this context looks quite natural both from the ‘‘Poisson’’ as well as from the ‘‘integrable’’ viewpoints. The ‘‘Poisson-Plücker’’ relations and their generalizations in the context of the Poisson polynomial structures were studied in [44]. On the other hand, the first links between integrability conditions (in the form of Hirota bilinear

identities for some elliptic difference many-body-like systems) and the Fay formulas were established in [17].

2. The Fay trisecant formulas on an elliptic curve are also related to a version of “triangle” relations known as the “associative” Yang-Baxter equation obtained by Polishchuk ([42]). This result gives additional evidence that the commutation relations (2), (3) could be interpreted as an algebraic kind of Yang-Baxter equation. An amusing appearance of the NC determinants in both constructions (the Cartier-Foata determinants defined above are a particular case of the quasi-determinants of Gelfand-Retakh [7]) shows that the ideas of [12] might be useful in “non-commutative integrability” constructions which involve quasi-determinants, quasi-Plücker relations etc.

## 5. DISCUSSION AND FUTURE PROBLEMS

We have proved that the elliptic algebras (under some mild restrictions) carry families of commuting elements which become in some cases examples of Integrable Systems.

Let us indicate some questions deserving future investigations.

We plan to construct an analogue of these commuting families in the algebras  $Q_{2m}(\mathcal{E}, \eta)$  which correspond to the maximal symplectic leaves of the elliptic algebras  $Q_{2m+1}(\mathcal{E}, \eta)$  using the bi-hamiltonian elliptic families. This analogue should quantize the bi-hamiltonian families in  $\mathbb{C}^n$  in the case of an odd  $n$ .

One of the main motivations for explicitly constructing the systems on  $\mathcal{V}_n$  in terms of the determinants of theta-functions of order  $n$  was the desire to find a confirmation to the hypothetical “separated” form of the hamiltonians of the  $n$ -point double-elliptic system, proposed in the paper [4].

The relevance of our construction to this circle of problems gained additional evidence from the paper [35]. Recent discussions around integrability in Dijkgraaf-Vafa and Seiberg-Witten theories provide us with “physical” insights supporting the relation between the Beauville-Mukai and double elliptic IS (see [6, 5]).

The future paper (in collaboration with A. Gorsky) which should clarify the place of the IS associated with the elliptic algebras and the integrability phenomena in SUSY gauge theories is in progress.

There are some open questions about the relations of these IS with the Beauville-Mukai Lagrangian fibrations on the Hilbert scheme  $(\mathbb{P}^2 \setminus \mathcal{E})^{[n]}$  as well as with their non-commutative (NC) counterparts proposed in [43]. It is likely that the quantization scheme for fraction fields proposed in [9] can be applied to the NC surface  $(\mathbb{P}_S \setminus \mathcal{E})$  to “extend” the deformation to the NC Hilbert scheme  $(\mathbb{P}_S \setminus \mathcal{E})^{[n]}$  introduced in [43]. Proposition 3.1 from [12] then should in principle give a NC integrable Beauville-Mukai system on  $(\mathbb{P}_S \setminus \mathcal{E})^{[n]}$  as a NC IS associated with the Cherednik algebras of [10].

Finally, we should mention an interesting question of generalization of the Beauville-Mukai IS which was introduced by Gelfand-Zakharevich on the Hilbert scheme  $(X_9)^{[n]}$  of the Del Pezzo surface  $X_9$  obtained by the blow-up of 9 points on  $\mathbb{P}^2$ . These 9 points are the intersection points of two cubic plane curves. Let  $\pi$  be a Poisson tensor on  $X_9$ , then it can be extended to the whole  $\mathbb{P}^2$  so that the extension  $\tilde{\pi}$  has 9 zeroes at these 9 points. The polynomial degree of the tensor  $\tilde{\pi}$  is 3 and it follows that this polynomial is a

linear combination of the polynomials corresponding to the initial elliptic curves. This bi-hamiltonian structure corresponds to the case  $n = 3$  studied in the paper [31]. The open question arises: what is the algebro-geometric construction behind the bi-hamiltonian elliptic Poisson algebra of [31] in the case of arbitrary  $n$ ?

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