Generalizations of Cauchy's Determinant Identity and Schur's Pfaffian Identity

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Abstract

We review several determinant and Pfaffian identities, which generalize the evaluation formulae of Cauchy's determinant $\det(1/(x_i + y_j))$ and Schur's Pfaffian $\mathrm{Pf}((x_j - x_i)/(x_j + x_i))$. As a multi-variable generalization, we consider Cauchy-type determinants and Schur-type Pfaffians of matrices with entries involving some generalized Vandermonde determinants. Also we give an elliptic generalization of Schur's Pfaffian identity, which is a Pfaffian counterpart of Frobenius' identity.

1 Introduction

Identities for determinants and Pfaffians are of great interest in many branches of mathematics and mathematical physics. Some people need relations among minors or subPfaffians of a general matrix (e.g., Plücker relations), others have to evaluate special determinants or Pfaffians (e.g, Vandermonde determinant). In combinatorics and representation theory, an important role is played by Cauchy's determinant identity [3]

$$\det\left(\frac{1}{x_i + y_j}\right)_{1\le i,j\le n} = \frac{\prod_{1\le i<j\le n}(x_j - x_i)(y_j - y_i)}{\prod_{i,j=1}^{n}(x_i + y_j)}, \tag{1.1}$$

or its equivalent form

$$\det\left(\frac{1}{1 - x_i y_j}\right)_{1\le i,j\le n} = \frac{\prod_{1\le i<j\le n}(x_j - x_i)(y_j - y_i)}{\prod_{i,j=1}^{n}(1 - x_i y_j)}. \tag{1.2}$$

And Schur's Pfaffian identity [22]

$$\mathrm{Pf}\left(\frac{x_j - x_i}{x_j + x_i}\right)_{1\le i,j\le 2n} = \prod_{1\le i<j\le 2n}\frac{x_j - x_i}{x_j + x_i}, \tag{1.3}$$

and its variation (given in [14] and [24])

$$\mathrm{Pf}\left(\frac{x_j - x_i}{1 - x_j x_i}\right)_{1\le i,j\le 2n} = \prod_{1\le i<j\le 2n}\frac{x_j - x_i}{1 - x_i x_j} \tag{1.4}$$

are of similar importance when we are working with Pfaffians. Here $\mathrm{Pf}\,X$ denotes the Pfaffian of a skew-symmetric matrix X. Since 19th century, many generalizations or variations of these determinant and Pfaffian identities have appeared in the literature. Besides, C. Krattenthaler [12], [13] has given a comprehensive survey of determinant evaluations.

In this article, we are interested in the following three types of generalizations or variations. The first type is a multi-variable generalization. The above identities contain only one (or two) set of variables, so it is natural to generalize these identities so that more sets of variables are involved. In this direction, the author [17], [18] and T. Sundquist [25] gave evaluation formulae for two-variable (or three-variable) Cauchy's determinants and Schur's Pfaffians such as

$$\det\left(\frac{b_j - a_i}{y_j - x_i}\right)_{1\le i,j\le n}, \quad \mathrm{Pf}\left(\frac{(a_j - a_i)(b_j - b_i)}{x_j - x_i}\right)_{1\le i,j\le 2n}, \quad \mathrm{Pf}\left(\frac{a_j - a_i}{1 + x_i x_j}\right)_{1\le i,j\le 2n}.$$

And their identities have remarkable applications in theory of symmetric functions, representation theory of classical groups, and enumerative combinatorics. One of our main results (Theorem 2.1) of this article provides a multi-variable generalization of their identities.

The second type is an elliptic generalization. G. Frobenius [4] gave the following determinant identity:

$$\det\left(\frac{\sigma(z + x_i + y_j)}{\sigma(z)\sigma(x_i + y_j)}\right)_{1\le i,j\le n} = \frac{\prod_{1\le i<j\le n} \sigma(x_j - x_i)\sigma(y_j - y_i)}{\prod_{i,j=1}^{n} \sigma(x_i + y_j)} \cdot \frac{\sigma(z + \sum_{i=1}^{n} x_i + \sum_{j=1}^{n} y_j)}{\sigma(z)}, \tag{1.5}$$

where $\sigma(x)$ is the Weierstrass sigma function. Note that, if we take the limit $z \to \infty$ in the rational case of this identity, we obtain Cauchy's determinant identity (1.1). Hence (1.5) can be regarded as an elliptic generalization of (1.1). Recently, in the study of elliptic hypergeometric series, Frobenius' identity plays a key role in proving their transformations (see [10], [21]), and several elliptic determinant and Pfaffian identities are discovered. In this article, we give an elliptic generalization of Schur's Pfaffian identity (Theorem 4.1). We expect that our elliptic generalization provides another tool in the theory of special functions.

The third type is a little different from the above two types. C. W. Borchardt [1] gave the following variation of Cauchy's determinant identities:

$$\det\left(\frac{1}{(x_i + y_j)^2}\right)_{1\le i,j\le n} = \frac{\prod_{1\le i<j\le n}(x_j - x_i)(y_j - y_i)}{\prod_{i,j=1}^{n}(x_i + y_j)} \cdot \mathrm{perm}\left(\frac{1}{x_i + y_j}\right)_{1\le i,j\le n}, \tag{1.6}$$

$$\det\left(\frac{1}{(1 - x_i y_j)^2}\right)_{1\le i,j\le n} = \frac{\prod_{1\le i<j\le n}(x_j - x_i)(y_j - y_i)}{\prod_{i,j=1}^{n}(1 - x_i y_j)} \cdot \mathrm{perm}\left(\frac{1}{1 - x_i y_j}\right)_{1\le i,j\le n}, \tag{1.7}$$

where $\mathrm{perm}\, X$ is the permanent of a square matrix X. We present a Borchardt-type variation of Schur's Pfaffian identities in Theorem 5.1. These Borchardt-type variations are used in [18] to evaluate the determinants and Pfaffians appearing in the 0-enumeration of symmetry classes of alternating sign matrices.

This paper is organized as follows. In Section 2, we review multi-variable generalizations of Cauchy's determinant identity and Schur's Pfaffian identity, and Section 3 is devoted to a survey of their specializations and applications. An elliptic generalization and a Borchardt-type variation of Schur's Pfaffian identity are given in Section 4 and 5 respectively.

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Notations: In this article, the bold-faced letters $\boldsymbol{x}$ etc. represent vectors of variables. We denote the all-one vector $(1, 1, \cdots, 1)$ by $\mathbf{1}$. For two vectors $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n)$, we use the following notations:

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \ldots, x_n + y_n), \quad \boldsymbol{x}\boldsymbol{y} = (x_1 y_1, \ldots, x_n y_n), \quad \boldsymbol{x}^k = (x_1^k, \ldots, x_n^k).$$

And we write $a\boldsymbol{x}$ for $(ax_1, \cdots, ax_n)$.

2 Multi-variable generalization

In this section, we present several identities of Cauchy-type determinants and Schur-type Pfaffians involving generalized Vandermonde determinants. Let $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{a} = (a_1, \cdots, a_n)$ be two vectors of variables of length n. For nonnegative integers p and q with $p + q = n$, we define a generalized Vandermonde matrix $V^{p,q}(\boldsymbol{x}; \boldsymbol{a})$ to be the $n \times n$ matrix with ith row

$$(1, x_i, \cdots, x_i^{p-1}, a_i, a_i x_i, \cdots, a_i x_i^{q-1}).$$

If $q = 0$, then $V^{n,0}(\boldsymbol{x}; \boldsymbol{a}) = \left(x_i^{j-1}\right)_{1 \le i,j \le n}$ and its determinant $\det V^{n,0}(\boldsymbol{x}; \boldsymbol{a}) = \prod_{1 \le i < j \le n}(x_j - x_i)$ is the usual Vandermonde determinant. We introduce another generalized Vandermonde matrix $W^n(\boldsymbol{x}; \boldsymbol{a})$ as the $n \times n$ matrix with ith row

$$(1 + a_i x_i^{n-1}, x_i + a_i x_i^{n-2}, \cdots, x_i^{n-1} + a_i).$$

If $\boldsymbol{a} = -\boldsymbol{x}^n$, $-\boldsymbol{x}^{n+1}$, or $\boldsymbol{x}^{n-1}$, then the determinants $\det W^n(\boldsymbol{x}; \boldsymbol{a})$ are factorized as follows:

$$\det W^n(\boldsymbol{x}; -\boldsymbol{x}^n) = \prod_{i=1}^{n}(1 - x_i) \prod_{1 \le i < j \le n}(x_j - x_i)(1 - x_i x_j),$$

$$\det W^n(\boldsymbol{x}; -\boldsymbol{x}^{n+1}) = \prod_{i=1}^{n}(1 - x_i^2) \prod_{1 \le i < j \le n}(x_j - x_i)(1 - x_i x_j),$$

$$\det W^n(\boldsymbol{x}; \boldsymbol{x}^{n-1}) = 2 \prod_{1 \le i < j \le n}(x_j - x_i)(1 - x_i x_j),$$

These are the Weyl's denominator formulae for type B_n, C_n and D_n.

One of our main results is the following theorem, which was conjectured by the author [19] and then proven in [7] with full generality.

Theorem 2.1. ([7, Theorem 1.1])

(a) Let n be a positive integer and let p and q be nonnegative integers. For six vectors of variables

$$\boldsymbol{x} = (x_1, \cdots, x_n),\ \boldsymbol{y} = (y_1, \cdots, y_n),\ \boldsymbol{a} = (a_1, \cdots, a_n),\ \boldsymbol{b} = (b_1, \cdots, b_n),$$
$$\boldsymbol{z} = (z_1, \cdots, z_{p+q}),\ \boldsymbol{c} = (c_1, \cdots, c_{p+q}),$$

we have

$$\det \left(\frac{\det V^{p+1,q+1}(x_i, y_j, \boldsymbol{z}; a_i, b_j, \boldsymbol{c})}{y_j - x_i} \right)_{1 \le i,j \le n}$$
$$= \frac{(-1)^{n(n-1)/2}}{\prod_{i,j=1}^{n}(y_j - x_i)} \det V^{p,q}(\boldsymbol{z}; \boldsymbol{c})^{n-1} \det V^{n+p,n+q}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}). \quad (2.1)$$

(b) Let n be a positive integer and let p, q, r, s be nonnegative integers. For seven vectors of variables

$$\boldsymbol{x} = (x_1, \cdots, x_{2n}),\ \boldsymbol{a} = (a_1, \cdots, a_{2n}),\ \boldsymbol{b} = (b_1, \cdots, b_{2n}),$$
$$\boldsymbol{z} = (z_1, \cdots, z_{p+q}),\ \boldsymbol{c} = (c_1, \cdots, c_{p+q}),$$
$$\boldsymbol{w} = (w_1, \cdots, w_{r+s}),\ \boldsymbol{d} = (d_1, \cdots, d_{r+s}),$$

we have

$$\begin{aligned}\operatorname{Pf}&\left(\frac{\det V^{p+1,q+1}(x_i, x_j, \boldsymbol{z}; a_i, a_j, \boldsymbol{c}) \det V^{r+1,s+1}(x_i, x_j, \boldsymbol{w}; b_i, b_j, \boldsymbol{d})}{x_j - x_i}\right)_{1 \le i,j \le 2n} \\ &= \frac{1}{\prod_{1 \le i < j \le 2n}(x_j - x_i)} \det V^{p,q}(\boldsymbol{z}; \boldsymbol{c})^{n-1} \det V^{r,s}(\boldsymbol{w}; \boldsymbol{d})^{n-1} \\ &\qquad \times \det V^{n+p,n+q}(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{a}, \boldsymbol{c}) \det V^{n+r,n+s}(\boldsymbol{x}, \boldsymbol{w}; \boldsymbol{b}, \boldsymbol{d}).\end{aligned} \tag{2.2}$$

(c) Let n be a positive integer and let p be a nonnegative integer. For six vectors of variables

$$\boldsymbol{x} = (x_1, \cdots, x_n),\ \boldsymbol{y} = (y_1, \cdots, y_n),\ \boldsymbol{a} = (a_1, \cdots, a_n),\ \boldsymbol{b} = (b_1, \cdots, b_n),$$
$$\boldsymbol{z} = (z_1, \cdots, z_p),\ \boldsymbol{c} = (c_1, \cdots, c_p),$$

we have

$$\begin{aligned}\det&\left(\frac{\det W^{p+2}(x_i, y_j, \boldsymbol{z}; a_i, b_j, \boldsymbol{c})}{(y_j - x_i)(1 - x_i y_j)}\right)_{1 \le i,j \le n} \\ &= \frac{1}{\prod_{i,j=1}^{n}(y_j - x_i)(1 - x_i y_j)} \det W^{p}(\boldsymbol{z}; \boldsymbol{c})^{n-1} \det W^{2n+p}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}).\end{aligned} \tag{2.3}$$

(d) Let n be a positive integer and let p and q be nonnegative integers. For seven vectors of variables

$$\boldsymbol{x} = (x_1, \cdots, x_{2n}),\ \boldsymbol{a} = (a_1, \cdots, a_{2n}),\ \boldsymbol{b} = (b_1, \cdots, b_{2n}),$$
$$\boldsymbol{z} = (z_1, \cdots, z_p),\ \boldsymbol{c} = (c_1, \cdots, c_p),$$
$$\boldsymbol{w} = (w_1, \cdots, w_q),\ \boldsymbol{d} = (d_1, \cdots, d_q),$$

we have

$$\begin{aligned}\operatorname{Pf}&\left(\frac{\det W^{p+2}(x_i, x_j, \boldsymbol{z}; a_i, a_j, \boldsymbol{c}) \det W^{q+2}(x_i, x_j, \boldsymbol{w}; b_i, b_j, \boldsymbol{d})}{(x_j - x_i)(1 - x_i x_j)}\right)_{1 \le i,j \le 2n} \\ &= \frac{1}{\prod_{1 \le i < j \le 2n}(x_j - x_i)(1 - x_i x_j)} \det W^{p}(\boldsymbol{z}; \boldsymbol{c})^{n-1} \det W^{q}(\boldsymbol{w}; \boldsymbol{d})^{n-1} \\ &\qquad \times \det W^{2n+p}(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{a}, \boldsymbol{c}) \det W^{2n+q}(\boldsymbol{x}, \boldsymbol{w}; \boldsymbol{b}, \boldsymbol{d}).\end{aligned} \tag{2.4}$$

We can unify the identities (2.1) and (2.3) (resp. (2.2) and (2.4)) by introducing a homogeneous version of the matrix $V^{p,q}(\boldsymbol{x}; \boldsymbol{a})$. For vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{a}$, $\boldsymbol{b}$ of length n and nonnegative integers p, q with $p+q=n$, we set $U^{p,q}\begin{pmatrix} \boldsymbol{x} & \boldsymbol{a} \\ \boldsymbol{y} & \boldsymbol{b} \end{pmatrix}$ to be the $n \times n$ matrix with ith row

$$(a_i x_i^{p-1}, a_i x_i^{p-2} y_i, \cdots, a_i y_i^{p-1}, b_i x_i^{q-1}, b_i x_i^{q-2} y_i, \cdots, b_i y_i^{q-1}).$$

Then we have

Theorem 2.2. [7, Theorem 3.2, Corollary 3.3]

(a) Let n be a positive integer and let p and q be fixed nonnegative integers. For vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, $\boldsymbol{w}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ of length n, and vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ of length $p+q$, we have

$$\det\left(\frac{\det U^{p+1,q+1}\left(\begin{array}{c|c} x_i, z_j, \boldsymbol{\xi} & a_i, c_j, \boldsymbol{\alpha} \\ y_i, w_j, \boldsymbol{\eta} & b_i, d_j, \boldsymbol{\beta} \end{array}\right)}{\det\begin{pmatrix} x_i & z_j \\ y_i & w_j \end{pmatrix}}\right)_{1\le i,j\le n}$$
$$= \frac{(-1)^{n(n-1)/2}}{\prod_{i,j=1}^{n}\det\begin{pmatrix} x_i & z_j \\ y_i & w_j \end{pmatrix}} \det U^{p,q}\left(\begin{array}{c|c} \boldsymbol{\xi} & \boldsymbol{\alpha} \\ \boldsymbol{\eta} & \boldsymbol{\beta} \end{array}\right)^{n-1} \det U^{n+p,n+q}\left(\begin{array}{c|c} \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi} & \boldsymbol{a}, \boldsymbol{c}, \boldsymbol{\alpha} \\ \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{\eta} & \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{\beta} \end{array}\right). \quad (2.5)$$

(b) Let n be a positive integer and let p, q, r and s be nonnegative integers. Suppose that the vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ have length $2n$, the vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ have length $p+q$, and the vectors $\boldsymbol{\zeta}$, $\boldsymbol{\omega}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ have length $r+s$. Then we have

$$\mathrm{Pf}\left(\frac{\det U^{p+1,q+1}\left(\begin{array}{c|c} x_i, x_j, \boldsymbol{\xi} & a_i, a_j, \boldsymbol{\alpha} \\ y_i, y_j, \boldsymbol{\eta} & b_i, b_j, \boldsymbol{\beta} \end{array}\right) \det U^{r+1,s+1}\left(\begin{array}{c|c} x_i, x_j, \boldsymbol{\zeta} & c_i, c_j, \boldsymbol{\gamma} \\ y_i, y_j, \boldsymbol{\omega} & d_i, d_j, \boldsymbol{\delta} \end{array}\right)}{\det\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}}\right)_{1\le i<j\le 2n}$$
$$= \frac{1}{\prod_{1\le i<j\le 2n}\det\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}} \det U^{p,q}\left(\begin{array}{c|c} \boldsymbol{\xi} & \boldsymbol{\alpha} \\ \boldsymbol{\eta} & \boldsymbol{\beta} \end{array}\right)^{n-1} \det U^{r,s}\left(\begin{array}{c|c} \boldsymbol{\zeta} & \boldsymbol{\gamma} \\ \boldsymbol{\omega} & \boldsymbol{\delta} \end{array}\right)^{n-1}$$
$$\times \det U^{n+p,n+q}\left(\begin{array}{c|c} \boldsymbol{x}, \boldsymbol{\xi} & \boldsymbol{a}, \boldsymbol{\alpha} \\ \boldsymbol{y}, \boldsymbol{\eta} & \boldsymbol{b}, \boldsymbol{\beta} \end{array}\right) \det U^{n+r,n+s}\left(\begin{array}{c|c} \boldsymbol{x}, \boldsymbol{\zeta} & \boldsymbol{c}, \boldsymbol{\gamma} \\ \boldsymbol{y}, \boldsymbol{\omega} & \boldsymbol{d}, \boldsymbol{\delta} \end{array}\right). \quad (2.6)$$

We give an outline of the proof of Theorems 2.1 and 2.2. (The details can be found in [7].) The proof consists of two parts: the first part is to reduce the proof of the six identities in these Theorems to that of one identity (2.2), and the second part is to prove (2.2) by using the Desnanot–Jacobi formula for Pfaffians and induction.

For the first part, we note the following relations among determinants of $U^{p,q}$, $V^{p,q}$ and W^n.

Lemma 2.3.

$$U^{p,q}\begin{pmatrix}\boldsymbol{x} & \boldsymbol{a}\\ \boldsymbol{y} & \boldsymbol{b}\end{pmatrix} = \prod_{k=1}^{p+q} a_k x_k^{p-1}\cdot V^{p,q}\left(\boldsymbol{x}^{-1}\boldsymbol{y}; \boldsymbol{a}^{-1}\boldsymbol{b}\boldsymbol{x}^{q-p}\right),$$

$$V^{p,q}(\boldsymbol{x};\boldsymbol{a}) = U^{p,q}\begin{pmatrix}\boldsymbol{1} & \boldsymbol{1}\\ \boldsymbol{x} & \boldsymbol{a}\end{pmatrix},$$

$$\det U^{n,n}\begin{pmatrix}\boldsymbol{x} & \boldsymbol{1}+\boldsymbol{a}\boldsymbol{x}\\ \boldsymbol{1}+\boldsymbol{x}^2 & \boldsymbol{x}+\boldsymbol{a}\end{pmatrix} = (-1)^{n(n-1)/2}\det W^{2n}(\boldsymbol{x};\boldsymbol{a}),$$

$$\det U^{n,n+1}\begin{pmatrix}\boldsymbol{x} & \boldsymbol{1}+\boldsymbol{a}\boldsymbol{x}^2\\ \boldsymbol{1}+\boldsymbol{x}^2 & \boldsymbol{1}+\boldsymbol{a}\end{pmatrix} = (-1)^{n(n-1)/2}\det W^{2n+1}(\boldsymbol{x};\boldsymbol{a}).$$

¿From these relations, we see that

(a) (2.1) is equivalent to (2.5),
(c) (2.2) is equivalent to (2.6),
(b) (2.3) follows from (2.5),
(d) (2.4) follows from (2.6).

And we can deduce (2.5) from (2.6), by taking $r = s = 0$ and

$$c_1 = \cdots = c_n = 1, \qquad c_{n+1} = \cdots = c_{2n} = 0,$$
$$d_1 = \cdots = d_n = 0, \qquad d_{n+1} = \cdots = d_{2n} = 1.$$

and using the following general relation between determinants and Pfaffians.

Lemma 2.4. If A is any $m \times (2n-m)$ matrix, then we have

$$\operatorname{Pf}\begin{pmatrix} O & A\\ -{}^tA & O\end{pmatrix} = \begin{cases}(-1)^{n(n-1)/2}\det A & \text{if } m = n,\\ 0 & \text{if } m \neq n.\end{cases}$$

Next we prove the identity (2.2). A tool in this step is an Pfaffian analogue of the Desnanot–Jacobi formula, which is given in [11] (see also [9]).

Lemma 2.5. Given a square matrix A and indices $i_1, \cdots, i_r, j_1, \cdots, j_r$, we denote by $A^{i_1,\cdots,i_r}_{j_1,\cdots,j_r}$ the matrix obtained by removing the rows $i_1, \cdots, i_r$ and the columns $j_1, \cdots, j_r$ of A. If A is a skew-symmetric matrix, then we have

$$\operatorname{Pf} A^{1,2}_{1,2}\cdot\operatorname{Pf} A^{3,4}_{3,4} - \operatorname{Pf} A^{1,3}_{1,3}\cdot\operatorname{Pf} A^{2,4}_{2,4} + \operatorname{Pf} A^{1,4}_{1,4}\cdot\operatorname{Pf} A^{2,3}_{2,3} = \operatorname{Pf} A\cdot\operatorname{Pf} A^{1,2,3,4}_{1,2,3,4}.$$

By applying this Desnanot–Jacobi formula for Pfaffians to the skew-symmetric matrix on the left hand side of (2.2) and using the induction on n, we can see that the desired equality is equivalent to the case $n = 2$ with $\boldsymbol{z}$, $\boldsymbol{c}$, $\boldsymbol{w}$, $\boldsymbol{d}$ replaced by

$$\boldsymbol{z} \leftarrow (\boldsymbol{x}^{(1,2,3,4)}, \boldsymbol{z}), \quad \boldsymbol{c} \leftarrow (\boldsymbol{a}^{(1,2,3,4)}, \boldsymbol{c}), \quad \boldsymbol{w} \leftarrow (\boldsymbol{x}^{(1,2,3,4)}, \boldsymbol{w}), \quad \boldsymbol{d} \leftarrow (\boldsymbol{b}^{(1,2,3,4)}, \boldsymbol{d}),$$

respectively, where $\boldsymbol{x}^{(1,2,3,4)}$ denotes the vector obtained by removing x_1, x_2, x_3, x_4 from $\boldsymbol{x}$. Then the identity (2.2) in the case $n = 2$ can be proven by the induction on $p+q+r+s$ with the help of the following relations between $\det V^{p,q}$ and $\det V^{p-1,q}$ (or $\det V^{q,p}$).

Lemma 2.6. (1) If $p \geq q$ and $p \geq 1$, then we have

$$\det V^{p,q}(\boldsymbol{x};\boldsymbol{a}) = \prod_{i=1}^{p+q-1} (x_{p+q} - x_i) \cdot \det V^{p-1,q}(x_1, \cdots, x_{p+q-1}; a'_1, \cdots, a'_{p+q-1}),$$

where we put

$$a'_i = \frac{a_i - a_{p+q}}{x_i - x_{p+q}} \quad (1 \leq i \leq p+q-1).$$

(2) For nonnegative integers p and q, we have

$$\det V^{p,q}(\boldsymbol{x};\boldsymbol{a}) = (-1)^{pq} \prod_{i=1}^{p+q} a_i \cdot \det V^{q,p}(\boldsymbol{x};\boldsymbol{a}^{-1}),$$

where $\boldsymbol{a}^{-1} = (a_1^{-1}, \cdots, a_{p+q}^{-1})$.

3 Specializations and applications

In this section, we give several specializations of the identities in Theorem 2.1 and 2.2, and review their applications.

3.1 Simplest case

First we consider the special case of (2.1) (resp. (2.2)) where $p = q = 0$ (resp. $p = q = r = s = 0$). Then the identities read

$$\det \left(\frac{b_j - a_i}{y_j - x_i} \right)_{1 \leq i,j \leq n} = \frac{(-1)^{n(n-1)/2}}{\prod_{i,j=1}^{n} (y_j - x_i)} \det V^{n,n}(\boldsymbol{x},\boldsymbol{y};\boldsymbol{a},\boldsymbol{b}), \tag{3.1}$$

$$\mathrm{Pf} \left(\frac{(a_j - a_i)(b_j - b_i)}{x_j - x_i} \right)_{1 \leq i,j \leq 2n} = \frac{1}{\prod_{1 \leq i < j \leq 2n} (x_j - x_i)} \det V^{n,n}(\boldsymbol{x};\boldsymbol{a}) \det V^{n,n}(\boldsymbol{x};\boldsymbol{b}). \tag{3.2}$$

These identities were first given by the author [17, Theorems 4.2, 4.7].

Substituting

$$x_i \leftarrow x_i^2, \quad y_i \leftarrow y_i^2, \quad a_i \leftarrow x_i, \quad b_i \leftarrow y_i \quad \text{in (3.1)},$$
$$x_i \leftarrow x_i^2, \quad a_i \leftarrow x_i, \quad b_i \leftarrow x_i \quad \text{in (3.2)},$$

and using

$$\det V^{n,n}(\boldsymbol{x}^2 : \boldsymbol{x}) = (-1)^{n(n-1)/2} \prod_{1 \leq i < j \leq 2n} (x_j - x_i),$$

we can recover Cauchy's determinant identity (1.1) from (3.1) and Schur's Pfaffian identity (1.3) from (3.2).

If we take another substitution

$$x_i \leftarrow x_i^3, \quad y_i \leftarrow y_i^3, \quad a_i \leftarrow x_i, \quad b_i \leftarrow y_i$$

in (3.1), we have

$$\det\left(\frac{1}{x_i^2+x_iy_j+y_i^2}\right)_{1\le i,j\le n}=\frac{\prod_{1\le i<j\le n}(x_j-x_i)(y_j-y_i)}{\prod_{i,j=1}^{n}(x_i^2+x_iy_j+y_j^2)}\cdot s_{\delta(n-1,n-1)}(x_1,\cdots,x_n,y_1,\cdots,y_n),$$

where $\delta(n-1,n-1)=\delta(n-1)\cup\delta(n-1)=(n-1,n-1,n-2,n-2,\cdots,2,2,1,1,0,0)$. This identity perfectly fits the Izergin–Korepin determinant formula for the partition function of the square ice model with domain wall boundary condition, and we can obtain the enumeration of alternating sign matrices. Recall that an *alternating sign matrix* (or ASM for short) is a square matrix satisfying the following three conditions :

(a) All entries are 1, -1 or 0.
(b) Every row and column have sum 1.
(c) In every row and column, the nonzero entries alternate in sign.

Proposition 3.1. The number of $n\times n$ alternating sign matrices is given by

$$\frac{1}{3^{n(n-1)/2}}s_{\delta(n-1,n-1)}(\mathbf{1}_{2n}),$$

where $\mathbf{1}_{2n}$ denotes the all-one vector of length $2n$.

In a similar way, we can use suitable specializations of the identities in Theorem 2.1 to evaluate the determinants and Pfaffians in the partition functions of square ice models associated to several symmetry classes of alternating sign matrices. For example, the special case of the Pfaffian identity (2.3) with $p=1$ enables us to enumerate the vertically and horizontally symmetric alternating sign matrices. See [18]. And the interested reader is referred to [2] for some stories about alternating sign matrices and plane partitions.

3.2 Identities of Sundquist and Ishikawa

Next we consider the special case of (2.5) (resp. (2.6)) where $p=q=0$ (resp. $p=q=r=s=0$). Then the identities are

$$\det\left(\frac{\det\begin{pmatrix}a_i & c_j\\ b_i & d_j\end{pmatrix}}{\det\begin{pmatrix}x_i & z_j\\ y_i & w_j\end{pmatrix}}\right)_{1\le i,j\le n}=\frac{(-1)^{n(n-1)/2}}{\prod_{i,j=1}^{n}\det\begin{pmatrix}x_i & z_j\\ y_i & w_j\end{pmatrix}}\det U^{n,n}\left(\begin{array}{c|c}\boldsymbol{x},\boldsymbol{z} & \boldsymbol{a},\boldsymbol{c}\\ \boldsymbol{y},\boldsymbol{w} & \boldsymbol{b},\boldsymbol{d}\end{array}\right),\tag{3.3}$$

$$\operatorname{Pf}\left(\frac{\det\begin{pmatrix}a_i & a_j\\ b_i & b_j\end{pmatrix}\det\begin{pmatrix}c_i & c_j\\ d_i & d_j\end{pmatrix}}{\det\begin{pmatrix}x_i & x_j\\ y_i & y_j\end{pmatrix}}\right)_{1\le i,j\le 2n}=\frac{1}{\prod_{1\le i<j\le 2n}\det\begin{pmatrix}x_i & x_j\\ y_i & y_j\end{pmatrix}}\times\det U^{n,n}\left(\begin{array}{c|c}\boldsymbol{x} & \boldsymbol{a}\\ \boldsymbol{y} & \boldsymbol{b}\end{array}\right)\det U^{n,n}\left(\begin{array}{c|c}\boldsymbol{x} & \boldsymbol{c}\\ \boldsymbol{y} & \boldsymbol{d}\end{array}\right).\tag{3.4}$$

This equation (3.4) is given by Ishikawa [5, Theorem 3.1], and proven there by using complex analysis.

If we substitute in Ishikawa's identity (3.4)

$$\begin{aligned} &x_i \leftarrow x_i^2, \quad y_i \leftarrow 1, \quad a_i \leftarrow a_i, \quad b_i \leftarrow 1, \quad c_i \leftarrow x_i, \quad d_i \leftarrow 1, \quad \text{or} \\ &x_i \leftarrow x_i, \quad y_i \leftarrow 1 + x_i^2, \quad a_i \leftarrow a_i, \quad b_i \leftarrow 1, \quad c_i \leftarrow x_i, \quad d_i \leftarrow 1, \end{aligned}$$

and use the factorization

$$\det U^{n,n} \left(\begin{array}{c|c} \boldsymbol{x}^2 & \boldsymbol{x} \\ \boldsymbol{1} & \boldsymbol{1} \end{array} \right) = \det U^{n,n} \left(\begin{array}{c|c} \boldsymbol{x} & \boldsymbol{x} \\ \boldsymbol{1} + \boldsymbol{x}^2 & \boldsymbol{1} \end{array} \right) = (-1)^{n(n-1)/2} \prod_{1 \le i < j \le 2n} (x_i - x_j),$$

then we have

$$\begin{aligned} \operatorname{Pf} \left(\frac{a_i - a_j}{x_i + x_j} \right)_{1 \le i,j \le 2n} &= \frac{(-1)^{n(n-1)/2}}{\prod_{1 \le i < j \le 2n} (x_i + x_j)} \det U^{n,n} \left(\begin{array}{c|c} \boldsymbol{x}^2 & \boldsymbol{a} \\ \boldsymbol{1} & \boldsymbol{1} \end{array} \right), \\ \operatorname{Pf} \left(\frac{a_i - a_j}{1 - x_i x_j} \right)_{1 \le i,j \le 2n} &= \frac{1}{\prod_{1 \le i < j \le 2n} (1 - x_i x_j)} \det U^{n,n} \left(\begin{array}{c|c} \boldsymbol{x} & \boldsymbol{a} \\ \boldsymbol{1} + \boldsymbol{x}^2 & \boldsymbol{1} \end{array} \right). \end{aligned}$$

It is not hard to show that these identities are equivalent to the ones obtained by Sundquist [25, Theorem 2.1]. (See also [7, §4].)

Ishikawa [5] uses the identity (3.4) (and other ingredients) to prove Stanley's conjecture [23] on a certain weighted summation of Schur functions.

Theorem 3.2. ([5, Theorem 1.1]) Given a partition λ, define $\omega(\lambda)$ by

$$\omega(\lambda) = a^{\sum_{i \ge 1} \lceil \lambda_{2i-1}/2 \rceil} b^{\sum_{i \ge 1} \lfloor \lambda_{2i-1}/2 \rfloor} c^{\sum_{i \ge 1} \lceil \lambda_{2i}/2 \rceil} d^{\sum_{i \ge 1} \lfloor \lambda_{2i}/2 \rfloor},$$

where a, b, c and d are indeterminates, and $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) stands for the smallest (resp. largest) integer greater (resp. less) than or equal to x for a given real number x. Then we have

$$\log \left(\sum_\lambda \omega(\lambda) s_\lambda \right) - \sum_{n \ge 1} \frac{1}{2n} a^n (b^n - c^n) p_{2n} - \sum_{n \ge 1} \frac{1}{4n} a^n b^n c^n d^n p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \dots]].$$

Here λ runs over all partitions and $p_r = \sum_{i \ge 1} x_i^r$ denotes the rth power sum symmetric function.

A key to the Ishikawa's proof is the following expression, which is obtained by using (3.4) and the minor-summation formula [8].

Proposition 3.3. ([5, Theorem 4.2]) For a vector of variables $\boldsymbol{x} = (x_1, \cdots, x_{2n})$, we have

$$\begin{aligned} &\sum_\lambda \omega(\lambda) s_\lambda(x_1, \cdots, x_{2n}) \\ &\quad = \frac{(-1)^{n(n+1)/2}}{\prod_{i=1}^{2n} (1 - a b x_i^2) \prod_{1 \le i < j \le 2n} (x_j - x_i)(1 - abcd x_i^2 x_j^2)} \\ &\qquad \det U^{n,n} \left(\begin{array}{c|c} \boldsymbol{x}^2 & \boldsymbol{x} + a\boldsymbol{x}^2 \\ \boldsymbol{1} + abcd\boldsymbol{x}^4 & \boldsymbol{1} - a(b+c)\boldsymbol{x}^2 - abc\boldsymbol{x}^3 \end{array} \right). \end{aligned}$$

3.3 Rectangular Schur functions

In this subsection, we are concerned with identities of Schur functions or relations of Littlewood–Richardson coefficients involving rectangular Young diagrams. We denote by $\mathrm{LR}^{\lambda}_{\mu,\nu}$ the Littlewood–Richardson coefficient associated to three partitions λ, μ and ν. (See [16, Chap. I].) Let $\square(a,b)$ denote the partition whose Young diagram is the rectangle $a \times b$, i.e.

$$\square(a,b) = (b^a) = (\underbrace{b,\dots,b}_{a}).$$

Then we have

$$\frac{1}{\prod_{1\le i<j\le n}(x_j - x_i)} \det V^{p,q}(\boldsymbol{x};\boldsymbol{x}^k) = \begin{cases} s_{\square(q,k-p)}(\boldsymbol{x}) & \text{if } k \ge p, \\ 0 & \text{if } k < p. \end{cases}$$

Hence, by specializing

$$a_i = x_i^{e+p+n}, \quad b_i = y_i^{e+p+n} \quad c_i = z_i^{e+p+n} \quad \text{in (2.1)},$$
$$a_i = x_i^{e+p+n}, \quad b_i = y_i^{f+r+n}, \quad c_i = z_i^{e+p+n}, \quad d_i = w_i^{f+r+n} \quad \text{in (2.2)},$$

we can derive the following identities for rectangular Schur functions from (2.1) and (2.2).

Proposition 3.4. (a) Let n be a positive integer and let e, p and q be nonnegative integers. For three vector of variables $\boldsymbol{x} = (x_1,\cdots,x_n)$, $\boldsymbol{y} = (y_1,\cdots,y_n)$ and $\boldsymbol{z} = (z_1,\cdots,z_{p+q})$, we have

$$\frac{1}{\prod_{1\le i<j\le n}(x_j - x_i)(y_j - y_i)} \det\left(s_{\square(q+1,e+n-1)}(x_i,y_j,\boldsymbol{z})\right)_{1\le i,j\le n}$$
$$= (-1)^{n(n-1)/2} s_{\square(q,e+n)}(\boldsymbol{z})^{n-1} s_{\square(q+n,e)}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}). \quad (3.5)$$

(b) Let n be a positive integer and let e, f, p, q, r and s be nonnegative integers. For three vector of variables $\boldsymbol{x} = (x_1,\cdots,x_{2n})$, $\boldsymbol{z} = (z_1,\cdots,z_{p+q})$ and $\boldsymbol{w} = (w_1,\cdots,z_{r+s})$, we have

$$\frac{1}{\prod_{1\le i<j\le 2n}(x_j - x_i)} \mathrm{Pf}\left((x_j - x_i)s_{\square(q+1,e+n-1)}(x_i,x_j,\boldsymbol{z})s_{\square(s+1,f+n-1)}(x_i,x_j,\boldsymbol{w})\right)_{1\le i,j\le 2n}$$
$$= s_{\square(q,e+n)}(\boldsymbol{z})^{n-1} s_{\square(s,f+n)}(\boldsymbol{w})^{n-1} s_{\square(n+q,e)}(\boldsymbol{x},\boldsymbol{z}) s_{\square(n+s,f)}(\boldsymbol{x},\boldsymbol{w}). \quad (3.6)$$

Here we note that the identity (3.5) with $q = e + n - 1$ appears in the context of orthogonal polynomials [15, Proposition 8.4.3].

In [17], the special cases of the identities (3.5) and (3.6) (i.e., the case of $p = q = 0$ and $p = q = r = s = 0$) were used to prove the following proposition. (This proposition itself can be proven by applying the Littlewood–Richardson rule.)

Proposition 3.5. Let n be positive integer and let e and f be nonnegative integers.

(a) We have

$$s_{\square(n,e)}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\mu \subset \square(n,e)} s_\mu(\boldsymbol{x}) s_{\mu^\dagger}(\boldsymbol{y}),$$

where μ runs over all partitions whose Young diagrams are contained in the rectangle $\square(n,e)$ and $\mu^\dagger = \mu^\dagger(n,e)$ denotes the complementary partition defined by

$$\mu_i^\dagger = e - \mu_{n+1-i} \quad (1 \le i \le n).$$

In other words,

$$\mathrm{LR}_{\mu,\nu}^{\square(n,e)} = \begin{cases} 1 & \text{if } \nu = \mu^\dagger(n,e), \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

(b) We have

$$s_{\square(n,e)}(\boldsymbol{x}) s_{\square(n,f)}(\boldsymbol{x}) = \sum_\lambda s_\lambda(\boldsymbol{x}),$$

where λ runs over all partitions with length $\le 2n$ and satisfying

$$\lambda_i + \lambda_{2n+1-i} = e + f \quad (1 \le i \le n), \quad \lambda_{n+1} \le \min(e,f).$$

In terms of Littlewood–Richardson coefficients, we have, for a partition λ with length $\le 2n$,

$$\mathrm{LR}_{\square(n,e),\square(n,f)}^{\lambda} = \begin{cases} 1 & \text{if } \lambda_{n+1} \le \min(e,f) \text{ and } \lambda_i + \lambda_{2n+1-i} = e+f \ (1 \le i \le n), \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

In [17], the author used the special cases of other identities (2.3) and (2.4) in Theorem 2.1, together with the minor-summation formulae, to obtain irreducible decompositions of some restrictions and tensor products for rectangular representations of classical groups.

Considering the more general case where $p \ge n$ and $q = r = s = 0$ in (3.6), we can prove the following theorem, which generalizes (3.8).

Theorem 3.6. ([7, Theorem 7.2]) Let n be a positive integer and let e and f be nonnegative integers. Let λ and μ be partitions such that $\lambda \subset \square(2n, e+f)$ and $\mu \subset \square(n,e)$. Then we have

(1) $\mathrm{LR}_{\mu,\square(n,f)}^{\lambda} = 0$ unless

$$\lambda_n \ge f \quad \text{and} \quad \lambda_{n+1} \le \min(e,f). \tag{3.9}$$

(2) If λ satisfies the above condition (3.9) and we define two partitions α and β by

$$\alpha_i = \lambda_i - f, \quad \beta_i = e - \lambda_{2n+1-i} \quad (1 \le i \le n), \tag{3.10}$$

then we have

$$\mathrm{LR}_{\mu,\square(n,f)}^{\lambda} = \mathrm{LR}_{\alpha,\mu^\dagger(n,e)}^{\beta}.$$

In particular, $\mathrm{LR}_{\mu,\square(n,f)}^{\lambda} = 0$ unless $\alpha \subset \beta$.

3.4 Staircase Schur functions

Lastly we consider the specialization which produce identities for the Schur functions associated to staircase Young diagrams. Let $\delta(k)$ be the staircase partition of length k, i.e.,

$$\delta(k) = (k, k-1, \cdots, 1).$$

We understand that $\delta(0)$ is the empty partition. Then we have

$$\frac{1}{\prod_{1\le i<j\le n}(x_j - x_i)} \det V^{p,q}(\boldsymbol{x}^2;\boldsymbol{x}) = \begin{cases} (-1)^{q(2p-q-1)/2} s_{\delta(p-q-1)}(\boldsymbol{x}) & \text{if } p > q, \\ (-1)^{p(p-1)/2} s_{\delta(q-p)}(\boldsymbol{x}) & \text{if } p \le q. \end{cases}$$

Hence, under the specialization

$$x_i \to x_i^2, \quad y_i \to y_i^2, \quad z_i \to z_i^2, \quad a_i \to x_i, \quad b_i \to y_i, \quad c_i \to z_i, \quad \text{in (2.1)},$$
$$x_i \to x_i^2, \quad z_i \to z_i^2, \quad w_i \to w_i^2, \quad a_i \to x_i, \quad b_i \to x_i, \quad c_i \to z_i, \quad d_i \to w_i \quad \text{in (2.2)},$$

one can deduce the following identities from (2.1) and (2.2).

Proposition 3.7. Let n be a positive integer and let k, r and s be nonnegative integers.

(a) For three vectors of variable $\boldsymbol{x} = (x_1, \cdots, x_n)$, $\boldsymbol{y} = (y_1, \cdots, y_n)$, and $\boldsymbol{z} = (z_1, \cdots, z_n)$, we have

$$\det\left(\frac{s_{\delta(k)}(x_i, y_j, \boldsymbol{z})}{x_i + y_j}\right)_{1\le i,j\le n} = \frac{\prod_{1\le i<j\le n}(x_j - x_i)(y_j - y_i)}{\prod_{i,j=1}^n (x_i + y_j)} s_{\delta(k)}(\boldsymbol{z})^{n-1} s_{\delta(k)}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}). \tag{3.11}$$

(b) For three vectors of variable $\boldsymbol{x} = (x_1, \cdots, x_{2n})$, $\boldsymbol{z} = (z_1, \cdots, z_r)$, and $\boldsymbol{w} = (w_1, \cdots, w_s)$, we have

$$\operatorname{Pf}\left(\frac{x_j - x_i}{x_j + x_i} s_{\delta(k)}(x_i, x_j \boldsymbol{z}) s_{\delta(l)}(x_i, x_j, \boldsymbol{w})\right)_{1\le i,j\le 2n}$$
$$= \prod_{1\le i<j\le 2n} \frac{x_j - x_i}{x_j + x_i} s_{\delta(k)}(\boldsymbol{z})^{n-1} s_{\delta(l)}(\boldsymbol{w})^{n-1} s_{\delta(k)}(\boldsymbol{x}, \boldsymbol{z}) s_{\delta(l)}(\boldsymbol{x}, \boldsymbol{w}). \tag{3.12}$$

If we take $k = 0$ in (3.11) and $k = l = 0$ in (3.12), we recover Cauchy's determinant identity (1.1) and Schur's Pfaffian identity (1.3). Another special case of (3.11) with $k = 1$ gives the rational case of Frobenius' identity (4.1):

$$\det\left(\frac{1}{x_i + y_j} \cdot \frac{z + x_i + y_j}{z}\right)_{1\le i,j\le n} = \frac{\prod_{1\le i<j\le n}(x_j - x_i)(y_j - y_i)}{\prod_{1\le i,j\le n}(x_i + y_j)} \cdot \frac{z + \sum_{i=1}^n x_i + \sum_{j=1}^n y_j}{z}.$$

On the other hand, if we take $k = l = 1$ in (3.12), we obtain

$$\operatorname{Pf}\left(\frac{x_j - x_i}{x_j + x_i} \cdot \frac{z + x_i + x_j}{z} \cdot \frac{w + x_i + x_j}{w}\right)_{1\le i,j\le 2n} = \prod_{1\le i<j\le 2n} \frac{x_j - x_i}{x_j + x_i} \cdot \frac{z + \sum_{i=1}^{2n} x_i}{z} \cdot \frac{w + \sum_{i=1}^{2n} x_i}{w}.$$

This identity suggests an elliptic generalization of Schur's Pfaffian identity. See the next section.

4 An elliptic generalization of Schur's Pfaffian identity

To deal with elliptic functions and their degeneration simultaneously, we introduce the following notation. Let $[x]$ denote a nonzero holomorphic function on the complex plane $\mathbb{C}$ in the variable x satisfying the following two conditions:

(i) $[x]$ is an odd function, i.e., $[-x] = -[x]$.

(ii) $[x]$ satisfies the Riemann relation:

$$[x+y][x-y][u+v][u-v]-[x+u][x-u][y+v][y-v]+[x+v][x-v][y+u][y-u] = 0.$$

It is known that such a function $[x]$ is obtained from one of the following functions by the transformation $[x] \to e^{ax^2+b}[cx]$ (see [26, Chap. XX, Misc. Ex. 38]):

(a) (elliptic case) $[x] = \sigma(x)$ (the Weierstrass sigma function).

(b) (trigonometric case) $[x] = e^x - e^{-x}$.

(c) (rational case) $[x] = x$.

Then Frobenius's determinant identity (1.5) takes the form

$$\det\left(\frac{[z+x_i+y_j]}{[z][x_i+y_j]}\right)_{1\le i,j\le n} = \frac{\prod_{1\le i<j\le n}[x_j-x_i][y_j-y_i]}{\prod_{i,j=1}^{n}[x_i+y_j]}\cdot\frac{[z+\sum_{i=1}^n x_i+\sum_{j=1}^n y_j]}{[z]}. \quad (4.1)$$

The following is our elliptic generalization of Schur's Pfaffian identity.

Theorem 4.1. ([20]) For complex variables $x_1, \cdots, x_{2n}$, z and w, we have

$$\operatorname{Pf}\left(\frac{[x_j-x_i]}{[x_j+x_i]}\cdot\frac{[z+x_i+x_j]}{[z]}\cdot\frac{[w+x_i+x_j]}{[w]}\right)_{1\le i,j\le 2n}$$
$$= \prod_{1\le i<j\le 2n}\frac{[x_j-x_i]}{[x_j+x_i]}\cdot\frac{[z+\sum_{i=1}^{2n}x_i]}{[z]}\cdot\frac{[w+\sum_{i=1}^{2n}x_i]}{[w]}. \quad (4.2)$$

If we take the limit $z\to\infty$, $w\to\infty$ in the rational case of (4.2), then we recover the Schur's Pfaffian identity (1.3).

Once the identity (4.2) is found, it is not hard to prove it. We can use the Desnanot–Jacobi formula for Pfaffians (Lemma 2.5) to reduce the proof of the general case to that of the $n=2$ case, and can show this case by applying the Riemann relation. (See [20] for the details.)

5 A Pfaffian–Hafnian analogue of Borchardt's identity

In this last section, we give an Borchardt-type variation of Schur's Pfaffian identity.

Recall the definition of Hafnians. For a symmetric matrix $A = (a_{ij})_{1\le i,j\le 2n}$, the Hafnian of A is defined by

$$\operatorname{Hf} A = \sum_{\sigma\in\mathcal{F}_{2n}} a_{\sigma(1)\sigma(2)}a_{\sigma(3)\sigma(4)}\cdots a_{\sigma(2n-1)\sigma(2n)},$$

where $\mathcal{F}_{2n}$ is the set of all permutations σ satisfying $\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)$ and $\sigma(2i-1) < \sigma(2i)$ for $1 \le i \le n$. Then we have

Theorem 5.1. ([6]) Let n be an positive integer. Then we have

$$\operatorname{Pf}\left(\frac{x_i - x_j}{(x_i + x_j)^2}\right)_{1 \le i,j \le 2n} = \prod_{1 \le i < j \le 2n} \frac{x_i - x_j}{x_i + x_j} \cdot \operatorname{Hf}\left(\frac{1}{x_i + x_j}\right)_{1 \le i,j \le 2n}, \tag{5.1}$$

$$\operatorname{Pf}\left(\frac{x_i - x_j}{(1 - x_i x_j)^2}\right)_{1 \le i,j \le 2n} = \prod_{1 \le i < j \le 2n} \frac{x_i - x_j}{1 - x_i x_j} \cdot \operatorname{Hf}\left(\frac{1}{1 - x_i x_j}\right)_{1 \le i,j \le 2n}. \tag{5.2}$$

Here we make a comment on the proof of this theorem. In [6], we prove the identity (5.2) by regarding the both sides as rational functions in x_n and comparing the poles and the residues. It would be interesting to find an algebraic proof of these identities (5.1) and (5.2).

References

[1] C. W. Borchardt, Bestimmung der symmetrischen Verbindungen vermittelst ihrer erzeugenden Funktion, J. Reine Angew. Math. **53** (1855), 193–198.

[2] D. M. Bressoud, "Proofs and Confirmations : The Story of the Alternating Sign Matrix Conjecture", Cambridge Univ. Press, 1999.

[3] A. L. Cauchy, Mémoire sur les fonctions altertées et sur les sommes alternées, Exercices Anal. et Phys. Math. **2** (1841), 151–159.

[4] G. Frobenius, Über die elliptischen Funktionen zweiter Art, J. Reine und Angew. Math. **93** (1882), 53–68.

[5] M. Ishikawa, Minor summation formula and a proof of Stanley's open problem, `arXiv:math.CO/0408204`.

[6] M. Ishikawa, H. Kawamuko, and S. Okada, A Pfaffian–Hafnian Analogue of the Borchardt's Identity, `arXiv:math.CO/0408364`.

[7] M. Ishikawa, S. Okada, H. Tagawa and J. Zeng Generalizations of Cauchy's determinant and Schur's Pfaffian", `arXiv:math.CO/0411280`.

[8] M. Ishikawa and M. Wakayama, Minor summation formulas of Pfaffians, Linear and Multilinear Algebra **39** (1995), 285–305.

[9] M. Ishikawa and M. Wakayama, Applications of the minor summation formula III: Plücker relations, lattice paths and Pfaffians, `arXiv:math.CO/0312358`.

[10] Y. Kajiwara and M. Noumi, Multiple elliptic hypergeometric series : an approach from the Cauchy determinant, Indag. Math. (N.S.) **14** (2003), 395–421.

[11] D. Knuth, Overlapping Pfaffians, Electron. J. Combin. **3** (2) ("The Foata Festschrift") (1996), 151–163.

[12] C. Krattenthaler, Advanced determinant calculus, Sem. Lothar. Combin. **42** (1999), B42q.

[13] C. Krattenthaler, Advanced determinant calculus : a complement, `arXiv:math.CO/0503507`.

[14] D. Laksov, A. Lascoux and A. Thorup, On Giambelli's theorem on complete correlations, Acta Math. **162** (1989), 143–199.

[15] A. Lascoux, Symmetric Functions and Combinatorial Operators on Polynomials, CBMS Reg. Conf. Ser. Math. **99**, AMS, 2003.

[16] I. G. Macdonald, "Symmetric Functions and Hall Polynomials (2nd ed.)," Oxford Univ. Press, 1995.

[17] S. Okada, Applications of minor-summation formulas to rectangular-shaped representations of classical groups, J. Alg. **205** (1998), 337–367.

[18] S. Okada, Enumeration of symmetry classes of alternating sign matrices and characters of classical groups, `arXiv:math.CO/0408234`.

[19] S. Okada, Determinant and Pfaffian formulae of Cauchy type and their applications, in "Aspects of Combinatorial Representation Theory," Sūrikaisekikenkyūsho Kōkyūroku **1382** (2004), 198–215 (in Japanese).

[20] S. Okada, An elliptic generalization of Schur's Pfaffian identity, `arXiv:math.CA/0412038`.

[21] E. M. Rains, Transformations of elliptic hypergeometric integrals, `arXiv:math.QA/0309252`.

[22] I. Schur, Über die Darstellung der symmetrischen und der alternirenden Gruppe durch gebrochene lineare Substitutionen, J. Reine Angew. Math. **139** (1911), 155–250.

[23] R. P. Stanley, Open problem, presented at International Conference on Formal Power Series and Algebraic Combinatorics (Vadstena 2003), June 23 – 27, 2003, available from `http://www-math.mit.edu/ rstan/trans.html`.

[24] J. R. Stembridge, Non-intersecting paths, Pfaffians and plane partitions, Adv. Math. **83** (1990), 96–131.

[25] T. Sundquist, Two variable Pfaffian identities and symmetric functions, J. Algebraic Combin. **5** (1996), 135–148.

[26] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis (Fourth edition), Cambridge Univ. Press, 1927 (reprinted in 1999).

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