On Deformation of Elliptic Quantum Planes

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Introduction

Elliptic Quantum Planes means here non-commutative deformations of the complex projective plane $\mathbb{P}^2(\mathbb{C})$. We consider deformations in the realm of non-commutative (complex) algebraic geometry. As we recall in the first section, elliptic modulus parameter enters into the game. Hence the adjective "elliptic" is used. Note also that, in that world, the complex projective line $\mathbb{P}^1(\mathbb{C})$, namely the Riemann sphere, does not admit even a non-trivial non-commutative deformation, unlike in the world of non-commutative differential geometry.

The aim of this article is twofold. In the first two sections, we briefly review the non-commutative projective geometry mainly focusing on the case of non-commutative deformation of the projective plane.

In the third section, we formulate a theorem pertaining to Hochschild cohomology of algebraic varieties, which generalizes a theorem of Keller [6].

In author's actual talk at the workshop on Elliptic Integrable Systems, he had only time to explain the first two sections of this article. The title of this article is left the same as that of the talk, while the half of the content is not specialized so much to the elliptic quantum plane.

The author would like to thank Profs. A.Kato and Y.Saito for discussion. He would also like to thank Prof. K.Ikeda of Keio Univ. for an occasion in the Algebraic Analysis Seminar about the content of the first half of this article. He would also like to thank Profs. K.Takasaki and M.Noumi, the organizers of the workshop, for giving him a chance to talk and for support.

1 Elliptic quantum planes – What are they ?

There are a few approaches to non-commutative generalization of algebraic geometry. Non-commutative projective geometry is one of these approaches. It is based on Serre's theorem characterizing the category of coherent sheaves on a projective variety recalled below.

Definition 1.1 *A non-commutative(=NC) projective variety is the quotient category $qgr(S)$ of the category of finitely generated graded S-modules by the epaisse subcategory consisting of those graded S-modules whose graded components in sufficiently high degrees are zero. Here S is a noetherian graded k-algebra with nice properties. We will assume that the ground field k is $\mathbb{C}$ for simplicity, but most of the definitions and results make sense for an algebraically closed field k.*

Remark 1.2 *Let X be a projective variety over a field k and $\mathcal{O}(1)$ an ample invertible $\mathcal{O}$-module. Then, after Serre, we have an equivalence of categories*

$$Coh(X) \simeq qgr(S), \qquad S := \oplus_{n \geq 0} H^0(X, \mathcal{O}(n))$$

where the left hand side means the category of coherent sheaves on X. For $X = \mathbb{P}^2$, we have $S = k[x, y, z]$.

Definition 1.3 *A NC projective plane is defined to be a category of the form $qgr(S)$, where S is an Artin-Schelter regular algebra of dimension 3 with Hilbert series $1/(1-t)^3$.*

Here a graded algebra $S = \oplus_{n \geq 0} S_n$ $(S_0 = \mathbb{C})$ is called Artin-Schelter regular algebra of dimension 3 if (1) S has homological dimension 3, (2) $\dim_{\mathbb{C}} S_n$ grows quadratically in n, and (3) one has

$$\mathrm{Ext}_S^i(\mathbb{C}, S) = \begin{cases} 0 & i \neq 3 \\ \mathbb{C}(m) & i = 3 \end{cases}$$

for some m, where (m) means the shift in degrees. The Hilbert series of S is by definition $\sum_{n \geq 0} (\dim S_n) t^n$.

M.Artin and his collaborators [1], [2] classified Artin-Schelter regular algebras S of dimension 3 with Hilbert series $1/(1-t)^3$. Their results say the following:

> If $qgr(S)$ is not equivalent to the category $Coh(\mathbb{P}^2)$, then there exists a cubic element $g \in S_3$ such that $gS = Sg$ and $S/gS \simeq B(E, \mathcal{L}, \sigma)$ for some elliptic curve E, a line bundle $\mathcal{L}$ on E, and an automorphism σ of E.

Remark 1.4 *Here $B(X, \mathcal{L}, \sigma)$ denotes the twisted homogeneous coordinate ring of the data $(X, \mathcal{L}, \sigma)$:*

$$B(X, \mathcal{L}, \sigma) = \oplus_{n \geq 0} H^0(X, \mathcal{L} \otimes \mathcal{L}^\sigma \cdots \mathcal{L}^{\sigma^{n-1}})$$

equipped with the naturally defined product and $\mathcal{L}^{\sigma^m} = (\sigma^m)^* \mathcal{L}$.

It is known that $qgr(B(X, \mathcal{L}, \sigma)) \simeq qgr(B(X, \mathcal{O}(1), id)) = Coh(X)$.

The algebra $B(X, \mathcal{L}, \sigma)$ itself is more subtle object. For example, one has $B(\mathbb{P}^1, \mathcal{O}(1), \sigma) = \mathbb{C}\{x, y\}/(xy - yx - x^2)$ for $\sigma(u) = u + 1$. Here $\mathbb{C}\{x, y\}$ denotes the non-commutative polynomial ring over $\mathbb{C}$.

By the classification [1], [2], a non-trivial NC $\mathbb{P}^2$ is determined by data $(E, \mathcal{L}, \sigma)$, hence the name "elliptic quantum plane."

The graded algebras for NC $\mathbb{P}^2$ include the following cases.
(1) Weyl algebra $\mathcal{D}(\mathbb{A}^1)$

$$S = \mathbb{C}\{x, y, z\}/(yx - xy - z^2, zx - xz, zy - yz)$$

(2) 3 dimensional Sklyanin algebra Skl_3

$$S = \mathbb{C}\{x, y, z\}/(axy + byx + cz^2, ayz + bzy + cx^2, azx + bxz + cy^2)$$

where $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$. In this case, E is given by the cubic

$$abc(x^3 + y^3 + z^3) - (a^3 + b^3 + c^3)xyz = 0$$

and σ is the translation by the point $(a : b : c)$.

Remark 1.5 *The above examples have some link with integrable systems.*

Cannings-Holland [4] found a one-to-one correspondence between the following two:

$$\{ \text{ right ideals of } \mathcal{D}(\mathbb{A}^1) \} \ \leftrightarrow \ Gr^{ad} \text{ (Wilson's adelic Grassmannian)}$$

This last Grassmannian is partitioned into the Calogero-Moser phase space $\mathcal{C}_n$: $Gr^{ad} = \coprod_{n \geq 0} \mathcal{C}_n$. The space $\mathcal{C}_n$ is also identified with a deformation of the Hilbert scheme of n points $(\mathbb{A}^2)^{[n]}$.

The elliptic version of this story is considered by Nevins-Stafford [8].

2   Hochschild cohomology of some varieties – a motivation

NC "deformations" of complex smooth algebraic vaieties appear in the context of stability conditions in the derived category of coherent sheaves $D^b(Coh(X))$ formulated by M. Douglas and T. Bridgeland [3]. The deformations of $Coh(X)$ arise as family of abelian subcategories of $D^b(Coh(X))$ parameterized by elements of the complexified ample cone in $H^2(X, \mathbb{R}) + \sqrt{-1} NS(X)_{\mathbb{R}}$.

According to the Homological Mirror Symmetry Conjecture by Kontsevich, deformations of $D^b(Coh(X))$ are parametrized by the so-called extended moduli space. Its tangent space is given by the Hochschild cohomology

$$HH^*(X) \simeq \oplus_{p,q} H^p(X, \Lambda^q T_X),$$

where T_X denotes the sheaf of holomorphic tangent vector fields on X.

The geometric meaning of deformations is not fully clear at his moment. The second Hochschild cohomology group

$$HH^2(X) \simeq H^2(X, \mathcal{O}_X) \oplus H^1(X, T_X) \oplus H^0(X, \Lambda^2 T_X)$$

contains the tangent space to the moduli sapce of complex structures $H^1(X, T_X)$. The remaining components seems to have also geometric meaning; $H^0(X, \Lambda^2 T_X)$ corresponds to deforming the product of the structure sheaf $\mathcal{O}_X$ [11] and $H^2(X, \mathcal{O}_X)$ corresponds to deforming the gluing compatibility. For the latter, the deformed objects are nothing but the twisted sheaves [5].

Let us look at the case $X = \mathbb{P}^2$. The total Hochschild cohomology is given by

$$HH^*(\mathbb{P}^2) \simeq H^0(\mathbb{P}^2, \mathcal{O}) \oplus H^0(\mathbb{P}^2, T) \oplus H^0(\mathbb{P}^2, \Lambda^2 T)$$

The right hand sides are the 0th, 1st and 2nd Hochschild cohomology groups. The dimension is 1, 8, 10 respectively.

The 1st group is the tangent space to the automprphism group of $\mathbb{P}^2$. Since $\Lambda^2 T = K^{-1} = \mathcal{O}(3)$, the 2nd group is isomorphic to $H^0(\mathbb{P}^2, \mathcal{O}(3))$ whch is the space of cubic equations. This suggests that the extended moduli classifies much finer objects with additional structure than the abelian subcategories of the derived category of coherent sheaves.

It would be desirable to find an appropriate framework of "non-commutative algebraic geometry" in which we can discuss deformations and analog of Kodaira-Spencer theory.

3 Derived Picard group of a quasi-coherent ringed scheme

In this section, we describe a generalization of Keller's interpretation of the Hochschild cohomology groups as the tangent spaces of the derived Picard group functor. It might shed some light on the consideration in the previous section.

Keller [6] defined, for an associative algebra A over a field k, the derived Picard group functor $DPic_A$ from the category $cdg(k)$ of (graded-)commutative differential graded k-algebras to the category of groups. He then proved that

$$\mathrm{Ker}\left(DPic_A(k[\epsilon_{-i}]/(\epsilon_{-i}^2)) \to DPic_A(k)\right) \simeq HH^{i+1}(A)$$

where ϵ_{-i} has degree $-i$.

We formulate a generalization of this result, leaving the detail to [9].

Definition 3.1 *Let $X = (X, \mathcal{O}_X)$ be a(n ordinary) scheme over a field k and $\mathcal{A}$ a quasi-coherent $\mathcal{O}_X$-algebra. Then a pair $(X, \mathcal{A})$ is called a quasi-coherent ringed scheme over k.*

We assume, from here on, that the product $\mathcal{A} \boxtimes \mathcal{A}^{op} =: \mathcal{A}^e$ on $X \times X = X^2$ exists.

Let $R \in cdg(k)$ be a commutative differential graded k-algebra. Keller [7] introduced R-relative derived category $D_R(E)$ for differential graded R-algebra E. The objects are differential graded E-modules. The morphisms in $D_R(E)$ are obtained by inverting the morphisms of dg E-modules which become homotopy equivalences when the scalars are restricted to R. This definition can be generalized to quasi-coherent ringed schemes.

In addition to R-relative derived category, Keller introduced the notion of relative derived tensor product $\otimes^{L,rel}$.

Definition 3.2 *Let $X = (X, \mathcal{A})$ be a quasi-coherent ringed scheme over k.*

A complex of $R \otimes \mathcal{A}^e$-modules U is called an invertible complex of bimodules in $D_R(R \otimes \mathcal{A}^e)$ if (1) its underlying complex of graded R-modules are locally R-free and if (2) there exists an object $V \in D_R(R \otimes \mathcal{A}^e)$ which is represented by locally R-free modules and is such that

$$U \otimes_{R \otimes \mathcal{A}}^{L,rel} V \simeq R \otimes \mathcal{A}, \quad V \otimes_{R \otimes \mathcal{A}}^{L,rel} U \simeq R \otimes \mathcal{A}.$$

An invertible complex of bimodules is also called as a tilting complex.

Definition 3.3 *Let $R \in cdg(k)$ be a commutative differential graded k-algebra and $(X, \mathcal{A})$ a quasi-coherent ringed scheme over k.*

The set of isomorphism classes in $D_R(R \otimes \mathcal{A}^e)$ of invertible complex of bimodules is denoted by $DPic_{X,\mathcal{A}}(R)$. It has a structure of group with the product being the derived tensor product.

$DPic_{X,\mathcal{A}}$ becomes a contravariant functor from $cdg(k)$ to the category of groups. We define its i-th piece of Lie superalgebra of $DPic_{X,\mathcal{A}}$ to be

$$LieDPic_{X,\mathcal{A}}^i := \mathrm{Ker}\left(DPic_{X,\mathcal{A}}(k[\epsilon_{-i}]/(\epsilon_{-i}^2)) \to DPic_{X,\mathcal{A}}(k)\right)$$

where ϵ_{-i} has degree $-i$ and the differential $d = 0$.

The totality of these pieces form a Lie superalgebra:

$$LieDPic^*_{X,\mathcal{A}} := \oplus_i LieDPic^i_{X,\mathcal{A}}$$

Definition 3.4 *The Hochschild cohomology of a quasi-coherent ringed scheme $(X, \mathcal{A})$ is defined to be :*

$$HH^i(X, \mathcal{A}) = HH^i(\mathcal{A}) := Ext^i_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A})$$

The totality of Hochschild cohomology groups equipped with the Gerstenhaber bracket form a Lie superalgebra:

$$HH^{*+1}(X, \mathcal{A}) := \oplus_i HH^{i+1}(X, \mathcal{A})$$

Theorem 3.5 *Let $(X, \mathcal{A})$ a quasi-coherent ringed scheme over k such that the product $\mathcal{A} \boxtimes \mathcal{A}^{op} =: \mathcal{A}^e$ on $X \times X = X^2$ exists.*

Then we have the following isomorphism of Lie superalgebras:

$$LieDPic^*_{X,\mathcal{A}} \simeq HH^{*+1}(X, \mathcal{A})$$

The theorem is more generally valid for quasi-coherent dg scheme.

References

[1] M. Artin and W. Schelter, Graded algebras of global dimension 3, Adv. in Math. **66** (1987) 171–216.

[2] M. Artin, J. Tate and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, The Grothendieck Festshrift, vol.1, Birkh"auser, Boston, (1990) 33–85.

[3] T. Bridgeland, Stability conditions on triangulated categories, math.AG/0212237; Stability conditions on K3 surfaces, math.AG/0307164.

[4] R.C. Cannings and M.P. Holland, Right ideals of rings of differential operators, J. Algebra 167 (1994), no. 1, 116–141.

[5] A. Călduraru, Derived categories of twisted sheaves on elliptic threefolds. J. Reine Angew. Math. 544 (2002), 161–179.

[6] B.Keller, Hochschild cohomology and derived Picard groups. J. Pure Appl. Algebra 190 (2004), no. 1-3, 177–196.

[7] B.Keller, Invariance and localization for cyclic homology of DG algebras, J. Pure Appl. Algebra 123 (1998), 223–273.

[8] T.A. Nevins and J.T. Stafford, Sklyanin algebras and Hilbert schemes of points, math.AG/0310045.

[9] Y. Shimizu, Derived Picard group of a quasi-coherent ringed scheme, In preparation.

[10] J. T. Stafford, Noncommutative projective geometry, Proceedings of the ICM, Beijing 2002, vol. 2, 93–104.

[11] A. Yekutieli, Deformation quantization of algebraic varieties, math.AG/0310399.

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