Isothermic Surfaces in Möbius and Lie Sphere Geometries

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Contents

1. Introduction

1.1. **Background.** Ω (Omega) surfaces are a part of the study of projective, Möbius and Lie sphere geometries. Classically, Demoulin and Eisenhart studied Ω surfaces, with recent renewed interest in them created by F. Burstall, D. Calderbank and U. Hertrich-Jeromin. These surfaces are a natural generalization of isothermic surfaces. Isothermic surfaces are those for which there exist conformal curvature line coordinates, and for which we can conclude the existence of useful additional associated structures. Even though a surface itself might not be isothermic, when some sphere congruence of the surface has conformal curvature line coordinates, and hence is *isothermic* (in a sense to be explained in this text), then we say that the surface is Ω. There are many more Ω surfaces than there are isothermic surfaces, but much of the additional associated structure for isothermic surfaces still exists for Ω surfaces as well.

We now describe two interesting characteristics of Ω surfaces. The first is not something that we focus on in this text, but is one motivation for the viewpoint taken here. The second illustrates an important theme in Lie sphere geometry.

(1) Ω surfaces are Legendre immersions, which can be regarded as 2-parameter families of null planes with certain regularity conditions in a 6-dimensional vector space with non-positive-definite metric, and as such, we will have a method to investige them without explicit concern for the singularities that result when these Ω surfaces are projected to surfaces in 3-dimensional spaceforms. Figure 1.1 shows examples of the singularities that can occur in the projections. We are interested in these kinds of singularities, but are also interested in having uniform methods of studying Ω surfaces that put regular points and singular points in the projections on an equal footing. Singularities in surfaces play an implicit background role, although they are never explicitly described here.

(2) A central idea in Lie sphere geometry is that surfaces and sphere congruences appear as objects of the same type. We will see this in Section 3.5. An example of an isothermic sphere congruence of an Ω surface is shown in Figure 1.2. In this figure, although the surface and the sphere congruence appear as very different types of objects, they will become the same type of object when lifted to Lie sphere geometry.

1.2. **Purpose of this text.** Three of the primary purposes of this text are as follows:

(1) **Purpose:**

To introduce Möbius geometry and Lie sphere geometry.

Of course, Möbius geometry and Lie sphere geometry themselves are classical fields, and we introduce Möbius geometry in the second chapter and Lie sphere geometry in the third chapter.

(2) **Purpose:**

To examine linear Weingarten surfaces.

All linear Weingarten surfaces in any of the 3-dimensional spaceforms are Ω, but generally are not isothermic. We will study characterizations of these surfaces in general, and will also study the special cases of constant mean curvature (CMC) surfaces in all 3-dimensional spaceforms, and also of flat surfaces in hyperbolic 3-space, in further detail.

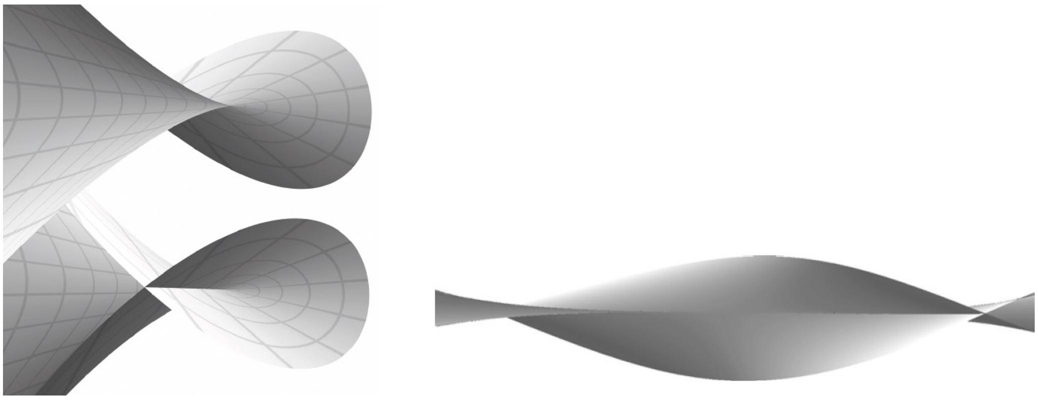

FIGURE 1.1. A swallowtail singularity on the left, and a more unusual singularity on the right. In the left-hand figure we see two cuspidal edge singularities converging to a single swallowtail singularity. This surface is shown twice, from opposite sides, and it lifts to an Ω surface in the 6-dimensional space. The right-hand figure shows a surface that again lifts to an Ω surface, and has four cuspidal edges converging to a more unusual singularity.

(3) **Purpose:**

To prepare for a study of discrete surfaces.

Having the additional mathematical structures available for isothermic surfaces extended to similar structures for Ω surfaces provides a framework for considering discrete versions of surfaces in a more general setting, which we intend to do in a separate text. This is the underlying viewpoint we take here. We will prove preparatory results that are especially useful for considering discrete surfaces, such as

- Lemmas 2.10, 2.11, 2.13-2.19, 2.27, 2.46, 2.57, 2.61-2.63, 2.73, 2.76, 3.17, 3.19, 3.20, 4.8, 4.13, 4.15, 4.19, 4.25, 4.29, 4.34, 4.57,
- Corollaries 2.29, 4.5, 4.14, 4.31, 4.43, 4.49, 4.60, 4.61, 4.69, and
- Theorems 2.53, 2.54, 2.71, 2.81, 4.42, 4.58, 4.67.

In that subsequent text, we will present an approach to discrete surfaces originating in significant part from a work of Burstall and Calderbank [20] for the case of smooth surfaces. For the time being, in this text, we consider smooth surfaces.

1.3. **Organization of this text.** This text is arranged as follows:

- In Chapter 2, we give an introduction to Möbius geometry, and a way to represent 3-dimensional spacefoms via Möbius geometry, and also a way to consider isothermic surfaces within the context of Möbius geometry. CMC surfaces in general spaceforms and flat surfaces in hyperbolic 3-space are given particular attention. Transformations of surfaces are considered as well, such as Christoffel transformations, Calapso (T-) transformations, Darboux transformations and Bäcklund transformations.
- In Chapter 3, we introduce Lie sphere geometry and Lie sphere transformations – then surfaces in spaceforms are considered in the context of Lie sphere geometry, and, in particular, lifts of those surfaces to Legendre immersions

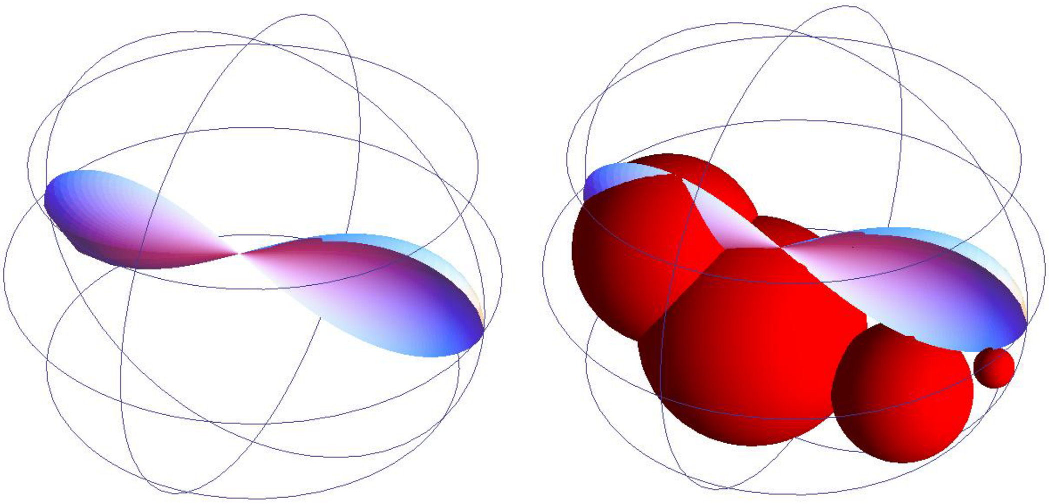

FIGURE 1.2. A surface of revolution (shown on the left) with a rotationally symmetric isothermic sphere congruence (shown on the right). The left side shows a surface of revolution with a cone-type singularity that lies within a fixed outer sphere. On the right side we have inserted spheres, which are members of the sphere congruence enveloped by both the surface and the outer fixed sphere. This surface is actually a flat surface in hyperbolic 3-space, in the Poincare ball model whose ideal boundary is that outer sphere, and this sphere congruence is isothermic.

in Lie sphere geometry are examined. After looking at the core example of Dupin cyclides, we describe Lie cyclides. Finally, as a further example, we describe a representation for flat surfaces in hyperbolic 3-space in terms of the language just used for Lie cyclides.

- In Chapter 4, we begin our study of Ω surfaces, starting with an explanation of the normal bundles of Legendre immersions in Lie sphere geometry, which is necessary for understanding the definition of Ω surfaces. We then look at other means for determining Ω surfaces:
  - Demoulin's equation,
  - existence of Moutard lifts,
  - harmonic separation of the principal curvature sphere congruences and isothermic sphere congurences,
  - existence of Christoffel dual lifts (as in Lemma 4.57 here, which is especially useful for the discretization of the theory).

  We also study Calapso transformations of Ω surfaces, and their properties. As examples of Ω surfaces, we look at Guichard surfaces, and at flat surfaces in hyperbolic 3-space, and then more generally at linear Weingarten surfaces. All the while, we compile results that have applications to discrete Ω surfaces (as noted in Purpose 3 above), as we will see in a subsequent text.

Acknowledgements. The author expresses his gratitute to Fran Burstall, David Calderbank, Udo Hertrich-Jeromin, Tim Hoffmann, Masatoshi Kokubu, Mason Pember, Yuta Ogata, Masaaki Umehara, Kotaro Yamada and Masashi Yasumoto for discussions that are the basis for the material in this text.

2. Isothermic and CMC surfaces, and their transformations

We begin by describing the 3-dimensional spaceforms using the 5-dimensional Minkowski space $\mathbb{R}^{4,1}$.

2.1. Minkowski 5-space.

$$\mathbb{R}^{4,1} = \left\{ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \middle| \; x_j \in \mathbb{R} \right\} \tag{2.1}$$

with Minkowski metric of signature $(+,+,+,+,-)$, that is, for

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} , \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}$$

in $\mathbb{R}^{4,1}$,

$$\langle X, Y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 - x_5 y_5 \; ,$$

and $||X||^2$ denotes $\langle X, X \rangle$. The 4-dimensional light cone is

$$L^4 = \{X \in \mathbb{R}^{4,1} \,|\, ||X||^2 = 0\} \; .$$

We can make the 3-dimensional spaceforms as follows: A spaceform M_κ is (see Figure 2.1)

$$M_\kappa = \{X \in L^4 \,|\, \langle X, Q_{M_\kappa} \rangle = -1\} \tag{2.2}$$

for any nonzero $Q_{M_\kappa} \in \mathbb{R}^{4,1}$. It will turn out (see the upcoming Lemma 2.5) that M_κ has constant sectional curvature κ, where $\kappa = -|Q_{M_\kappa}|^2$, so without loss of generality we can obtain any spaceform by choosing

$$Q_{M_\kappa} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2}(1-\kappa) \\ \frac{1}{2}(1+\kappa) \end{pmatrix} . \tag{2.3}$$

Remark 2.1. There is no real computational advantage to restricting to the form in (2.3). However, we frequently do this, as it gives us a convenient way to explicitly separate out 3-dimensional spaceforms, surfaces, normal vector fields, and such.

Letting $\mathbb{R}^3 \cup \{\infty\}$ denote the one point compactification of

$$\mathbb{R}^3 = \{x = (x_1, x_2, x_3) \,|\, x_j \in \mathbb{R}\} \; ,$$

with $|x|^2 = x_1^2 + x_2^2 + x_3^2$, we can write

$$M_\kappa = \left\{ X = \frac{1}{1+\kappa |x|^2} \cdot \begin{pmatrix} 2x^t \\ |x|^2 - 1 \\ |x|^2 + 1 \end{pmatrix} \middle| \; x \in \mathbb{R}^3 \cup \{\infty\} \, , \; |x|^2 \neq \tfrac{-1}{\kappa} \right\} , \tag{2.4}$$

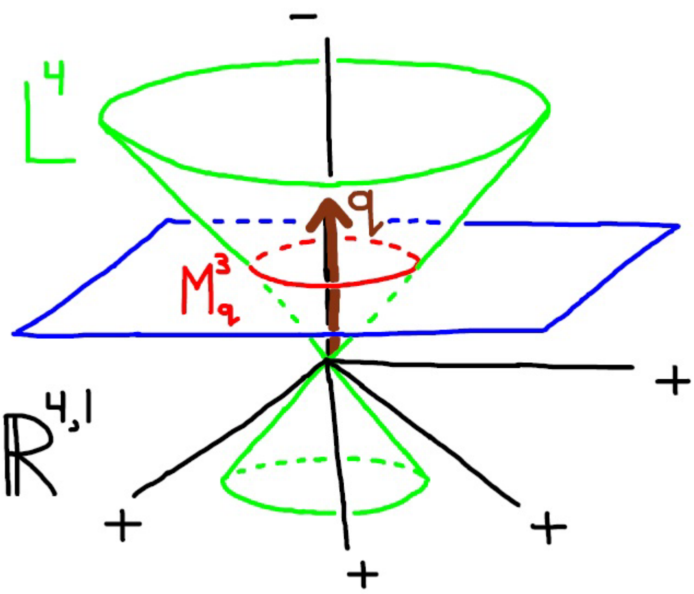

FIGURE 2.1. The Möbius geometric model for 3-dimensional space-forms $M_\kappa = M_q^3$

which is in 1-1 correspondence with

$$\mathfrak{R} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \cup \{\infty\} \,|\, |x|^2 \neq -\kappa^{-1}\} \;.$$

The form in (2.4) follows from this:

Lemma 2.2. *Any point $X \in L^4$ such that $\langle X, Q_{M_\kappa} \rangle = -1$ can be written as*

$$X = \frac{1}{1+\kappa|x|^2} \cdot \begin{pmatrix} 2x^t \\ |x|^2 - 1 \\ |x|^2 + 1 \end{pmatrix}$$

for some $x \in \mathfrak{R}$.

Proof. Write

$$X = \begin{pmatrix} y^t \\ y_4 \\ y_5 \end{pmatrix}$$

with $y = (y_1, y_2, y_3)$.

Suppose that $y_5 \neq y_4$. Setting $x = (y_5 - y_4)^{-1} y$, one can confirm that

$$(1+\kappa|x|^2)^{-1} \cdot \begin{pmatrix} 2x^t \\ |x|^2 - 1 \\ |x|^2 + 1 \end{pmatrix} = \begin{pmatrix} y^t \\ y_4 \\ y_5 \end{pmatrix} \;,$$

since Q_{M_κ} is as in (2.3) and $\langle X, X \rangle = 0$ and $\langle X, Q_{M_\kappa} \rangle = -1$, giving us the equations

$$\kappa(y_5 + y_4) = 2 - (y_5 - y_4) \;,$$

$$2(y_5 - y_4) = (y_5 - y_4)^2 + \kappa|y|^2 \qquad \text{when } \kappa \neq 0 \;,$$

$$2(y_5 + y_4) = |y|^2 \qquad \text{when } \kappa = 0 \;.$$

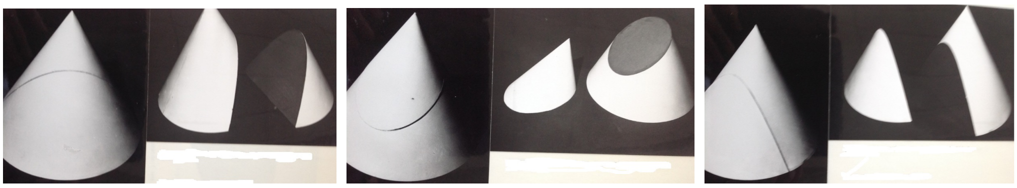

FIGURE 2.2. Physical models of lightcone cuts, representing Euclidean space on the left, spherical space in the middle, and hyperbolic space on the right (owned by the geometry group at the Technical University of Vienna)

If $y_5 = y_4$, then $||X||^2 = 0$ gives $y = (0,0,0)$, and then $\langle X, Q_{M_\kappa}\rangle = -1$ gives

$$y_5 = y_4 = \kappa^{-1} \ ,$$

so $x = \infty$ works when $\kappa \neq 0$, and the case $\kappa = 0$ does not occur. $\square$

When $\kappa < 0$, M_κ becomes two copies of hyperbolic 3-space with sectional curvature κ. Also, note that

$$1 + \kappa|x|^2 \ \text{ is never zero for points in } M_\kappa .$$

M_κ is called a *quadric*, because it is determined by a quadratic equation (for the light cone L^4) and a linear equation ($\langle X, Q_{M_\kappa}\rangle = -1$). (We sometimes abbreviate M_κ to just M.)

Remark 2.3. We will often regard X as living in the projectivized light cone PL^4, so X can be equivalently considered as $\alpha \cdot X$ for any nonzero real scalar α.

The tangent space of M_κ at X is

$$T_X M_\kappa = \left\{ \mathcal{T}_a = \frac{2}{(1+\kappa|x|^2)^2} \begin{pmatrix} a^t + \kappa|x|^2 a^t - 2\kappa(x\cdot a)x^t \\ (1+\kappa)(x\cdot a) \\ (1-\kappa)(x \cdot a) \end{pmatrix} \right\} ,$$

for $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, where $x \cdot a$ is the standard $\mathbb{R}^3$ inner product. When $X = X(t) \in M$ is a smooth function of a real variable t, and when $\prime$ denotes differentiation with respect to t, we have

$$X' = \mathcal{T}_{x'} \ .$$

A computation gives

$$\langle \mathcal{T}_a, \mathcal{T}_b \rangle = \frac{4}{(1+\kappa|x|^2)^2}(a\cdot b) \ , \tag{2.5}$$

$$||\mathcal{T}_a|| = 1 \Leftrightarrow |a| = \tfrac{1}{2}|1+\kappa|x|^2| \ .$$

Remark 2.4. Note that the $\mathbb{R}^{4,1}$ metric $\langle \mathcal{T}_a, \mathcal{T}_b\rangle$ is 4 times the usual $\mathbb{R}^3$ metric $a \cdot b$ of a and b, when $\kappa = 0$. We distinguish between these two metrics by using $\langle , \rangle$ in the first case, and a dot $\cdot$ in the second case. We denote norm squared of a vector with respect to the $\mathbb{R}^{4,1}$ metric with a pair of doubled lines as in $||\mathcal{T}_a||^2$, and norm squared of a vector with respect to the usual $\mathbb{R}^3$ metric with a pair of single lines as in $|a|^2$.

We tolerate this difference by a factor of 4 in order to conform with the usual expression of the metric in the cases when $\kappa \neq 0$.

Also,

$$X'' = \mathcal{T}_{\frac{-4\kappa(x\cdot x')}{1+\kappa|x|^2}\cdot x' + x''} - \frac{4|x'|^2}{(1+\kappa|x|^2)^2}\begin{pmatrix} \kappa x^t \\ \frac{1}{2}(-1-\kappa) \\ \frac{1}{2}(-1+\kappa) \end{pmatrix} . \tag{2.6}$$

Note that generally X'' is not contained in $T_X M_\kappa$.

The following lemma follows from (2.5).

Lemma 2.5. *M_κ as determined by Q_{M_κ} in* (2.3) *has constant sectional curvature κ.*

Proof. We can make the standard computations (see [94], for example): Take

$$D_{\partial_i}\partial_j = \sum_k \Gamma^k_{ij}\partial_k ,$$

where

$$\Gamma^k_{ij} = \tfrac{1}{2}\Sigma_m g^{km}(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij})$$

with

$$g_{ij} = 4\delta_{ij}(1+\kappa|x|^2)^{-2} ,$$

and

$$[D_v, D_w] = D_v D_w - D_w D_v , \quad [\partial_i, \partial_j] = 0 .$$

We can then compute the sectional curvature via

$$K(v,w) = \frac{-\langle D_{[v,w]}v + [D_v, D_w]v, w\rangle}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2} .$$

□

Remark 2.6. To avoid the somewhat long computations in the above proof of Lemma 2.5, we could instead do the following:

When we have a vector space V with metric $\langle\cdot,\cdot\rangle$ given by $(g_{ij}) = (\pm\delta_{ij})$ with respect to some rectangular coordinates $(x_1, ..., x_n)$ of V determined by some choice of orthonormal basis, and a quadric (a submanifold) $\{\sigma \in V \,|\, |\sigma|^2 = c_0\}$, one can compute the sectional curvature of the quadric without using Christoffel symbols. In our case, V will be $\mathbb{R}^4$ when $\kappa > 0$, and will be $\mathbb{R}^{3,1}$ with signature $(-,+,+,+)$ when $\kappa < 0$.

For example, take

$$M = \mathbb{H}^3 = \{\sigma \in \mathbb{R}^{3,1} \,|\, |\sigma|^2 = -1\} \subset \mathbb{R}^{3,1} .$$

Take independent $\mathcal{U}, \mathcal{V} \in T_\sigma M$ and a local coordinate chart

$$\phi : (u,v,w) \to \mathbb{H}^3$$

in a neighborhood of $\sigma = \sigma_0 = \phi(u_0, v_0, w_0) \in M$ so that $\mathcal{U} = \partial_u$ and $\mathcal{V} = \partial_v$ at (u_0, v_0, w_0). We then have $\mathcal{U}$ identified with $\sigma_u = d\phi(\mathcal{U})$, and $\mathcal{V}$ with $\sigma_v = d\phi(\mathcal{V})$, at the point σ_0. Then, for the connection ∇ induced by the metric of $\mathbb{H}^3 \subset \mathbb{R}^{3,1}$,

$$\nabla_{\mathcal{U}}\nabla_{\mathcal{V}}\mathcal{U} - \nabla_{\mathcal{V}}\nabla_{\mathcal{U}}\mathcal{U} =$$

$$\nabla_{\partial_u} d\phi^{-1}(\sigma_{uv} - \frac{\langle\sigma_{uv}, n\rangle}{\langle n,n\rangle}n) - \nabla_{\partial_v} d\phi^{-1}(\sigma_{uu} - \frac{\langle\sigma_{uu}, n\rangle}{\langle n,n\rangle}n) ,$$

where $n = \sigma \perp T_\sigma M$. Thus

$$\nabla_{\mathcal{U}}\nabla_{\mathcal{V}}\mathcal{U} - \nabla_{\mathcal{V}}\nabla_{\mathcal{U}}\mathcal{U} =$$

$$\nabla_{\partial_u} d\phi^{-1}(\sigma_{uv} + \langle \sigma_{uv}, \sigma \rangle \sigma) - \nabla_{\partial_v} d\phi^{-1}(\sigma_{uu} + \langle \sigma_{uu}, \sigma \rangle \sigma) =$$
$$d\phi^{-1}(A - B - (\text{part of } A \text{ normal to } \mathbb{H}^3) + (\text{part of } B \text{ normal to } \mathbb{H}^3)) \ ,$$

where

$$A = \sigma_{uvu} + \langle \sigma_{uvu}, \sigma \rangle \sigma + \langle \sigma_{uv}, \sigma_u \rangle \sigma + \langle \sigma_{uv}, \sigma \rangle \sigma_u$$

and

$$B = \sigma_{uuv} + \langle \sigma_{uuv}, \sigma \rangle \sigma + \langle \sigma_{uu}, \sigma_v \rangle \sigma + \langle \sigma_{uu}, \sigma \rangle \sigma_v \ .$$

Since the normal direction to M is σ, we have

$$\nabla_{\mathcal{U}} \nabla_{\mathcal{V}} \mathcal{U} - \nabla_{\mathcal{V}} \nabla_{\mathcal{U}} \mathcal{U} = d\phi^{-1}(-\langle \sigma_u, \sigma_v \rangle \sigma_u + \langle \sigma_u, \sigma_u \rangle \sigma_v)$$

and then

$$\langle d\phi(\nabla_{\mathcal{U}} \nabla_{\mathcal{V}} \mathcal{U} - \nabla_{\mathcal{V}} \nabla_{\mathcal{U}} \mathcal{U}), d\phi(\mathcal{V}) \rangle = \langle \sigma_u, \sigma_u \rangle \langle \sigma_v, \sigma_v \rangle - \langle \sigma_u, \sigma_v \rangle^2 \ .$$

This implies the sectional curvature of M is constantly -1, justifying our naming $\mathbb{H}^3$ for M.

In this way, we now know that

$$\mathbb{H}^3 = \{\sigma \in \mathbb{R}^{3,1} \,|\, |\sigma|^2 = -1\}$$

has constant sectional curvature $\kappa = -1$, and similarly that

$$\mathbb{S}^3 = \{\sigma \in \mathbb{R}^4 \,|\, |\sigma|^2 = 1\}$$

has constant sectional curvature $\kappa = 1$. $\mathbb{H}^3$ can be inserted into $\mathbb{R}^{4,1}$ with signature $(+,+,+,+,-)$ by replacing $\sigma = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ with $(\sigma_1, \sigma_2, \sigma_3, -1, \sigma_0)^t$. Similarly, $\mathbb{S}^3$ can be inserted into $\mathbb{R}^{4,1}$ by replacing $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ with $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, 1)^t$.

General values κ for the sectional curvature can be dealt with in the same way.

We see from (2.5) that the collection of M_κ given by the above choice (2.3) for Q, for various κ, are all conformally equivalent (or Möbius equivalent). In fact, the map $M_\kappa \ni X \to x \in \mathfrak{R}$ is stereographic projection when $\kappa \neq 0$.

2.2. **Surfaces in spaceforms.** We now consider surfaces in the spaceforms. Let

$$x = (x_1(u,v), x_2(u,v), x_3(u,v)) \leftrightarrow X = X(u,v) \in M_\kappa$$

be a surface in M_κ without umbilic points. Assume x has conformal curvature-line coordinates (u,v). We call such coordinates *isothermic* coordinates. Thus x is an isothermic surface (where "isothermic surface" means any surface for which isothermic coordinates exist).

Note that the surface x can be defined before the space form M_κ is chosen, and only once M_κ is chosen do we know the form of X as in (2.4).

Notation: With Q_{M_κ} as in (2.3), we let n denote the unit normal vector for x, once M_κ is chosen. n_0 denotes the unit normal with respect to Euclidean 3-space M_0. We sometimes write H_κ for the mean curvature of the surface x with respect to the spaceform M_κ, because the mean curvature depends on the choice of κ.

Lemma 2.7. *The mean curvature H_κ of x with respect to the space form M_κ given by Q_{M_κ} as in* (2.3)*, with $\triangle x = \partial_u \partial_u x + \partial_v \partial_v x$, is*

$$H_\kappa = \tfrac{1}{2}|x_u|^{-2}(\triangle x \cdot n) + 2\frac{\kappa}{1+\kappa|x|^2}(x \cdot n) =$$
$$= \tfrac{1}{2}(1+\kappa|x|^2)|x_u|^{-2}(\triangle x \cdot n_0) + 2\kappa(x \cdot n_0) =$$
$$(1+\kappa|x|^2)H_0 + 2\kappa(x \cdot n_0) \ .$$

Then H_κ is constant exactly when $\partial_u H_\kappa = \partial_v H_\kappa = 0$, which is equivalent to

$$(2.7)\quad (\partial_u H_0)\cdot(1+\kappa|x|^2) = \kappa\frac{k_1-k_2}{2}\partial_u(|x|^2)\ , \quad (\partial_v H_0)\cdot(1+\kappa|x|^2) = \kappa\frac{k_2-k_1}{2}\partial_v(|x|^2)\ ,$$

where the $k_j \in \mathbb{R}$ are the principal curvatures with respect to the Euclidean spaceform M_0, i.e. $\partial_u n_0 = -k_1\partial_u x$ and $\partial_v n_0 = -k_2 \partial_v x$. Furthermore, the Gaussian curvature of x with respect to the spaceform M_κ is

$$K_\kappa = (1+\kappa|x|^2)^2 K_0 + 4H_0\kappa(x\cdot n_0) + 4\kappa^2(x\cdot n_0)^2\ .$$

Proof. Letting x_{1u} denote $\frac{d}{du}(x_1)$, and similarly taking other notations, the unit normal vector to the surface is $\mathcal{T}_n$, where $n = (1+\kappa|x|^2)n_0$ and

$$n_0 = \frac{1}{2}\cdot\frac{(x_{2u}x_{3v}-x_{3u}x_{2v}, x_{3u}x_{1v}-x_{1u}x_{3v}, x_{1u}x_{2v}-x_{2u}x_{1v})}{\sqrt{(x_{2u}x_{3v}-x_{3u}x_{2v})^2+(x_{3u}x_{1v}-x_{1u}x_{3v})^2+(x_{1u}x_{2v}-x_{2u}x_{1v})^2}}\ .$$

The first fundamental form (g_{ij}) satisfies $\langle \mathcal{T}_{x_u}, \mathcal{T}_{x_v}\rangle = 0 = g_{12} = g_{21}$, and

$$g_{11} = \langle \mathcal{T}_{x_u}, \mathcal{T}_{x_u}\rangle = \frac{4|x_u|^2}{(1+\kappa|x|^2)^2} = \frac{4|x_v|^2}{(1+\kappa|x|^2)^2} = \langle \mathcal{T}_{x_v}, \mathcal{T}_{x_v}\rangle = g_{22}\ .$$

Then using (2.6), with the symbol $'$ denoting either ∂_u or ∂_v, we have (where the superscript "T" denotes the perpendicular projection to a vector tangent to M_κ, i.e in $T_X M_\kappa$)

$$b_{11} = \langle X_{uu}^T, \mathcal{T}_n\rangle = \langle X_{uu}, \mathcal{T}_n\rangle = \frac{4}{(1+\kappa|x|^2)^2}(x_{uu}\cdot n) + \frac{8\kappa|x_u|^2}{(1+\kappa|x|^2)^3}(x\cdot n)\ ,$$

$$b_{12} = b_{21} = \langle X_{uv}^T, \mathcal{T}_n\rangle = \langle X_{uv}, \mathcal{T}_n\rangle = 0\ ,$$

$$b_{22} = \langle X_{vv}^T, \mathcal{T}_n\rangle = \langle X_{vv}, \mathcal{T}_n\rangle = \frac{4}{(1+\kappa|x|^2)^2}(x_{vv}\cdot n) + \frac{8\kappa|x_v|^2}{(1+\kappa|x|^2)^3}(x\cdot n)\ .$$

The result about H_κ follows, using $H_0 = (k_1+k_2)/2$. The form of K_κ is similarly obtained. □

2.3. **Möbius transformations.** The Möbius transformations are the maps from $\mathbb{S}^3$ to $\mathbb{S}^3$ that take 2-spheres to 2-spheres, and they are equivalent to the orthogonal transformations $O_{4,1}$ of $\mathbb{R}^{4,1}$. We will not prove that here, and instead just give some examples. Identifying $\mathbb{S}^3$ with $\mathbb{R}^3\cup\{\infty\}$, the group of Möbius transformations (including orientation reversing maps), i.e. the group of maps that preserve the collection of spheres and planes in $\mathbb{R}^3\cup\{\infty\}$, is generated (via repeated composition of maps) by:

$$(y_1,y_2,y_3)\to(ry_1,ry_2,ry_3)\ ,$$

$$(y_1,y_2,y_3)\to(y_1,y_2,y_3)+(y_{0,1},y_{0,2},y_{0,3})\ ,$$

$$(y_1,y_2,y_3)\to(-y_1,y_2,y_3)\ ,$$

$$(y_1,y_2,y_3)\to(y_1\cos\theta - y_2\sin\theta, y_1\sin\theta + y_2\cos\theta, y_3)\ ,$$

$$(y_1,y_2,y_3)\to(y_1\cos\theta - y_3\sin\theta, y_2, y_1\sin\theta + y_3\cos\theta)\ ,$$

$$(y_1,y_2,y_3)\to(y_1, y_2\cos\theta - y_3\sin\theta, y_2\sin\theta + y_3\cos\theta)\ ,$$

$$(y_1,y_2,y_3)\to(y_1,y_2,y_3)/(y_1^2+y_2^2+y_3^2)\ ,$$

where $\theta, r, y_{0,1}, y_{0,2}, y_{0,3}$ are any real constants. These seven maps are a dilation, a translation, a reflection, three rotations, and an inversion, respectively.

The Möbius transformations are given by the application of $O_{4,1}$ matrices to the vectors X in (2.4). The identity map (a Möbius transformation of course) is given by left multiplication to X by the identity matrix in $O_{4,1}$. The map

$$(y_1, y_2, y_3) \to (-y_1, y_2, y_3)$$

is similarly given by the diagonal matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} .$$

The map

$$(y_1, y_2, y_3) \to (y_1, y_2, y_3)/(y_1^2 + y_2^2 + y_3^2)$$

is given by the diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} ,$$

after invoking Remark 2.3. The map

$$(y_1, y_2, y_3) \to (ry_1, ry_2, ry_3)$$

for positive constants r is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cosh\theta & \sinh\theta \\ 0 & 0 & 0 & \sinh\theta & \cosh\theta \end{pmatrix} ,$$

where θ satisfies $\cosh\theta + \sinh\theta = r$, again invoking Remark 2.3. The map

$$(y_1, y_2, y_3) \to (y_1, y_2, y_3) + (y_{0,1}, y_{0,2}, y_{0,3})$$

is given by

$$\begin{pmatrix} 1 & 0 & 0 & -y_{0,1} & y_{0,1} \\ 0 & 1 & 0 & -y_{0,2} & y_{0,2} \\ 0 & 0 & 1 & -y_{0,3} & y_{0,3} \\ y_{0,1} & y_{0,2} & y_{0,3} & 1 - \frac{1}{2}|y_0|^2 & \frac{1}{2}|y_0|^2 \\ y_{0,1} & y_{0,2} & y_{0,3} & -\frac{1}{2}|y_0|^2 & 1 + \frac{1}{2}|y_0|^2 \end{pmatrix} ,$$

where $|y_0|^2 = y_{0,1}^2 + y_{0,2}^2 + y_{0,3}^2$, once again using Remark 2.3. The map

$$(y_1, y_2, y_3) \to (y_1 \cos\theta - y_2 \sin\theta, y_1 \sin\theta + y_2 \cos\theta, y_3)$$

is given by

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} .$$

When $\kappa \neq 0$, i.e. when Q as in (2.3) is not null, M_κ has a particular Möbius transformation called the antipodal map, which we now describe: A point X in M_κ can be decomposed as

$$X = \mathcal{A} + \kappa^{-1} Q ,$$

where $\mathcal{A} \perp Q$. The antipodal map is

$$\mathcal{A} + \kappa^{-1} Q \to -\mathcal{A} + \kappa^{-1} Q ,$$

that is, we are moving the point X to another point in M_κ that is on the opposite side of Q. In detail, X as in (2.4) is

$$\frac{1}{1+\kappa|x|^2} \begin{pmatrix} 2x^t \\ \frac{1}{2}(\kappa^{-1}+1)(\kappa|x|^2-1) \\ \frac{1}{2}(\kappa^{-1}-1)(\kappa|x|^2-1) \end{pmatrix} + \kappa^{-1} Q$$

and is mapped to

$$\frac{-1}{1+\kappa|x|^2} \begin{pmatrix} 2x^t \\ \frac{1}{2}(\kappa^{-1}+1)(\kappa|x|^2-1) \\ \frac{1}{2}(\kappa^{-1}-1)(\kappa|x|^2-1) \end{pmatrix} + \kappa^{-1} Q = \frac{1}{1+\kappa|y|^2} \begin{pmatrix} 2y^t \\ |y|^2-1 \\ |y|^2+1 \end{pmatrix} ,$$

where $y = -x/(\kappa|x|^2)$. Hence this map is represented by

$$x \to -x/(\kappa|x|^2) .$$

Remark 2.8. Möbius transformations of the ambient space preserve the conformal structure of the space, so will preserve the conformal structure of any surface inside the space as well. Furthermore, Möbius transformations will preserve contact orders of any spheres tangent to the surface, and so will preserve the principal curvature spheres. It follows that if $x(u,v)$ is an isothermic parametrization of a surface, it will remain an isothermic parametrization even after a Möbius transformation is applied. Furthermore, an umbilic point of x will remain an umbilic point after the Möbius transformation is applied.

2.4. **Cross ratios.** There is no evidently simple geometric interpretation for the cross ratio, but it is still very useful. It is an invariant of projective geometry. (See [124], for example.) We use cross ratios in this text, and they will be vital to understanding discrete Ω surfaces (in a subsequent text).

Definition 2.9. *The cross ratio of four points x_p, x_q, x_r and x_s in $\mathbb{R}^3$ is*

$$cr_{x_p x_q x_r x_s} = \frac{(z_p - z_q)(z_r - z_s)}{(z_q - z_r)(z_s - z_p)} = \frac{\det\begin{pmatrix} z_p & z_q \\ 1 & 1 \end{pmatrix} \det\begin{pmatrix} z_r & z_s \\ 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} z_q & z_r \\ 1 & 1 \end{pmatrix} \det\begin{pmatrix} z_s & z_p \\ 1 & 1 \end{pmatrix}} ,$$

where the points z_p, z_q, z_r and z_s are complex numbers whose real and imaginary parts are given by the first two coordinates in $\mathbb{R}^3$ of the four image points of x_p, x_q, x_r and

x_s under a Möbius transformation of $\mathbb{R}^3 \cup \{\infty\}$ that takes all four points x_p, x_q, x_r and x_s to points with third coordinate zero in $\mathbb{R}^3$.

Lemma 2.10. *The cross ratio given in Definition 2.9 is well defined up to complex conjugation.*

Proving Lemma 2.10 amounts to showing that the resulting value for $\mathrm{cr}_{x_p x_q x_r x_s}$ will be the same, up to complex conjugation, regardless of which Möbius transformation we choose. Lemma 2.10 follows because any two choices of Möbius transformation in Definition 2.9 will differ, when restricted to

$$\{y_3 \equiv 0\} \cap \{\mathbb{R}^3 \cup \{\infty\}\}\ ,$$

by a Möbius transformation of the complex plane in which z_p, z_q, z_r and z_s lie.

For $x_\beta, x_\gamma \in \mathbb{R}^3$, taking corresponding $X_\beta, X_\gamma \in M_\kappa$ as in (2.4), we have

$$\langle X_\beta, X_\gamma \rangle_{\mathbb{R}^{4,1}} = \frac{-2|x_\beta - x_\gamma|^2}{(1+\kappa|x_\beta|^2)(1+\kappa|x_\gamma|^2)}\ . \tag{2.8}$$

As in Remark 2.3, we can freely scale X_β and X_γ to $\alpha_\beta X_\beta$ and $\alpha_\gamma X_\gamma$, and then $\langle X_\beta, X_\gamma\rangle$ will scale to $\alpha_\beta\alpha_\gamma\langle X_\beta, X_\gamma\rangle$. However, writing the cross ratio in terms of such inner products, we find it is invariant under such scalings, because, using the fact that Möbius transformations can be represented by matrices in $O_{4,1}$, and using (2.8), a direct computation gives:

Lemma 2.11. *For $x_p, x_q, x_r, x_s \in \mathbb{R}^3$, we have*

$$cr_{x_p x_q x_r x_s} = \frac{s_{pq}s_{rs} - s_{pr}s_{qs} + s_{ps}s_{qr} \pm \sqrt{\mathcal{E}}}{2 s_{ps} s_{qr}}\ ,$$

where

$$s_{\beta\gamma} := \langle X_\beta, X_\gamma\rangle$$

and

$$\mathcal{E} = s_{pq}^2 s_{rs}^2 + s_{pr}^2 s_{qs}^2 + s_{ps}^2 s_{qr}^2 - 2 s_{pr}s_{ps}s_{qr}s_{qs} - 2 s_{pq}s_{ps}s_{qr}s_{rs} - 2 s_{pq}s_{pr}s_{qs}s_{rs} \le 0\ .$$

Remark 2.12. Because the X_p, X_q, X_r, X_s all lie in the light cone, the induced metric on the subspace $\mathrm{span}\{X_p, X_q, X_r, X_s\}$ is not positive definite. Therefore, we can choose a basis e_1, e_2, e_3, e_4 of this subspace so that

$$||e_1||^2 = ||e_2||^2 = ||e_3||^2 = -||e_4||^2 = 1 \text{ and } \langle e_i, e_j\rangle = 0 \ \ (i \neq j)\ .$$

Writing $X_\beta = a_{1\beta}e_1 + a_{2\beta}e_2 + a_{3\beta}e_3 + a_{4\beta}e_4$ in terms of the basis e_1, e_2, e_3, e_4, we have

$$\mathcal{E} = \det(\langle X_\beta, X_\gamma\rangle_{\beta,\gamma=p,q,r,s}) =$$

$$= \det\left(\begin{pmatrix} a_{1p} & a_{1q} & a_{1r} & a_{1s} \\ a_{2p} & a_{2q} & a_{2r} & a_{2s} \\ a_{3p} & a_{3q} & a_{3r} & a_{3s} \\ a_{4p} & a_{4q} & a_{4r} & a_{4s}\end{pmatrix}^t \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1\end{pmatrix} \begin{pmatrix} a_{1p} & a_{1q} & a_{1r} & a_{1s} \\ a_{2p} & a_{2q} & a_{2r} & a_{2s} \\ a_{3p} & a_{3q} & a_{3r} & a_{3s} \\ a_{4p} & a_{4q} & a_{4r} & a_{4s}\end{pmatrix}\right)\ .$$

This provides a reason why $\mathcal{E} \le 0$.

We finish this section with some further comments on how we can interpret cross ratios:

Comment 1: Given four points $p_1 < p_2 < p_3 < p_4$ along the real line $\mathbb{R}$ in the complex plane $\mathbb{C}$, we can consider the two half circles from p_1 to p_3 and from p_2 to

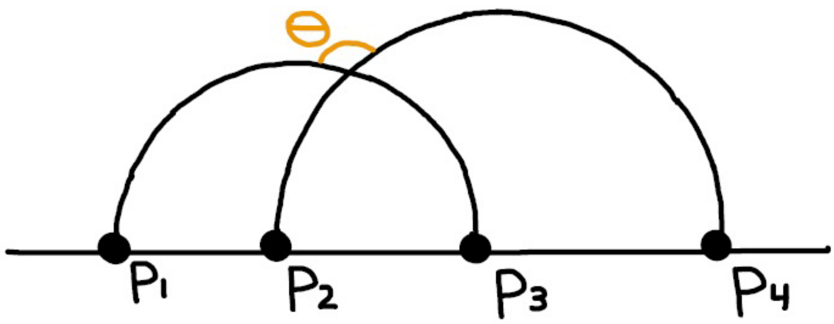

FIGURE 2.3. The situation in Comment 1

p_4, respectively, in the upper half plane $\mathbb{C}^{i+} = \{x + iy \,|\, y > 0\}$. These are images of geodesics (straight lines, and "diagonals" of the "quadrilateral" $\{p_1, p_2, p_3, p_4\}$) in $\mathbb{C}^{i+}$ when $\mathbb{C}^{i+}$ is given the hyperbolic metric. At the point where these two circles intersect, we can consider the angle θ between the two circles with respect to the uppermost adjacent region. This angle θ can be regarded as a geometric quantity, since it is a notion that is invariant under Möbius transformations of $\mathbb{C} \cup \{\infty\}$ that preserve the set $\mathbb{R} \cup \{\infty\}$. See Figure 2.3. Let

$$\text{cr} = \frac{p_2 - p_1}{p_3 - p_2} \cdot \frac{p_4 - p_3}{p_1 - p_4} \tag{2.9}$$

be the cross ratio of p_1, p_2, p_3 and p_4. Here we determine the precise relationship between θ and cr, thus giving cr also a geometric meaning.

Lemma 2.13. *For four points $p_1 < p_2 < p_3 < p_4$ in the real line $\mathbb{R}$ in $\mathbb{C}$, and for θ and cr as defined above,*

$$cr = \frac{\cos\theta + 1}{\cos\theta - 1} = -\cot^2 \tfrac{\theta}{2} \ .$$

Proof. Without loss of generality, applying a Möbius transformation fixing $\mathbb{R} \cup \{\infty\}$ as a set if necessary, we may assume that

$$p_4 = \infty \ , \quad p_1 = 1 > p_2 > p_3 = -1 \ .$$

Then one half-circle is

$$\{e^{it} \,|\, t \in [0, \pi]\}$$

and the other is the half-line

$$\{p_2 + it \,|\, t \geq 0\} \ .$$

Then θ is the angle between these two half-circles with respect to the adjacent region to the upper right. It can be immediately checked that $\cos\theta = -p_2$ and that

$$\text{cr} = \frac{p_2 - 1}{-1 - p_2} \cdot \frac{\infty - (-1)}{1 - \infty} = \frac{p_2 - 1}{p_2 + 1} \ .$$

The result follows. □

Comment 2: Consider $\mathbb{R}^{2,1} = \{(x_1, x_2, x_0) \,|\, x_j \in \mathbb{R}\}$ with metric of signature $(+, +, -)$ and with 2-dimensional light cone L^2. Let ℓ_1, ℓ_2, ℓ_3, ℓ_4 be four lines in L^2 that are pairwise nonparallel.

Let P be a plane in $\mathbb{R}^{2,1}$ that does not contain the origin. Then $\mathcal{C} = L^2 \cap P$ will be a conic section. There will be four intersection points $y_j = \ell_j \cap \mathcal{C}$ (possibly including one or two points at infinity of the conic section $\mathcal{C}$). Let $\mathcal{L}$ be a line in P and Y a line in L^2. Then one can stereographically project y_j within P though the point $Y \cap P \in \mathcal{C}$ to a point $p_j \in \mathcal{L}$. See Figure 2.4. We have the following fact:

Lemma 2.14. *Regarding $\mathcal{L}$ as the real line, and then computing the cross ratio cr of the four points p_1, p_2, p_3, p_4 as in* (2.9)*, the value of cr is independent of the choices of P and Y and $\mathcal{L}$.*

Proof. We first show that cr is independent of choice of P. Suppose we have chosen one line Y in L^2 and have made two different choices P_1 and P_2 for the plane. Choosing the line $\mathcal{L}$ to be $\mathcal{L} = P_1 \cap P_2$, stereographic projection of either

$$\ell_j \cap P_1 \text{ to } \mathcal{L} \text{ though } Y \cap P_1 \text{ within } P_1$$

or

$$\ell_j \cap P_2 \text{ to } \mathcal{L} \text{ though } Y \cap P_2 \text{ within } P_2$$

will produce the same four points p_j in $\mathcal{L}$, and so cr does not depend on whether P_1 or P_2 was used.

Thus, without loss of generality, we may fix one choice of P. Then elementary calculations will show that cr also does not depend on the choices of Y and $\mathcal{L}$. □

Comment 3: When the plane P in Comment 2 is chosen to be

$$P = \{(x_1, x_2, 1) \,|\, x_1, x_2 \in \mathbb{R}\} ,$$

clearly P can be regarded as a plane in the standard Euclidean $\mathbb{R}^3$, and it follows that cr $\in \mathbb{R}$ can be computed using the formula in Lemma 2.11, with X_β replaced by y_j. More explicitly, we could consider

$$y_j = (y_{1,j}, y_{2,j}, 1) \in L^2$$

and translate these points to $(y_{1,j}, y_{2,j}, 0) \in \mathbb{R}^3$ with $y_{1,j}^2 + y_{2,j}^2 = 1$, which would then lift to

$$Y_j = \begin{pmatrix} 2y_{1,j} \\ 2y_{2,j} \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

via the L^4 representation in (2.4). We already know we can insert Y_j into the formula in Lemma 2.11 to determine the cross ratio. However, $Y_j \approx 2y_j$, so we know it is legitimate to use y_j in the formula in Lemma 2.11 as well.

By the homogeneous nature of the formula in Lemma 2.11, we know that formula applies regardless of which points

$$\hat{y}_j = r_j y_j \quad (r_j \in \mathbb{R} \setminus \{0\})$$

in ℓ_j are chosen.

Comment 4: When the plane P in Comment 2 is again chosen as in Comment 3, the points y_j can be regarded as unitary complex numbers

$$\hat{y}_j := y_{1,j} + iy_{2,j} = e^{i\theta_j}$$

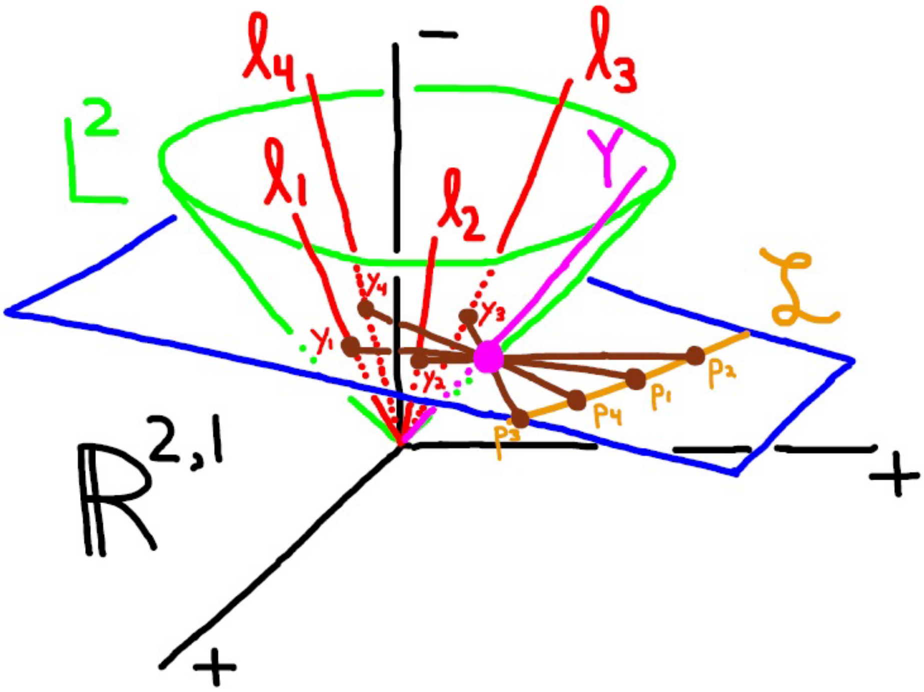

FIGURE 2.4. The situation in Comment 2

for $\theta_j \in \mathbb{R}$, and then

$$\text{cr} = \frac{\hat{y}_2 - \hat{y}_1}{\hat{y}_3 - \hat{y}_2} \cdot \frac{\hat{y}_4 - \hat{y}_3}{\hat{y}_1 - \hat{y}_4} = \frac{e^{i\theta_2} - e^{i\theta_1}}{e^{i\theta_3} - e^{i\theta_2}} \cdot \frac{e^{i\theta_4} - e^{i\theta_3}}{e^{i\theta_1} - e^{i\theta_4}} \tag{2.10}$$

will give the same value for cr.

Comment 5: Here we give a way to compute cr using matrices in the Lie algebra $\text{su}_{1,1}$. Identifying $\mathbb{R}^{2,1}$ with $\text{su}_{1,1}$ via

$$x_\beta = (x_1, x_2, x_0) \leftrightarrow \begin{pmatrix} ix_0 & x_1 + ix_2 \\ x_1 - ix_2 & -ix_0 \end{pmatrix} =: X_\beta \ ,$$

and then taking $X_p, X_q, X_r, X_s \in \text{su}_{1,1} \approx \mathbb{R}^{2,1}$, let us denote the eigenvalues of

$$(X_p - X_q)(X_q - X_r)^{-1}(X_r - X_s)(X_s - X_p)^{-1} \tag{2.11}$$

by λ_1, λ_2.

Lemma 2.15. *λ_1, λ_2 are invariant under isometries and homotheties of $\mathbb{R}^{2,1}$.*

Proof. It is evident that a homothety $X_\beta \to rX_\beta$ for some $r \in \mathbb{R} \setminus \{0\}$ will not change λ_1, λ_2.

A rotation of $\mathbb{R}^{2,1}$ is represented by

$$\text{su}_{1,1} \ni X \to FXF^{-1} \in \text{su}_{1,1}$$

for some $F \in \text{SU}_{1,1}$. This transformation also will not change λ_1, λ_2. □

One can also check the following:

Lemma 2.16. *The λ_1 and λ_2 in Lemma 2.15 are either real or complex conjugate, that is,*

$$\lambda_1, \lambda_2 \in \mathbb{R} \quad \text{or} \quad \overline{\lambda_1} = \lambda_2 \ .$$

Furthermore, when $X_\beta \in L^2$, i.e. when $\det X_\beta = 0$ for $\beta = p, q, r, s$, then

$$\lambda := \lambda_1 = \lambda_2 \in \mathbb{R} \ .$$

Lemma 2.17. *When x_0 has the same value for all four points $X_p, X_q, X_r, X_s \in L^2$, then λ is equal to $\mathrm{cr}_{x_p x_q x_r x_s}$.*

Proof. Setting $X_1 = X_p$, $X_2 = X_q$, $X_3 = X_r$ and $X_4 = X_s$ and rewriting corresponding x_j as $(x_{1,j}, x_{2,j}, x_0)$, and noting that x_0 is independent of j, we have

$$(X_1 - X_2)(X_2 - X_3)^{-1}(X_3 - X_4)(X_4 - X_1)^{-1} =$$

$$\prod_{j=1}^{4} \begin{pmatrix} 0 & x_{1,1j} - x_{1,j+1} + i(x_{2,j} - x_{2,j+1}) \\ x_{1,j} - x_{1,j+1} - i(x_{2,j} - x_{2,j+1}) & 0 \end{pmatrix}^{j+1} =$$

$$\begin{pmatrix} \mathrm{cr} & 0 \\ 0 & \overline{\mathrm{cr}} \end{pmatrix} ,$$

where

$$\mathrm{cr} = \frac{x_{1,1} - x_{1,2} + i(x_{2,1} - x_{2,2})}{x_{1,2} - x_{1,3} + i(x_{2,2} - x_{2,3})} \cdot \frac{x_{1,3} - x_{1,4} + i(x_{2,3} - x_{2,4})}{x_{1,4} - x_{1,1} + i(x_{2,4} - x_{2,1})} \ .$$

Applying Comment 4, it follows that $\lambda = \mathrm{cr}$. □

A computation shows:

Lemma 2.18. *When*

$$X_1 = X_p, X_2 = X_q, X_3 = X_r, X_4 = X_s \in L^2 \ ,$$

then, for any $r_j \in \mathbb{R} \setminus \{0\}$,

$$(r_1 X_1 - r_2 X_2)(r_2 X_2 - r_3 X_3)^{-1}(r_3 X_3 - r_4 X_4)(r_4 X_4 - r_1 X_1)^{-1}$$

and (2.11) will have the same eigenvalue $\lambda = cr_{x_p x_q x_r x_s}$.

We conclude that cr will be the eigenvalue of (2.11), regardless of how the X_j are scaled, when $X_j \in L^2$.

Comment 6: Similarly to Comment 5, we can use the Lie algebra su_2 to compute the cross ratio cr in the case of four points $(x_{1,j}, x_{2,j}, x_{3,j}) \in \mathbb{R}^3$ $(j = 1, ..., 4)$. We identify $\mathbb{R}^3$ with su_2 via

$$(x_1, x_2, x_3) \leftrightarrow \begin{pmatrix} ix_3 & x_1 + ix_2 \\ -x_1 + ix_2 & -ix_3 \end{pmatrix} =: X \ .$$

Then cr will again be the eigenvalues of (2.11), with the X_j now in su_2. The analog of Lemma 2.15 will still hold. However, there will be no analog of Lemma 2.18 here, as $\mathbb{R}^3$ has no comparable light cone, so we will not be free to scale the X_j in this case.

FIGURE 2.5. A physical model of an ellipsoid in $\mathbb{R}^3$, with curvature lines carved into it (owned by the geometry group at the Technical University of Vienna)

2.5. **Isothermicity.** Take a smooth surface x in $\mathbb{R}^3$ with unit normal n_0. Away from umbilics, there exist curvature line coordinates (u, v) of $x = x(u, v)$, i.e.

$$x_u \perp x_v \quad \text{and} \quad (n_0)_u || x_u \ , \quad (n_0)_v || x_v \ .$$

Then the first and second fundamental forms are, like in the proof of Lemma 2.7 with $\kappa = 0$,

$$I = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix} \ , \quad II = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix} \ .$$

One can always stretch the coordinates, so that $x = x(u, v) = x(\tilde{u}(u), \tilde{v}(v))$ for any strictly monotonic functions $\tilde{u}$ depending only on u, and $\tilde{v}$ depending only on v. Note that $x_{\tilde{u}} \cdot x_{\tilde{v}} = 0$, and $x_{\tilde{u}\tilde{v}} = x_{uv} \frac{du}{d\tilde{u}} \frac{dv}{d\tilde{v}}$ implies $x_{\tilde{u}\tilde{v}} \cdot n_0 = 0$, so $(\tilde{u}, \tilde{v})$ are also curvature line coordinates. The surface is then isothermic if and only if there exist $\tilde{u}$, $\tilde{v}$ such that the metric becomes conformal, i.e. $x_{\tilde{u}} \cdot x_{\tilde{u}} = x_{\tilde{v}} \cdot x_{\tilde{v}}$, and this is equivalent to

$$\frac{g_{11}}{g_{22}} = \frac{a(u)}{b(v)} \ ,$$

where the function a depends only on u, and b depends only on v.

Now consider the cross ratio cr_ϵ of the four points $x(u, v)$, $x(u+\epsilon, v)$, $x(u+\epsilon, v+\epsilon)$ and $x(u, v + \epsilon)$. A computation gives

$$\lim_{\epsilon \to 0} \mathrm{cr}_\epsilon = -\frac{g_{11}}{g_{22}} \ . \tag{2.12}$$

So x is isothermic if and only if

$$\lim_{\epsilon \to 0} \mathrm{cr}_\epsilon = -\frac{a(u)}{b(v)} \ . \tag{2.13}$$

This description of isothermicity does not involve any stretching by $\tilde{u}$ or $\tilde{v}$, which we would not be able to do in the case of discrete surfaces anyways, and discrete surfaces, as noted in the introduction, are one of our primary motivations here. The corresponding statement for discrete surfaces, where stretching of coordinates is no longer possible, is that the surface is discrete isothermic if and only if the cross ratio factorizing function can be chosen so that $a_{pq} = a_{rs}$ and $a_{ps} = a_{qr}$ for vertices p, q, r, s (in order) about a given quadrilateral, but this will be explained in a separate text.

2.6. **The third fundamental form.** When x has curvature line coordinates, the first and second fundamental forms are

$$I = (g_{ij})_{i,j=1}^2 = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix} \; , \quad II = (b_{ij})_{i,j=1}^2 = \begin{pmatrix} k_1 E & 0 \\ 0 & k_2 G \end{pmatrix} \; .$$

Because $(n_0)_u = -k_1 x_u$ and $(n_0)_v = -k_2 x_v$, the third fundamental form is

$$III = \begin{pmatrix} 4(n_0)_u \cdot (n_0)_u & 4(n_0)_u \cdot (n_0)_v \\ 4(n_0)_v \cdot (n_0)_u & 4(n_0)_v \cdot (n_0)_v \end{pmatrix} = \begin{pmatrix} k_1^2 E & 0 \\ 0 & k_2^2 G \end{pmatrix} \; ,$$

and then it is immediate that

$$III - 2H \cdot II + K \cdot I = 0 \; .$$

In fact, for any (not necessarily curvature line) choice of coordinates u and v,

$$III = \frac{g_{22}b_{11} - 2g_{12}b_{12} + g_{11}b_{22}}{g_{11}g_{22} - g_{12}^2} (b_{ij})_{i,j=1}^2 - \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2} (g_{ij})_{i,j=1}^2 \; .$$

2.7. **Moutard lifts.** Given an immersion $x(u,v)$ in $\mathbb{R}^3$, the light cone lift

$$X = X(u,v) \in M_0$$

could also be represented by $\alpha \cdot X$ for any choice of nonzero real-valued function $\alpha = \alpha(u,v)$ (see Remark 2.3). If we can choose α and coordinates u, v so that

$$\partial_u \partial_v (\alpha X) || X \; , \qquad (2.14)$$

or equivalently $\alpha_u x_v + \alpha_v x_u + \alpha x_{uv} = 0$, then we say that αX is a *Moutard lift.* We will see the usefulness of Moutard lifts later.

Lemma 2.19. *Moutard lifts exist for any isothermic immersion.*

Proof. Let $x(u,v)$ be a smooth isothermic immersion with isothermic coordinates u, v. Because $b_{12} = 0$, there exist real-valued functions A, B so that

$$x_{uv} = A x_u + B x_v \; . \qquad (2.15)$$

Taking the inner product of this with x_u and with x_v, and using that

$$x_u \cdot x_u = x_v \cdot x_v > 0 \quad \text{and} \quad x_u \cdot x_v = 0 \; ,$$

we find that

$$A = \partial_v(\tfrac{1}{2} \log(x_u \cdot x_u)) \; , \quad B = \partial_u(\tfrac{1}{2} \log(x_u \cdot x_u)) \; , \qquad (2.16)$$

and thus

$$A_u = B_v \; . \qquad (2.17)$$

The existence of a solution α to the equation

$$\alpha_u x_v + \alpha_v x_u + \alpha x_{uv} = 0$$

is equivalent to solving the system

$$\alpha_u = -\alpha B \; , \quad \alpha_v = -\alpha A \; ,$$

because of (2.15). The compatibility condition of this system is (2.17), seen as follows:

$$\alpha_{uv} = \alpha_{vu} \to (-\alpha B)_v = (-\alpha A)_u \to$$

$$\alpha_v B + \alpha B_v = \alpha_u A + \alpha A_u \rightarrow$$
$$(-\alpha A)B + \alpha B_v = (-\alpha B)A + \alpha A_u \rightarrow B_v = A_u \ .$$

This proves the lemma. □

From the above proof, we see that the isothermic coordinates u, v are the same as the coordinates for which the Moutard equation (2.14) holds.

2.8. **Spheres.** The spheres in any of the spaceforms M_κ are the surfaces x such that $4|x - C_0|^2$ (the square of the radius r_0 in the case of Euclidean space) is constant for some constant $C_0 \in \mathbb{R}^3$, generically. In the case $\kappa = 0$, with a sphere has radius r_0, then $r_0 H_0 = \pm 1$. The sphere can be parametrized, and then

$$x = x(u, v) = (-1/H_0) n_0(u, v) + C_0$$

for some constant C_0. Then the equation $H_\kappa = (1 + \kappa|x|^2)H_0 + 2\kappa(x \cdot n_0)$ in Lemma 2.7 implies the following formula (note that $|n_0|^2 = \frac{1}{4}$)

$$H_\kappa = H_0 - \frac{\kappa}{4H_0} + H_0 \kappa |C_0|^2 \tag{2.18}$$

for the relationships between the different mean curvatures for a sphere considered in the different spaceforms M_κ.

A point

$$\mathcal{S} = \begin{pmatrix} z^t \\ z_4 \\ z_5 \end{pmatrix} \ , \quad z = (z_1, z_2, z_3) \ ,$$

in $\mathbb{R}^{4,1}$ with positive norm

$$||\mathcal{S}||^2 = |z|^2 + z_4^2 - z_5^2 > 0$$

determines a sphere $\tilde{\mathcal{S}}$ in the spaceform M_κ as follows: Set

$$\tilde{\mathcal{S}} = \{ Y \in M_\kappa \,|\, \langle Y, \mathcal{S} \rangle = 0 \} \ . \tag{2.19}$$

See Figure 2.6.

Note that $Y \in \tilde{\mathcal{S}}$ implies Y is perpendicular to $\mathcal{S} - Y$, so $\tilde{\mathcal{S}}$ is the base of the tangent cone from $\mathcal{S}$ to M_κ, once $\mathcal{S}$ is scaled so that $\langle \mathcal{S}, Q_{M_\kappa} \rangle = -1$. (Note that $\tilde{\mathcal{S}}$ is invariant under scalings of $\mathcal{S}$.) In fact, $Y - \mathcal{S}$ is then a normal vector field to $\tilde{\mathcal{S}}$ within the tangent space of M_κ.

So we have now seen how both points *and* spheres in the spaceforms can be described by using vectors in the single space $\mathbb{R}^{4,1}$, respectively lightlike and spacelike vectors, which is a valuable property from the viewpoint of Möbius geometry.

If $\mathcal{S}$ satisfies $z_5 = 0$, then $\tilde{\mathcal{S}}$ is a great sphere in $M_1 = \mathbb{S}^3$. Also, note that if $||\mathcal{S}||^2 = 0$, then $\mathcal{S}$ is a point in $\mathbb{S}^3$ and $\tilde{\mathcal{S}}$ consists of just a real scalar multiple of $\mathcal{S}$, hence $\tilde{\mathcal{S}}$ simply gives back the same point $\mathcal{S}$.

Let ℓ be the horizontal line segment from $\mathcal{S}$ to the timelike axis

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ t \end{pmatrix} \middle| \ t \in \mathbb{R} \right\} \ .$$

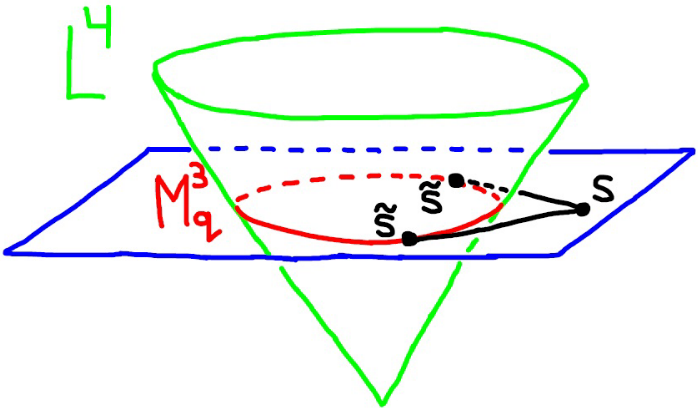

FIGURE 2.6. The sphere $\tilde{\mathcal{S}}$

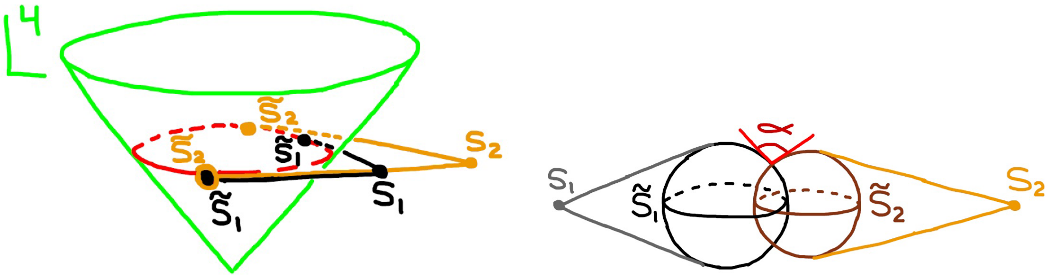

FIGURE 2.7. Two depictions of the setting in Lemma 2.20

Then $m = \ell \cap L^4$ is a single point, which, when considered as being in $\mathbb{S}^3 = M_1$, gives the center of $\tilde{\mathcal{S}}$ in $\mathbb{S}^3$.

Lemma 2.20. *Let $\tilde{\mathcal{S}}_1, \tilde{\mathcal{S}}_2$ be two intersecting spheres in $\mathbb{S}^3$ produced from $\mathcal{S}_1, \mathcal{S}_2$, respectively, and suppose $||\mathcal{S}_1||^2 = ||\mathcal{S}_2||^2 = 1$. Let α be the intersection angle between $\tilde{\mathcal{S}}_1$ and $\tilde{\mathcal{S}}_2$. Then $\cos\alpha = \pm\langle \mathcal{S}_1, \mathcal{S}_2\rangle$, where the sign on the right hand side depends on the orientations of $\tilde{\mathcal{S}}_1$ and $\tilde{\mathcal{S}}_2$. (See Figure 2.7.)*

Proof. As $\kappa = 1$, any $p \in \mathbb{S}^3 = M_1$ has x_5 coordinate equal to 1. Take $p \in \tilde{\mathcal{S}}_1 \cap \tilde{\mathcal{S}}_2 \subset M_1$. Scale $\mathcal{S}_1$ and $\mathcal{S}_2$ so that they also have x_5 coordinates equal to 1. Then $\mathcal{S}_1 - p$ and $\mathcal{S}_2 - p$ are normals (in the tangent space of $\mathbb{S}^3$) to $\tilde{\mathcal{S}}_1$ and $\tilde{\mathcal{S}}_2$, respectively, at p. So

$$\cos\alpha = \left\langle \frac{\mathcal{S}_1 - p}{||\mathcal{S}_1 - p||}, \frac{\mathcal{S}_2 - p}{||\mathcal{S}_2 - p||} \right\rangle =$$

$$\frac{1}{||\mathcal{S}_1 - p||}\frac{1}{||\mathcal{S}_2 - p||}\langle \mathcal{S}_1, \mathcal{S}_2\rangle = \frac{1}{||\mathcal{S}_1||}\frac{1}{||\mathcal{S}_2||}\langle \mathcal{S}_1, \mathcal{S}_2\rangle \ .$$

Returning to the scalings for $\mathcal{S}_1$ and $\mathcal{S}_2$ so that $||\mathcal{S}_1||^2 = ||\mathcal{S}_2||^2 = 1$, the lemma is proved. □

Remark 2.21. Lemma 2.20 implies that if $\mathcal{S}$ gives a sphere $\tilde{\mathcal{S}}$ containing $Y \in M_\kappa$, then $\{\mathcal{S} + tY \,|\, t \in R\}$ gives a pencil of spheres at Y, i.e. the collection of spheres of arbitrary radius through Y and tangent to $\tilde{\mathcal{S}}$. See Figure 2.8.

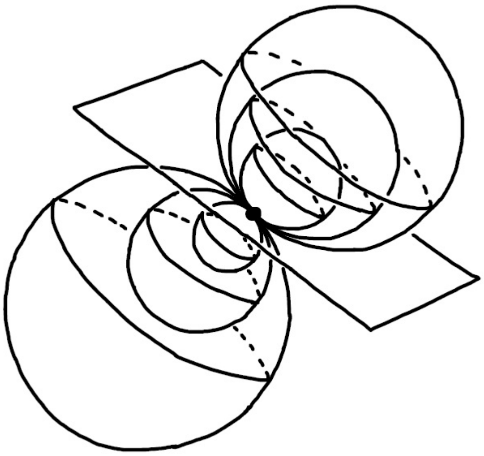

FIGURE 2.8. A pencil of spheres

Lemma 2.22. *Inversion through $\tilde{\mathcal{S}}$ is the map*

$$f : p \to p - 2\langle p, \mathcal{S}\rangle \mathcal{S} \ ,$$

when $||\mathcal{S}||^2 = 1$.

Proof. First note that $p \in L^4$ implies $p - 2\langle p, \mathcal{S}\rangle\mathcal{S} \in L^4$. Now let C be a circle that intersects $\tilde{\mathcal{S}}$ perpendicularly. We wish to show that $p \in C$ implies $f(p) \in C$. Note that $C = \tilde{\mathcal{S}}_1 \cap \tilde{\mathcal{S}}_2$ for some spheres $\tilde{\mathcal{S}}_1$ and $\tilde{\mathcal{S}}_2$. Then $\tilde{\mathcal{S}}_1 \perp \tilde{\mathcal{S}}$ and $\tilde{\mathcal{S}}_2 \perp \tilde{\mathcal{S}}$, and so

$$\langle \mathcal{S}, \mathcal{S}_1\rangle = \langle \mathcal{S}, \mathcal{S}_2\rangle = 0 \ ,$$

by Lemma 2.20. Then $p \in C$ implies $p \in \tilde{\mathcal{S}}_1 \cap \tilde{\mathcal{S}}_2$, which implies $\langle p, \mathcal{S}_1\rangle = \langle p, \mathcal{S}_2\rangle = 0$. Thus

$$\langle p - 2\langle p, \mathcal{S}\rangle\mathcal{S}, \mathcal{S}_1\rangle = \langle p - 2\langle p, \mathcal{S}\rangle\mathcal{S}, \mathcal{S}_2\rangle = 0 \ ,$$

and so $f(p) \in C$. □

Lemma 2.23. *$\tilde{\mathcal{S}}$ is a sphere with mean curvature*

$$H_0 = \pm\frac{z_5 - z_4}{2||\mathcal{S}||}$$

and center

$$\frac{z}{z_5 - z_4}$$

in M_0, and is a sphere in M_κ with mean curvature

$$H_\kappa = \pm\frac{(z_5 - z_4)^2 + \kappa(|z|^2 - ||\mathcal{S}||^2)}{2(z_5 - z_4)||\mathcal{S}||} \ .$$

Proof. Consider the case $\kappa = 0$. Take

$$Y = \begin{pmatrix} 2y^t \\ |y|^2 - 1 \\ |y|^2 + 1 \end{pmatrix} \in \tilde{S} \ , \quad y = (y_1, y_2, y_3) \ .$$

Then $\langle \mathcal{S}, Y\rangle = 0$ implies

$$4|y - (z_5 - z_4)^{-1}z|^2 = (2(z_5 - z_4)^{-1}||\mathcal{S}||)^2 \ ,$$

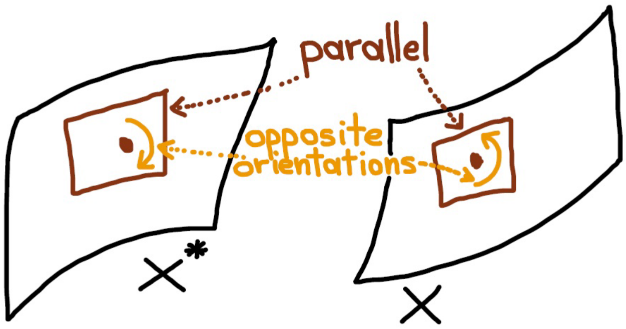

FIGURE 2.9. The Christoffel transformation

and thus $\tilde{\mathcal{S}}$ is a sphere of radius $2||\mathcal{S}||/|z_5 - z_4|$ and center $z/(z_5 - z_4)$. (By Remark 2.4, the coefficient 4 on the right-hand side of the above equation is needed.) Hence H_0 is as in the lemma. The final statement of the lemma now follows from Equation (2.18). □

Remark 2.24. If x_p, x_q, x_r and x_s in $\mathbb{R}^3$ (with associated lifts X_p, X_q, X_r and X_s in M_0) all lie in the circle of positive radius determined by the intersection of two distinct spheres $\tilde{\mathcal{S}}_1$ and $\tilde{\mathcal{S}}_2$ given by spacelike vectors $\mathcal{S}_1$ and $\mathcal{S}_2$, see (2.19), then the fact that $X_p, X_q, X_r, X_s \in \tilde{\mathcal{S}}_1 \cap \tilde{\mathcal{S}}_2$ is equivalent to

$$X_p, X_q, X_r, X_s \perp \text{span}\{\mathcal{S}_1, \mathcal{S}_2\} .$$

Because $\tilde{\mathcal{S}}_1 \cap \tilde{\mathcal{S}}_2$ is a circle of positive radius, the $\mathbb{R}^{4,1}$ metric restricted to $\text{span}\{\mathcal{S}_1, \mathcal{S}_2\}$ is positive definite. This implies, in particular, that X_p, X_q, X_r and X_s all lie in a 3-dimensional space.

Remark 2.25. Looking back at Section 2.4, let us take points x_p, x_q, x_r, x_s that are concircular, with corresponding $X_p, X_q, X_r, X_s \in M_\kappa$. This makes the cross ratio $\text{cr} = \text{cr}_{x_p x_q x_r x_s}$ real-valued, by Remark 2.24 (i.e., $\mathcal{E} = 0$ by Remark 2.12). Then, once cr is real, the value cr, along with the values of X_p and X_q and X_s, determine X_r via

$$X_r = \alpha \left(X_p + \frac{1}{\langle X_q, X_s \rangle} \{ (\text{cr} - 1) \langle X_p, X_s \rangle X_q + (\text{cr}^{-1} - 1) \langle X_p, X_q \rangle X_s \} \right) \tag{2.20}$$

for some real scalar α, by Lemma 2.11. In this way, the cross ratio is a parameter for parametrizing the circle containing x_p, x_q and x_s.

2.9. **Christoffel transformations.** We now define the Christoffel transformation x^*, or "dual surface". For a CMC (constant mean curvature) surface in $\mathbb{R}^3$, this gives the parallel CMC surface.

Let x be a surface in $\mathbb{R}^3$ with mean curvature H_0 and unit normal n_0 (with respect to the metric in (2.5)). The Christoffel transformation x^* satisfies that (see Figure 2.9)

- x^* is defined on the same domain as x (to avoid issues related to global behavior of x^*, we consider only simply-connected domains here),
- x^* has the same conformal structure as x,
- and x and x^* have parallel tangent planes with opposite orientations at corresponding points.

One can check that automatically the principal curvature directions at corresponding points of x and x^* will themselves also be parallel.

This description above of the Christoffel transformations turns out to be equivalent to the following definition, and the existence of the integrating factor ρ below is equivalent to the existence of isothermic coordinates. Then, we will see in Corollary 2.29 that we can choose x^* so that

$$dx^* = -\frac{x_u}{|x_u|^2}du + \frac{x_v}{|x_v|^2}dv$$

We avoid umbilic points in this discussion.

Definition 2.26. *A Christoffel transformation x^* of an umbilic-free surface x in $\mathbb{R}^3$ is a surface that satisfies*

$$dx^* = \rho(dn_0 + H_0 dx)$$

for some nonzero real-valued function ρ on the surface x (here x^* is determined only up to translations and dilations).

Lemma 2.27. *Away from umbilics of x, the Christoffel transform x^* exists if and only if x is isothermic.*

Proof. First we prove one direction, by assuming x is isothermic and then showing x^* exists.

Take x to be isothermic, and take isothermic coordinates u, v for x, so $x_{uv} = Ax_u + Bx_v$ for some A, B. Then

$$d(-\tfrac{x_u}{|x_u|^2}du + \tfrac{x_v}{|x_v|^2}dv) =$$

$$16g_{11}^{-2}((|x_u|^2 + |x_v|^2)x_{uv} - 2(x_u \cdot x_{uv})x_u - 2(x_v \cdot x_{uv})x_v)du \wedge dv = 0 \ .$$

This implies that there exists an x^* such that

$$dx^* = -\tfrac{x_u}{|x_u|^2}du + \tfrac{x_v}{|x_v|^2}dv \ . \tag{2.21}$$

Also,

$$dn_0 + H_0 dx = \tfrac{1}{8}(b_{11} - b_{22})(-\tfrac{x_u}{|x_u|^2}du + \tfrac{x_v}{|x_v|^2}dv) \ ,$$

implying that x^* is a Christoffel transform, since $b_{11} - b_{22} \neq 0$ at non-umbilic points.

Now we prove the other direction, by assuming x^* exists and then showing that x has isothermic coordinates.

For any choice of coordinates u, v for $x = x(u, v)$, the Codazzi equations are

$$(b_{11})_v - (b_{12})_u = \Gamma^1_{12}b_{11} + (\Gamma^2_{12} - \Gamma^1_{11})b_{12} - \Gamma^2_{11}b_{22} \ ,$$

$$(b_{12})_v - (b_{22})_u = \Gamma^1_{22}b_{11} + (\Gamma^2_{22} - \Gamma^1_{21})b_{12} - \Gamma^2_{21}b_{22} \ .$$

Here the Christoffel symbols are $\Gamma^h_{ij} = \frac{1}{2}\sum_{k=1}^2 g^{hk}(\partial_{u_j}g_{ik} + \partial_{u_i}g_{jk} - \partial_{u_k}g_{ij})$, where $u_1 = u$ and $u_2 = v$. Because we are avoiding any umbilic points of x, we may assume that u and v are curvature line coordinates for x, and so $g_{12} = b_{12} = 0$. It follows that

$$\Gamma^1_{11} = \frac{\partial_u g_{11}}{2g_{11}} \ , \ \Gamma^2_{22} = \frac{\partial_v g_{22}}{2g_{22}} \ , \ \Gamma^2_{11} = -\frac{\partial_v g_{11}}{2g_{22}} \ ,$$

$$\Gamma^1_{22} = -\frac{\partial_u g_{22}}{2g_{11}} \ , \ \Gamma^1_{12} = \Gamma^1_{21} = \frac{\partial_v g_{11}}{2g_{11}} \ , \ \Gamma^2_{12} = \Gamma^2_{21} = \frac{\partial_u g_{22}}{2g_{22}} \ .$$

Denoting the principal curvatures by k_j, the Codazzi equations simplify to

$$2(k_1)_v = \frac{\partial_v g_{11}}{g_{11}} \cdot (k_2 - k_1) \ , \quad 2(k_2)_u = \frac{\partial_u g_{22}}{g_{22}} \cdot (k_1 - k_2) \ . \tag{2.22}$$

Then existence of x^* gives

$$d(\rho dn_0 + \rho H_0 dx) = 0 \ ,$$

from which it follows that

$$\begin{pmatrix} 0 & \frac{b_{11}}{g_{11}} - \frac{b_{22}}{g_{22}} \\ \frac{b_{22}}{g_{22}} - \frac{b_{11}}{g_{11}} & 0 \end{pmatrix} \begin{pmatrix} \rho_u \\ \rho_v \end{pmatrix} = \rho \cdot \begin{pmatrix} \left(\frac{b_{11}}{g_{11}} + \frac{b_{22}}{g_{22}}\right)_v \\ \left(\frac{b_{11}}{g_{11}} + \frac{b_{22}}{g_{22}}\right)_u \end{pmatrix} \ . \tag{2.23}$$

Then because $\rho_{uv} = \rho_{vu}$, we have

$$\left(\frac{(k_2 + k_1)_v}{k_1 - k_2}\right)_u = \left(\frac{(k_1 + k_2)_u}{k_2 - k_1}\right)_v \ ,$$

which implies

$$\frac{2(((k_1)_v)_u + ((k_2)_u)_v)}{k_1 - k_2} + 2(k_2 - k_1)^{-2} \left((k_1)_v (k_2 - k_1)_u + (k_2)_u (k_2 - k_1)_v\right) = 0 \ .$$

Using the Codazzi equations (2.22), we have

$$\left(\log \frac{g_{11}}{g_{22}}\right)_{uv} = 0 \ .$$

In particular, there exist positive functions $a(u)$ and $b(v)$ depending only on u and v, respectively, so that

$$(a(u))^2 g_{11} = (b(v))^2 g_{22} \ .$$

Writing $u = u(\hat{u})$ and $v = v(\hat{v})$ for new curvature line coordinates $\hat{u}$ and $\hat{v}$, we have $\hat{g}_{12} = \hat{b}_{12} = 0$ and $\hat{g}_{11} = (u_{\hat{u}})^2 g_{11}$ and $\hat{g}_{22} = (v_{\hat{v}})^2 g_{22}$, for the fundamental form entries $\hat{g}_{ij}$ and $\hat{b}_{ij}$ in terms of $\hat{u}$ and $\hat{v}$. We can choose $\hat{u}$ and $\hat{v}$ so that $u_{\hat{u}} = a(u(\hat{u}))$ and $v_{\hat{v}} = b(v(\hat{v}))$ hold. Then $\hat{g}_{11} = \hat{g}_{22}$ and so $\hat{u}, \hat{v}$ are isothermic coordinates. □

The last part of the above proof is reminiscent of an argument used in Section 2.5.

Proposition 2.28. *The form* (2.22) *of the Codazzi equations, in the case of curvature line coordinates, is invariant under different choices of the spaceform M_κ.*

Proof. Following the proof of Lemma 2.7, when changing spaceforms, i.e. when changing $\kappa = 0$ to general κ, we have, for $s = 1 + \kappa |x|^2$:

$$g_{11} \to \hat{g}_{11} = s^{-2} g_{11} \ , \quad g_{12} = 0 \to \hat{g}_{12} = 0 \ , \quad g_{22} \to \hat{g}_{22} = s^{-2} g_{22} \ .$$

$$b_{11} \to \hat{b}_{11} = s^{-1} b_{11} + 2\kappa g_{11} s^{-2} (x \cdot n_0) \ , \quad b_{12} = 0 \to \hat{b}_{12} = 0 \ ,$$

$$b_{22} \to \hat{b}_{22} = s^{-1} b_{22} + 2\kappa g_{22} s^{-2} (x \cdot n_0) \ .$$

Here, $b_{11} = 4(x_{uu} \cdot n_0)$ and $b_{22} = 4(x_{vv} \cdot n_0)$. These transformations were seen in the proof of Lemma 2.7 using $x = x(u, v)$ with isothermic coordinates (u, v), but they still hold with (u, v) that are just curvature line coordinates.

So

$$k_1 = \frac{b_{11}}{g_{11}} \to \hat{k}_1 = \frac{\hat{b}_{11}}{\hat{g}_{11}} = s k_1 + 2\kappa (x \cdot n_0) \ .$$

Similarly, $\hat{k}_2 = sk_2 + 2\kappa(x \cdot n_0)$. The Codazzi equations for curvature line coordinates when $\kappa = 0$ are as in (2.22). Then

$$\frac{2\hat{k}_{1,v}}{\hat{k}_2 - \hat{k}_1} - \frac{(\hat{g}_{11})_v}{\hat{g}_{11}} =$$

$$\frac{4\kappa(x \cdot x_v)k_1 + 2sk_{1,v} + 4\kappa(x \cdot n_{0,v})}{s(k_2 - k_1)} - \frac{4s^{-3}(-\kappa)(x \cdot x_v)g_{11} + s^{-2}(g_{11})_v}{s^{-2}g_{11}} =$$

$$\frac{4\kappa}{s}\left((x \cdot x_v)(\frac{k_1}{k_2 - k_1} + 1) + (x \cdot n_{0,v})\frac{1}{k_2 - k_1}\right) =$$

$$\frac{4\kappa}{s(k_2 - k_1)}\left(x \cdot (k_2 x_v + n_{0,v})\right) = \frac{4\kappa}{s(k_2 - k_1)}\left(x \cdot \vec{0}\right) = 0 \ .$$

Similarly,

$$\frac{2\hat{k}_{2,u}}{\hat{k}_1 - \hat{k}_2} = \frac{(\hat{g}_{22})_u}{\hat{g}_{22}} \ .$$

□

The proof of Lemma 2.27 gives the following corollary:

Corollary 2.29. *Away from umbilic points, one Christoffel transformation x^* of an isothermic surface $x = x(u, v)$ can be taken as a solution of*

$$dx^* = -\frac{x_u}{|x_u|^2}du + \frac{x_v}{|x_v|^2}dv \ .$$

With respect to isothermic coordinates (u, v), Equation (2.23) implies

$$\rho_u = -\frac{g_{11}\partial_u H_0}{g_{11}H_0 - b_{22}} \cdot \rho \ , \quad \rho_v = -\frac{g_{11}\partial_v H_0}{g_{11}H_0 - b_{11}} \cdot \rho \ . \tag{2.24}$$

The existence of x^* then automatically implies the compatibility condition $(\rho_u)_v = (\rho_v)_u$, with ρ_u and ρ_v as in the right-hand sides of the equations in (2.24).

This pair of equations (2.24) tells us that ρ is uniquely determined once its value is chosen at a single point, and thus the solution ρ is unique up to scalar multiplication by a constant factor. Thus the Christoffel transformation in Corollary 2.29 is essentially the unique choice, up to homothety and translation in $\mathbb{R}^3$. As a result of this, with no loss of generality, we can now simply take the definition of x^* as follows:

Definition 2.30. *The Christoffel transformation of a surface x in $\mathbb{R}^3$ with isothermic coordinates (u, v) is any x^* (defined in $\mathbb{R}^3$ up to translation) such that*

$$dx^* = -\frac{x_u}{|x_u|^2}du + \frac{x_v}{|x_v|^2}dv \ .$$

The constant scalar factor freedom that is allowed for ρ still implicitly exists in Definition 2.30, because of the constant scalar factor freedom allowed for the coordinates u, v.

Remark 2.31. The function ρ in Definition 2.26 is a constant scalar multiple of the multiplicative inverse of the mean curvature of x^*, seen as follows: The Christoffel transform of the Christoffel transform $(x^*)^*$, with respect to Definition 2.30, satisfies

$$d((x^*)^*) = -\tfrac{x_u^*}{|x_u^*|^2}du + \tfrac{x_v^*}{|x_v^*|^2}dv = x_u du + x_v dv = dx \ ,$$

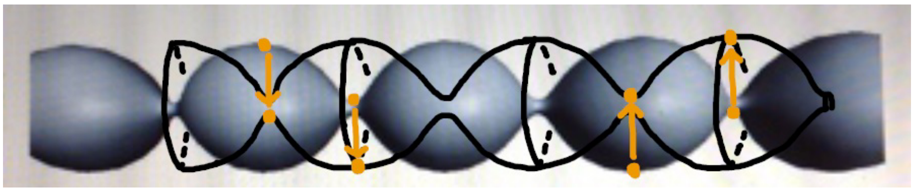

FIGURE 2.10. A Delaunay unduloid and its dual surface (Christoffel transform), which is a translated copy of the same surface

so $(x^*)^*$ should be the original surface x, up to translation and homothety, with respect to Definition 2.26. Thus, by scaling and translating appropriately, we may assume $(x^*)^* = x$. Also, if the normal of x is n_0, then the normal of x^* is $-n_0$. Thus

$$dx = d((x^*)^*) = \rho^*(dn_0^* + H_0^* dx^*) = \rho^*(-dn_0 + H_0^*\rho(dn_0 + H_0 dx)) ,$$

and so

$$(1 - \rho\rho^* H_0 H_0^*)dx = (H_0^*\rho\rho^* - \rho^*)dn_0 .$$

Since dx and dn_0 are linearly independent away from umbilic points, it follows that

$$\rho H_0^* = \rho^* H_0 = 1 .$$

Remark 2.32. When H_0 is a nonzero constant and we have isothermic coordinates, Equation (2.24) implies ρ is constant. Then x^* and the parallel CMC surface $x^{||} = x + H_0^{-1} n_0$ differ by only a homothety and translation of $\mathbb{R}^3$. Thus the Christoffel transformation is essentially the same as the parallel CMC surface to x, as expected.

Example 2.33. The round cylinder gives one simple example of a Christoffel transform's orientation reversing property. For the cylinder $x(u, v) = (\cos u, \sin u, v)$ in $\mathbb{R}^3$ (with metric as in (2.5)), the normal vector is $n_0 = \frac{1}{2}(\cos u, \sin u, 0)$, and the Christoffel transform is $x^*(u, v) = (-\cos u, -\sin u, v)$ with its normal vector $n_0^* = \frac{1}{2}(-\cos u, -\sin u, 0)$. Thus $n_0^* = -n_0$.

Example 2.34. Delaunay surfaces are CMC surfaces of revolution in $\mathbb{R}^3$. They can be either embedded (unduloids) or nonembedded (nodoids). The Christoffel transforms of Delaunay surfaces are again Delaunay surfaces. See Figure 2.10.

Lemma 2.35. *We have the relation*

$$dx^* = \frac{2}{(k_1 - k_2)|x_u|^2}(dn_0 + H_0 dx) .$$

Proof. This proof is a direct computation:

$$\left(\frac{2}{(k_1 - k_2)|x_u|^2}(dn_0 + H_0 dx) + \tfrac{x_u}{|x_u|^2} du - \tfrac{x_v}{|x_v|^2} dv \right) |x_u|^2 =$$

$$\frac{2}{k_1 - k_2}(-k_1 x_u du - k_2 x_v dv + \tfrac{k_1+k_2}{2}(x_u du + x_v dv)) + x_u du - x_v dv = 0 .$$

□

The Hopf differential (see [49], for example) for a surface in $\mathbb{R}^3$ is defined as (the inner products $\langle \cdot, \cdot \rangle$ and "$\cdot$" are bilinearly extended to apply to complex vectors)

$$\hat{Q} dz^2 , \quad \hat{Q} = \langle \mathcal{T}_{n_0}, (X_0)_{zz} \rangle = 4(n_0 \cdot x_{zz}) \qquad (z = u + iv) .$$

Corollary 2.36. *If H_0 is constant for the surface $x = x(u, v)$ in $\mathbb{R}^3$ with isothermic coordinates (u, v), then the factor $\hat{Q}$ of the Hopf differential is a real constant.*

Proof. The factor

$$\hat{Q} = 4(n_0 \cdot \tfrac{1}{4}(x_{uu} - x_{vv})) = (k_1 - k_2)|x_u|^2$$

is constant by Lemma 2.35 and Remark 2.32. It is clearly also real. □

2.10. **Flat connections on the tangent bundle.** Let us first review what a connection is. Later we will see how isothermic surfaces have a 1-parameter family of flat connections. Although we do not show it here (see [29] for such an argument), the converse is also true: existence of a family of flat connections implies that the surface is isothermic.

Recall that the Riemannian connection of a Riemannian manifold is the unique connection satisfying

$$\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z \ , \tag{2.25}$$

$$\nabla_X (fY + Z) = X(f)Y + f\nabla_X Y + \nabla_X Z \ , \tag{2.26}$$

$$\nabla_X Y - \nabla_Y X = [X, Y] \ , \tag{2.27}$$

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \ , \tag{2.28}$$

where X, Y, Z are any smooth tangent vector fields of the manifold, and f is any smooth function from the manifold to $\mathbb{R}$. The first two relations (2.25), (2.26) define affine connections, and inserting the last two conditions (2.27), (2.28) makes the connection a Riemannian connection.

Taking an n-dimensional manifold M^n with affine connection ∇, and taking a basis X_1, X_2, ... , X_n of vector fields for the tangent spaces, we define Γ^k_{ij} and R^k_{lij} by

$$\nabla_{X_i} X_j = \sum_{k=1}^n \Gamma^k_{ij} X_k \ ,$$

$$\nabla_{X_i}\nabla_{X_j} X_l - \nabla_{X_j}\nabla_{X_i} X_l - \nabla_{[X_i,X_j]} X_l = \sum_{k=1}^n R^k_{lij} X_k \ . \tag{2.29}$$

We define the one forms ω^i and ω^i_j by (here δ^i_j is the Kronecker delta function)

$$\omega^i(X_j) = \delta^i_j \ , \quad \omega^i_j = \sum_{k=1}^n \Gamma^i_{kj}\omega^k \ .$$

The one forms ω^i_j are called the connection one forms. Then

$$d\omega^i_l + \sum_{p=1}^n \omega^i_p \wedge \omega^p_l = \tfrac{1}{2} \sum_{j,k=1}^n R^i_{ljk}\omega^j \wedge \omega^k \ .$$

When the connection is the Riemannian connection, the R^i_{ljk} give the Riemannian curvature tensor. When, for an affine connection, all of the R^i_{ljk} are zero, then we say that ∇ is a flat connection.

Connections are equivalent to having a notion of parallel transport along each given curve in the base manifold, and a connection is flat if and only if the parallel transport map depends only on the homotopy class of each curve (with fixed endpoints). In particular, if the surface x is simply connected, parallel transport is independent of path if and only if the connection is flat, which can be seen as follows: One direction is immediately clear from Equation (2.29), by choosing the X_i there to be constant vector fields (that is, by choosing X_i by using parallel translation, i.e. $\nabla_* X_i = 0$), and then all R^k_{lij} become 0. To see the other direction, let us restrict to the case that M^n is a 2-dimensional simply-connected manifold given by a surface $x = x(u, v)$ in $\mathbb{R}^3$ as in Section 2.2. Suppose that the connection is flat. Then Equation (2.29) implies

$$\nabla_{\partial_u} \nabla_{\partial_v} Y = \nabla_{\partial_v} \nabla_{\partial_u} Y$$

for any tangent vector field Y. Then we can apply an argument like in the proof of Proposition 3.1.2 in [49] to conclude that if Y is constructed so that $\nabla_{\partial_u} Y = 0$ along one curve where $v = v_0$ is constant and so that $\nabla_{\partial_v} Y = 0$ everywhere, then also $\nabla_{\partial_u} Y = 0$ everywhere, and so Y is a vector field that is parallel on any curve in x.

2.11. **The wedge product.** In the next section, unlike the previous section, we will consider flat connections on a bundle over a surface that is not the tangent bundle. To prepare for that, we consider the wedge product here.

Given a vector space V over $\mathbb{R}$ with an inner product $\langle , \rangle$ and fixed A, B and variable v lieing in V, we can define the wedge operator, a map from V to V, by

$$(A \wedge B)(v) = \langle A, v \rangle B - \langle B, v \rangle A \ . \tag{2.30}$$

Note that

(1) $(A + \alpha_1 B) \wedge B = A \wedge B$ for any $\alpha_1 \in \mathbb{R}$,
(2) $(\alpha_2 A) \wedge (\alpha_3 B) = \alpha_2 \alpha_3 (A \wedge B)$ for any $\alpha_2, \alpha_3 \in \mathbb{R}$.

Thus it suffices to consider A and B that form the edges of a unit square, and $A \wedge B$ becomes the operator that is rotation by 90 degrees in the plane spanned by A and B. With the inner product of $A \wedge B$ and $\hat{A} \wedge \hat{B}$ defined as

$$\langle A \wedge B, \hat{A} \wedge \hat{B} \rangle = \det \begin{pmatrix} \langle A, \hat{A} \rangle & \langle A, \hat{B} \rangle \\ \langle B, \hat{A} \rangle & \langle B, \hat{B} \rangle \end{pmatrix} \ ,$$

we find that when A and B form the edges of a unit square, we have

$$|A \wedge B|^2 = \langle A, A \rangle \langle B, B \rangle - \langle A, B \rangle^2 = 1 \ .$$

Incidentally, the properties (1) and (2) above give that $A \wedge B$ really depends only on the plane spanned by A and B, and on the area of the parallelogram determined by A and B. Note also that

$$\langle v, (A \wedge B) v \rangle = 0$$

for all $v \in V$. A computation gives also

$$|(A \wedge B) v|^2 = -\langle v, (A \wedge B)^2 v \rangle \ .$$

Remark 2.37. The wedge product just defined here is a different type of object than the wedge product $du \wedge dv$ of the two 1-forms du and dv. Sometimes both types of wedge products can appear in the same equation, and we can distinguish between them simply by noting what types of objects they are applied to.

We now give two ways to define wedge products that do not use a metric:

Second definition of $\wedge$: One can define $\wedge$, involving neither a metric nor a choice of basis, as a map from the alternating bilinear forms $\mathcal{B}$ to $\mathbb{R}$, so that

$$(A \wedge B)(\mathcal{B}) = \mathcal{B}(A, B) \ .$$

Before we defined the wedge product as in (2.30) (which we now denote by $\wedge_m$ since it used the *m*etric), which is a map taking vectors to vectors. Setting

$$\mathcal{B}_{x,y}(A, B) = \langle (x \wedge_m y)A, B\rangle = \langle x, A\rangle\langle y, B\rangle - \langle y, A\rangle\langle x, B\rangle \ ,$$

then

$$\mathcal{B}_{x,y}(B, A) = -\mathcal{B}_{x,y}(A, B) \ ,$$

so it is alternating (skew). Also,

$$\mathcal{B}_{x,y}(A, B) = \mathcal{B}_{A,B}(x, y)$$

and

$$\mathcal{B}_{y,x}(A, B) = -\mathcal{B}_{x,y}(A, B) \ .$$

If x, y can be any pair of vectors, then the $\mathcal{B}_{x,y}$ generate all alternating bilinear forms (we do not prove this here), so $A \wedge B$ is determined by

$$(A \wedge B)(\mathcal{B}_{x,y}) = \mathcal{B}_{x,y}(A, B) = \mathcal{B}_{A,B}(x, y) = \langle (A \wedge_m B)x, y\rangle \ .$$

This gives a one-to-one correspondence between $A \wedge B$ and $A \wedge_m B$.

For a basis of V we can take $e_1 = A$, $e_2 = B$, e_3, ..., e_n with $e_j \perp \mathrm{span}\{A, B\}$ for all $j \geq 3$. Then

(1) $((A \wedge_m B)e_j, e_k) = 0$ if $j \geq 3$ or $k \geq 3$,
(2) $((A \wedge_m B)e_j, e_j) = 0$ for all j,
(3) $((A \wedge_m B)e_j, e_k) = -((A \wedge_m B)e_k, e_j)$ for all j, k, and
(4) $((A \wedge_m B)e_1, e_2) = \langle A, A\rangle\langle B, B\rangle - \langle A, B\rangle^2$.

Thus

$$(A \wedge B)(\mathcal{B}_{e_j,e_k}) = 0$$

if $j \geq 3$ or $k \geq 3$ or $j = k$, and

$$(A \wedge B)(\mathcal{B}_{e_1,e_2}) = \langle A, A\rangle\langle B, B\rangle - \langle A, B\rangle^2 \ .$$

Third definition of $\wedge$: One can define the wedge product by the following properties:

(1) It is a map
$$V \times V \to \wedge^2 V \ , \quad (A, B) \mapsto A \wedge B$$
($\wedge^2 V$ is the collection generated by general $A \wedge B$).
(2) $A \wedge B$ is alternating, i.e. $A \wedge B = -B \wedge A$.
(3) $A \wedge B$ is bilinear, that is, separately linear in A and B.
(4) $A \wedge B$ is universal, i.e. for any vector space W and any alternating bilinear map
$$\mathcal{B} : V \times V \to W \ ,$$
there exists a unique linear map $\beta : \wedge^2 V \to W$ such that
$$\mathcal{B}(A, B) = \beta(A \wedge B)$$
for all $A, B \in V$.

For the $\mathcal{B}_{x,y} : V \times V \to \mathbb{R}$ in the second definition above, for example, β would be

$$\beta(A \wedge B) = (A \wedge B)(\mathcal{B}_{x,y}) = \mathcal{B}_{x,y}(A, B) \ ,$$

or equivalently

$$\beta(A \wedge_m B) = \langle (A \wedge_m B)x, y \rangle \ .$$

The $A \wedge B$ and $A \wedge_m B$ in the second definition above are both universal.

Lemma 2.38. *If $\wedge : V \times V \to \wedge^2 V$ and $\tilde{\wedge} : V \times V \to \tilde{\wedge}^2 V$ are both universal alternating bilinear maps, then there exists a unique linear isomorphism $\theta : \wedge^2 V \to \tilde{\wedge}^2 V$ such that $\theta(A \wedge B) = A \tilde{\wedge} B$ for all $A, B \in V$.*

Proof. Because $\wedge$ is universal, there exists a unique linear map $\theta : \wedge^2 V \to \tilde{\wedge}^2 V$ such that $A \tilde{\wedge} B = \theta(A \wedge B)$. Similarly, $\tilde{\wedge}$ being universal implies there exists a unique linear map $\phi : \tilde{\wedge}^2 V \to \wedge^2 V$ such that

$$A \wedge B = \phi(A \tilde{\wedge} B) \ .$$

We have

$$\phi \circ \theta(A \wedge B) = A \wedge B \ .$$

However, does not yet mean that θ and ϕ are inverse to each other, since $A \wedge B$ is not a general element of $\wedge^2 V$ (the $A \wedge B$ are only a set of generators of $\wedge^2 V$). Using the definition of universality of $\wedge$ in item (4) above with $\mathcal{B}$ equal to $\wedge$ itself and $W = \wedge^2 V$, there exists a unique linear map $\beta : \wedge^2 V \to \wedge^2 V$ such that

$$\beta(A \wedge B) = A \wedge B \ .$$

Since one possible such map is the identity map, it must be that $\phi \circ \theta$ is the identity map on $\wedge^2 V$. (Similarly, $\theta \circ \phi$ is the identity map on $\tilde{\wedge}^2 V$.) □

We still need to show that this definition of the wedge is not empty:

Lemma 2.39. *There exists a map $V \times V \to \wedge^2 V$ that is universal, alternating and bilinear.*

Proof. Let $\mathbb{R}[V \times V]$ be the vector space with basis

$$\{A \times B \,|\, A, B \in V\} \ .$$

$\mathbb{R}[V \times V]$ is called the free vector space generated by $V \times V$. We can regard $A \times B \in \mathbb{R}[V \times V]$ as the function with

$$(A \times B)(x, y) = 0$$

unless $x = A$ and $y = B$, and

$$(A \times B)(A, B) = 1 \ ,$$

and then $\mathbb{R}[V \times V]$ equals the set of those functions from $V \times V$ to $\mathbb{R}$ that are 0 at all but a finite number of points in $V \times V$. Thus the vector space $\mathbb{R}[V \times V]$ exists, as it is a subset of the space $\mathbb{R}^{V \times V}$ of all functions from $V \times V$ to $\mathbb{R}$. Consider the subspace $\mathcal{R}$ of $\mathbb{R}[V \times V]$ consisting of elements of the form

(1) $(\lambda A_1 + \mu A_2) \times B - \lambda(A_1 \times B) - \mu(A_2 \times B)$ for $\lambda, \mu \in \mathbb{R}$,
(2) $A \times (\lambda B_1 + \mu B_2) - \lambda(A \times B_1) - \mu(A \times B_2)$ and
(3) $A \times B + B \times A$.

Both $\mathbb{R}[V \times V]$ and $\mathcal{R}$ are infinite dimensional. However, the quotient space $\wedge^2 V = \mathbb{R}[V \times V]/\mathcal{R}$, with $A \wedge B = [A \times B]_{\text{mod } \mathcal{R}}$, is finite dimensional. The map $(A, B) \to A \wedge B$ is alternating and bilinear. It is also universal, since it was constructed without imposing anything more than the bilinear and alternating conditions. $\square$

For a basis e_1, ..., e_n of V we can define $\wedge^2 V$ to be $\mathbb{R}[\{e_i \wedge e_j : i < j\}]$, and for

$$A = \sum_{i=1}^{n} a_i e_i \ , \quad B = \sum_{j=1}^{n} b_j e_j \ ,$$

we have

$$A \wedge B = \sum_{i,j=1}^{n} a_i b_j e_i \wedge e_j \ ,$$

where $e_j \wedge e_i$ denotes $-e_i \wedge e_j$ when $j > i$, and $e_i \wedge e_i = 0$.

The wedge product for 1-forms. Suppose $A = A(u_1, u_2)$ and $B = B(u_1, u_2)$ are vectors in V depending on the real parameters u_1 and u_2. Then the wedge product of the 1-forms Adu_i and Bdu_j is defined by

$$(Adu_i \wedge Bdu_j)v = ((A \wedge B)v)du_i \wedge du_j \ .$$

Here we can think of the du_i and du_j as scalars to be taken outside the wedge product of A and B, and then we are seeing two different types of wedge products on the right-hand side, that is, one type of wedge in $du_i \wedge du_j$ and another type in $A \wedge B$.

Suppose that $C = C(u_1, u_2)$ and $D = D(u_1, u_2)$ are vectors in V depending on u_1 and u_2, as well. We can also apply the commutator wedge to objects of the form $A \wedge Bdu_i$ and $C \wedge Ddu_j$, as follows:

$$[(A \wedge Bdu_i) \wedge (C \wedge Ddu_j)](v) = \tag{2.31}$$
$$((A \wedge B)((C \wedge D)v) - (C \wedge D)((A \wedge B)v))du_i \wedge du_j \ .$$

This commutator wedge is symmetric, not skew symmetric, that is,

$$[(A \wedge Bdu_i) \wedge (C \wedge Ddu_j)] = [(C \wedge Ddu_j) \wedge (A \wedge Bdu_i)] \ .$$

The wedge product for $\mathbb{R}^{4,1}$. For vectors A and B in $\mathbb{R}^{4,1}$, $A \wedge B$ can be regarded as a matrix in the Lie algebra $o_{4,1}$ defined by (2.30). We make this more explicit in the proof of the following lemma:

Lemma 2.40. *$A \wedge B$ lies in the Lie algebra $o_{4,1}$.*

Proof. For

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} , \quad A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{pmatrix} , \ B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{pmatrix} ,$$

the matrix $\mathfrak{A}$ that represents $A \wedge B$ is

$$\begin{pmatrix} 0 & B_1A_2 - A_1B_2 & B_1A_3 - A_1B_3 & B_1A_4 - A_1B_4 & B_5A_1 - A_5B_1 \\ B_2A_1 - A_2B_1 & 0 & B_2A_3 - A_2B_3 & B_2A_4 - A_2B_4 & B_5A_2 - A_5B_2 \\ B_3A_1 - A_3B_1 & B_3A_2 - A_3B_2 & 0 & B_3A_4 - A_3B_4 & B_5A_3 - A_5B_3 \\ B_4A_1 - A_4B_1 & B_4A_2 - A_4B_2 & B_4A_3 - A_4B_3 & 0 & B_5A_4 - A_5B_4 \\ B_5A_1 - A_5B_1 & B_5A_2 - A_5B_2 & B_5A_3 - A_5B_3 & B_5A_4 - A_5B_4 & 0 \end{pmatrix} ,$$

which does satisfy the condition $\mathfrak{A} \cdot D + D \cdot \mathfrak{A}^t = 0$ to lie in $o_{4,1}$. □

Lemma 2.41. *If $T \in o_{4,1}$ and A and B are vectors in $\mathbb{R}^{4,1}$, then*

$$[T, A \wedge B] = (TA) \wedge B + A \wedge (TB) \ .$$

Proof. For $C \in \mathbb{R}^{4,1}$,

$$\langle A, TC \rangle = A^t DTC = -A^t T^t DC = -\langle TA, C \rangle \ .$$

Then

$$[T, A \wedge B]C = T(A \wedge B)C - (A \wedge B)TC =$$
$$T(\langle A, C \rangle B - \langle B, C \rangle A) - (\langle A, TC \rangle B - \langle B, TC \rangle A) =$$
$$\langle A, C \rangle TB - \langle B, C \rangle TA - \langle A, TC \rangle B + \langle B, TC \rangle A =$$
$$\langle A, C \rangle TB - \langle B, C \rangle TA + \langle TA, C \rangle B - \langle TB, C \rangle A =$$
$$((TA) \wedge B)C + (A \wedge (TB))C \ .$$

□

Remark 2.42. A second proof of Lemma 2.41 can be given as follows: Let $g(t)$ be a curve in $O_{4,1}$ such that $g(0) = I$ and $g'(0) = T$. Then

$$(g(A \wedge B)g^{-1})C = g(\langle A, g^{-1}C \rangle B - \langle B, g^{-1}C \rangle A) =$$
$$\langle gA, C \rangle (gB) - \langle gB, C \rangle (gA) = ((gA) \wedge (gB))C \ ,$$

so

$$\partial_t (g(A \wedge B)g^{-1})|_{t=0} = \partial_t ((gA) \wedge (gB))|_{t=0} \ ,$$

which implies

$$[g'(0), A \wedge B] = (g'(0)A) \wedge B + A \wedge (g'(0)B) \ .$$

Remark 2.43. Later, when we consider Lie sphere geometry, we will again use wedge products as defined in (2.30), and will again have two lemmas analogous to those just above. All arguments are identical, except that $\mathbb{R}^{4,1}$ becomes $\mathbb{R}^{4,2}$ with metric signature $(-, +, +, +, +, -)$, and $o_{4,1}$ becomes $o_{4,2}$, and $O_{4,1}$ becomes $O_{4,2}$, and D becomes

$$D = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \ ,$$

and the vectors A and B become

$$A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{pmatrix} \ , \quad B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \end{pmatrix} \ ,$$

and the matrix in the proof of Lemma 2.40 becomes

$$\begin{pmatrix} 0 & B_1A_2-A_1B_2 & B_1A_3-A_1B_3 & B_1A_4-A_1B_4 & B_1A_5-A_1B_5 & B_6A_1-A_6B_1 \\ B_1A_2-A_1B_2 & 0 & B_2A_3-A_2B_3 & B_2A_4-A_2B_4 & B_2A_5-A_2B_5 & B_6A_2-A_6B_2 \\ B_1A_3-A_1B_3 & B_3A_2-A_3B_2 & 0 & B_3A_4-A_3B_4 & B_3A_5-A_3B_5 & B_6A_3-A_6B_3 \\ B_1A_4-A_1B_4 & B_4A_2-A_4B_2 & B_4A_3-A_4B_3 & 0 & B_4A_5-A_4B_5 & B_6A_4-A_6B_4 \\ B_1A_5-A_1B_5 & B_5A_2-A_5B_2 & B_5A_3-A_5B_3 & B_5A_4-A_5B_4 & 0 & B_6A_5-A_6B_5 \\ B_1A_6-A_1B_6 & B_6A_2-A_6B_2 & B_6A_3-A_6B_3 & B_6A_4-A_6B_4 & B_6A_5-A_6B_5 & 0 \end{pmatrix} .$$

2.12. **Conserved quantities and CMC surfaces.** For a smooth isothermic surface $x = x(u, v)$, we can regard $\mathbb{R}^{4,1}$ as the 5-dimensional fibers of a trivial vector bundle defined on x. To define a flat connection on this bundle, like in Section 2.10, we can instead just define a path-independent notion (within homotopy classes) of parallel vector fields. Such a vector field can be given by a section $Y = Y(u, v) \in \mathbb{R}^{4,1}$ solving (here we rename ∇ to Γ because we now fix a specific connection), for some $\lambda \in \mathbb{R}$,

$$\Gamma Y = 0 \ , \quad \Gamma := d + \lambda\tau \ ,$$

$$\tau = \frac{-2}{\langle (X_0)_u, (X_0)_u \rangle} X_0 \wedge X_{0,u} du + \frac{2}{\langle (X_0)_v, (X_0)_v \rangle} X_0 \wedge X_{0,v} dv \tag{2.32}$$

$$= \frac{-1}{2|x_u|^2} X_0 \wedge X_{0,u} du + \frac{1}{2|x_v|^2} X_0 \wedge X_{0,v} dv \ ,$$

where X_0 is the particular lift into the light cone that lies in M_0 (i.e. X_0 is as in Lemma 2.2 with $\kappa = 0$). We call τ a *retraction form* of X_0.

Remark 2.44. For the Riemannian connection $\nabla_Z Y$, Z and Y are both vector fields in the tangent bundle of the manifold. In Γ, however, Y will be a vector field in the trivial $\mathbb{R}^{4,1}$ vector bundle, and Z will still be in the tangent bundle of the surface. The $X_0 \wedge X_{0,u}$ and $X_0 \wedge X_{0,v}$ parts of Γ apply to Y, and the 1-forms du and dv in Γ apply to Z (as ∂_u and ∂_v form a basis of the fibers of the tangent bundle).

Remark 2.45. Note that τ is invariant of the choice of lift X of x. We chose the particular lift X_0 when defining τ above, but we could have chosen any lift X. We will use X to denote a general choice of lift.

We claimed above that the solution Y of $\Gamma Y = 0$ is independent of path. We now give two proofs of this, the first without using a Moutard lift, and the second using such a lift, to illustrate the usefulness of Moutard lifts. The second proof is shorter.

Lemma 2.46. *For any initial condition, there exists a solution Y of $\Gamma Y = 0$ independent of choice of path (within homotopy classes), that is, Γ is a flat connection for any choice of λ.*

Proof. For this proof, one can of course simply check that the curvature tensor of Γ is zero (equivalent to Equation (2.33) below), and then employ the well-known fact that parallel transport will then be independent of path. However, with a future study of discrete surfaces in mind, we start this proof from a different viewpoint here, as follows:

The condition for Γ to be flat is that the solution Y exists independently of choice of path, and an argument shows this to be equivalent to

$$\lambda d\tau + \tfrac{1}{2}\lambda^2 [\tau \wedge \tau] = 0 \ . \tag{2.33}$$

One direction of this argument showing equivalence is straightforward, and the other direction requires an argument like the one in the proof of Proposition 3.1.2 in [49].

By (2.33), it suffices to show

$$d\tau = [\tau \wedge \tau] = 0 \ .$$

That is, we need

$$\partial_v(\tau(\partial_u)) = \partial_u(\tau(\partial_v)) \tag{2.34}$$

and

$$\tau(\partial_u)\tau(\partial_v) = \tau(\partial_v)\tau(\partial_u) \ . \tag{2.35}$$

Equation (2.35) holds, by a computation that shows, for $Z \in \mathbb{R}^{4,1}$,

$$\tau(\partial_u)\tau(\partial_v)(Z) = \tau(\partial_v)\tau(\partial_u)(Z) = \tfrac{1}{4}|x_u|^{-4}\langle X_0, Z\rangle\langle X_{0,u}, X_{0,v}\rangle X_0 \ .$$

The condition (2.34) can be shown with the following consequences of Equation (2.15):

$$X_{0,uv} = AX_{0,u} + BX_{0,v} \ , \quad \partial_v(|x_u|^{-2}) = -2A|x_u|^{-2} \ , \quad \partial_u(|x_v|^{-2}) = -2B|x_v|^{-2} \ .$$

□

Another consequence of Equation (2.15) is

$$\partial_v(|x_u|^{-1}) = -A|x_u|^{-1} \quad \text{and} \quad \partial_u(|x_v|^{-1}) = -B|x_v|^{-1} \ ,$$

and from this it follows that

$$s := \tfrac{1}{\sqrt{2}}|x_u|^{-1}X_0$$

is a Moutard lift. Also,

$$\tau = -s \wedge s_u du + s \wedge s_v dv \ .$$

We make use of this in the next second proof of Lemma 2.46.

Proof. We start by noting that

$$-\partial_v(\tau(\partial_u)) + \partial_u(\tau(\partial_v)) = (s \wedge s_u)_v + (s \wedge s_v)_u =$$

$$s_v \wedge s_u + s \wedge s_{uv} + s_u \wedge s_v + s \wedge s_{uv} = 2s \wedge s_{uv} = 0 \ .$$

For $Z \in \mathbb{R}^{4,1}$,

$$(\tau(\partial_u)\tau(\partial_v) - \tau(\partial_v)\tau(\partial_u))Z = (-(s \wedge s_u)(s \wedge s_v) + (s \wedge s_v)(s \wedge s_u))Z =$$

$$-(s \wedge s_u)(\langle Z, s\rangle s_v - \langle Z, s_v\rangle s) + (s \wedge s_v)(\langle Z, s\rangle s_u - \langle Z, s_u\rangle s) =$$

$$\langle Z, s\rangle(-\langle s, s_v\rangle s_u + \langle s_u, s_v\rangle s + \langle s, s_u\rangle s_v - \langle s_u, s_v\rangle s)+$$

$$\langle Z, s_v\rangle(\langle s, s\rangle s_u - \langle s, s_u\rangle s) - \langle Z, s_u\rangle(\langle s, s\rangle s_v - \langle s, s_v\rangle s) \ ,$$

and all terms after the final equal sign above are zero. □

Remark 2.47. Noting that the compatibility condition for ($T = T(u, v)$ a 5×5 matrix)

$$dT = T \cdot \lambda\tau$$

is once again (2.33) (see the upcoming Lemma 2.61), solutions exist for any choice of initial condition, and solving this for $T \in O_{4,1}$ with initial condition $T = \mathrm{I}$ at some point (u_0, v_0) in the (u, v)-domain, we have that $\Gamma Y = 0$ with $Y = Y_0$ at (u_0, v_0) is equivalent to

$$Y = T^{-1}Y_0 \ , \tag{2.36}$$

since $\Gamma Y = 0$ means $d(TY) = 0$. We will see later that T is a Calapso transformation (Section 2.13), and when $Y_0 \in L^4$, then Y represents a Darboux transform (Section 2.14) of the surface x.

In the next definition, we are once again considering general spaceforms M, so the normalization (2.3) is not assumed.

Definition 2.48. *If there exist Q and Z in $\mathbb{R}^{4,1}$ depending smoothly on u and v such that, for $P = Q + \lambda Z$,*

$$\Gamma P = 0 \quad (\textit{or equivalently } d(TP) = 0) \tag{2.37}$$

holds for all $\lambda \in \mathbb{R}$, then we call P a linear conserved quantity *of x.*

Remark 2.49. By Definition 2.48, a linear conserved quantity exists if and only if there exist $Q = Q(u, v)$ and $Z = Z(u, v)$ in $\mathbb{R}^{4,1}$ such that the following three conditions hold:

(1) Q is constant,
(2) $dZ = -\tau Q$,
(3) $\tau Z = 0$.

Necessary and sufficient conditions for existence of a linear conserved quantity can be stated without ever referring to τ if we wish, as follows:

(1) Q is constant,
(2) $\langle X_0, Q\rangle X_{0,u} - \langle X_{0,u}, Q\rangle X_0 = 2|x_u|^2 Z_u$,
(3) $\langle X_{0,v}, Q\rangle X_0 - \langle X_0, Q\rangle X_{0,v} = 2|x_v|^2 Z_v$,
(4) $\langle X_0, Z\rangle X_{0,u} = \langle X_{0,u}, Z\rangle X_0$,
(5) $\langle X_0, Z\rangle X_{0,v} = \langle X_{0,v}, Z\rangle X_0$.

Some properties related to linear conserved quantities are immediate. For example, Q is constant (as noted in Remark 2.49) and $\tau X = 0$ (by the definition of τ). Some other immediate properties are given in the next lemmas, the first of which follows directly from items (4) and (5) in Remark 2.49:

Lemma 2.50. *X is perpendicular to both Z and dZ.*

Lemma 2.51. *$\langle Z, Z\rangle$ is constant.*

Proof. We have that

$$d(\langle Z, Z\rangle) = 2\langle Z, dZ\rangle = -2\langle Z, \tau Q\rangle = 0 \ ,$$

because τQ lies in $\mathrm{span}\{X_0, X_{0,u}, X_{0,v}\} \perp Z$. □

Corollary 2.52. *We have $\langle Z, Z\rangle \geq 0$, and if $\langle Z, Z\rangle = 0$, then Z is parallel to X.*

Proof. Because Z is perpendicular to X, and because X is lightlike, Z is either spacelike, or is a scalar multiple of X. □

Furthermore, when $Z \neq 0$, the upcoming Equation (2.41) will imply $\langle Z, Z\rangle > 0$, i.e. $\langle Z, Z\rangle \neq 0$.

Properties like these will be utilized to prove Theorems 2.53 and 2.54 below. The first of these two theorems characterizes the case that x is a part of a sphere.

Theorem 2.53. *The surface x is part of a sphere (in any spaceform) if and only if it has a constant conserved quantity $P = Q + \lambda \cdot 0$.*

Proof. Suppose that x has a conserved quantity Q of order 0. Let M be the spaceform defined by Q, and let X be the lift of x lieing in M. Then $\langle X, Q\rangle = 0$, and therefore Q is either spacelike or parallel to X. Thus X lies in the sphere given by $\mathcal{S} = Q$, as in (2.19). If Q is lightlike, then X would be a single point, not a surface, so Q must be spacelike (so the curvature κ of M is strictly negative). In fact, X is part of the virtual boundary sphere at infinity of M. Thus X will detemine a part of a finite sphere in other choices for the spaceform.

Conversely, in the case that x is part of a sphere, with lift X in M, then there exists a constant $\mathcal{S}$ that is perpendicular to X, by (2.19). Taking $Q = \mathcal{S}$, it follows that $Q = Q + \lambda \cdot 0$ is a constant (and linear) conserved quantity, by the conditions at the end of Remark 2.49. □

Theorem 2.54. [20] *An isothermic immersion $x = x(u, v)$ without umbilic points has constant mean curvature in a spaceform M (produced by $Q \neq 0$) if and only if there exists (for that Q) a linear conserved quantity $P = Q + \lambda Z$.*

Proof. Assume that x has a linear conserved quantity. Applying a single $O_{4,1}$ isometry to $\mathbb{R}^{4,1}$ if necessary, we can take Q as in (2.3), and we denote the components of Z by z_j, i.e.

$$Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} \in \mathbb{R}^{4,1} \ .$$

Applying $\langle X_0, dZ \rangle = 0$ to $d(\langle X_0, Z \rangle)$, we have, when $z = (z_1, z_2, z_3)$,

$$z \cdot dx + (z_4 - z_5)(x \cdot dx) = 0 \ ,$$

which implies z must be of the form

$$z = (z_5 - z_4)x + hn_0$$

for some real-valued function h. Then $\langle X_0, Z \rangle = 0$ implies

$$(z_5 - z_4)|x|^2 + 2h(x \cdot n_0) - z_4 - z_5 = 0 \ .$$

We can then compute that $\langle Z, Z \rangle = h^2/4$, so h is constant. In particular $\langle Z, Z \rangle \geq 0$.

The relation $dZ = -\tau Q$ from (2.37) gives that

$$d(z_5 - z_4) + 2\kappa(x \cdot dx^*) = 0 \text{ and } (z_5 - z_4)dx + hdn_0 - (1 + \kappa|x|^2)dx^* = 0 \ .$$

The second of these two equations gives us a pair of (real) equations that are linear with respect to both h and $z_5 - z_4$. Solving simultaneously for h and $z_5 - z_4$ gives

$$h = \frac{2(1 + \kappa|x|^2)}{|x_u|^2(k_1 - k_2)} \ , \tag{2.38}$$

which we know to be constant, and

$$z_5 - z_4 = \tfrac{1}{2}h(k_2 + k_1) = h \cdot H_0 \ . \tag{2.39}$$

Equations (2.38), (2.39) and h being constant then imply

$$d(z_5 - z_4) = hdH_0 = \frac{2(1 + \kappa|x|^2)}{|x_u|^2(k_1 - k_2)} dH_0 \ .$$

Using $d(z_5 - z_4) = -2\kappa(x \cdot dx^*)$ and (2.21), we find that (2.7) holds, and so H_κ is constant. One direction of the theorem now follows.

To prove the converse direction, assume that x is a CMC surface with isothermic coordinate $z = u+iv$. The Hopf differential is a constant multiple of dz^2 (see Corollary

2.36 here for the case when the spaceform is $\mathbb{R}^3$ and Equations (5.1.1) and (5.2.1) in [49] for other spaceforms). Thus, by the end of the proof of Lemma 2.7, we see that

$$b_{11} - b_{22} = \frac{4|x_u|^2(k_1 - k_2)}{1 + \kappa|x|^2}$$

is constant, and so, defining h as in (2.38), this h is also constant. We may assume Q is as in (2.3), and then take

$$Z = \tfrac{1}{2} h H_0 X_0 + h \begin{pmatrix} n_0^t \\ x \cdot n_0 \\ x \cdot n_0 \end{pmatrix} . \tag{2.40}$$

We then set the candidate for the conserved quantity to be $P = Q + \lambda Z$. A computation gives $\Gamma P = 0$, by Equation (2.7), and the definitions of k_1 and k_2, and the properties

$$\langle X_0, Z \rangle = \langle dX_0, Z \rangle = \langle X_0, dZ \rangle = 0$$

and

$$\langle X_0, Q \rangle = -(1 + \kappa|x|^2) ,$$

so P is a linear conserved quantity. □

In Theorem 2.54, when x is of constant mean curvature in the spaceform for a given Q and not totally umbilic, then Z is unique. In fact, in the proof above we saw that Z has the unique form (2.40), where h is the constant as in (2.38). Furthermore, because $1 + \kappa|x|^2$ is never zero, h cannot be zero, so the norm of Z satisfies

$$||Z|| = \tfrac{1}{2}|h| > 0 . \tag{2.41}$$

Also, by Lemma 2.7, the mean curvature satisfies

$$H_\kappa = -2h^{-1}\langle Z, Q \rangle = -\text{sgn}(h)\frac{1}{||Z||}\langle Z, Q \rangle . \tag{2.42}$$

Any constant scaling of the linear conserved quantity is still a linear conserved quantity, and will change the mean curvature by a constant multiple.

Next, noting that $z_5 - z_4 = hH_0$, Lemma 2.23 tells us that Z determines a sphere, as in (2.19), in M_0 with mean curvature

$$\pm\frac{|z_5 - z_4|}{2||Z||} = \pm\frac{|h||H_0|}{2 \cdot \frac{1}{2}|h|} = \pm|H_0| ,$$

so this sphere has the same mean curvature as the mean curvature at the corresponding point of the surface. For this, it is not necessary that H_0 be constant.

By Lemma 2.50, the spheres determined by Z contain the corresponding points X in the surface and are tangent to the surface, so Lemma 2.7 implies that Z determines a sphere congruence for which each sphere has mean curvature equalling that of the corresponding point on the surface, regardless of the choice of spaceform (i.e. the choice of value κ). Thus Z is the *mean curvature sphere congruence.* (One must check that Z and X have common orientation as well, which is left to the reader.)

Lemma 2.55. *The mean curvature sphere congruence Z can be characterized as the* conformal Gauss map *of the surface X, i.e. the unique sphere congruence with the same induced conformal structure as X.*

FIGURE 2.11. Left: a mean curvature sphere. Right: the situation described in Lemma 2.56, where the two principal curvature spheres are related by inversion through the sphere Z

Proof. That Z is the conformal Gauss map can be seen from the following computation (we do not show uniqueness here):

$$\langle dZ, dZ\rangle = h^2(H_0^2 dx\cdot dx + 2H_0(dx\cdot dn_0) + dn_0\cdot dn_0) = h^2|H_0dx + dn_0|^2 = \frac{1}{4}h^2(k_1-k_2)^2 + |-x_u du + x_v dv|^2 = \frac{1}{4}h^2|x_u|^2(k_1-k_2)^2(du^2+dv^2)\ .$$

□

Lemma 2.56. *The mean curvature sphere congruence Z can also be characterized as the* central sphere congruence*, i.e. the sphere congruence whose spheres exchange the principal curvature spheres via inversion.*

Proof. Let $X = X(u,v)$ be the lift of the surface in M_κ. Take

$$\Lambda = \Lambda(u,v) = \begin{pmatrix} \ell^t \\ \ell_4 \\ \ell_5 \end{pmatrix} \in \mathbb{R}^{4,1}$$

such that $||\Lambda|| = 1$ (i.e. Λ lies in the de Sitter space $\mathbb{S}^{3,1}$) and

$$\langle \Lambda, Q\rangle = \langle \Lambda, X\rangle = \langle \Lambda, dX\rangle = 0\ ,$$

with Q as in (2.3). This makes Λ the tangent geodesic plane to the surface. These conditions are equivalent to

- $|\ell|^2 + \ell_4^2 - \ell_5^2 = 1$,
- $(\ell_4+\ell_5)\kappa + \ell_5 - \ell_4 = 0$,
- $-2x\cdot\ell + \ell_5 + \ell_4 + |x|^2(\ell_5-\ell_4) = 0$,
- $-\ell\cdot dx + (x\cdot dx)(\ell_5-\ell_4) = 0$.

Define S_t, z, z_4 and z_5 by

$$S_t = \Lambda + tX = \Lambda + \frac{t}{1+\kappa|x|^2}X_0 = \begin{pmatrix} z^t \\ z_4 \\ z_5 \end{pmatrix}\ .$$

Then S_t also lies in $\mathbb{S}^{3,1}$ and is perpendicular to both X and dX. By Remark 2.21, the S_t represent all of the tangent spheres to X. Then, by Lemma 2.23, the mean curvature of the sphere S_t with respect to the spaceform M_κ is

$$\frac{z_5-z_4}{2} - \frac{\kappa}{2(z_5-z_4)} + \kappa\frac{|z|^2}{2(z_5-z_4)} = t\ .$$

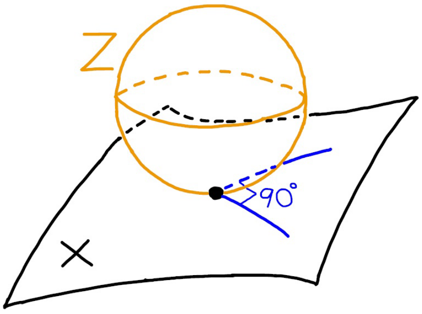

FIGURE 2.12. The situation described in Lemma 2.57, where the directions of second order contact between Z and X are marked

Then, if k_j are the principal curvatures of X, S_{k_1} and S_{k_2} are the principal curvature spheres. So by Lemma 2.22, when Z is the central sphere congruence, we have

$$-S_{k_2} = S_{k_1} - 2\langle S_{k_1}, Z\rangle \cdot Z \ . \tag{2.43}$$

Note that, as we wish to have an inversion that preserves orientation rather than reversing it, we changed S_{k_2} to $-S_{k_2}$ here. This does not change the sphere itself, as S_{k_2} is defined only projectively anyways. Now the image of S_{k_1} under inversion and S_{k_2} itself will have the same orientation.

We have that $Z = S_t$ for some t, and so we can now compute from (2.43) that

$$t = \frac{1}{2}(k_1 + k_2) \ ,$$

i.e. t is the mean curvature. Thus the central sphere congruence is the same as the mean curvature sphere congruence. □

Lemma 2.57. *The mean curvature sphere congruence Z can be characterized as the sphere congruence that has second order contact with the surface in orthogonal directions.*

Proof. Principal curvature spheres, second order contact and orthogonality are examples of notions that are invariant under Möbius transformations. Because only Möbius invariant notions appear in this proof, without loss of generality we may assume that the surface $X(u, v)$ lies in $M_0 = \mathbb{R}^3$.

Let Z be the mean curvature sphere at a point $X(u_0, v_0)$ of the surface. Then $X(u_0, v_0)$ is one point of the sphere Z. Let p be a different point in Z and let S be a sphere with center p that intersects Z transversally. We apply inversion f_S of $\mathbb{R}^3$ through the sphere S, so that the point p is mapped to infinity and the sphere Z is thus mapped to a flat plane $f_S(Z)$. The image $f_S(X(u, v))$ of $X(u, v)$ under inversion will satisfy $H = 0$ at the point $f_S(X(u_0, v_0))$. Thus the asymptotic directions of $f_S(X(u, v))$ at that point are perpendicular to each other, and are also the directions of second order contact with $f_S(Z)$. Because we have been working only with Möbius invariant notions, the lemma follows. □

2.13. **Calapso transformations.** In the following definition, the surface x has lift X in some spaceform M, but since we are dealing with a Möbius geometric notion, the choice of spaceform will not matter.

Definition 2.58. *Let $x = x(u,v)$, with lift $X = X(u,v) \in M$, be an immersed surface with isothermic coordinates u, v. A* Calapso transformation *$T \in O_{4,1}$ is a solution of*

$$T^{-1}dT = \lambda\tau \ . \tag{2.44}$$

Then

$$L^4 \ni X \to TX \in L^4$$

is a Calapso transform*, also called a* T-transform *or* conformal deformation*.*

Remark 2.59. If the initial condition for the solution T lies in $O_{4,1}$, then T will lie in $O_{4,1}$ for all u and v.

Remark 2.60. Because of (2.44), $\lambda\tau$ can be thought of as the logarithmic derivative of the Calapso transformation.

The Calapso transformation is classical, and was studied by Calapso, Bianchi and Cartan. It preserves the conformal structure and thus is naturally of interest in Möbius geometry. In the case that the initial surface is CMC, it is the same as the Lawson correspondence (see Section 2.20).

The following result has already been noted in Remark 2.47:

Lemma 2.61. *If x is isothermic, then Calapso transformations exist.*

Proof. The compatibility condition for the system

$$T^{-1}T_u = \lambda U \ , \quad U = \tfrac{-1}{2|x_u|^2} X_0 \wedge X_{0,u} \ , \tag{2.45}$$

$$T^{-1}T_v = \lambda V \ , \quad V = \tfrac{1}{2|x_v|^2} X_0 \wedge X_{0,v} \tag{2.46}$$

to have a solution T is

$$\lambda(UV - VU) + V_u - U_v = 0$$

for all $\lambda \in \mathbb{R}$, which means

$$(X_0 \wedge X_{0,v})(X_0 \wedge X_{0,u}) = (X_0 \wedge X_{0,u})(X_0 \wedge X_{0,v})$$

and

$$(|x_u|^2)_v(X_0 \wedge X_{0,u}) - |x_u|^2 X_0 \wedge X_{0,uv} =$$
$$-(|x_v|^2)_u(X_0 \wedge X_{0,v}) + |x_v|^2 X_0 \wedge X_{0,vu} \ .$$

The first of these two relations is easily checked, and the second follows from the facts that

$$|x_u|^2 = |x_v|^2 \ , \ \ x_u \cdot x_v = 0 \ , \ \ x_{uv} = Ax_u + Bx_v \tag{2.47}$$

for some functions A, B, and then, for the same A, B, that

$$X_{0,uv} = AX_{0,u} + BX_{0,v} \ . \tag{2.48}$$

□

In Möbius geometry (that is, in the space $\mathbb{R}^{4,1}$), isothermic surfaces are deformable (Calapso transformations with deformation parameter λ), and this deformation preserves second order invariants in Möbius geometry, such as the conformal class (as the next lemma shows). Note that for surfaces in Euclidean geometry, a nontrivial deformation will never preserve the second order invariants of Euclidean differential geometry, i.e. the first and second fundamental forms.

Lemma 2.62. *If $x(u,v)$ and associated lift $X(u,v)$ have isothermic coordinates (u,v), then (u,v) will also be isothermic coordinates for any Calapso transform TX.*

For the map TX in this lemma taking values in the lightcone, we have not yet formally defined the notion of "isothermic coordinates". However, a working definition implicitly appears in the proof below, and a proper justification of that working definition will be given in Section 4.2 (see also Corollary 4.31).

Proof. Without loss of generality, we may assume M is of the form M_κ in (2.2) and (2.3). The lift X takes the form

$$X = s_x \begin{pmatrix} 2x^t \\ |x|^2 - 1 \\ |x|^2 + 1 \end{pmatrix}$$

for some scalar s_x, and the normal n lifts to

$$\mathcal{T}_n = s_n \begin{pmatrix} n^t \\ n \cdot x \\ n \cdot x \end{pmatrix}$$

for some scalar s_n. Then

$$\langle X_u, X_u \rangle = \langle X_v, X_v \rangle = 4s_x^2 (x_u \cdot x_u) \ , \quad \langle X_u, X_v \rangle = 0 \ ,$$

$$\langle \mathcal{T}_n, X_u \rangle = \langle \mathcal{T}_n, X_v \rangle = \langle \mathcal{T}_n, X \rangle = 0 \ .$$

From (2.45) and (2.46) and the fact that $T \in O_{4,1}$, we can see that

$$(TX)_u = TX_u \quad \text{and} \quad (TX)_v = TX_v \ ,$$

and then it follows that X and TX have the same metric. We also have that $T\mathcal{T}_n$ is perpendicular to $(TX)_u$ and $(TX)_v$, so $T\mathcal{T}_n$ is a normal of the Calapso transformation. Finally, as (u,v) are isothermic coordinates for x and so $\langle X_{uv}, \mathcal{T}_n \rangle = 0$, we can compute that $\langle (TX)_{uv}, T\mathcal{T}_n \rangle = 0$, completing the proof. □

Lemma 2.63. *If X is a Moutard lift of $x(u,v)$, then TX is a Moutard lift as well.*

Proof. As in the proof of Lemma 2.62, we have $(TX)_{uv} = TX_{uv}$, and so $X_{uv}||X$ implies $(TX)_{uv} = TX_{uv}||TX$. □

Let us write $T = T^\lambda$, since the solution T^λ in $dT^\lambda = T^\lambda \cdot \lambda\tau$ depends on λ. Then, the Calapso transformation $X^\lambda := T^\lambda X$ has an associated

$$\tau^\lambda = \frac{-2}{\langle X_u^\lambda, X_u^\lambda \rangle} X^\lambda \wedge X_u^\lambda du + \frac{2}{\langle X_v^\lambda, X_v^\lambda \rangle} X^\lambda \wedge X_v^\lambda dv \tag{2.49}$$

as well, and we can in turn determine the Calapso transformations of $T^\lambda X$ by solving

$$d(T^{\lambda,\mu}) = T^{\lambda,\mu} \cdot \mu\tau^\lambda$$

for $T^{\lambda,\mu}$. The next lemma is about this $T^{\lambda,\mu}$.

Lemma 2.64. *With suitable choice of initial conditions for the solutions $T^{\lambda,\mu}$ and T^λ, we have*

$$T^{\mu+\lambda} = T^{\lambda,\mu} T^\lambda \ .$$

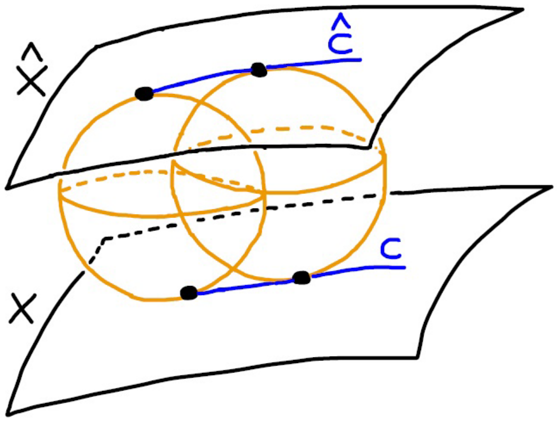

FIGURE 2.13. A Darboux transform: when the curve c in the surface x is a curvature line, then the corresponding curve $\hat{c}$ in the surface $\hat{x}$ is also a curvature line

Proof. Let us take the initial conditions to be (here "I" denotes the identity matrix)

$$T^{\lambda,\mu} = T^{\lambda} = I$$

for all $\lambda, \mu \in \mathbb{R}$. We need to show that $d(T^{\lambda,\mu}T^{\lambda})$ equals $T^{\lambda,\mu}T^{\lambda} \cdot (\mu + \lambda)\tau$. For this we can first verify that $\tau^{\lambda}T^{\lambda} = T^{\lambda}\tau$. Then we can compute that

$$d(T^{\lambda,\mu}T^{\lambda}) = d(T^{\lambda,\mu})T^{\lambda} + T^{\lambda,\mu}dT^{\lambda} = T^{\lambda,\mu}(\mu\tau^{\lambda}T^{\lambda} + T^{\lambda}\lambda\tau) =$$

$$T^{\lambda,\mu}(\mu T^{\lambda}\tau + T^{\lambda}\lambda\tau) = T^{\lambda,\mu}T^{\lambda}(\mu + \lambda)\tau \ .$$

□

2.14. **Darboux transformations.** Geometrically, a Darboux transformation of an isothermic surface is one such that

- there exists a sphere congruence enveloped by the original surface and the transform,
- the correspondence, given by the sphere congruence, from the original surface to the other enveloping surface (i.e. the transform), preserves curvature lines,
- this correspondence preserves conformality.

However, we define Darboux transformations differently:

Definition 2.65. *Let T be a Calapso transformation of X. Then $\hat{X}$ in the projectivized light cone PL^4 is a Darboux transformation of X if $T\hat{X}$ is constant in PL^4 for some choice of λ.*

The equation that $T\hat{X}$ is constant is called *Darboux's linear system.*

Take a spaceform M_κ as in (2.2) and (2.3). Let $\hat{x}$ be the surface in $\mathbb{R}^3 = M_0$ that $\hat{X}$ is a lift of. Then

$$T\hat{X} = T\left(\hat{\alpha}\begin{pmatrix} 2\hat{x}^t \\ |\hat{x}|^2 - 1 \\ |\hat{x}|^2 + 1 \end{pmatrix}\right)$$

being constant in PL^4 means that

$$d\left(rT\begin{pmatrix}2\hat{x}^t\\|\hat{x}|^2-1\\|\hat{x}|^2+1\end{pmatrix}\right)=0 \tag{2.50}$$

for some nonzero function $r \in \mathbb{R}$. A computation then shows the following lemma. An equation of the form $y' = f(y)$, where $f(y)$ is a quadratic polynomial, is called a Riccati equation, so Equation (2.51) below is a Riccati-type partial differential equation (where y becomes $\hat{x}$).

Lemma 2.66. *Equation* (2.50) *is equivalent to the Riccati-type equation*

$$d\hat{x} = \lambda|\hat{x}-x|^2 dx^* - 2\lambda(\hat{x}-x)((\hat{x}-x)\cdot dx^*)\ . \tag{2.51}$$

Remark 2.67. From Equation (2.51) we have

$$|\hat{x}_u|^2 = |\hat{x}_v|^2 = \lambda^2|x_u|^{-2}|\hat{x}-x|^4\ , \quad \hat{x}_u\cdot\hat{x}_v = 0\ ,$$

so $\hat{x}$ is conformally parametrized by u and v. Also, we can check that

$$\hat{n}\,||\,(-|\hat{x}-x|^2 n + 2((\hat{x}-x)\cdot n)(\hat{x}-x))$$

and then that

$$\hat{x}_{uv}\cdot\hat{n} = 0\ ,$$

and therefore the parametrization of $\hat{x}$ by u and v is isothermic.

Lemma 2.68. *If $\hat{X}$ is a Darboux transform of X, then X is also a Darboux transform of $\hat{X}$, and both are Darboux transforms with respect to the same choice of λ.*

Proof. Equation (2.51) implies

$$|x_u|^2|\hat{x}_u|^2 = \lambda^2|\hat{x}-x|^4\ .$$

and

$$x_u\cdot(\hat{x}-x) = \frac{\lambda|\hat{x}-x|^2}{|\hat{x}_u|^2}(\hat{x}_u\cdot(\hat{x}-x))\ ,$$

$$x_v\cdot(\hat{x}-x) = -\frac{\lambda|\hat{x}-x|^2}{|\hat{x}_u|^2}(\hat{x}_v\cdot(\hat{x}-x))\ .$$

Using these equations, we can rewrite Equation (2.51) as

$$\hat{x}_u = -\frac{|\hat{x}_u|^2 x_u}{\lambda|\hat{x}-x|^2} + 2(\hat{x}-x)|\hat{x}-x|^{-2}(\hat{x}_u\cdot(\hat{x}-x))\ ,$$

$$\hat{x}_v = \frac{|\hat{x}_u|^2 x_v}{\lambda|\hat{x}-x|^2} + 2(\hat{x}-x)|\hat{x}-x|^{-2}(\hat{x}_v\cdot(\hat{x}-x))\ .$$

Isolating the x_u and x_v terms, we find that

$$x_u = \lambda|x-\hat{x}|^2\hat{x}_u^* - 2\lambda(x-\hat{x})(\hat{x}_u^*\cdot(x-\hat{x}))\ , \tag{2.52}$$

$$x_v = \lambda|x-\hat{x}|^2\hat{x}_v^* - 2\lambda(x-\hat{x})(\hat{x}_v^*\cdot(x-\hat{x}))\ , \tag{2.53}$$

and these are the same as Equation (2.51), but with the x and $\hat{x}$ switched, proving the lemma. □

To prove that Darboux transformations as we have defined them have all the required geometric properties mentioned at the beginning of this section, it remains only to find a common sphere congruence $\mathcal{S}(u,v)$ to x and $\hat{x}$, i.e. $\mathcal{S}(u,v)$ such that

(1) $\mathcal{S}(u,v) \perp X(u,v)$,
(2) $\mathcal{S}(u,v) \perp dX(u,v)$,
(3) $\mathcal{S}(u,v) \perp \hat{X}(u,v)$, and
(4) $\mathcal{S}(u,v) \perp d\hat{X}(u,v)$.

Taking

$$\mathcal{S}(u,v) = \begin{pmatrix} 2\hat{x}^t \\ |\hat{x}|^2 - 1 \\ |\hat{x}|^2 + 1 \end{pmatrix} + \frac{(x-\hat{x})\cdot(x-\hat{x})}{(x-\hat{x})\cdot\hat{n}} \begin{pmatrix} \hat{n}^t \\ \hat{x}\cdot\hat{n} \\ \hat{x}\cdot\hat{n} \end{pmatrix} ,$$

direct computations show that items (1), (3) and (4) above hold. Then showing that item (2) holds amounts to showing

$$2((x-\hat{x})\cdot\hat{n})((\hat{x}-x)\cdot dx) + ((x-\hat{x})\cdot(x-\hat{x}))(\hat{n}\cdot dx) = 0,$$

and this follows from Equations (2.52), (2.53).

Remark 2.69. When the surface x has a linear conserved quantity $Q + \lambda Z$, one possibility for a Darboux transform is to take $\hat{X} = Q + \lambda_0 Z$ with $\lambda = \lambda_0$ chosen so that $||\hat{X}||^2 = 0$. Note that $||Q + \lambda_0 Z||^2$ is constant with respect to u and v, since

$$d(\langle Q, Z\rangle) = \langle Q, dZ\rangle = \langle Q, -\tau Q\rangle = 0 .$$

This is a special case of a Darboux transform, and even of a Bäcklund transform, called a "complementary surface", and we will come back to this in Section 2.16, after defining polynomial conserved quantities.

Remark 2.70. We define Bäcklund transforms only after defining polynomial conserved quantities, in Section 2.16. But for now, we just mention that Bäcklund transforms (more generally than just complementary surfaces) can be obtained by this recipe:

- take a surface x with a linear conserved quantity $P = Q + \lambda Z$,
- pick a value $\lambda = \mu$,
- pick an initial condition $\hat{x}_p$, at some point p in the domain of x, such that

$$\hat{X}_{0,p} = \begin{pmatrix} 2\hat{x}_p^t \\ |\hat{x}_p|^2 - 1 \\ |\hat{x}_p|^2 + 1 \end{pmatrix} \perp P(\mu)_p ,$$

- solve the Riccati equation (2.51) for $\hat{x}$.

Actually, we can choose either μ or $\hat{x}_p$ first, and then choose the other. Also, one can check that $\hat{X}$ will remain perpendicular to $P(\mu)$ at all points in the domain of x. This gives a 3-parameter family of Bäcklund transformations, generally not preserving topology of the surface x when x is not simply connected.

Also, note that we need to choose μ so that $P(\mu)$ is not timelike, for otherwise no possible choice of $\hat{X}_{0,p}$ would exist. In the case that $P(\mu)$ is lightlike, the solution $\hat{X}$ becomes $P(\mu)$ itself, and the Bäcklund transform will be a complementary surface as described in Remark 2.69 and Section 2.16.

We now give a characterization of CMC surfaces in terms of Christoffel and Darboux transformations:

Theorem 2.71. *A smooth surface x in $\mathbb{R}^3$ has constant mean curvature if and only if some scaling and translation of the Christoffel transform x^* equals a Darboux transform $\hat{x}$ (for some value of λ).*

Proof. Assume x is a CMC surface in $\mathbb{R}^3$. Then $x^* = x + H_0^{-1} n_0$ is the parallel CMC surface, by Remark 2.32. To show x^* is a Darboux transformation, we can compute that Equation (2.51) holds, for $\lambda = H_0^2/|n_0|^2 \in \mathbb{R}$.

Now we show the converse direction, proven in [63]. Assume $\hat{x}$ is a Darboux transform of x, and that $\hat{x} = a \cdot x^* + \vec{b}$ for some constants $a \in \mathbb{R} \setminus \{0\}$ and $\vec{b} \in \mathbb{R}^3$. So there exists λ such that (2.51) holds, that is,

$$(\lambda |ax^* + \vec{b} - x|^2 - a) dx^* = 2\lambda ((ax^* + \vec{b} - x) \cdot dx^*)(ax^* + \vec{b} - x) \ .$$

It follows that

$$(ax^* + \vec{b} - x) \cdot x_u^* = (ax^* + \vec{b} - x) \cdot x_v^* = 0 \ .$$

Since

$$|ax^* + \vec{b} - x|^2 = a\lambda^{-1}$$

is constant,

$$ax^* + \vec{b} - x = r \cdot n_0$$

for some constant $r \in \mathbb{R}$. So

$$dx^* = a^{-1} dx + r a^{-1} dn_0 \ .$$

Definition 2.26 implies x has CMC $H_0 = r^{-1}$. □

Corollary 2.72. *Let x be a CMC surface in $\mathbb{R}^3$. Suppose $\hat{x}$ is both a Christoffel and Darboux transform of x, as in Theorem 2.71. Then, $|\hat{x} - x|^2$ is constant, and $\hat{x} - x$ is perpendicular to dx, so $\hat{x}$ is a parallel surface to x.*

2.15. **Other transformations.** Here we make some brief remarks about two other transformations. The interested reader can find other sources for more complete information about them.

Ribaucour transformations: If we disregard some degenerate cases, Ribaucour transforms (like Darboux transforms) preserve curvature lines, but (unlike Darboux transforms) they do not necessarily preserve the conformal structure. A simple example of a Ribaucour transform of a surface in $\mathbb{R}^3$ is its reflection across a plane, which is not a Darboux transform. So Ribaucour transformations are more general than Darboux transforms.

Goursat transformations: In the case of a CMC $H \neq 0$ surface, a Goursat transformation is the composition of three transformations, first a Christoffel transformation, second a Möbius transformation, and third another Christoffel transformation.

In the case of a minimal surface, a Goursat transformation is as follows: lift the minimal surface to a null curve in $\mathbb{C}^3$, apply a complex orthogonal transformation to that null curve, and then project back to $\mathbb{R}^3$. It is a Möbius transformation for the Gauss map. One example of this is a catenoid being transformed into a minimal surface that is defined only on the universal cover of the annular domain, and a picture of this can be found in Section 5.3 of [60].

2.16. **Polynomial conserved quantities.** Definition 2.48 and Equation (2.37) can be extended to define smooth surfaces with polynomial conserved quantities of order n simply by replacing $P = Q + \lambda Z$ with

$$P = Q + \lambda P_1 + \lambda^2 P_2 + \dots + \lambda^{n-1} P_{n-1} + \lambda^n Z$$

in that definition, where Q, Z and the P_j (we define $P_0 = Q$ and $P_n = Z$) are maps from the domain for $x = x(u,v)$ to $\mathbb{R}^{4,1}$. When x has a conserved quantity of order n with $\langle Z, Z\rangle > 0$, we say that x is *special of type n*.

We now state a result about polynomial conserved quantities of Darboux transforms of special surfaces of type n:

Lemma 2.73. *If the initial isothermic surface $x = x(u,v)$ has a polynomial conserved quantity P of order n, then any Darboux transform $\hat{x} = \hat{x}(u,v)$ has a polynomial conserved quantity $\hat{P}$ of order at most $n+1$.*

Proof. Let X_0 be the lift as in Lemma 2.2 with $\kappa = 0$ of the initial surface x with Calapso transformation $T = T^\lambda$ and polynomial conserved quantity $P = P(\lambda)$ of order n. Then $T^\lambda P$ is constant with respect to u and v. Let $\hat{X}_0$ be the lift as in Lemma 2.2 with $\kappa = 0$ of the Darboux transform $\hat{x}$ of x, i.e. $T^\lambda \hat{X}_0$ is constant in PL^4 for some particular choice of λ, and let us refer to that choice of λ as $\lambda = \mu$, i.e. $T^\mu \hat{X}_0$ is constant with respect to u and v. From now on we take μ to be that fixed value, and λ will denote a free real parameter.

Let A be the matrix in $O_{4,1}$ representing the map

$$\mathbb{R}^{4,1} \ni Y \to Y + (\langle X_0, \hat{X}_0\rangle)^{-1} \left\{ \frac{-\lambda}{\mu} \langle Y, \hat{X}_0\rangle X_0 + \frac{\lambda}{\mu - \lambda} \langle Y, X_0\rangle \hat{X}_0 \right\} \in \mathbb{R}^{4,1} \ .$$

Then the map that A^{-1} represents will be

$$\mathbb{R}^{4,1} \ni Y \to Y + (\langle X_0, \hat{X}_0\rangle)^{-1} \left\{ \frac{-\lambda}{\mu} \langle Y, X_0\rangle \hat{X}_0 + \frac{\lambda}{\mu - \lambda} \langle Y, \hat{X}_0\rangle X_0 \right\} \in \mathbb{R}^{4,1} \ .$$

We note, as an aside, that by Equation (2.51) we have

$$\hat{x}_u = -\mu |\hat{x} - x|^2 \frac{x_u}{|x_u|^2} + 2\mu(\hat{x} - x) \left((\hat{x} - x) \cdot \frac{x_u}{|x_u|^2} \right) \ ,$$

$$\hat{x}_v = \mu |\hat{x} - x|^2 \frac{x_v}{|x_v|^2} - 2\mu(\hat{x} - x) \left((\hat{x} - x) \cdot \frac{x_v}{|x_v|^2} \right) \ .$$

We now wish to show

$$\lambda \hat{\tau}(\partial_u) = A^{-1} \lambda \tau(\partial_u) A + A^{-1} dA(\partial_u) \ , \tag{2.54}$$

$$\lambda \hat{\tau}(\partial_v) = A^{-1} \lambda \tau(\partial_v) A + A^{-1} dA(\partial_v) \ . \tag{2.55}$$

For this purpose, we will take convenient scalar multiples of X_0 and $\hat{X}_0$. Because $T^\mu \hat{X}_0$ is constant in PL^4, we have that $T^\mu \hat{X}_0 = \hat{r} \hat{Y}$, where $\hat{Y}$ is a constant vector and $\hat{r} = \hat{r}(u,v)$ is a real scalar factor. Then

$$T^\mu (\hat{r}^{-1} \hat{X}_0) = \hat{Y} \ .$$

Because

$$0 = d(T^\mu (\hat{r}^{-1} \hat{X}_0)) = (dT^\mu)(\hat{r}^{-1} \hat{X}_0) + T^\mu d(\hat{r}^{-1} \hat{X}_0) = \\ T^\mu (\mu \tau (\hat{r}^{-1} \hat{X}_0) + d(\hat{r}^{-1} \hat{X}_0)) \ ,$$

we have

$$(d+\mu\tau)(\hat{r}^{-1}\hat{X}_0) = 0 \ .$$

Similarly, there exists a real scalar factor $r = r(u,v)$ so that $\hat{T}^{\mu}(r^{-1}X_0) = Y$ is a constant vector, and

$$(d+\mu\hat{\tau})(r^{-1}X_0) = 0 \ .$$

We use the above two equations repeatedly in the following computations. Also,

$$d\langle r^{-1}X_0, \hat{r}^{-1}\hat{X}_0\rangle = \langle d(r^{-1}X_0), \hat{r}^{-1}\hat{X}_0\rangle + \langle r^{-1}X_0, d(\hat{r}^{-1}\hat{X}_0)\rangle =$$

$$\mu(\langle r^{-1}X_0, \hat{\tau}(\hat{r}^{-1}\hat{X}_0)\rangle\langle \tau(r^{-1}X_0), \hat{r}^{-1}\hat{X}_0\rangle) = 0 \ ,$$

because $\tau X_0 = \hat{\tau}\hat{X}_o = 0$. Thus, without loss of generality,

$$\langle X, \hat{X}\rangle = 1 \ ,$$

where we define

$$X := r^{-1}X_0 \ , \quad \hat{X} := \hat{r}^{-1}\hat{X}_0 \ .$$

Using these special normalizations X and $\hat{X}$, we can now make a cleaner computation showing Equations (2.54) and (2.55), as follows: First we note that we can rewrite the maps that A and A^{-1} represent as

$$Y \to Y + \frac{-\lambda}{\mu}\langle Y, \hat{X}\rangle X + \frac{\lambda}{\mu-\lambda}\langle Y, X\rangle \hat{X} \ ,$$

$$Y \to Y + \frac{-\lambda}{\mu}\langle Y, X\rangle \hat{X} + \frac{\lambda}{\mu-\lambda}\langle Y, \hat{X}\rangle X \ .$$

Defining

$$\mathcal{A}^{(u)} := -\lambda\hat{\tau}(\partial_u) + A^{-1}\lambda\tau(\partial_u)A + A^{-1}dA(\partial_u) \ ,$$

$$\mathcal{A}^{(v)} := -\lambda\hat{\tau}(\partial_v) + A^{-1}\lambda\tau(\partial_v)A + A^{-1}dA(\partial_v) \ .$$

It suffices to check that $\mathcal{A}^{(*)}Z = 0$, for $Z = X, \hat{X}$ and for

$$Z = Z(u,v) \in (\mathrm{span}\{X, \hat{X}\})^{\perp} \ .$$

An useful fact will be

$$A^{-1}X_u = X_u + \tfrac{\lambda}{\mu-\lambda}\langle X_u, \hat{X}\rangle X = X_u + \tfrac{\lambda}{\mu-\lambda}\langle -\mu\hat{\tau}(\partial_u)X, \hat{X}\rangle X = X_u \ ,$$

and similarly

$$A^{-1}X_v = X_v \ , \quad A^{-1}\hat{X}_u = \hat{X}_u \ , \quad A^{-1}\hat{X}_v = \hat{X}_v \ .$$

If $Z = X$, then

$$\mathcal{A}^{(u)}Z = (A^{-1}A_u - \lambda\hat{\tau}(\partial_u))X = A^{-1}A_uX + \tfrac{\lambda}{\mu}X_u =$$

$$A^{-1}((1-\tfrac{\lambda}{\mu})X)_u + (\tfrac{\lambda}{\mu}-1)X_u = 0 \ .$$

Now suppose $Z \in (\mathrm{span}\{X, \hat{X}\})^{\perp}$. Then

$$A^{-1}Z_u - Z_u = -\tfrac{\lambda}{\mu}\langle Z_u, X\rangle \hat{X} + \tfrac{\lambda}{\mu-\lambda}\langle Z_u, X\rangle X =$$

$$\tfrac{\lambda}{\mu}\langle Z, X_u\rangle \hat{X} - \tfrac{\lambda}{\mu-\lambda}\langle Z, X_u\rangle X =$$

$$\tfrac{\lambda}{\mu}\langle Z, -\mu\hat{\tau}(\partial_u)X\rangle \hat{X} - \tfrac{\lambda}{\mu-\lambda}\langle Z, -\mu\tau(\partial_u)\hat{X}\rangle X =$$

$$-\lambda\langle Z, \tfrac{-2}{\langle \hat{X}_u, \hat{X}_u\rangle}\hat{X}_u\rangle \hat{X} + \tfrac{\mu\lambda}{\mu-\lambda}\langle Z, \tfrac{-2}{\langle X_u, X_u\rangle}X_u\rangle X =$$

$$\frac{2\lambda}{\langle \hat{X}_u, \hat{X}_u\rangle}\langle Z, \hat{X}_u\rangle \hat{X} - \frac{2\mu\lambda}{(\mu-\lambda)\langle X_u, X_u\rangle}\langle Z, X_u\rangle X \ ,$$

and so

$$\mathcal{A}^{(u)}Z = A^{-1}\lambda\tau(\partial_u)Z + A^{-1}A_uZ - \lambda\hat{\tau}(\partial_u)Z =$$
$$\frac{2\lambda}{\langle X_u, X_u\rangle}\langle X_u, Z\rangle A^{-1}X + A^{-1}(AZ)_u - Z_u - \frac{2\lambda}{\langle \hat{X}_u, \hat{X}_u\rangle}\langle \hat{X}_u, Z\rangle \hat{X} = 0 \ ,$$

by substituting in the relation for $A^{-1}Y_u - Y_u$ just above. These and other similar computations show Equations (2.54), (2.55).

Using Equations (2.54) and (2.55), we see that

$$d(T^\lambda A) = T^\lambda A \cdot \lambda\hat{\tau} \ ,$$

and so

$$\hat{T}^\lambda = T^\lambda A$$

solves

$$(\hat{T}^\lambda)^{-1}d\hat{T}^\lambda = \lambda\hat{\tau} \ .$$

Then we define

$$\hat{P} = \mu(\mu-\lambda)A^{-1}P \ .$$

$\hat{T}^\lambda\hat{P}$ is constant, since

$$d(\hat{T}^\lambda\hat{P}) = \mu(\mu-\lambda)d(T^\lambda A\cdot A^{-1}P) = \mu(\mu-\lambda)d(T^\lambda P) = 0 \ .$$

Thus $\hat{P}$ is a polynomial conserved quantity for $\hat{x}$ of degree at most $n+2$. To show that the degree is actually at most $n+1$, it suffices to show that the leading coefficient

$$\hat{P}_{n+2} = \langle X_0, \hat{X}_0\rangle^{-1}\langle P_n, X_0\rangle \hat{X}_0$$

is zero. But, like in item (3) of Remark 2.49, we have

$$\tau P_n = \tau Z = 0 \ ,$$

so $\langle P_n, X_0\rangle = 0$ (see Lemma 2.50), and $\hat{P}$ has order at most $n+1$. □

Remark 2.74. Although A and A^{-1} in the above proof have poles at $\lambda = \mu$, note that $\hat{P}$ itself does not.

The Darboux transform in Lemma 2.73 is a Bäcklund transform exactly when it is of type at most n. See Remarks 2.69 and 2.70. We can take this as a definition of Bäcklund transformations. One can think of Bäcklund transformations are Darboux transformations that preserve special properties. For example, they will preserve the property of being CMC.

For an isothermic surface with a polynomial conserved quantity of order n, we define a complementary surface, like in Remark 2.69, as follows: take a value μ so that

$$||P(\mu)||^2 = ||Q + \mu P_1 + \mu^2 P_2 + ... + \mu^{n-1}P_{n-1} + \mu^n Z||^2 = 0$$

and define the complementary surface to be $P(\mu)$. (Clearly, choices of μ for which $P(\mu)$ is not lightlike are not allowed.) This will be a Bäcklund transformation, i.e. of type at most n.

Complementary surfaces can be of type n. But if a Bäcklund transform is of type $n-1$ (Darboux transforms must be of type at least $n-1$, by Lemmas 2.68 and 2.73), then it must be a complementary surface, by Lemma 4.10 of [25]. Examples of type

$n-1$ Bäcklund transforms can come from CMC 1 surfaces in $\mathbb{H}^3$ and minimal surfaces in $\mathbb{R}^3$. In fact, we have the following lemma:

Lemma 2.75. *In the case $n = 1$ (i.e. CMC surfaces), CMC $\pm\sqrt{-\kappa}$ surfaces in M_κ are the only cases where a type $n-1 = 0$ Bäcklund transform can exist. In particular, if such a Bäcklund transform exists, then $\kappa \le 0$.*

Proof. When the linear conserved quantity is normalized so that $||Z||^2 = 1$, we have (see (2.41), (2.42) and Lemma 2.5)

$$||\lambda Z + Q||^2 = \lambda^2 - 2H\lambda - \kappa \ ,$$

and the discriminant is

$$2\sqrt{H^2 + \kappa} \ .$$

When a type 0 Bäcklund transform exists, we have a higher order zero of $\lambda^2 - 2H\lambda - \kappa$ (by Lemma 4.10 in [25]), so

$$H^2 + \kappa = 0 \ ,$$

i.e. $H^2 = -\kappa$. (See [25] for further details.) □

2.17. **Bianchi permutability.** In this section, we will see how two Darboux transformations of a surface can themselves have a common Darboux transformation. Consider an isothermic surface, in a spaceform M^3,

$$X(u, v) \subset M^3 \subset L^4 \subset \mathbb{R}^{4,1}$$

with isothermic coordinates u, v, and with associated 1-parameter family of flat connections

$$\Gamma^\lambda = d + \lambda\tau \ ,$$

where τ is as defined in Equation (2.32).

Lemma 2.76. *$\hat{X}(u, v)$ is a Darboux transform of $X(u, v)$ with Darboux parameter $\lambda = \mu$ if and only if $\Gamma^\mu \hat{X} || \hat{X}$.*

Proof. $\hat{X}$ is a Darboux transform with Darboux parameter μ if and only if $T^\mu \hat{X} = r(u, v)Y_0$ for some constant vector Y_0 and some scalar function $r(u, v)$. This condition can be restated in derivate form as

$$(dT^\mu)\hat{X} + T^\mu d\hat{X} = dr \cdot Y_0 \ ,$$

which is equivalent to

$$\Gamma^\mu \hat{X} = \tfrac{dr}{r} r (T^\mu)^{-1} Y_0 = \tfrac{dr}{r} \hat{X} \ .$$

This proves the lemma. □

Definition 2.77. *We define the orthogonal transformation $\Gamma_v^w(s)$ of $\mathbb{R}^{4,1}$, for $v, w \in L^4$, by*

$$\Gamma_v^w(s)Y = sY \quad \textit{if } Y||w \ ,$$

$$\Gamma_v^w(s)Y = s^{-1}Y \quad \textit{if } Y||v \ ,$$

$$\Gamma_v^w(s)Y = Y \quad \textit{if } Y \perp \textit{span}\{v, w\} \ ,$$

for each $s \in \mathbb{R}$.

The explicit form for $\Gamma_v^w(s)$ as in the next lemma can be easily verified:

Lemma 2.78. *$\Gamma_v^w(s)Y$, for $Y \in \mathbb{R}^{4,1}$, can be written as*

$$\Gamma_v^w(s)Y = Y + \langle v, w\rangle^{-1}\{(s-1)\langle Y, v\rangle w + (s^{-1}-1)\langle Y, w\rangle v\} \ .$$

The next lemma, however, requires a bit more work to prove:

Lemma 2.79. *Taking lifts X, $\hat{X}$ such that $\langle X, \hat{X}\rangle = 1$, $\Gamma^\mu \hat{X} = \hat{\Gamma}^\mu X = 0$, for $\Gamma^\lambda = d + \lambda\tau$ and $\hat{\Gamma}^\lambda = d + \lambda\hat{\tau}$, as we did in the proof of Lemma 2.73, we have*

1. $\Gamma_X^{\hat{X}}(s) = \exp((\log(s))X \wedge \hat{X})$,
2. *for all $\lambda \neq \mu$,*

$$\Gamma_X^{\hat{X}}(1 - \tfrac{\lambda}{\mu}) \circ \Gamma^\lambda = \hat{\Gamma}^\lambda \ ,$$

where the symbol $\circ$ denotes the composition of maps

$$\Gamma_X^{\hat{X}}(1 - \tfrac{\lambda}{\mu}) \circ \Gamma^\lambda := (\Gamma_X^{\hat{X}}(1 - \tfrac{\lambda}{\mu}))(\Gamma^\lambda)(\Gamma_X^{\hat{X}}(1 - \tfrac{\lambda}{\mu}))^{-1} \ .$$

(The second of the two equations above is actually independent of the choices of lifts X and $\hat{X}$.)

Proof. The proof of the first item is straightforward. A proof of the second item can be accomplished in the following three steps:

1. Insert X into both the left and right hand sides of the equation, and show

$$(1 - \tfrac{\lambda}{\mu})\Gamma_X^{\hat{X}}(1 - \tfrac{\lambda}{\mu})(-\mu\hat{\tau}X) = (\lambda - \mu)\hat{\tau}X \ .$$

The conditions on the choices of lifts X and $\hat{X}$ imply that X_u, X_v, $\hat{X}_u$, $\hat{X}_v$ are all perpendicular to span$\{X, \hat{X}\}$. We need to show, for the coefficient of the du part for example, that $-\mu(1 - \frac{\lambda}{\mu})\hat{X}_u = (\lambda - \mu)\hat{X}_u$, which of course is true.

2. Insert $\hat{X}$ into both the left and right hand sides of the equation, and argue similarly to the previous case.
3. Insert $Y \perp \text{span}\{X, \hat{X}\}$, and show, for example,

$$\Gamma_X^{\hat{X}}(1 - \tfrac{\lambda}{\mu})Y_u + \tfrac{2\lambda}{|X_u|^2}\langle X_u, Y\rangle \tfrac{\mu}{\mu - \lambda} X = Y_u + \tfrac{2\lambda}{|\hat{X}_u|^2}\langle \hat{X}_u, Y\rangle \hat{X} \ .$$

Writing

$$Y_u = -\langle \hat{X}_u, Y\rangle X - \langle X_u, Y\rangle \hat{X} + Z \qquad (Z \perp \text{span}\{X, \hat{X}\}) \ ,$$

we need to show statements like

$$\langle \hat{X}_u, Y\rangle\langle X_u, X_u\rangle = 2\mu\langle X_u, Y\rangle \ ,$$

which follows from $\hat{X}_u = \frac{2\mu}{|X_u|^2}X_u$.

The proof is then completed by noting that for any scalar functions $f_1(u,v)$ and $f_2(u,v)$,

$$(\Gamma_X^{\hat{X}}(1 - \tfrac{\lambda}{\mu}) \circ \Gamma^\lambda)(f_1 X + f_2\hat{X} + Y) = f_1(\Gamma_X^{\hat{X}}(1 - \tfrac{\lambda}{\mu}) \circ \Gamma^\lambda)X +$$

$$f_2(\Gamma_X^{\hat{X}}(1 - \tfrac{\lambda}{\mu}) \circ \Gamma^\lambda)\hat{X} + (\Gamma_X^{\hat{X}}(1 - \tfrac{\lambda}{\mu}) \circ \Gamma^\lambda)Y + df_1 X + df_2\hat{X} =$$

$$f_1\hat{\Gamma}^\lambda X + f_2\hat{\Gamma}^\lambda\hat{X} + \hat{\Gamma}^\lambda Y + df_1 X + df_2\hat{X} = \hat{\Gamma}^\lambda(f_1 X + f_2\hat{X} + Y) \ .$$

□

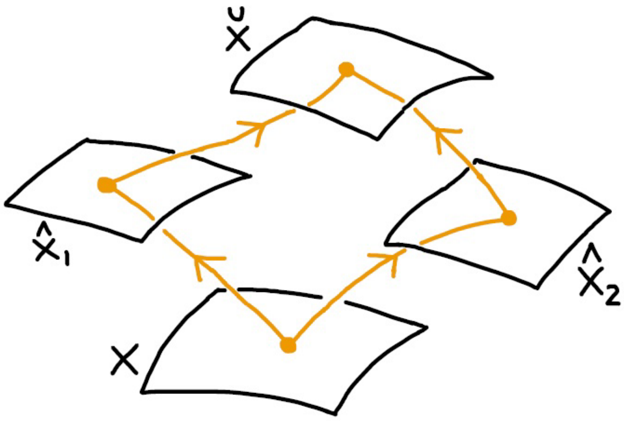

FIGURE 2.14. Bianchi permutatibility

Now suppose that $\hat{X}_1$ and $\hat{X}_2$ are both Darboux transforms of X, with Darboux parameters μ_1 and μ_2 respectively,

$$\mu_1 \neq \mu_2 ,$$

and with associated 1-parameter families of flat connections $\hat{\Gamma}_1^\lambda$ and $\hat{\Gamma}_2^\lambda$ respectively.

Bianchi permutability is the existence of another surface $\check{X}$ that is both a Darboux transform of $\hat{X}_1$ with Darboux parameter μ_2 and a Darboux transform of $\hat{X}_2$ with Darboux parameter μ_1. We define $\check{X}$ as

$$\check{X} = \Gamma_{\hat{X}_1}^{\hat{X}_2}(t)X \subset L^4 \cap \text{span}\{X, \hat{X}_1, \hat{X}_2\} .$$

Noting that $\dim(\text{span}\{X, \hat{X}_1, \hat{X}_2, \check{X}\}) \leq 3$, it follows that $X, \hat{X}_1, \hat{X}_2, \check{X}$ will all lie on one circle, once they are projected down to some spaceform M^3. In particular, their cross ratio will be real, and Lemma 2.11 implies

$$\text{cr}(X, \hat{X}_2, \check{X}, \hat{X}_1) = \frac{\langle X, \hat{X}_2\rangle\langle \hat{X}_1, \check{X}\rangle - \langle X, \check{X}\rangle\langle \hat{X}_1, \hat{X}_2\rangle + \langle X, \hat{X}_1\rangle\langle \hat{X}_2, \check{X}\rangle}{2\langle X, \hat{X}_1\rangle\langle \hat{X}_2, \check{X}\rangle} .$$

Then, using the facts that

$$\langle \hat{X}_1, \check{X}\rangle = t\langle X, \hat{X}_1\rangle ,$$
$$\langle \hat{X}_2, \check{X}\rangle = t^{-1}\langle X, \hat{X}_2\rangle ,$$
$$\langle X, \check{X}\rangle = (t + t^{-1} - 2)\frac{\langle X, \hat{X}_1\rangle\langle X, \hat{X}_2\rangle}{\langle \hat{X}_1, \hat{X}_2\rangle} ,$$

we can prove the following lemma, which was actually the motivation for the choice of definition of $\check{X}$ above.

Lemma 2.80. *The cross ratio of $X, \hat{X}_2, \check{X}, \hat{X}_1$ is*

$$cr_{X,\hat{X}_2,\check{X},\hat{X}_1} = t .$$

The next theorem shows that, when t is chosen correctly, $\check{X}$ is a Darboux transform of both $\hat{X}_1$ and $\hat{X}_2$, with Darboux parameters μ_2 and μ_1 respectively.

Theorem 2.81. *Taking $t = \mu_1\mu_2^{-1}$, we have $\hat{\Gamma}_1^{\mu_2}\check{X}||\check{X}$, $\hat{\Gamma}_2^{\mu_1}\check{X}||\check{X}$.*

FIGURE 2.15. Physical models of the helicoid in $\mathbb{R}^3$, a minimal surface (owned by the geometry group at the Technical University of Vienna)

Proof. Let us prove the first of the two claims $\hat{\Gamma}_1^{\mu_2}\breve{X}||\breve{X}$ here. The second claim is proven similarly. The first of the two claims is equivalent to

$$\Gamma_X^{\hat{X}_1}(1-\mu_2\mu_1^{-1})\Gamma^{\mu_2}\Gamma_{\hat{X}_1}^{X}(1-\mu_2\mu_1^{-1})\Gamma_{\hat{X}_1}^{\hat{X}_2}(\mu_1\mu_2^{-1})X||\Gamma_{\hat{X}_1}^{\hat{X}_2}(\mu_1\mu_2^{-1})X \ ,$$

i.e.

$$\Gamma^{\mu_2}\Gamma_{\hat{X}_1}^{X}(1-\mu_2\mu_1^{-1})\Gamma_{\hat{X}_1}^{\hat{X}_2}(\mu_1\mu_2^{-1})X||\Gamma_{\hat{X}_1}^{X}(1-\mu_2\mu_1^{-1})\Gamma_{\hat{X}_1}^{\hat{X}_2}(\mu_1\mu_2^{-1})X \ .$$

Since $\hat{X}_2$ is a Darboux transform of X with Darboux parameter μ_2, it will suffice to show that

$$\Gamma_{\hat{X}_1}^{X}(1-\mu_2\mu_1^{-1})\Gamma_{\hat{X}_1}^{\hat{X}_2}(\mu_1\mu_2^{-1})X$$

is parallel to $\hat{X}_2$, and it can be shown by straightforward computation that

$$\Gamma_{\hat{X}_1}^{X}(1-\mu_2\mu_1^{-1})\Gamma_{\hat{X}_1}^{\hat{X}_2}(\mu_1\mu_2^{-1})X = \frac{\mu_1-\mu_2}{\mu_2}\cdot\frac{\langle X,\hat{X}_1\rangle}{\langle \hat{X}_1,\hat{X}_2\rangle}\hat{X}_2 \ .$$

□

In the next sections 2.18, 2.19, 2.21, 2.22, we consider particular surfaces in ways only loosely connected with previous sections, but in later chapters we wish to consider their relations with Lie sphere geometry, and also consider their discretizations in a separate text, so we include these sections here. Section 2.20 is about the connection between the surfaces in Section 2.18 and those in Section 2.19, and makes use of conserved quantities in Möbius geometry, as described in previous sections.

2.18. **Minimal surfaces in $\mathbb{R}^3$.** We can always take a CMC surface to have local isothermic coordinate $z=u+iv$, $u,v\in\mathbb{R}$, $i=\sqrt{-1}$ (away from umbilic points), and then the Hopf differential becomes $\hat{Q}dz^2$ for some real constant $\hat{Q}$. Rescaling the coordinate z by a constant real factor, we may assume $\hat{Q}=1$. So now assume we have an isothermic minimal surface in $\mathbb{R}^3$ with Hopf differential function $\hat{Q}=1$. Then

$$\frac{\hat{Q}dz^2}{dg}=\frac{dz}{g'} \ ,$$

where g is the stereographic projection of the Gauss map to the complex plane, and $g' = dg/dz$. The map g taking z in the domain of the immersion (of the surface) to $\mathbb{C}$ is holomorphic. Because we are avoiding umbilics, we have $g' \neq 0$. When we are only concerned with local behavior of the surface, we can ignore the possibility that g has poles or other singularities. Then the Weierstrass representation is

$$x = \mathrm{Re} \int_{z_0}^{z} (2g, 1-g^2, i+ig^2)\frac{dz}{g'} \ .$$

The explanation of the Weierstrass representation for minimal surfaces here is extremely brief. More complete explanations of this representation can be found in many places, including [49], and many of the more standard references can be found in the bibliographies of [49] and [103].

2.19. **CMC 1 surfaces in $\mathbb{H}^3$.** We can now similarly describe CMC 1 surfaces in $\mathbb{H}^3$. Construction of CMC 1 surfaces with isothermic coordinates starts with the Bryant equation (g is an arbitrary holomorphic function such that $g' \neq 0$)

$$dF = F \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \frac{dz}{g'}$$

with solution $F \in \mathrm{SL}_2\mathbb{C}$, and the surface is then

$$F \cdot \bar{F}^t \in \mathbb{H}^3 \ .$$

Here, hyperbolic 3-space is

$$\mathbb{H}^3 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^{3,1} \,|\, x_0 > 0, x_0^2 - x_1^2 - x_2^2 - x_3^2 = 1\} =$$
$$\left\{ \mathcal{A} = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \middle| \det(\mathcal{A}) = 1,\, x_0 > 0 \right\} =$$
$$\{a \cdot \bar{a}^t \,|\, a \in \mathrm{SL}_2\mathbb{C}\} \ .$$

Example 2.82. Take any constant $q \in \mathbb{C} \setminus \{0\}$. Then for $g = qz$, one solution to the Bryant equation is

$$F = q^{-1/2} \begin{pmatrix} \cosh(z) & q\sinh(z) - qz\cosh(z) \\ \sinh(z) & q\cosh(z) - qz\sinh(z) \end{pmatrix} ,$$

and $F\bar{F}^t$ is a CMC 1 Enneper cousin in $\mathbb{H}^3$.

Example 2.83. To make CMC 1 surfaces of revolution, called catenoid cousins, one can use $g = e^{\mu z}$ for μ either real or purely imaginary.

Once again, the explanation of the Weierstrass representation here, for CMC 1 surfaces in $\mathbb{H}^3$, is brief. More complete explanations of this representation can be found in [49], [103] and references therein.

2.20. **The Lawson correspondence.** For a surface x with isothermic coordinates u and v, and with lift X in L^4 and τ as in (2.32), X gives a surface in the spaceform determined by Q, and we have the Calapso transformation $X^\lambda := T^\lambda X$, with $(T^\lambda)^{-1} dT^\lambda = \lambda\tau$. For any fixed real μ, the τ^μ for X^μ is similarly defined as in (2.49) (with λ there replaced by μ), and it is readily shown (see the proof of Lemma 2.64) that

$$\tau^\mu = T^\mu \tau (T^\mu)^{-1} \ .$$

Now suppose X has a polynomial conserved quantity P. Then

$$(d + \lambda\tau^\mu)(T^\mu P(\lambda + \mu)) = 0 ,$$

and therefore

$$P^\mu = P^\mu(\lambda) := T^\mu P(\lambda + \mu)$$

is a polynomial conserved quantity for X^μ of the same order as $P = P(\lambda)$.

For later use, we note that

$$dX^\mu = d(T^\mu X) = T^\mu(dX + (T^\mu)^{-1}dT^\mu X) = T^\mu(dX + \mu\tau X) = T^\mu dX , \tag{2.56}$$

and we similarly have, for the top term coefficient $Z^\mu = T^\mu Z$ of P^μ,

$$dZ^\mu = T^\mu(dZ + \mu\tau Z) = T^\mu dZ , \tag{2.57}$$

because $\tau Z = 0$.

When P is a linear conserved quantity, x^μ gives a CMC surface in the spaceform determined by the constant term Q^μ in P^μ. Let us rescale X so that it lies in the spaceform M determined by the constant term Q of P. Since

$$Q^\mu = T^\mu(\mu Z + Q) ,$$

and

$$\langle X^\mu, Q^\mu\rangle = \langle X, Q\rangle = -1 ,$$

X^μ lies in the spaceform M^μ determined by Q^μ. Then $||dX^\mu||^2 = ||dX||^2$ implies the metrics of the two surfaces in M and M^μ coming from X and X^μ, respectively, are the same. Using (without loss of generality) the case (2.3) and Equation (2.40) and that h is constant, we have

$$\left\langle \mathcal{T}_n, \left(\frac{1}{1+\kappa|x|^2}X_0\right)_{zz} \right\rangle = \frac{1}{1+\kappa|x|^2}(n_0 \cdot (x_{uu} - x_{vv})) =$$

$$h^{-1}\left\langle \left(\frac{1}{1+\kappa|x|^2}X_0\right)_v, Z_v \right\rangle - h^{-1}\left\langle \left(\frac{1}{1+\kappa|x|^2}X_0\right)_u, Z_u \right\rangle .$$

We also have, by (2.56) and (2.57),

$$\langle dX^\mu, dZ^\mu\rangle = \langle T^\mu dX, T^\mu dZ\rangle = \langle dX, dZ\rangle .$$

Furthermore, $Z^\mu = T^\mu Z$, and so $||Z^\mu||^2 = ||Z||^2$ and so Equation (2.41) implies h is the same constant for the two surfaces. It follows that the Hopf differentials of the two surfaces are the same. Also, assuming we have normalized P so that $||Z||^2 = 1$, the mean curvatures H and H^μ of x and x^μ are related by

$$H^\mu = -\langle T^\mu Z, T^\mu(\mu Z + Q)\rangle = -\mu + H .$$

We conclude that we have the Lawson correspondence between the surface given by X in the spaceform M determined by Q with constant sectional curvature $-||Q||^2$ and the surface given by X^μ in the spaceform M^μ determined by Q^μ with constant sectional curvature

$$-||Q^\mu||^2 = -\mu^2 + 2\mu H - ||Q||^2 .$$

(See Section 5.5.1 of [49].)

FIGURE 2.16. Spacelike constant mean curvature surfaces of revolution in Minkowski 3-space with timelike axes (see [16])

Remark 2.84. Minimal surfaces in $\mathbb{R}^3$ and CMC 1 surfaces in $\mathbb{H}^3$, as in Sections 2.18 and 2.19, are related to each other by the Lawson correspondence. In this case,

$$||Q||^2 = H = 0 \quad \text{and} \quad \mu = -1 ,$$

so

$$-||Q^\mu||^2 = -1 \quad \text{and} \quad H^\mu = 1 .$$

2.21. **Spacelike CMC surfaces in $\mathbb{R}^{2,1}$.** In this section, we consider maximal surfaces and spacelike CMC surfaces in $\mathbb{R}^{2,1}$, in preparation for considering discrete versions of these surfaces later. Consider a smooth surface

$$x(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$$

in $\mathbb{R}^3$ or $\mathbb{R}^{2,1}$ (with metric of signature $(+,+,-)$ in the case of $\mathbb{R}^{2,1}$), with unit normal n_0, which we simply write as n here. Suppose the surface is spacelike, in the case of $\mathbb{R}^{2,1}$. Also, suppose that the coordinates u, v are isothermic, with first and second fundamental forms

$$I = \begin{pmatrix} x_u \cdot x_u & 0 \\ 0 & x_v \cdot x_v \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} ,$$

$$II = \begin{pmatrix} n \cdot x_{uu} & n \cdot x_{vu} \\ n \cdot x_{uv} & n \cdot x_{vv} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} k_1 E & 0 \\ 0 & k_2 E \end{pmatrix} .$$

where "$\cdot$" denotes the inner product associated with $\mathbb{R}^3$ or $\mathbb{R}^{2,1}$. We have $n_u = -k_1 x_u$ and $n_v = -k_2 x_v$, where k_1 and k_2 are the principal curvatures. For the Hopf differential $\hat{Q} dz^2$ with $z = u + iv$, the Hopf differential function $\hat{Q}$ is, with "$\cdot$" linearly extended to complex vectors,

$$\hat{Q} = n \cdot x_{zz} = \frac{E}{4}(k_1 - k_2) .$$

If the mean curvature H is constant, then

$$\hat{Q} = (E/4)(k_1 - k_2) \in \mathbb{R}$$

is constant. (See Corollary 2.36, for example.)

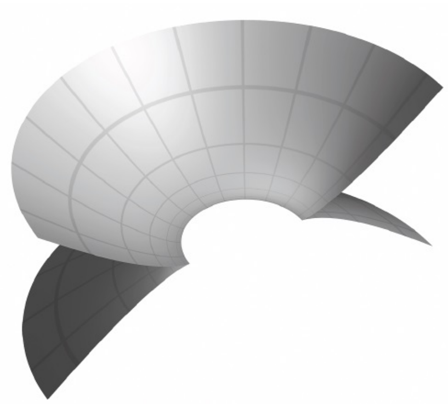

FIGURE 2.17. A constant Gaussian curvature surface of revolution in Minkowski 3-space that is a parallel surface to the CMC surface shown on the righthand side of Figure 2.16 (see [16])

FIGURE 2.18. Spacelike constant mean curvature surfaces of revolution in Minkowski 3-space with lightlike axes (see [16])

Lemma 2.85. *If x is isothermic in $\mathbb{R}^3$ or $\mathbb{R}^{2,1}$ with isothermic coordinates u, v, then x^* exists, solving*

$$dx^* = -\frac{x_u}{E}du + \frac{x_v}{E}dv \ .$$

Proof. This was already proven in the case of $\mathbb{R}^3$ in Lemma 2.27, so let us be brief here: We want to show "$d^2x^* = 0$", i.e.

$$d(-x_u E^{-1} du + x_v E^{-1} dv) = 0 \ ,$$

i.e.

$$2x_{uv}E - x_u E_v - x_v E_u = 0 \ .$$

We can see this by noting that $b_{12} = 0$ implies $x_{uv} = Ax_u + Bx_v$ for some reals A and B, and that $\langle x_u, x_v \rangle = 0$. □

The x^* in Lemma 2.85 is the same as the x^* in Definition 2.30, but scaled by a factor of 1/4. This is a non-essential change.

Proposition 2.86. *Let x be an isothermic immersion in $\mathbb{R}^3$ or $\mathbb{R}^{2,1}$, with x^* as in Lemma 2.85. Then x is CMC H if and only if*

$$dx^* = h(H dx + dn)$$

for some constant h.

FIGURE 2.19. Spacelike CMC surfaces of revolution in Minkowski 3-space with spacelike axes, the righthand surface being a cylinder (see [16])

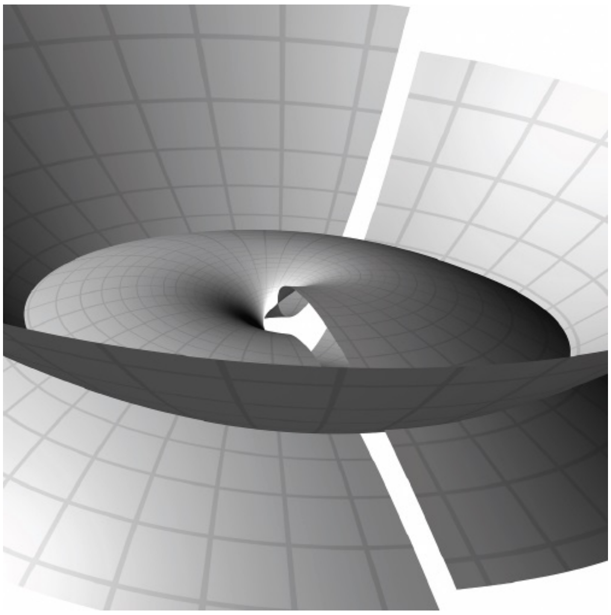

FIGURE 2.20. A spacelike CMC surface in Minkowski 3-space lying in the associate family of the spacelike CMC surface of revolution shown on the righthand side of Figure 2.16, and itself not being a surface of revolution (see [16])

Proof. Let us again be brief, because the $\mathbb{R}^3$ case was already dealt with in Remark 2.32:

$$-x_u E^{-1} du + x_v E^{-1} dv = h(H dx + dn) \ , \quad h \text{ constant} \tag{2.58}$$

is equivalent to $k_1 + k_2 = 2H$ with

$$h = 2E^{-1}(k_1 - k_2)^{-1}$$

constant. The first of these last two equalities is clearly true, and h is constant if and only if the Hopf differential function $\hat{Q}$ is constant, which is true if and only if x has constant mean curvature. □

Because, even without assuming H is the mean curvature, Equation (2.58) forces H to be the mean curvature, we have:

Corollary 2.87. *An isothermic immersion x in $\mathbb{R}^3$ or $\mathbb{R}^{2,1}$ is CMC if and only if*

$$-\frac{x_u}{E}du + \frac{x_v}{E}dv = h(Hdx + dn)$$

for some real constants h and H.

2.22. **Weierstrass representation for flat surfaces in $\mathbb{H}^3$.** We denote by $\mathbb{R}^{3,1}$ the Minkowski 4-space with the inner product $\langle\cdot,\cdot\rangle$ of signature $(-,+,+,+)$. The hyperbolic 3-space $\mathbb{H}^3$ is considered as the upper-half component of the "radius 1" two-sheeted hyperboloid in $\mathbb{R}^{3,1}$ with the induced metric. Identifying $\mathbb{R}^{3,1}$ with the set of 2×2 Hermitian matrices Herm(2) via

$$\mathbb{R}^{3,1} \ni (x_0, x_1, x_2, x_3) \leftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \in \mathrm{Herm}(2),$$

$\mathbb{H}^3$ is represented, like in Section 2.19, as

$$\begin{aligned} \mathbb{H}^3 &= \{x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^{3,1} \,|\, \langle x, x\rangle = -1, x_0 > 0\} \\ &= \{X \in \mathrm{Herm}(2) \,|\, \det X = 1,\ \mathrm{trace} X > 0\} \\ &= \{a\bar{a}^t \,|\, a \in \mathrm{SL}_2\,\mathbb{C}\} = \mathrm{SL}_2\,\mathbb{C}/\,\mathrm{SU}_2\ . \end{aligned}$$

Any isometry of $\mathbb{H}^3$ is given by the map on the collection of points $a\bar{a}^t$ in $\mathbb{H}^3$, as follows:

$$a\bar{a}^t \to (a_0 a)\overline{a_0 a}^t\ ,$$

for some $a_0 \in \mathrm{SL}_2\,\mathbb{C}$.

We can consider the projection to the upper half-space model

$$\pi : H^3 \ni (x_0, x_1, x_2, x_3) \longmapsto \frac{1}{x_0 - x_3}(x_1 + ix_2, 1) \in \mathbb{R}^3_+, \tag{2.59}$$

where $\mathbb{R}^3_+ := \{(\zeta, h) \in \mathbb{C}\times\mathbb{R}\,;\, h > 0\}$. This is an isometry when the target is given the metric

$$\left(\mathbb{R}^3_+, \frac{|d\zeta|^2 + dh^2}{h^2}\right). \tag{2.60}$$

The ideal boundary $\partial\mathbb{H}^3$ is identified with $\mathbb{C}\cup\{\infty\}$. We can write

$$\pi(a\bar{a}^t) = \frac{(a_{11}\overline{a_{21}} + a_{12}\overline{a_{22}}, 1)}{a_{21}\overline{a_{21}} + a_{22}\overline{a_{22}}}, \qquad \text{where} \quad a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SL}_2\,\mathbb{C}. \tag{2.61}$$

Similarly, 3-dimensional de Sitter space $\mathbb{S}^{2,1}$ can be described as

$$\mathbb{S}^{2,1} = \left\{ a \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \bar{a}^t \,\middle|\, a \in \mathrm{SL}_2\,\mathbb{C} \right\}.$$

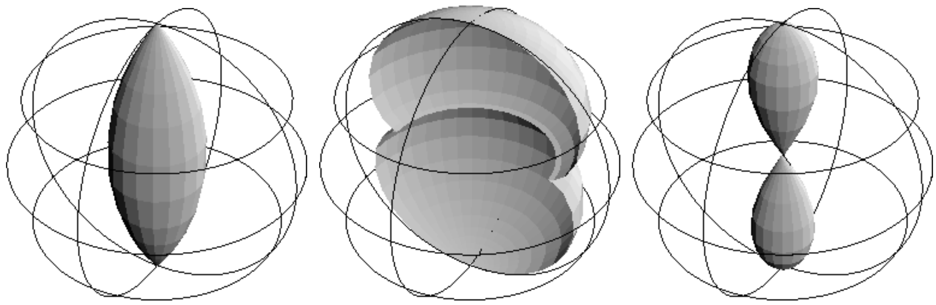

FIGURE 2.21. Flat surfaces in $\mathbb{H}^3$: a hyperbolic cylinder on the left, a "snowman" in the middle, an "hourglass" on the right (see Examples 2.92, 3.36 and 3.38)

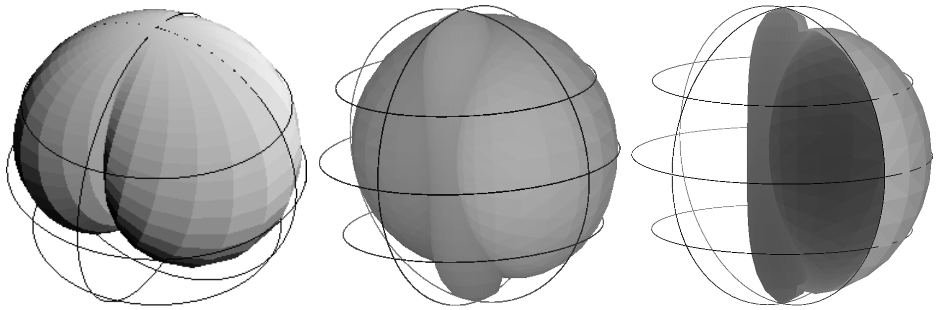

FIGURE 2.22. Flat surfaces in $\mathbb{H}^3$, the surface on the left being a "peach front" (see Example 3.37), the surface in the middle having Weierstrass data $G = z$ and $G_* = z^2$ on the Riemann surface $\mathbb{C} \setminus \{0, 1\}$, the surface on the right being a portion of the surface in the middle (see [78])

Remark 2.88. A point $(x_0, x_1, x_2, x_3) \in \mathbb{H}^3$ in the Minkowski model becomes

$$\frac{(x_1, x_2, x_3)}{1 + x_0}$$

in the Poincare model. (See [103].)

Definition 2.89. *A smooth map $x\colon M^2 \to \mathbb{H}^3$ with unit normal vector field n in $\mathbb{S}^{2,1}$ from a 2-manifold M^2 is called a* front *if the map $(x, n)\colon M^2 \to T_1\mathbb{H}^3$ is an immersion. Here, $T_1\mathbb{H}^3$ denotes the unit tangent bundle to $\mathbb{H}^3$.*

The *parallel front* x_t at distance t of a front x is given by

$$x_t = (\cosh t)x + (\sinh t)n, \quad n_t = (\cosh t)n + (\sinh t)x, \tag{2.62}$$

where n_t is the unit normal vector field of x_t.

Definition 2.90. *A front $x\colon M^2 \to \mathbb{H}^3$ is called a* flat front *if, for each $p \in M^2$, there exists $t \in \mathbb{R}$ such that the parallel front x_t is a flat immersion at p, where an immersion is* flat *if its intrinsic Gaussian curvature K is identically zero.*

If any one parallel front is a flat immersion, then all parallel fronts that are immersions are flat (see the comments on parallel flat fronts below). So Definition 2.90 is sensible.

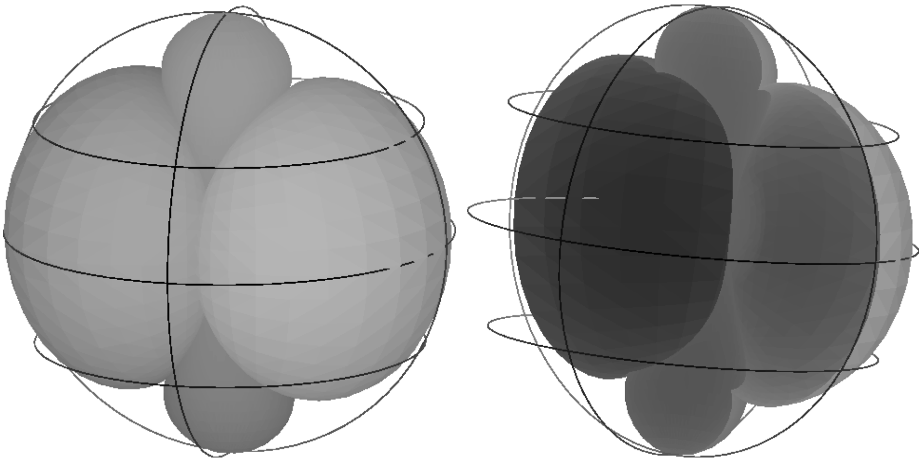

FIGURE 2.23. Flat surfaces in $\mathbb{H}^3$, the surface on the left having Weierstrass data $G = z$ and $G_* = z^3$ on the Riemann surface $\mathbb{C} \setminus \{0, 1, -1\}$, the surface on the right being a portion of the surface on the left (see [78])

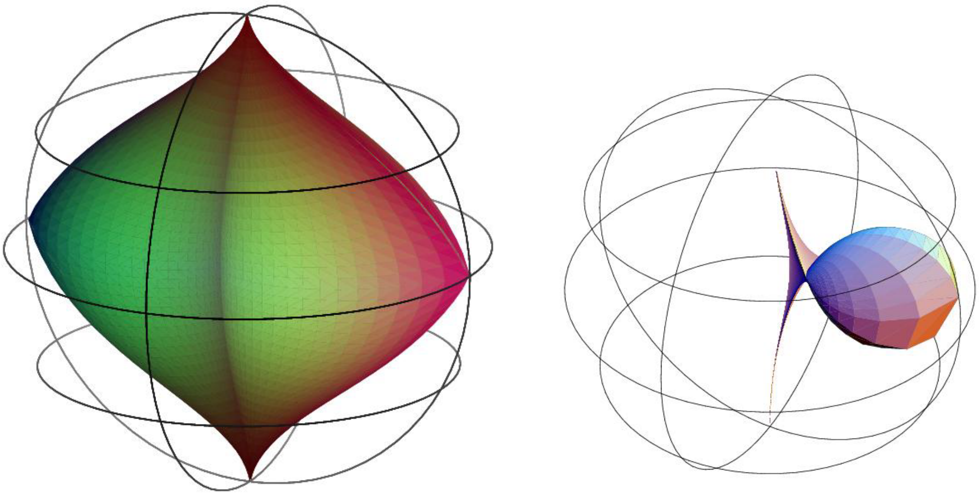

FIGURE 2.24. Flat surfaces in $\mathbb{H}^3$, the surface on the left being the caustic of the surface shown in Figure 2.23 (see [78]), the surface on the right being an example of a p-front that is not globally a caustic (see [79])

The following result is in [55] (a proof can also be found in [49]), and is similar in spirit to both the Weierstrass representation for minimal surfaces in $\mathbb{R}^3$ and the representation of Bryant for CMC 1 surfaces in $\mathbb{H}^3$:

Theorem 2.91. *A flat front x from a Riemann surface M^2 with local coordinate z to $\mathbb{H}^3$ can be locally constructed from two complex analytic one-forms $\omega = \hat{\omega} dz$ and $\theta = \hat{\theta} dz$ on M^2 (z is a local complex coordinate on M^2) as follows:*

$$x = E\bar{E}^t \ , \quad \textit{where} \quad E = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_2\,\mathbb{C} \quad \textit{solves} \quad dE = E \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix} \ ,$$

and its normal vector field in $\mathbb{S}^{2,1}$ *is*

$$n = E \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{E}^t .$$

Example 2.92. Let $\alpha \in \mathbb{R} \setminus \{0, 1\}$ be a constant and define holomorphic functions $\hat{\omega}$, $\hat{\theta}$ on the universal cover of $M^2 = \mathbb{C} \setminus \{0\}$ as follows:

$$\hat{\theta} = -\frac{1}{c^2} z^{-2/(1-\alpha)} \qquad \hat{\omega} = \frac{c^2\alpha}{(1-\alpha)^2} z^{2\alpha/(1-\alpha)}$$

for some constant $c \in \mathbb{R} \setminus \{0\}$. Then we have surfaces called *hyperbolic cylinders* if $\alpha = -1$, *hourglasses* if $\alpha < 0$ ($\alpha \neq -1$) and *snowmen* if $\alpha > 0$. All of these surfaces are well-defined on M^2 itself. See Figure 2.21. If we set $\alpha = 0$ and replace M^2 by $\mathbb{C}$, then we have a horosphere.

The first and second fundamental forms of x are

$$\begin{aligned} ds^2 &= |\omega + \bar{\theta}|^2 = \hat{Q}dz^2 + \overline{\hat{Q}}d\bar{z}^2 + (|\omega|^2 + |\theta|^2), \qquad \hat{Q}dz^2 = \omega\theta, \\ II &= |\theta|^2 - |\omega|^2. \end{aligned} \tag{2.63}$$

Immersed umbilic points occur at points where the Hopf differential $\hat{Q}dz^2$ (a holomorphic 2-differential) is zero but ds^2 is not degenerate, i.e. exactly one of the two 1-forms ω and θ is zero.

Although ω and θ are generally only defined on the universal cover $\widetilde{M}^2$ of M^2, $|\omega|^2$ and $|\theta|^2$ are well-defined on M^2 itself, as is $\hat{Q}dz^2$. Furthermore, the zeros of $\hat{Q}$ are the umbilic points of x. Defining a meromorphic function

$$\rho = \frac{\theta}{\omega} \tag{2.64}$$

on $\widetilde{M}^2$, we have that $|\rho| \colon M^2 \to [0, +\infty]$ is well-defined on M^2, and $p \in M^2$ is a singular point of x exactly when $|\rho(x)| = 1$.

The hyperbolic Gauss maps. The hyperbolic Gauss maps are

$$G = \tfrac{A}{C} , \quad G_* = \tfrac{B}{D} .$$

Geometrically, G and G_* represent the intersection points in the ideal boundary $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$ of $\mathbb{H}^3$ of the two oppositely-directed normal geodesics emanating from x in the n and $-n$ directions, respectively.

For $a \in \mathrm{SL}_2\,\mathbb{C}$, the transformation $E_x \mapsto aE_x$ corresponds to the rigid motion $x \mapsto ax\bar{a}^t$ in $\mathbb{H}^3$. Then the hyperbolic Gauss maps change, with $a = (a_{ij})_{i,j=1}^2$, by a Möbius transformation:

$$G \mapsto a \star G = \frac{a_{11}G + a_{12}}{a_{21}G + a_{22}} , \qquad G_* \mapsto a \star G_* = \frac{a_{11}G_* + a_{12}}{a_{21}G_* + a_{22}} . \tag{2.65}$$

Because $\det E = 1$, we have

$$dG = \frac{-\omega}{C^2} , \quad dG_* = \frac{\theta}{D^2} , \quad G - G_* = (CD)^{-1} ,$$

so the Hopf differential is

$$\hat{Q}dz^2 = -(CD)^2 dGdG_* = \frac{-dGdG_*}{(G - G_*)^2} . \tag{2.66}$$

Another computation gives

$$\text{(2.67)}\qquad \frac{G''}{G'} - \frac{2G'}{G-G_*} = \frac{\hat\omega'}{\hat\omega} , \quad \frac{G''_*}{G'_*} - \frac{2G'_*}{G_*-G} = \frac{\hat\theta'}{\hat\theta} .$$

Defining

$$S(G) = \left(\frac{G''}{G'}\right)' - \frac{1}{2}\left(\frac{G''}{G'}\right)^2$$

as the Schwarzian derivative, and defining $s(\hat\omega) = (\hat\omega'/\hat\omega)' - (1/2)(\hat\omega'/\hat\omega)^2$, we have

$$\text{(2.68)}\qquad S(g) - S(G) = 2\hat{Q}dz^2 \ \text{ if and only if } \ s(\hat\omega) - \{G, z\} = 2\hat{Q} ,$$

$$S(g_*) - S(G_*) = 2\hat{Q}dz^2 \ \text{ if and only if } \ s(\hat\theta) - \{G_*, z\} = 2\hat{Q} .$$

We know that $S(g) - S(G) = 2\hat{Q}dz^2$ and $S(g_*) - S(G_*) = 2\hat{Q}dz^2$ hold, by using

$$\frac{G''}{G'} = \frac{\hat\omega'}{\hat\omega} - \frac{2D\hat\omega}{C} , \quad \frac{G''_*}{G'_*} = \frac{\hat\theta'}{\hat\theta} - \frac{2C\hat\theta}{D} .$$

U_1-ambiguity. Changing E to

$$E \cdot \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix}$$

does not change the surface if $\gamma \in \mathbb{R}$, but does change θ and ω to $e^{i\gamma}\theta$ and $e^{-i\gamma}\omega$. So θ and ω have a U_1-ambiguity.

Dual flat fronts. Also,

$$\text{(2.69)}\qquad E_x^\natural := E_x \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

gives the same front x, but the unit normal $E_f^\natural e_3 \bar{E}_f^{\natural\,t} = -n$ is reversed. $E_f^\natural$ is called the *dual* of E_f. The hyperbolic Gauss maps $G^\natural$, $G_*^\natural$, the canonical forms $\omega^\natural$, $\theta^\natural$ and Hopf differential $\hat{Q}^\natural dz^2$ satisfy

$$G^\natural = G_* , \quad G_*^\natural = G , \quad \omega^\natural = \theta , \quad \theta^\natural = \omega , \quad \hat{Q}^\natural dz^2 = \hat{Q}dz^2 .$$

Parallel flat fronts. Replacing $\hat\theta$ and $\hat\omega$ with $e^t\hat\theta$ and $e^{-t}\hat\omega$ for some $t \in \mathbb{R}$, we find that

$$E \cdot \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}$$

becomes a solution of the equation in Theorem 2.91, and that x and n change to the x_t and n_t in (2.62). It follows that all the parallel surfaces of a flat front are flat wherever they are immersions.

Remark 2.93. It was shown in [82] that E can be written as (the Small-type formula [123])

$$E = \begin{pmatrix} A & dA/\omega \\ C & dC/\omega \end{pmatrix} ,$$

where $A = CG$ and $C = i\sqrt{\frac{\omega}{dG}}$.

A holomorphic Legendrian lift E can be expressed in terms of a pair (G, ω), see [82]:

$$E = \begin{pmatrix} GC & d(GC)/\omega \\ C & dC/\omega \end{pmatrix}, \qquad \text{where} \quad C = i\sqrt{\frac{\omega}{dG}}. \tag{2.70}$$

Another representation formula for E is given in [82]:

$$E = \begin{pmatrix} G/\xi & \xi G_*/(G - G_*) \\ 1/\xi & \xi/(G - G_*) \end{pmatrix}, \quad \xi := \delta \exp \int_{z_0}^{z} \frac{dG}{G - G_*}, \tag{2.71}$$

where $z_0 \in M^2$ is a base point of integration and $\delta \in \boldsymbol{C} \setminus \{0\}$ is a constant. Here δ corresponds to the choices in the U_1-ambiguity and in the family of parallel fronts. We then have

$$\omega = -\frac{dG}{\xi^2}, \qquad \hat{Q} dz^2 = -\frac{dG\, dG_*}{(G - G_*)^2}, \tag{2.72}$$

and the second of these two equations was already seen in Equation (2.66).

Remark 2.94. The normal n gives a flat spacelike surface in $\mathbb{S}^{2,1}$, and the normal to n is x.

Remark 2.95. We have a relation with a Riccati-type equation here. Given the form of $\hat{Q}$ in (2.72) and the fact that we can locally take

$$\hat{Q} = 1/4$$

wherever the Hopf differential is not zero (by a conformal change of coordinate z), we have

$$(G - G_*)^2 = -4G'G'_*. \tag{2.73}$$

This is a Riccati-type equation, which means we have a means to solve it. Taking G as given, we can find G_*. The solution G_* is determined by an initial condition in $\mathbb{C}$, so there is a real 2-dimensional choice of solutions G_*.

Example 2.96. Here are some examples of determining a G_*, given a G:

(1) $G = 0$ implies we can take $G_* = 0$,
(2) $G = c$ implies we can take $G_* = c$,
(3) $G = az + b$ implies we can take

$$G_* = az + b + 2ia - \frac{4a}{ce^{iz} - i},$$

(4) $G = az^3 + b$ implies we can take

$$G_* = b + az(z^2 - 12) + 6iaz^2 \cdot \frac{e^{iz} - ic}{e^{iz} + ic},$$

(5) $G = a/z$ implies we can take

$$G_* = \frac{a(c\cos(z/2) + \sin(z/2))}{(cz + 2)\cos(z/2) + (z - 2c)\sin(z/2)},$$

(6) $G = e^{z/\sqrt{2}}$ implies we can take

$$G_* = -e^{z/\sqrt{2}} \cdot \tan\left(\frac{z}{2\sqrt{2}} - c\right),$$

where a, b and c are constants. Some other examples for which one can find G_* explicitly are

$$G = z^3 \pm \sqrt{3}z^2 + z\ , \quad G = z^n\ , \quad G = \cos z\ ,$$
$$G = \sin z\ , \quad G = (z-1)e^z \ \text{ and } \ G = a\log z\ .$$

We now consider how to find all solutions G_* for a given G. Define $u = u(z)$ by

$$G_* = 4G'\left(\frac{u'}{u} - \frac{2G'' - G}{4G'}\right)\ ,$$

and then (2.73) becomes

$$u'' + u \cdot \frac{(G')^2 - 2G'G''' + (G'')^2}{4(G')^2} = 0\ . \tag{2.74}$$

Then, once we have one solution G_* with associated solution u_0 to (2.74), we have all solutions G_*, as follows: Given one solution u_0 to (2.74), one other independent solution is

$$u = u_0 \int u_0^{-2} dz\ .$$

Thus we know all solutions to (2.74), and in turn all solutions G_*.

3. Lie sphere geometry

We now "lift" Möbius geometry up to Lie sphere geometry, which is the natural setting for considering Ω surfaces.

3.1. **Lie sphere transformations.** The first part of this section reiterates material found in [31], so we will keep the descriptions here brief.

Consider $\mathbb{R}^{4,2}$ with the metric $(\cdot,\cdot)$ of signature $(-,+,+,+,+,-)$ and

$$\mathbb{R}^{4,1} = \text{span}\left\{\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}\right\}^{\perp} \subset \mathbb{R}^{4,2}$$

with induced metric $\langle\cdot,\cdot\rangle$ of signature $(-,+,+,+,+)$. (This $\mathbb{R}^{4,1}$ has timelike direction in the first coordinate rather than the last, unlike the $\mathbb{R}^{4,1}$ we used in previous sections. We make this non-essential change to conform here with the notations used in [31] and other sources.) Let L^4, resp. L^5, be the light cone in $\mathbb{R}^{4,1}$, resp. $\mathbb{R}^{4,2}$. Slicing L^4 by hyperplanes $\mathcal{P}$ of $\mathbb{R}^{4,1}$ gives 3-dimensional spaceforms M^3. Spheres in these spaceforms are given by

$$\{\vec{x} \in M^3 = L^4 \cap \mathcal{P} \mid \langle \vec{x}, \xi\rangle = 0\}$$

for spacelike vectors $\xi \in \mathbb{R}^{4,1}$ (point spheres result when ξ is lightlike). See Section 2.8.

Möbius transformations are given via $A \in O_{4,1}$ applied to $\mathbb{R}^{4,1}$ by

$$\xi \to A \cdot \xi \,, \tag{3.1}$$

telling us how points (point spheres) and spheres transform. See Section 2.3.

Example 3.1. As an example of this, like in Section 2.3, take $v = (v_1, v_2, v_3)$ and $p = (p_1, p_2, p_3)$ in $\mathbb{R}^3$ and $r \in \mathbb{R}$, and consider the transformation

$$\xi_{p,r} \to A\xi_{p,r} = \xi_{p+v,r} \,,$$

where

$$A := \begin{pmatrix} 1+\frac{1}{2}v\cdot v & \frac{1}{2}v\cdot v & v_1 & v_2 & v_3 \\ -\frac{1}{2}v\cdot v & 1-\frac{1}{2}v\cdot v & -v_1 & -v_2 & -v_3 \\ v_1 & v_1 & 1 & 0 & 0 \\ v_2 & v_2 & 0 & 1 & 0 \\ v_3 & v_3 & 0 & 0 & 1 \end{pmatrix} \in O_{4,1} \,,$$

$$\xi_{p,r} := \begin{pmatrix} \frac{1}{2}(1+p\cdot p - r^2) \\ \frac{1}{2}(1-p\cdot p+r^2) \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} .$$

The vector $\xi_{p,r}$ represents a sphere with center p and radius r (when we take $Q_{M_0} = (1,-1,0,0,0)^t$). The map (3.1) translates spheres of radius $|r|$ and center p by v in $\mathbb{R}^3$. Note that $||\xi_{p,r}||^2 = r^2 > 0$ when the radius $|r|$ is nonzero.

In Lie sphere geometry, on the other hand, the Lie sphere transformations are given by matrices in $O_{4,2}$, and the group of Lie sphere transformations is isomorphic to $O_{4,2}/\{\pm I\}$. The objects under consideration are oriented spheres, and for this we use $\mathbb{R}^{4,2}$, i.e. we give a sphere by

$$\bar{\xi}_{p,r} := \begin{pmatrix} \xi_{p,r} \\ \pm r \end{pmatrix} \in L^5 \ ,$$

where the $\xi_{p,r}$ part determines a sphere (with center p and radius $|r|$) in $\mathbb{R}^3$ (and also determines a sphere in any other spaceform) and the sign in front of the final coordinate r determines the orientation of the sphere (see Section 2.5 of [31]).

To determine the form of $\bar{\xi}_{p,r}$ when $r = \infty$, we can consider the family of spheres

$$r^{-1} \cdot \bar{\xi}_{p+rn,r} = \begin{pmatrix} (2r)^{-1}(1 + p \cdot p + 2rp \cdot n) \\ (2r)^{-1}(1 - p \cdot p - 2rp \cdot n) \\ r^{-1}p^t + n^t \\ 1 \end{pmatrix}$$

all containing the point p and all having the unit normal n at p, and then take the limit as $r \to \infty$, to obtain the following vector representing the plane through p with unit normal n:

$$\bar{\xi}_{p,n} := \begin{pmatrix} p \cdot n \\ -p \cdot n \\ n^t \\ 1 \end{pmatrix} \ .$$

When x is a surface and n is a unit normal vector field of x, $\xi_{x,n}$ represents the tangent planes to x.

Example 3.2. With A as in Example 3.1, take

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in O_{4,2}$$

and take the map

$$\begin{pmatrix} \xi_{p,r} \\ \pm r \end{pmatrix} \to \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \bar{\xi}_{p,r} = \bar{\xi}_{p+v,r} \ .$$

This is an example of a Möbius transformation becoming a Lie sphere transformation. Orientation of the spheres could be reversed by instead choosing

$$\begin{pmatrix} A & 0 \\ 0 & -1 \end{pmatrix} \in O_{4,2} \ .$$

All Möbius transformations can be included in the collection of Lie sphere transformations by taking arbitrary $A \in O_{4,1}$ in Example 3.2.

The next example is a Lie sphere transformation that does not reduce to a Möbius transformation, because it takes point spheres to true spheres ("true sphere" means that the sphere has a strictly positive radius), and takes some true spheres to point spheres as well:

Example 3.3. This will be an example of a Laguerre transformation. Laguerre transformations are Lie sphere transformations that preserve the point at ∞ for $\mathbb{R}^3 \cup \{\infty\}$, i.e. that preserve planes in $\mathbb{R}^3$.

If a Lie sphere transformation (or just "Lie transformation" for short) in $\mathbb{R}^{4,2}$ fixes a spacelike vector, then it reduces to a Lie transformation of $\mathbb{R}^{3,2}$. If it fixes a timelike vector, then it will produce a Möbius transformation. If it fixes a lightlike vector, then it gives a Laguerre transformation.

Möbius transformations preserve the angle of intersection between spheres. Lie sphere transformations preserve oriented tangential contact between spheres. Laguerre transformations preserve ratios amongst tangential distances between spheres, and we will see in the example here how Laguerre transformations do this. Set

$$W = \begin{pmatrix} 1-\frac{1}{2}s^2 & -\frac{1}{2}s^2 & 0 & 0 & 0 & -s \\ \frac{1}{2}s^2 & 1+\frac{1}{2}s^2 & 0 & 0 & 0 & s \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ s & s & 0 & 0 & 0 & 1 \end{pmatrix} \in O_{4,2} \; ,$$

$$\bar{\xi}_{p,r} \to W\bar{\xi}_{p,r} = \bar{\xi}_{p,r+s} \; . \tag{3.2}$$

This map preserves a lightlike vector, which means it preserves a sphere S. We can choose a unit timelike vector $\hat{p}$ so $\mathbb{R}^{4,1} = \text{span}\{\hat{p}\}^\perp$ and S is a point-sphere in that $\mathbb{R}^{4,1}$. Then we can choose $\mathbb{R}^3 \subset L^4 \subset \mathbb{R}^{4,1}$ such that S is the point at infinity of $\mathbb{R}^3$. More explicitly,

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is an eigenvector of W, and we can take

$$\mathbb{R}^{4,1} = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}^\perp$$

and then

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{projects to} \quad q := \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

in $\mathbb{R}^{4,1}$. We use this q to make

$$\mathbb{R}^3 = \{X \in L^4 \,|\, \langle X, q \rangle_{\mathbb{R}^{4,1}} = -1\} \; .$$

(We are now using a lowercase "q" to denote the vector that determines the spaceform, rather than the uppercase "Q" that we used before. This is for compatibility with

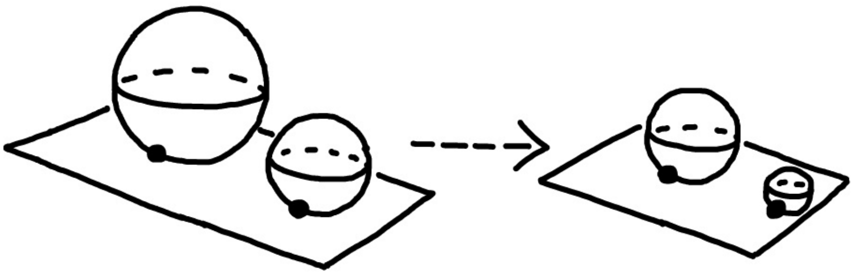

FIGURE 3.1. A Laguerre transformation

the "p, q" notation soon to be used in Section 3.3.) Then q is the point at infinity of $\mathbb{R}^3$, and finite points $p = (p_1, p_2, p_3) \in \mathbb{R}^3$ are of the form

$$\begin{pmatrix} \frac{1}{2}(1 + p \cdot p) \\ \frac{1}{2}(1 - p \cdot p) \\ p^t \end{pmatrix} .$$

A configuration consisting of two spheres tangent to one plane will be mapped by this Laguerre transformation to a configuration of the same type in that $\mathbb{R}^3$. (See page 49 of [31], and Corollary 4.4 on page 55 of [31].) This is depicted in Figure 3.1. In particular, planes are mapped to planes. If the two spheres touch the plane (tangentially of course) at the two points p and q, then the tangential distance between the two spheres is the distance between p and q inside the plane. We will now illustrate how the ratios of such distances are preserved by this map (3.2).

The spacelike vector

$$P := \begin{pmatrix} a \\ -a \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

is perpendicular to q, so it gives a plane $\tilde{P}$ in $\mathbb{R}^3$. A sphere tangent to this plane is given by

$$\xi := r^{-1} \begin{pmatrix} \frac{1}{2}(1 + 2ra + a^2 + p_2^2 + p_3^2) \\ \frac{1}{2}(1 - 2ra - a^2 - p_2^2 - p_3^2) \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} ,$$

where $p_1 = r + a$, for some $r \in \mathbb{R}$. This is because

$$||P||^2 = ||\xi||^2 = \langle P, \xi \rangle = 1 ,$$

see Lemma 2.20. This plane and sphere intersect at the unique point

$$L := \begin{pmatrix} \frac{1}{2}(1 + a^2 + p_2^2 + p_3^2) \\ \frac{1}{2}(1 - a^2 - p_2^2 - p_3^2) \\ a \\ p_2 \\ p_3 \end{pmatrix}$$

because $||L||^2 = \langle L, P\rangle = \langle L, \xi\rangle = 0$. Now,

$$W \cdot \begin{pmatrix} P \\ 1 \end{pmatrix} = \begin{pmatrix} a-s \\ -(a-s) \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} ,$$

$$W \cdot \begin{pmatrix} \xi \\ 1 \end{pmatrix} = \frac{1}{r} \begin{pmatrix} \frac{1}{2}(1+2(r+s)(a-s)+(a-s)^2+p_2^2+p_3^2) \\ \frac{1}{2}(1-2(r+s)(a-s)-(a-s)^2-p_2^2-p_3^2) \\ (r+s)+(a-s) \\ p_2 \\ p_3 \\ r+s \end{pmatrix} = \frac{1}{r} \begin{pmatrix} \xi_{(p_1,p_2,p_3),r+s} \\ r+s \end{pmatrix} ,$$

so the intersection point of the image plane and image sphere under this map is

$$\begin{pmatrix} \frac{1}{2}(1+(a-s)^2+p_2^2+p_3^2) \\ \frac{1}{2}(1-(a-s)^2-p_2^2-p_3^2) \\ a-s \\ p_2 \\ p_3 \end{pmatrix} .$$

Computing similarly for more general P perpendicular to q, we see that, in this case (3.2), tangential distance between spheres is preserved. In general, homotheties of $\mathbb{R}^3$ are also allowed amongst the Laguerre transformations, since they also preserve planes, and so only ratios of tangential distances between spheres are preserved.

Fact: ([31], Theorem 3.16) The group of Lie sphere transformations is equal to the union of the group of Möbius transformations and group of Laguerre transformations, and subsequent compositions of transformations.

3.2. **Lifting surfaces to Lie sphere geometry, parallel transformations.** For illustrating the process of lifting surfaces, let us first define the quadric PL^5, and then take the case of $\mathbb{H}^3$.

The quadric: The Lie quadric PL^5 is projectivized L^5. Each line in PL^5, or equivalently, each null plane in L^5, will give a collection of spheres making oriented tangential contact at some point, in any choice of 3-dimensional spaceform M^3 (this is related to Remark 2.21, and we also come back to this in Section 3.5).

Consider a surface in $\mathbb{H}^3$ and its normal:

$$x(u,v) = (x_0, x_1, x_2, x_3) : M^2 \to \mathbb{H}^3 \subset \mathbb{R}^{3,1} ,$$

$$n(u,v) = (n_0, n_1, n_2, n_3) : M^2 \to \mathbb{S}^{2,1} \subset \mathbb{R}^{3,1} ,$$

i.e. $n \perp x$ and $n \perp dx$. Here $\mathbb{H}^3$ and $S^{2,1}$ are defined the standard way, and the metric of $\mathbb{R}^{3,1}$ has signature $(-,+,+,+)$. Let

$$\Lambda = \Lambda(u,v) = \mathrm{span}\{X, N\} ,$$

where X and N are the lifts

$$X = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \\ 0 \\ 1 \end{pmatrix}$$

to $\mathbb{R}^{4,2}$. Then Λ is a null plane in $\mathbb{R}^{4,2}$, and a line in PL^5. The projection

$$\phi : P\left(\Lambda \cap \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}^{\perp}\right) = P(\Lambda \cap \mathbb{R}^{4,1}) \to \mathbb{H}^3 , \tag{3.3}$$

$$\phi\left[\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ 0 \end{pmatrix}\right] = \left(\frac{y_0}{y_4}, \frac{y_1}{y_4}, \frac{y_2}{y_4}, \frac{y_3}{y_4}\right)$$

returns us to x.

We have the following properties:

$$(X, X) = (N, N) = (N, X) = (X, dX) = (N, dX) = (X, dN) = 0 .$$

Before application of the above projection ϕ, we could first apply an isometric transformation $A \in O_{4,2}$ of $\mathbb{R}^{4,2}$ to the null plane Λ, obtaining the null plane $A\Lambda$. Then, after projecting by ϕ to $\mathbb{R}^{4,1}$, we would get some kind of transform $\hat{x}$ of x, still in $\mathbb{H}^3$. This is the viewpoint we take in the next lemma ($I_{j\times j}$ denotes the $j \times j$ identity matrix):

Lemma 3.4. *If*

$$A = \begin{pmatrix} I_{4\times 4} & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{pmatrix} ,$$

the transform

$$\hat{x} = \hat{x}(u, v) = \phi\left(A \cdot \Lambda(x, n) \cap \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}^{\perp}\right)$$

is a parallel surface of $x = x(u, v)$.

Note that we can use

$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

to project to the Möbius geometric space $\mathrm{span}\{p\}^\perp \approx \mathbb{R}^{4,1}$ and then use

$$q = Q_{\mathbb{H}^3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

to get the spaceform $\mathbb{H}^3$, similarly to (2.2). The map given by left multiplication by A fixes the 2-plane $\mathrm{span}\{p, q\}$.

Proof. Using the abbreviations $c := \cosh\theta$ and $s := \sinh\theta$, define

$$S := A\left(\left\{ a \begin{pmatrix} x^t \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} n^t \\ 0 \\ 1 \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}\right) = \left\{ a \begin{pmatrix} x^t \\ c \\ s \end{pmatrix} + b \begin{pmatrix} n^t \\ s \\ c \end{pmatrix} \middle| a, b \in \mathbb{R} \right\} .$$

This implies

$$\hat{S} := S \cap \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}^\perp = \left\{ rc \begin{pmatrix} x^t \\ c \\ s \end{pmatrix} - rs \begin{pmatrix} n^t \\ s \\ c \end{pmatrix} \middle| r \in \mathbb{R} \right\}$$

$$= \left\{ r \begin{pmatrix} cx^t - sn^t \\ 1 \\ 0 \end{pmatrix} \middle| r \in \mathbb{R} \right\} .$$

Thus

$$\hat{x} := \phi(\hat{S}) = x \cdot \cosh\theta - n \cdot \sinh\theta ,$$

a parallel surface. □

Remark 3.5. We can give the analogous results to Lemma 3.4 for $\mathbb{R}^3$ or $\mathbb{S}^3$ instead of $\mathbb{H}^3$, as well.

Fact: ([31], Theorem 3.18) All Lie sphere transformations are generated (by composition) from Möbius transformations and parallel surface transformations in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$.

Example 3.6. Now we give simple examples involving geodesic planes and spheres:
Take the following geodesic plane and its normal:

$$x(u, v) = (\sqrt{1 + u^2 + v^2}, u, v, 0) \subset \mathbb{H}^3 , \quad n(u, v) = (0, 0, 0, 1) \in \mathbb{S}^{2,1} .$$

Then

$$\Lambda = \left\{ \begin{pmatrix} a\sqrt{1+u^2+v^2} \\ au \\ av \\ b \\ a \\ b \end{pmatrix} \middle| \; a, b \in \mathbb{R} \right\} .$$

Consider these eight cases:

(1)

$$A = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & I_{4\times 4} & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

implies

$$\hat{x} = \left(\frac{1}{\cos\theta} \cdot \sqrt{1+u^2+v^2}, u, v, \tan\theta \cdot \sqrt{1+u^2+v^2} \right) .$$

Here the starting point (before the transformation is applied) is the vector in Λ with $a = 1$ and $b = 0$, which is the choice that gives a single point in $\mathbb{H}^3$. The image of the transformation, applied to Λ, is

$$\left\{ \begin{pmatrix} \cos\theta \cdot a\sqrt{1+u^2+v^2} + \sin\theta \cdot b \\ au \\ av \\ b \\ a \\ -\sin\theta \cdot a\sqrt{1+u^2+v^2} + \cos\theta \cdot b \end{pmatrix} \middle| \; a, b \in \mathbb{R} \right\} ,$$

with $a = 1$ and $b = \frac{\sin\theta}{\cos\theta} \cdot \sqrt{1+u^2+v^2}$ being the choice that gives the single point $\hat{x}$ above. The other seven examples below operate similarly.

(2) With A as in Lemma 3.4, we have that

$$\hat{x} = (\cosh\theta \cdot \sqrt{1+u^2+v^2}, \cosh\theta \cdot u, \cosh\theta \cdot v, -\sinh\theta)$$

is a parallel unbounded sphere in $\mathbb{H}^3$ having the same limit at the ideal boundary $\partial\mathbb{H}^3$ as x has.

(3)

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\theta & 0 & \sinh\theta \\ 0 & 0 & I_{3\times 3} & 0 \\ 0 & \sinh\theta & 0 & \cosh\theta \end{pmatrix}$$

implies

$$\hat{x} = \left(\sqrt{1+u^2+v^2}, \frac{1}{\cosh\theta} \cdot u, v, -\tanh\theta \cdot u \right) .$$

(4)

$$A = \begin{pmatrix} I_{3\times 3} & 0 & 0 & 0 \\ 0 & \cosh\theta & 0 & \sinh\theta \\ 0 & 0 & 1 & 0 \\ 0 & \sinh\theta & 0 & \cosh\theta \end{pmatrix}$$

implies $\hat{x}(u,v) = x(u,v)$.

(5)

$$A = \begin{pmatrix} * & 0 \\ 0 & I_{2\times 2} \end{pmatrix} ,$$

where $*$ denotes any matrix in $O_{3,1}$, implies $\hat{x}$ is isometric to x.

(6)

$$A = \begin{pmatrix} \cosh\theta & 0 & \sinh\theta & 0 \\ 0 & I_{3\times 3} & 0 & 0 \\ \sinh\theta & 0 & \cosh\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

implies

$$\hat{x} = \frac{1}{\sinh\theta\sqrt{1+u^2+v^2}+\cosh\theta}\left(\cosh\theta\sqrt{1+u^2+v^2}+\sinh\theta, u, v, 0\right) .$$

Note that this $\hat{x}$ is simply a reparametrization of x.

(7)

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta & 0 \\ 0 & 0 & I_{2\times 2} & 0 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

implies

$$\hat{x} = \frac{1}{\cos\theta - u\sin\theta}\left(\sqrt{1+u^2+v^2}, \sin\theta + u\cdot\cos\theta, v, 0\right) .$$

This $\hat{x}$ is also just a reparametrization of x.

(8)

$$A = \begin{pmatrix} I_{3\times 3} & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

implies

$$\hat{x} = \frac{1}{\cos\theta}\left(\sqrt{1+u^2+v^2}, u, v, \sin\theta\right) .$$

This $\hat{x}$ is an unbounded sphere in $\mathbb{H}^3$.

In cases (1), (3)-(7) above, $\hat{x}$ is a geodesic plane, and is a sphere in cases (2), (8).

3.3. **Different ways to project to a spaceform.** In Section 3.2 we used the vector

$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

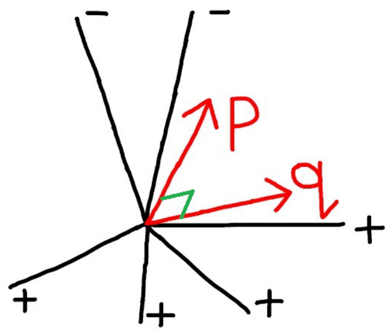

FIGURE 3.2. The setting in Section 3.3

to project to Möbius geometry in $\mathbb{R}^{4,1}$, and we used

$$q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

to define the 3-dimensional hyperbolic space

$$\mathbb{H}^3 = \{X \in L^5 \,|\, X \perp p,\ (X,q) = -1\} ,$$

like the definition in (2.4).

We could, however, make more other choices for p and q. In Section 3.1, we had in effect chosen

$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

When choosing p and q, we only need that p and q are both nonzero, and perpendicular to each other, and to avoid a certain distinctly different (but interesting) special case let us also assume p is not lightlike. We call p the *point sphere complex*, and we call q the *spaceform vector*.

To get positive definite spaceforms as in the first part of these notes, we could take p with $||p||^2 = -1$, and then take any nonzero $q \perp p$ to define the spaceform M, as

$$M = \{X \in \mathbb{R}^{4,2} \,|\, (X,X) = 0,\ (X,p) = 0,\ (X,q) = -1\} , \tag{3.4}$$

which will have sectional curvature $-||q||^2$.

Specific examples that we might choose are:

- an $\mathbb{R}^3$ is produced using

$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

- an $\mathbb{S}^3$ is produced using

$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

- an $\mathbb{H}^3$ is produced using

$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} .$$

Remark 3.7. In Example 3.6 we were using

$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} , \quad q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} .$$

to choose the spaceform $\mathbb{H}^3$. Then, in case (2) of Example 3.6 we had that $\hat{x}$ is an unbounded sphere. (To see it is unbounded, one could, for example, project to the upperhalf space model for $\mathbb{H}^3$, via

$$(x_0, x_1, x_2, x_3) \to \frac{(x_1, x_2, 1)}{x_0 - x_3} , \tag{3.5}$$

and see that projection of this sphere into that model is unbounded.)

The points ρ in this sphere $\hat{x}$ satisfy

$$(\rho, \rho) = 0 , \quad (\rho, p) = 0 , \quad (\rho, q) = -1 , \quad (\rho, \mathcal{S}_\theta) = 0 ,$$

where

$$\mathcal{S}_\theta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \sinh\theta \\ \cosh\theta \end{pmatrix}$$

in $\mathbb{R}^{4,2}$. In this way the sphere parametrized by $\hat{x}$ is associated with the vector $\mathcal{S}_\theta$, analogous to the description in Section 2.8.

Another example is the spherical piece parametrized by coordinates (u, v) via ($b \in \mathbb{R}$ is a constant)

$$(\sqrt{1+b^2}, u, v, \sqrt{b^2 - u^2 - v^2}) ,$$

a part of a bounded sphere, which again can be seen by projecting to the upperhalf space model for $\mathbb{H}^3$ (using the same p and q, and projection map (3.5), as above). This sphere is associated with the vector

$$\mathcal{S}_b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \sqrt{1+b^2} \\ b \end{pmatrix} \in \mathbb{R}^{4,2} .$$

In the first case, where we have unbounded spheres in $\mathbb{H}^3$,

$$\left(\frac{(\mathcal{S}_\theta, q)}{(\mathcal{S}_\theta, p)} \right)^2 < 1 .$$

In the second case, where we have bounded spheres in $\mathbb{H}^3$,

$$\left(\frac{(\mathcal{S}_b, q)}{(\mathcal{S}_b, p)} \right)^2 > 1 .$$

In fact, one can show that arbitrary choices of $\mathcal{S}$ produce

- unbounded spheres with asymptotic boundary a circle when

$$\left(\frac{(\mathcal{S}, q)}{(\mathcal{S}, p)} \right)^2 < 1 .$$

- horospheres or the ideal boundary sphere $\partial\mathbb{H}^3$ (with one orientation or the other) when

$$\left(\frac{(\mathcal{S}, q)}{(\mathcal{S}, p)} \right)^2 = 1 ,$$

  with $\partial\mathbb{H}^3$ occuring when the projection of $\mathcal{S}$ to $(\text{span}\{p\})^\perp$ is parallel to q, and a horosphere occuring otherwise.
- bounded spheres when

$$\left(\frac{(\mathcal{S}, q)}{(\mathcal{S}, p)} \right)^2 > 1 .$$

The ideal boundary $\partial\mathbb{H}^3$ for $\mathbb{H}^3$ is identified with points ρ in the projectivized light cone of $\mathbb{R}^{3,1}$ extended to $\hat{\rho} \in \mathbb{R}^{4,2}$ so that

$$(\hat{\rho}, p) = (\hat{\rho}, q) = 0 .$$

3.4. **Lorentzian spaceform case.** We could also choose p so that

$$||p||^2 = +1 ,$$

and then the 5-dimensional space

$$\{p\}^\perp := (\text{span}\{p\})^\perp$$

perpendicular to p will have metric with signature such that some orthogonal basis has three spacelike vectors and two timelike vectors. ($\{p_1, p_2, ...\}$ is shorthand notation for span$\{p_1, p_2, ...\}$.) The resulting M (using the definition (3.4)) will be a spaceform with a Lorentzian metric.

To be explicit, let us take

$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \quad q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2}(1-\kappa) \\ 0 \\ \frac{1}{2}(1+\kappa) \end{pmatrix}$$

for some $\kappa \in \mathbb{R}$. Then points in M can be represented as

$$X = \frac{1}{1+\kappa|y|^2} \begin{pmatrix} 2y^t \\ |y|^2 - 1 \\ 0 \\ |y|^2 + 1 \end{pmatrix} , \quad |y|^2 = |y|^2_{\mathbb{R}^{2,1}} ,$$

where $y = (y_0, y_1, y_2) \in \mathbb{R}^{2,1} \cup \{\infty\}$ such that $|y|^2 \neq -\kappa^{-1}$, analogous to Equation (2.4) and Lemma (2.2). Note that $\kappa = -||q||^2$.

As in Section 2.1, the metric induced on M is then

$$\frac{4}{(1+\kappa(-y_0^2+y_1^2+y_2^2))^2}(-dy_0^2 + dy_1^2 + dy_2^2) .$$

To see that the curvature of such a signature space M is κ, one could make the standard computations for this pseudo-Riemannian case as well, just as we did in the proof of Lemma 2.5 (again, see [94], for example), using the same formulas. Alternatively, one could also proceed as we did in Remark 2.6, in this pseudo-Riemannian case, now checking that anti-de Sitter space

$$\mathbb{H}^{2,1} = \{\sigma \in \mathbb{R}^{2,2} \,|\, |\sigma|^2 = -1\}$$

has constant sectional curvature -1, and de Sitter space

$$\mathbb{S}^{2,1} = \{\sigma \in \mathbb{R}^{3,1} \,|\, |\sigma|^2 = 1\}$$

has constant sectional curvature 1, instead of using the $\mathbb{H}^3$ and $\mathbb{S}^3$ as in Remark 2.6.

Remark 3.8. We can refer to the case $||p||^2 > 0$ as Lorentz Möbius geometry.

3.5. **Contact elements and Legendre immersions.** First we give the definition of Legendre immersions in the context of Lie sphere geometry:

Definition 3.9. *Let*

$$\Lambda \subset L^5 \subset \mathbb{R}^{4,2}$$

be a null plane, which projectivizes to a line in PL^5 called a contact element. *This line represents a family of spheres (a pencil) that are all tangent (with same orientation) at one point.*

If Λ is a (smooth) map from $M = M^2$ to the collection of null planes in $\mathbb{R}^{4,2}$, where M is a 2-dimensional manifold, then Λ is a Legendre immersion *if,*

$$\textit{for any } m \in M \textit{ and any choice of } Y \in T_m M,$$

$$dX(Y) \in \Lambda \textit{ for all sections } X \textit{ of } \Lambda \textit{ implies } Y = 0$$

$$\textit{(immersion condition)} ,$$

and if, for any pair of sections X_1, X_2 of Λ,

$$dX_1 \perp X_2$$

$$\textit{(contact condition)} .$$

Remark 3.10. The immersion condition in Definition 3.9 can be restated in terms of a basis of sections for the null planes Λ as follows: If

$$\Lambda = \text{span}\{X_1, X_2\} ,$$

with basis $X_1, X_2 : M^2 \to L^5$, then the immersion condition is equivalent to

$$dX_1(Y), dX_2(Y) \in \Lambda \quad \text{implies} \quad Y = 0$$

for all $Y \in T_m M$, one can then check that this condition is independent of the choice of basis X_1, X_2.

Next we consider what this definition means in the context of the conformal 3-sphere:

Definition 3.11. *Let*

$$x : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$$

be a smooth map to $\mathbb{S}^3$. Suppose there exists a smooth map

$$n : M^2 \to T_x \mathbb{S}^3$$

(so $n \perp x$) such that n has norm identically 1 and

$$n \perp dx .$$

We say that x is a front *if x and n considered together form an immersion, that is, if, for $Y \in T_m M^2$,*

$$dx(Y) = dn(Y) = 0 \Rightarrow Y = 0 .$$

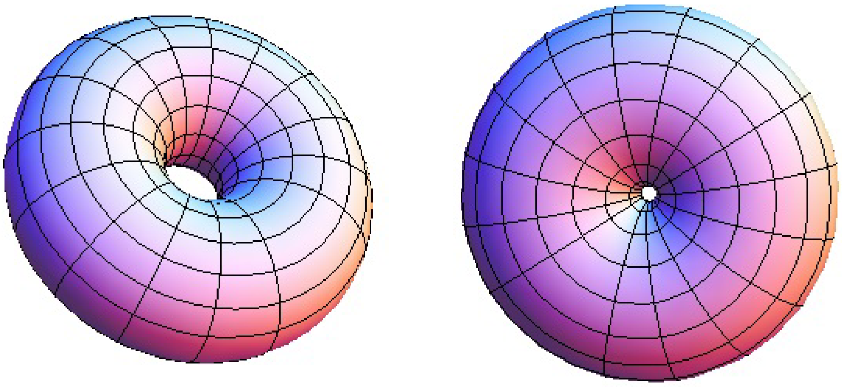

FIGURE 3.3. Projections of the Dupin cyclide to $\mathbb{R}^3$

In the next definition, let $\mathbb{S}^3$ be identified with the 3-dimensional spaceform determined by

$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

A straightforward computation (or simply applying Theorem 4.2 in [31]) implies that the following Definition 3.12 is compatible with Definition 3.9 above.

Definition 3.12. *Let x and n define a front, as in Definition 3.11. Let*

$$\Lambda = span\{X_1, X_2\}$$

with

$$X_1 = X = \begin{pmatrix} 1 \\ x^t \\ 0 \end{pmatrix} \perp p \quad and \quad X_2 = N = \begin{pmatrix} 0 \\ n^t \\ 1 \end{pmatrix} \perp q\ ,$$

then Λ is the Legendre immersion *induced by the pair $(x, n) \subset T_1\mathbb{S}^3$. ($T_1\mathbb{S}^3$ denotes the unit tangent bundle to $\mathbb{S}^3$.) We call Λ the congruence of contact elements of the surface x in $\mathbb{S}^3$.*

Remark 3.13. Note that $(X, q) = (N, p) = -1$ and $(N, q) = (X, p) = 0$.

3.6. **Dupin cyclides.** Dupin cyclides are Legendre immersions, and can be constructed in this way:

(1) Split $\mathbb{R}^{4,2}$ into two fixed orthogonal $V_1 \approx \mathbb{R}^{2,1}$ and $V_2 \approx \mathbb{R}^{2,1}$.
(2) Each V_j has a light cone L^2, and each projectivized L^2 is a circle C_j, $j = 1, 2$.
(3) Taking $p_1 \in C_1$ and $p_2 \in C_2$, we have a line $\ell_{p_1p_2}$ through p_1 and p_2 lieing in the quadric PL^5.
(4) Taking various p_1 and p_2, we get a two-dimensional family of lines in the quadric. This is the Dupin cyclide, with curvature spheres represented by p_1 and p_2. (Curvature spheres are explained in the upcoming Section 3.8.)

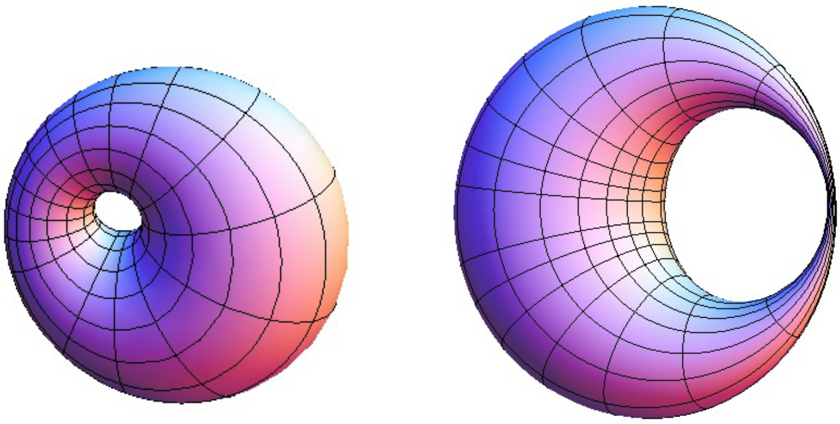

FIGURE 3.4. Further projections of the Dupin cyclide to $\mathbb{R}^3$

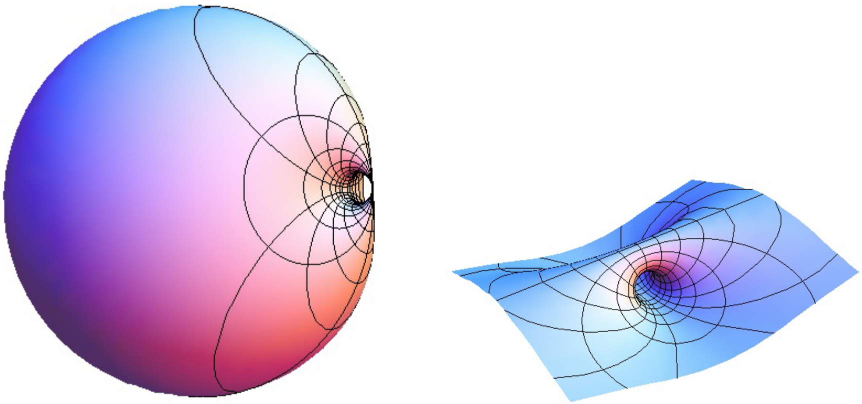

FIGURE 3.5. Further projections of the Dupin cyclide to $\mathbb{R}^3$, the projection on the left being one that extends out to ∞ in $\mathbb{R}^3 \cup \{\infty\}$

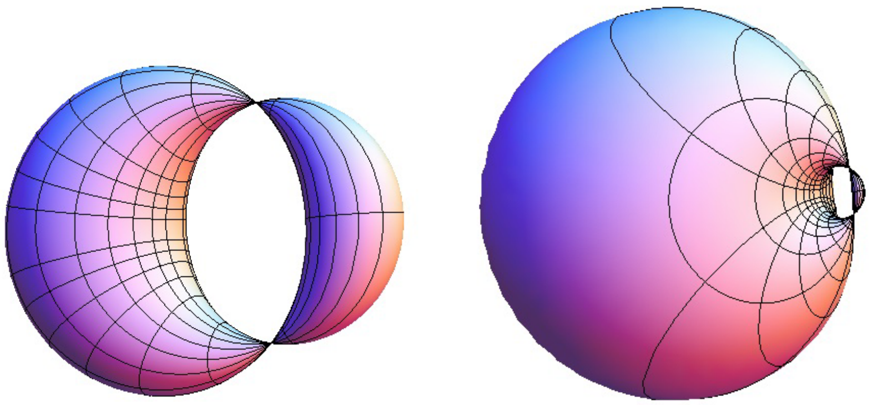

FIGURE 3.6. Further projections of the Dupin cyclide to $\mathbb{R}^3$

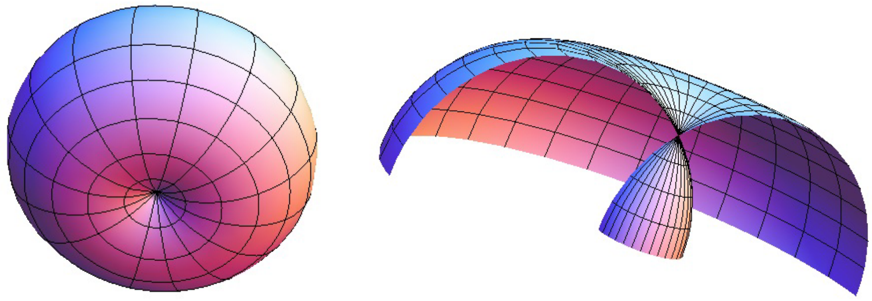

FIGURE 3.7. Another projection of the Dupin cyclide to $\mathbb{R}^3$, with a partial piece shown on the right

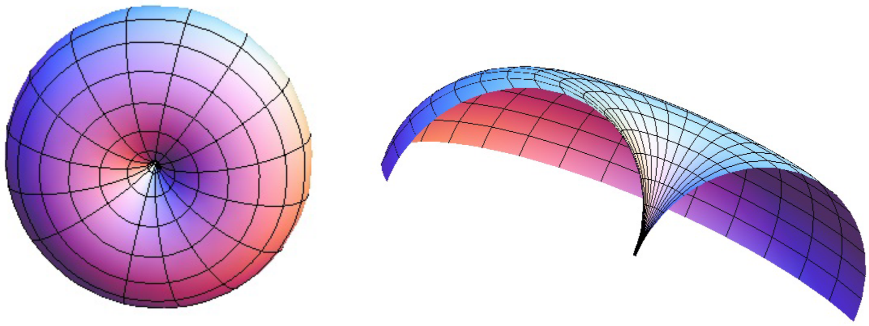

FIGURE 3.8. Yet another projection of the Dupin cyclide to $\mathbb{R}^3$, with a partial piece shown on the right

(5) Various projections of the $\ell_{p_1 p_2}$, through the Möbius geometry in $\mathbb{R}^{4,1}$, to $\mathbb{R}^3$, give all the Dupin cyclides in $\mathbb{R}^3$.

However, we define them this way, regarding $\mathbb{S}^3$ as Möbius equivalent to $\mathbb{R}^3 \cup \{\infty\}$:

Definition 3.14. *$f : M^2 \to \mathbb{S}^3$ is a* Dupin cyclide *if the principal curvature function associated to each curvature line is constant along that curvature line.*

Examples of Dupin cyclides considered in $\mathbb{R}^3$, in addition to cylinders and cones, are the following three:

(1) Take a nonintersecting line and circle in a plane and rotate the circle about the line to get a donut-shaped embedded surface of revolution in $\mathbb{R}^3$.
(2) Take a line and circle intersecting tangentially at one point in a plane and rotate the circle about the line to get a surface of revolution with one singular point in $\mathbb{R}^3$.
(3) Take a line and circle intersecting transversally at two points in a plane and rotate the circle about the line to get a surface of revolution with two cone-like singular points in $\mathbb{R}^3$.

Theorem 3.15. *(Pinkall,* [95], [96]*) All Dupin cyclides are Möbius transformations of the above surfaces of revolution. In particular, this includes Clifford tori and cylinders and cones.*

See Figures 3.3-3.8.

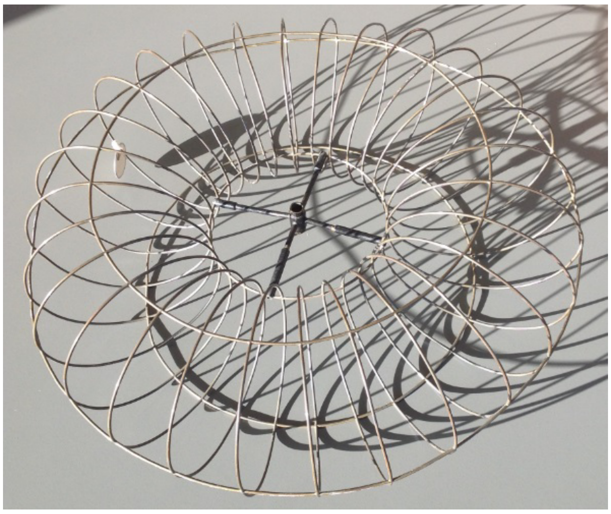

FIGURE 3.9. A wire frame model of a Clifford torus in $\mathbb{R}^3$ (owned by the geometry group at the Technical University of Vienna)

Theorem 3.16. *(Pinkall,* [95], [96]*) Up to the freedom of Lie sphere transformations, there is only one Dupin cyclide.*

So, for example, in Theorem 3.16 the transformation to parallel surfaces in $\mathbb{R}^3$ of the first (embedded "donut") of the three examples above is allowed.

Pinkall gave various equivalent descriptions of Dupin cyclides in $\mathbb{R}^3$ in his thesis ([95], [96]):

(1) Principal curvatures along corresponding lines of curvature are constant (this is a restatement of Definition 3.14).
(2) All lines of curvature are circular arcs.
(3) Principal curvature spheres along lines of curvature are constant.
(4) Focal surfaces degenerate to curves.
(5) The surface is a channel surface in two ways, i.e. it envelopes two different 1-parameter families of spheres.

To illustrate Theorem 3.16, we could start with the Clifford torus

$$x(u,v) = (\cos v \cdot (\sqrt{2} + \cos u), \sin v \cdot (\sqrt{2} + \cos u), \sin u) \ , \tag{3.6}$$

considered in $\mathbb{R}^3$. Under the metric induced on $\mathbb{R}^3 \cup \{\infty\}$ by the usual stereographic projection of $\mathbb{S}^3$ to $\mathbb{R}^3 \cup \{\infty\}$, this surface x is a minimal surface, which can be seen, for example, by noting that, with respect to the $\mathbb{S}^3$ metric,

$$\text{dist}((0,0,0),(1,0,0)) = \text{dist}((1,0,0),(\infty,0,0)) = \tfrac{\pi}{2} \ ,$$

$$\text{dist}((0,0,0),(\sqrt{2}-1,0,0)) = \text{dist}((\sqrt{2}-1,0,0),(1,0,0)) = \tfrac{\pi}{4} \ ,$$

$$\text{dist}((\tfrac{1}{\sqrt{2}},0,\tfrac{1}{\sqrt{2}}),(1,0,0)) = \text{dist}((\tfrac{1}{\sqrt{2}},0,\tfrac{1}{\sqrt{2}}),(0,0,1)) = \tfrac{\pi}{4} \ ,$$

and that the surface has two distinct rotational symmetries.

Then the induced Legendre immersion in PL^5 is given by the span of

$$\xi_1 = \bar{\xi}_{p=(\sqrt{2}\cos v, \sqrt{2}\sin v, 0), r=1} = \begin{pmatrix} 1 \\ 0 \\ \sqrt{2}\cos v \\ \sqrt{2}\sin v \\ 0 \\ 1 \end{pmatrix} ,$$

$$\xi_2 = \bar{\xi}_{p=(0,0,-\sqrt{2}\tan u), r=1+\frac{\sqrt{2}}{\cos u}} \,||\, \begin{pmatrix} -1 \\ \frac{2\cos u+\sqrt{2}}{\cos u+\sqrt{2}} \\ 0 \\ 0 \\ \frac{-\sqrt{2}\sin u}{\cos u+\sqrt{2}} \\ 1 \end{pmatrix} .$$

In particular, $(\xi_1, \xi_1) = (\xi_2, \xi_2) = (\xi_1, \xi_2) = 0$. Also, $\text{span}\{\xi_1, \xi_2\} = \text{span}\{\bar{\xi}_{x,0}, \bar{\xi}_{-n}\}$. Now

$$\xi_1 \in V_1 := \left\{ \begin{pmatrix} A \\ 0 \\ B \\ C \\ 0 \\ A \end{pmatrix} \middle| \; A, B, C \in \mathbb{R} \right\} ,$$

$$\xi_2 \in V_2 := \left\{ \begin{pmatrix} A \\ B \\ 0 \\ 0 \\ C \\ -A \end{pmatrix} \middle| \; A, B, C \in \mathbb{R} \right\}$$

and $V_1 \approx \mathbb{R}^{2,1}$, $V_2 \approx \mathbb{R}^{2,1}$ and $V_1 \perp V_2$. Then, for any $\mathcal{A} \in O_{4,2}$, the projection of

$$\mathcal{A}(\text{span}\{\xi_1, \xi_2\}) \cap \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}^{\perp}$$

to $\mathbb{R}^3$ gives any arbitrary Dupin cyclide in $\mathbb{R}^3$.

The Clifford torus can also be described in $\mathbb{S}^3 \subset \mathbb{R}^4$ as

$$x(u, v) = \tfrac{1}{\sqrt{2}}(\cos u, \sin u, \cos v, \sin v) ,$$

with normal

$$n(u, v) = \tfrac{1}{\sqrt{2}}(\cos u, \sin u, -\cos v, -\sin v) ,$$

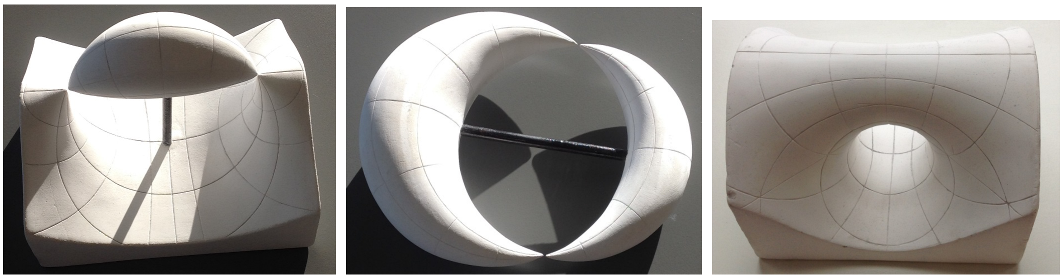

FIGURE 3.10. Physical models of three different projections of the Dupin cyclide to $\mathbb{R}^3$ (owned by the geometry group at the Technical University of Vienna)

and the associated lines in the quadric PL^5 are given by

$$\mathrm{span}\left\{\gamma^{\pm} = \tfrac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\x^t\\0\end{pmatrix} \pm \begin{pmatrix}0\\n^t\\1\end{pmatrix}\right)\right\} ,$$

with

$$\gamma^+ = \gamma^+(u) = \begin{pmatrix}\sqrt{2}^{-1}\\ \cos u\\ \sin u\\ 0\\ 0\\ \sqrt{2}^{-1}\end{pmatrix} \quad \text{and} \quad \gamma^- = \gamma^-(v) = \begin{pmatrix}\sqrt{2}^{-1}\\ 0\\ 0\\ \cos v\\ \sin v\\ -\sqrt{2}^{-1}\end{pmatrix} .$$

Taking

$$A = \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix} \in O(4) ,$$

we have

$$A \cdot x^t = \tfrac{1}{\sqrt{2}}\begin{pmatrix}\cos v\\ \sin v\\ \cos u\\ \sin u\end{pmatrix} ,$$

so there exists an isometry of $\mathbb{S}^3 = (\mathbb{R}^3 \cup \{\infty\}, ds^2_{\mathbb{S}^3})$ preserving the set of points in the Clifford torus that switches the two geodesic circles $\{(\cos\theta, \sin\theta, 0)\,|\,\theta \in [0, 2\pi)\}$ and $\{(0, 0, t)\,|\,t \in \mathbb{R}\}$ (the second of these is not parametrized by arc length). Thus, the two 3-dimensional regions bounded by the Clifford torus in (3.6), considered in $\mathbb{S}^3$, are congruent to each other.

For the Clifford torus in (3.6), regarded as a surface in $\mathbb{R}^3$, one focal curve is traversed only once, while the other is traversed twice. However, when this torus is regarded as a minimal surface in $\mathbb{S}^3$, both focal curves are traversed only once.

3.7. **Surfaces in various spaceforms.** Suppose we have a Legendre immersion Λ. Let us make a choice of $p, q \in \mathbb{R}^{4,2}$, $||p||^2 \neq 0$, $q \neq 0$ and $p \perp q$, to define a particular 3-dimensional spaceform M^3 in $\{p\}^{\perp}$, like in (2.2). Here we collect some facts regarding the choices of p and q.

Generically we have a lift $X \in \Lambda$ of a surface x in M^3 and a lift $N \in \Lambda$ of the normal n to x (that is, N represents the tangent geodesic planes of x in M^3) so that $\Lambda = \text{span}\{X, N\}$ and

$$(X, p) = (N, q) = 0 \ , \quad (X, q) = (N, p) = -1 \ .$$

Lemma 3.17. *Scaling p by a factor $\gamma \in \mathbb{R}$ scales the normal to the surface (the normal determined by N) in the tangent space TM of the spaceform M by the factor γ^{-1}, and so scales the principle curvatures κ_i of the surface by the factor γ^{-1}.*

Proof. Because q is unchanged, x is also unchanged. However, n changes to $\gamma^{-1}n$, and then the Rodrigues equations become

$$(\gamma^{-1}\kappa_1)\partial_u x + \partial_u(\gamma^{-1}n) = 0 \ , \quad (\gamma^{-1}\kappa_2)\partial_v x + \partial_v(\gamma^{-1}n) = 0 \ ,$$

and so the κ_i change to $\gamma^{-1}\kappa_i$. (We soon introduce the Rodrigues equations, in (3.7).) □

Remark 3.18. Scaling p does not change M^3, nor the surface in M^3. In Lemma 3.17 we are regarding the principle curvatures as depending on both the surface itself and also on the choice of normal in TM^3.

One can similarly prove:

Lemma 3.19. *Scaling q by a factor $\gamma \in \mathbb{R}$ scales the principle curvatures of the surface by the same factor γ.*

Lemma 3.20. *Scaling both p and q by the same factor will leave the principle curvatures of the surface unchanged.*

Remark 3.21. For any $A \in O_{4,2}$ (i.e. any isometry of $\mathbb{R}^{4,2}$) that preserves $\text{span}\{p, q\}$, the two surfaces coming from p, q and Ap, Aq will be parallel surfaces of each other, like in Lemma 3.4.

Remark 3.22. As we will see in Lemma 3.25 below, the curvature spheres of the surfaces are represented by, for $j = 1, 2$,

$$N + \kappa_j X \ .$$

Remark 3.23. When $(p, p) > 0$, so the spaceform is Lorentzian, and when considering a spacelike surface in the spaceform, the definition of the Gaussian curvature is sometimes taken with the opposite sign from the Riemannian spaceform case:

$$K = -\det A \ ,$$

where A is the shape operator. See, for example, [3].

Remark 3.24. In $\mathbb{R}^{4,1}$ (Möbius geometry), surfaces and the sphere congruences they envelop are both 2-parameter families of vectors, the only difference being that the former lie in L^4, while the latter lie in $\mathbb{S}^{3,1}$. However, in $\mathbb{R}^{4,2}$ (Lie sphere geometry), Legendre maps and the sphere congruences they envelop are fundamentally different things, each of the latter being merely a single "section" of the former.

3.8. **Lie cyclides.** Consider a Legendre immersion

$$\Lambda = \mathrm{span}\{X, N\}$$

for a surface x in $\mathbb{H}^3$ with normal $n \in \mathbb{S}^{2,1}$. In particular, let us choose

$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} ,$$

and so

$$X = \begin{pmatrix} x^t \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} n^t \\ 0 \\ 1 \end{pmatrix} .$$

The principal curvatures k_i satisfy (the ∂_i are in principal curvature directions, that is, when u, v are curvature line coordinates for $x = x(u, v)$, we can take $\partial_1 = \partial_u$ and $\partial_2 = \partial_v$)

$$\partial_i n + k_i \partial_i x = 0 . \tag{3.7}$$

This is the Rodrigues equation, which holds for any spaceform. Now set

$$K_i := N + k_i X .$$

Lemma 3.25. *K_i is a curvature sphere of x.*

Proof. We have

$$N + k_i X = \begin{pmatrix} n^t + k_i x^t \\ k_i \\ 1 \end{pmatrix}$$

and $X, dX \in \mathrm{span}\{p\}^\perp$. Because

$$(N + k_i X, X) = (N, X) + k_i (X, X) = 0 + k_i 0 = 0 ,$$

we know that x lies in the sphere determined by K_i. Because

$$(N + k_i X, dX) = (N, dX) + k_i (X, dX) = 0 + k_i 0 = 0 ,$$

we know K_i is tangent to the surface x. Now, as ∂_i is a directional derivative in the direction associated with k_i, we have

$$(K_i, \partial_i \partial_i X) = \partial_i (N + k_i X, \partial_i X) - (\partial_i (N + k_i X), \partial_i X) =$$

$$\partial_i(0) - \left(\begin{pmatrix} \partial_i k_i \cdot x^t \\ \partial_i k_i \\ 0 \end{pmatrix} , \begin{pmatrix} \partial_i x^t \\ 0 \\ 0 \end{pmatrix} \right) = \partial_i k_i \cdot \langle x, \partial_i x \rangle_{\mathbb{R}^{3,1}} =$$

$$\tfrac{1}{2} \partial_i k_i \cdot \partial_i (\langle x, x \rangle_{\mathbb{R}^{3,1}}) = \tfrac{1}{2} \partial_i k_i \cdot \partial_i (-1) = 0 .$$

Noting that now

$$(tK_i, sX) = (tK_i, d(sX)) = (tK_i, \partial_i \partial_i (sX)) = 0$$

for any scalar factors t and s considered as functions on the domain of x, we see that K_i is a principal curvature sphere. □

Lemma 3.26. *$K = N + kX$ is a principal curvature sphere in some principal curvature direction if and only if there exists a direction $\vec{v}$ such that $\partial_{\vec{v}} K \in \Lambda$, and then $\vec{v}$ will be the principal curvature direction for the principal curvature k.*

Proof. We have $\partial_i K_i = \partial_i k_i \cdot X \in \Lambda$. Conversely, for some function k on the surface and some directional derivative ∂,

$$\partial(N + kX) = \begin{pmatrix} \partial n^t + k\partial x^t \\ 0 \\ 0 \end{pmatrix} + \partial k \cdot X \in \Lambda$$

would give that $\partial n + k\partial x = 0$ (as $\partial k \cdot X$ is in Λ, and

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ 0 \\ 0 \end{pmatrix} \in \Lambda$$

implies $\vec{y} = \vec{0}$), i.e. k is the principal curvature with principal curvature direction ∂. □

Remark 3.27. A Legendre immersion has an umbilic point at a domain point (u_0, v_0) (umbilic points on Legendre immersions are the points where the two curvature spheres coincide) if and only if its projection to a conformal $\mathbb{S}^3$ has an umbilic at (u_0, v_0). Umbilics on Legendre immersions will cause problems when we consider Ω surfaces, where isothermicity is involved, so we exclude them.

Remark 3.28. Curvature spheres are preserved under Lie sphere transformations.

Lemma 3.29. *If Γ be a section of Λ, then $d\Gamma$ is perpendicular to Λ.*

Proof. A section is of the form $\Gamma = aX + bN$ for some scalar functions a and b. Then

$$(\Gamma, X) = (aX + bN, X) = a \cdot 0 + b \cdot 0 = 0 \ ,$$
$$(\Gamma, N) = (aX + bN, N) = a \cdot 0 + b \cdot 0 = 0 \ ,$$

so

$$(d\Gamma, X) = -(aX + bN, dX) = -a(X, dX) - b(N, dX) = -a \cdot 0 - b \cdot 0 = 0$$

and

$$(d\Gamma, N) = -(aX + bN, dN) = a(dX, N) - b(N, dN) = 0 \ .$$

□

The K_i can be used to determine the contact elements

$$\Lambda = \text{span}\{K_1, K_2\} \ ,$$

when excluding umbilics. Also, $dK_i \perp \Lambda$ and $\partial_i K_i \in \Lambda$, by Lemmas 3.29 and 3.26.

Set

$$V = \text{span}\{K_1, \partial_2 K_1, \partial_2^2 K_1\} \ , \quad W = \text{span}\{K_2, \partial_1 K_2, \partial_1^2 K_2\} \ .$$

Proposition 3.30. *V and W are perpendicular, and each is a Minkowski 3-space, i.e. has metric with $(+, +, -)$ signature.*

Proof. First we note the following properties:

(1) X and N are perpendicular, because the 3-dimensional tangent space $T_x\mathbb{H}^3$ to $\mathbb{H}^3$ at x in $\mathbb{R}^{3,1}$ equals $\{x\}^\perp$.
(2) Note that we cannot allow the surface X to have umbilic points, for K_1 and K_2 to remain independent vectors, i.e. to have $\Lambda = \text{span}\{K_1, K_2\}$ remain a 2-dimensional space.
(3) $(\partial_2 K_1, K_2) = -(K_1, \partial_2 K_2) = 0$, since $\partial_2 K_2 \in \Lambda$.
(4) $(\partial_2^2 K_1, K_2) = -(\partial_2 K_1, \partial_2 K_2) = 0$, since $\partial_2 K_2 \in \Lambda$ and Lemma 3.29 implies $\partial_2 K_1$ is perpendicular to both K_1 and K_2.
(5) $\partial_1 K_1 \in \Lambda$ implies

$$\partial_1 K_1 = a_1 K_1 + b_1 K_2$$

for some scalar functions a_1 and b_1. So, for $j = 1, 2$, we have

$$(K_j, \partial_2\partial_1 K_1) = (K_j, \partial_2(a_1 K_1 + b_1 K_2)) =$$

$$a_1(K_j, \partial_2 K_1) + b_1(K_j, \partial_2 K_2) = 0 \ .$$

(6) Lemma 3.29 implies $\partial_2 K_1$ is perpendicular to Λ, and also $\partial_j K_j \in \Lambda$ for $j = 1, 2$, so

$$(\partial_2 K_1, \partial_j K_j) = 0 \ .$$

(7) $(\partial_1 K_2, \partial_2 K_1) = \partial_1(K_2, \partial_2 K_1) - (K_2, \partial_1\partial_2 K_2) = -(K_2, \partial_2\partial_1 K_1) = 0$.
(8) $(\partial_2^2 K_1, \partial_1 K_2) = \partial_2(\partial_2 K_1, \partial_1 K_2) - (\partial_2 K_1, \partial_2\partial_1 K_2)$ and then

$$(\partial_2^2 K_1, \partial_1 K_2) = -(\partial_2 K_1, \partial_1(a_2 K_1 + b_2 K_2)) =$$

$$-a_2(\partial_2 K_1, \partial_1 K_1) - b_2(\partial_2 K_1, \partial_1 K_2) = 0 \ .$$

(9) $(\partial_2^2 K_1, \partial_1^2 K_2) = \partial_2(\partial_2 K_1, \partial_1^2 K_2) - (\partial_2 K_1, \partial_2\partial_1^2 K_2) = -(\partial_2 K_1, \partial_1^2 \partial_2 K_2)$ and then

$$(\partial_2^2 K_1, \partial_1^2 K_2) = -(\partial_2 K_1, \partial_1^2(a_2 K_1 + b_2 K_2)) =$$

$$-\partial_1(\partial_2 K_1, \partial_1(a_2 K_1 + b_2 K_2)) + (\partial_2\partial_1 K_1, \partial_1(a_2 K_1 + b_2 K_2)) =$$

$$-\partial_1(a_2(\partial_2 K_1, \partial_1 K_1) + b_2(\partial_2 K_1, \partial_1 K_2)) +$$

$$(\partial_2(a_1 K_1 + b_1 K_2), \partial_1(a_2 K_1 + b_2 K_2)) =$$

$$(\partial_2(a_1 K_1 + b_1 K_2), \partial_1(a_2 K_1 + b_2 K_2)) =$$

$$(a_1\partial_2 K_1 + b_1 \partial_2 K_2, a_2 \partial_1 K_1 + b_2 \partial_1 K_2) = 0 \ .$$

(10) $\Lambda \subset \{\partial_2 K_1\}^\perp$ and Λ is 2-dimensional and totally null, so the metric of $\mathbb{R}^{4,2}$ restricted to $\{\partial_2 K_1\}^\perp$ cannot be of signature $(+,+,+,+,-)$. So $\partial_2 K_1$ is either spacelike or lightlike.
(11) We now know

$$(\partial_2 K_1, \partial_2 K_1) \geq 0 \ ,$$

but it is actually strictly positive, seen as follows: if $||\partial_2 K_1||^2 = 0$, then $\text{span}\{\partial_2 K_1, \Lambda\}$ is totally null in $\mathbb{R}^{4,2}$, so must be only at most 2-dimensional, which implies K_1 and K_2 are parallel, contradicting the fact that we have excluded umbilics.

Using the above properties, we can show the following:

$$(K_j, K_j) = 0\ ,\quad j = 1, 2\ ,$$
$$(K_1, \partial_2 K_1) = (K_2, \partial_1 K_2) = 0\ ,$$
$$(K_1, \partial_2^2 K_1) = -a\ ,\quad a > 0\ ,$$
$$(K_2, \partial_1^2 K_2) = -b\ ,\quad b > 0\ ,$$
$$(K_1, K_2) = (K_1, \partial_1 K_2) = (K_1, \partial_1^2 K_2) = 0\ ,$$
$$(\partial_2 K_1, \partial_2 K_1) = a\ ,$$
$$(\partial_1 K_2, \partial_1 K_2) = b\ ,$$
$$(\partial_2 K_1, K_2) = (\partial_2 K_1, \partial_1 K_2) = (\partial_2 K_1, \partial_1^2 K_2) = 0\ ,$$
$$(\partial_2^2 K_1, K_2) = (\partial_2^2 K_1, \partial_1 K_2) = (\partial_2^2 K_1, \partial_1^2 K_2) = 0\ .$$

Here, a and b are functions (not constants).

Now, to complete the proof, one can just notice that the matrix

$$\begin{pmatrix} (K_1, K_1) & (\partial_2 K_1, K_1) & (\partial_2^2 K_1, K_1) \\ (K_1, \partial_2 K_1) & (\partial_2 K_1, \partial_2 K_1) & (\partial_2^2 K_1, \partial_2 K_1) \\ (K_1, \partial_2^2 K_1) & (\partial_2 K_1, \partial_2^2 K_1) & (\partial_2^2 K_1, \partial_2^2 K_1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -a \\ 0 & a & * \\ -a & * & * \end{pmatrix}$$

has strictly negative determinant, so V has a nondegenerate metric. Thus the metric on V has either $(+,+,-)$ or $(-,-,-)$ signature. But $(-,-,-)$ signature is not possible in $\mathbb{R}^{4,2}$, so V is a Minkowski 3-space, and similarly so is W. □

Remark 3.31. Note that everything in Proposition 3.30 and its proof still holds if we replace K_1 and K_2 by any scalar multiples of them.

Remark 3.32. We could rename V and W to V and $V^\perp$ if we like, now that we know they are perpendicular.

The splitting $V \oplus W$ of $\mathbb{R}^{4,2}$ by the perpendicular 3-planes V and W is called a *Lie cyclide*, giving a congruence of Lie cyclides over the surface. Furthermore, each Lie cyclide produces a Dupin cyclide (as explained at the beginning of Section 3.6), which makes second order contact with the surface at that point on the surface. This is the Dupin cyclide congruence of the surface. (This is somewhat akin to attaching quadratic surfaces with second order tangential contact at points of a surface in $\mathbb{R}^3$.)

Proposition 3.33. *A congruence of Lie cyclides (equivalently, Dupin cyclide congruence) along a surface is constant if and only if the surface is a Dupin cyclide.*

Proof. Lemma 3.26 implies there exist real scalars a and b so that

$$\partial_1 K_1 = a K_1 + b K_2\ ,$$

and the Lie cyclide being constant implies

$$\partial_1 K_1 \in \text{span}\{K_1, \partial_2 K_1, \partial_2^2 K_1\}\ ,$$

and so $b = 0$. Thus $\partial_1 K_1 || K_1$, and similarly $\partial_2 K_2 || K_2$. Therefore

$$K_1 || \partial_1 K_1 = \partial_1 n + k_1 \partial_1 x + \partial_1 k_1 \cdot x = \partial_1 k_1 \cdot x\ ,$$

and similarly $K_2 || \partial_1 k_2 \cdot x$, giving that

$$\partial_1 k_1 \cdot x || (n + k_1 x)$$

and

$$\partial_2 k_2 \cdot x || (n + k_2 x) \ .$$

Hence $\partial_1 k_1 = \partial_2 k_2 = 0$, and we have a Dupin cyclide. □

Lemma 3.34. *The Dupin cyclide congruence of a lift of a surface of revolution in a 3-dimensional Riemannian spaceform is constant along the rotational directions of the surface. In particular, the congruence consists of only a 1-parameter family of Dupin cyclides.*

Proof. Without loss of generality, we can take the surface of revolution to be in $\mathbb{R}^3$, as

$$x(u,v) = (f(u)\cos v, f(u)\sin v, g(u)) \ ,$$

where

$$f_u^2 + g_u^2 = f^2 \ ,$$

and the unit normal is

$$n(u,v) = f^{-1} \cdot (-g_u \cos v, -g_u \sin v, f_u) \ ,$$

and the first and second fundamental forms become

$$I = \begin{pmatrix} f^2 & 0 \\ 0 & f^2 \end{pmatrix} \ , \quad II = \begin{pmatrix} f^{-1}(f_u g_{uu} - f_{uu} g_u) & 0 \\ 0 & g_u \end{pmatrix} \ .$$

Thus the principal curvatures are

$$k_1 = f^{-3}(f_u g_{uu} - f_{uu} g_u) \ , \quad k_2 = f^{-2} g_u \ .$$

Now, with

$$X = \begin{pmatrix} \frac{1}{2}(1 + x \cdot x) \\ \frac{1}{2}(1 - x \cdot x) \\ x^t \\ 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} x \cdot n \\ -x \cdot n \\ n^t \\ 1 \end{pmatrix} \ ,$$

we find that $K_2 = k_2 X + N$ is independent of v. So

$$W = \text{span}\{K_2, K_{2,u}, K_{2,uu}\}$$

is independent of v, as desired. Because V is perpendicular to W, V must also be independent of v. (We can also check by direct computation that V is independent of v). □

3.9. **General frame equations for surfaces in $\mathbb{H}^3 \subset \mathbb{R}^{3,1}$.** Take $x \in \mathbb{H}^3$ and $n \in \mathbb{S}^{2,1}$ as in the beginning of Section 3.2. Set

$$F = (x, x_u, x_v, n) \ ,$$

where (u,v) are curvature line coordinates for x, so

$$n_u + k_1 x_u = n_v + k_2 x_v = 0 \ .$$

Thus the first and fundamental forms have the forms

$$|dx|^2 = E du^2 + G dv^2 \ , \quad II = k_1 E du^2 + k_2 G dv^2 \ .$$

Defining ϕ and ψ by

$$F_u = F\phi \ , \quad F_v = F\psi \ ,$$

a computation gives

$$\phi = \begin{pmatrix} 0 & E & 0 & 0 \\ 1 & \frac{1}{2}\frac{E_u}{E} & \frac{1}{2}\frac{E_v}{E} & -k_1 \\ 0 & -\frac{1}{2}\frac{E_v}{G} & \frac{1}{2}\frac{G_u}{G} & 0 \\ 0 & k_1 E & 0 & 0 \end{pmatrix} , \quad \psi = \begin{pmatrix} 0 & 0 & G & 0 \\ 0 & \frac{1}{2}\frac{E_v}{E} & -\frac{1}{2}\frac{G_u}{E} & 0 \\ 1 & \frac{1}{2}\frac{G_u}{G} & \frac{1}{2}\frac{G_v}{G} & -k_2 \\ 0 & 0 & k_2 G & 0 \end{pmatrix} .$$

Then $F_{uv} = F_{vu}$ implies

$$\phi_v + \psi\phi = \psi_u + \phi\psi ,$$

which in turn gives the Gauss and Codazzi equations

$$(\log E)_v = \frac{2(k_1)_v}{k_2 - k_1} , \quad (\log G)_u = \frac{2(k_2)_u}{k_1 - k_2} ,$$

$$k_1 k_2 - 1 = \frac{-1}{2\sqrt{EG}} \left(\left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right) .$$

3.10. **Frame equations for flat surfaces in $\mathbb{H}^3$ via Lie cyclides.** Because $k_1 k_2 = 1$ for flat surfaces, away from umbilics we can set, for some real-valued function $\varphi = \varphi(u, v)$,

$$k_1 = \tanh\varphi , \quad k_2 = \coth\varphi .$$

Then, by integrating the Codazzi equations, we have

$$E = f_E(u)\cosh^2\varphi , \quad G = f_G(v)\sinh^2\varphi ,$$

for some functions f_E and f_G depending only on u and v, respectively. So, changing coordinates via $u = u(\hat{u})$ and $v = v(\hat{v})$ appropriately, without loss of generality we may assume the first fundamental form is

$$|dx|^2 = \cosh^2\varphi du^2 + \sinh^2\varphi dv^2 ,$$

and then the second fundamental form becomes

$$II = \sinh\varphi\cosh\varphi \cdot (du^2 + dv^2)$$

and the Gauss equation becomes simply

$$\Delta\varphi = \varphi_{uu} + \varphi_{vv} = 0 .$$

We also have

$$k_1 - k_2 = \frac{-1}{\sinh\varphi\cosh\varphi} ,$$

$$\frac{\partial_u k_1}{k_1 - k_2} = -k_1 \partial_u\varphi , \quad \partial_v k_2 = \frac{-\partial_v\varphi}{\sinh^2\varphi} ,$$

$$\frac{\partial_v k_2}{k_1 - k_2} = k_2 \partial_v\varphi , \quad \partial_u k_1 = \frac{\partial_u\varphi}{\cosh^2\varphi} .$$

Note that (here we find the notation $\tilde{K}_i$, rather than K_i, convenient, because of a rescaling of the $\tilde{K}_i$ we make just below)

$$\tilde{K}_i = \begin{pmatrix} n^t + k_i x^t \\ k_i \\ 1 \end{pmatrix} ,$$

$$\partial_u \tilde{K}_1 = \frac{\partial_u k_1}{k_1 - k_2}(\tilde{K}_1 - \tilde{K}_2) = -k_1(\tilde{K}_1 - \tilde{K}_2)\partial_u\varphi ,$$

$$\partial_v \tilde{K}_2 = \frac{\partial_v k_2}{k_1 - k_2}(\tilde{K}_1 - \tilde{K}_2) = k_2(\tilde{K}_1 - \tilde{K}_2)\partial_v \varphi \ .$$

Then we have

$$\partial_u \left(\cosh\varphi \cdot \tilde{K}_1\right) = \partial_u \varphi \cdot (\sinh\varphi \cdot \tilde{K}_2) \ ,$$
$$\partial_v \left(\sinh\varphi \cdot \tilde{K}_2\right) = \partial_v \varphi \cdot (\cosh\varphi \cdot \tilde{K}_1) \ ,$$
$$\left|\partial_v(\cosh\varphi \cdot \tilde{K}_1)\right|^2 = \left|\partial_u(\sinh\varphi \cdot \tilde{K}_2)\right|^2 = 1 \ .$$

Now we set

$$K_1 := \cosh\varphi \tilde{K}_1 \ , \quad K_2 := \sinh\varphi \tilde{K}_2 \ .$$

This will not change the 3-dimensional spaces V and W, and will not change the truth of Proposition 3.30, by Remark 3.31. We then have

$$K_1 = \cosh\varphi \cdot \begin{pmatrix} n^t + k_1 x^t \\ k_1 \\ 1 \end{pmatrix} \ , \quad K_2 = \sinh\varphi \cdot \begin{pmatrix} n^t + k_2 x^t \\ k_2 \\ 1 \end{pmatrix} \ ,$$

and the consequent properties

$$\partial_u K_1 = \partial_u \varphi \cdot K_2 \ ,$$
$$\partial_v K_2 = \partial_v \varphi \cdot K_1 \ ,$$
$$\partial_u \partial_v K_2 = \partial_u \partial_v \varphi \cdot K_1 + \partial_u \varphi \partial_v \varphi \cdot K_2 \ ,$$
$$\partial_v \partial_u K_1 = \partial_u \partial_v \varphi \cdot K_2 + \partial_u \varphi \partial_v \varphi \cdot K_1 \ ,$$
$$\partial_u^2 \partial_v K_2 = \partial_u^2 \partial_v \varphi \cdot K_1 + (2\partial_u \partial_v \varphi \partial_u \varphi + \partial_v \varphi \partial_u^2 \varphi) \cdot K_2 + \partial_u \varphi \partial_v \varphi \cdot \partial_u K_2 \ ,$$
$$\partial_v^2 \partial_u K_1 = \partial_v^2 \partial_u \varphi \cdot K_2 + (2\partial_u \partial_v \varphi \partial_v \varphi + \partial_u \varphi \partial_v^2 \varphi) \cdot K_1 + \partial_u \varphi \partial_v \varphi \cdot \partial_v K_1 \ ,$$
$$\partial_u K_2 = \partial_u \varphi \cdot K_1 + \sinh\varphi \cdot (k_2 - k_1) \cdot \begin{pmatrix} \partial_u x^t \\ 0 \\ 0 \end{pmatrix} \ ,$$
$$\partial_u^2 K_2 = \partial_u^2 \varphi \cdot K_1 + (\partial_u \varphi)^2 \cdot K_2 - \frac{\sinh\varphi}{\cosh^2\varphi} \cdot \partial_u \varphi \cdot \begin{pmatrix} \partial_u x^t \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\cosh\varphi} \cdot \begin{pmatrix} \partial_u^2 x^t \\ 0 \\ 0 \end{pmatrix} \ ,$$
$$\partial_v K_1 = \partial_v \varphi \cdot K_2 + \cosh\varphi \cdot (k_1 - k_2) \cdot \begin{pmatrix} \partial_v x^t \\ 0 \\ 0 \end{pmatrix} \ ,$$
$$\partial_v^2 K_1 = \partial_v^2 \varphi \cdot K_2 + (\partial_v \varphi)^2 \cdot K_1 + \frac{\cosh\varphi}{\sinh^2\varphi} \cdot \partial_v \varphi \cdot \begin{pmatrix} \partial_v x^t \\ 0 \\ 0 \end{pmatrix} - \frac{1}{\sinh\varphi} \cdot \begin{pmatrix} \partial_v^2 x^t \\ 0 \\ 0 \end{pmatrix} \ ,$$
$$|\partial_v K_1|^2 = |\partial_u K_2|^2 = 1 \ .$$

Furthermore,

$$\partial_u^2 K_2 \perp \partial_u K_2 \ , \quad \partial_v^2 K_1 \perp \partial_v K_1$$

and

$$(\partial_u^2 K_2, K_2) = -|\partial_u K_2|^2 = (\partial_v^2 K_1, K_1) = -|\partial_v K_1|^2 = -1 \ .$$

Also,

$$(\partial_u^2 K_2, \partial_v^2 K_1) = 0 \ ,$$

seen as follows:

$$(\partial_u^2 K_2, \partial_v^2 K_1) = \partial_u(\partial_u K_2, \partial_v^2 K_1) - (\partial_u K_2, \partial_u \partial_v^2 K_1) =$$
$$\partial_u(\partial_v(\partial_u K_2, \partial_v K_1) - (\partial_u \partial_v K_2, \partial_v K_1)) - \partial_v(\partial_u K_2, \partial_u \partial_v K_1) +$$
$$(\partial_u \partial_v K_2, \partial_u \partial_v K_1) = \partial_u(\partial_v 0 - 0) - \partial_v 0 + 0 = 0 \ .$$

Define

$$\hat{K}_1 = \partial_v^2 K_1 + \alpha K_1 \ , \quad \hat{K}_2 = \partial_u^2 K_2 + \beta K_2 \ ,$$

such that $\hat{K}_j$ are null for both $j = 1$ and $j = 2$. This means taking

$$\alpha = \tfrac{1}{2}|\partial_v^2 K_1|^2 \ , \quad \beta = \tfrac{1}{2}|\partial_u^2 K_2|^2 \ .$$

Noting that

$$\partial_v^2 x = (\sinh\varphi)^2 x - (\tanh\varphi)\partial_u\varphi\partial_u x + (\coth\varphi)\partial_v\varphi\partial_v x + k_2(\sinh\varphi)^2 n \ ,$$

we find that

$$|\partial_v^2 x|^2_{\mathbb{R}^{3,1}} = (k_2^2 - 1)(\sinh\varphi)^4 + (\sinh\varphi)^2(\partial_u\varphi)^2 + (\cosh\varphi)^2(\partial_v\varphi)^2 \ ,$$

and then that

$$\alpha = \tfrac{1}{2}(\partial_u\varphi)^2 - (\partial_v\varphi)^2 + \tfrac{1}{2} \ .$$

Similarly, we have

$$\beta = \tfrac{1}{2}(\partial_v\varphi)^2 - (\partial_u\varphi)^2 - \tfrac{1}{2} \ .$$

One can then check that

$$\partial_v\varphi \cdot \partial_u\beta + \partial_u\varphi \cdot \partial_v\alpha + 2(\alpha + \beta) \cdot \partial_u\partial_v\varphi = 0 \ . \tag{3.8}$$

Now all inner products amongst the elements of the basis

$$F = \{K_1, \partial_v K_1, \hat{K}_1, K_2, \partial_u K_2, \hat{K}_2\} \tag{3.9}$$

for $\mathbb{R}^{4,2}$ are zero, except for

$$-\langle K_1, \hat{K}_1 \rangle = \langle \partial_v K_1, \partial_v K_1 \rangle = -\langle K_2, \hat{K}_2 \rangle = \langle \partial_u K_2, \partial_u K_2 \rangle = 1 \ .$$

One can check that

$$2\partial_u\partial_v\varphi\partial_v\varphi + \partial_u\alpha + \partial_u\varphi\partial_v^2\varphi = 0 \ , \tag{3.10}$$

$$2\partial_u\partial_v\varphi\partial_u\varphi + \partial_v\beta + \partial_v\varphi\partial_u^2\varphi = 0 \ . \tag{3.11}$$

Using Equations (3.10) and (3.11), we have the properties

$$dK_1 = \partial_v K_1 dv + \partial_u\varphi K_2 du \ ,$$
$$d(\partial_v K_1) = K_1(\partial_u\varphi\partial_v\varphi du - \alpha dv) + \hat{K}_1 dv + \partial_u\partial_v\varphi \cdot K_2 du \ ,$$
$$d\hat{K}_1 = \partial_v K_1(\partial_u\varphi\partial_v\varphi du - \alpha dv) + K_2((\partial_u\partial_v^2\varphi + \alpha\partial_u\varphi)du -$$
$$(\partial_u^2\partial_v\varphi + \beta\partial_v\varphi)dv) + \partial_u\partial_v\varphi\partial_u K_2 dv - \hat{K}_2\partial_v\varphi dv \ ,$$
$$dK_2 = K_1\partial_v\varphi dv + \partial_u K_2 du \ ,$$
$$d(\partial_u K_2) = K_1\partial_u\partial_v\varphi dv + K_2(\partial_u\varphi\partial_v\varphi dv - \beta du) + \hat{K}_2 du \ ,$$
$$d\hat{K}_2 = K_1((\partial_u^2\partial_v\varphi + \beta\partial_v\varphi)dv - (\partial_u\partial_v^2\varphi + \alpha\partial_u\varphi)du) + \partial_v K_1\partial_u\partial_v\varphi du -$$
$$\hat{K}_1\partial_u\varphi du + \partial_u K_2(\partial_u\varphi\partial_v\varphi dv - \beta du) \ .$$

With F as defined in (3.9), we have the system

$$\partial_u F = FU \ , \quad \partial_v F = FV \ ,$$

with U and V defined as

$$U = \begin{pmatrix} 0 & \partial_u\varphi\partial_v\varphi & 0 & 0 & 0 & -\partial_u\partial_v^2\varphi - \alpha\partial_u\varphi \\ 0 & 0 & \partial_u\varphi\partial_v\varphi & 0 & 0 & \partial_u\partial_v\varphi \\ 0 & 0 & 0 & 0 & 0 & -\partial_u\varphi \\ \partial_u\varphi & \partial_u\partial_v\varphi & \partial_u\partial_v^2\varphi + \alpha\partial_u\varphi & 0 & -\beta & 0 \\ 0 & 0 & 0 & 1 & 0 & -\beta \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} 0 & -\alpha & 0 & \partial_v\varphi & \partial_u\partial_v\varphi & \partial_u^2\partial_v\varphi + \beta\partial_v\varphi \\ 1 & 0 & -\alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\partial_u^2\partial_v\varphi - \beta\partial_v\varphi & 0 & \partial_u\varphi\partial_v\varphi & 0 \\ 0 & 0 & \partial_u\partial_v\varphi & 0 & 0 & \partial_u\varphi\partial_v\varphi \\ 0 & 0 & -\partial_v\varphi & 0 & 0 & 0 \end{pmatrix}.$$

The compatibility condition $\partial_u\partial_v F = \partial_v\partial_u F$ is

$$VU + \partial_v U = \partial_u V + UV \ ,$$

and this turns out to be equivalent to three equations. The first two are the equations established in (3.10) and (3.11). The third equation, using $\Delta\varphi = 0$, simplifies to Equation (3.8).

3.11. **Converting to the Weierstrass representation for flat surfaces in $\mathbb{H}^3$.** The Weierstrass-type representation found in [55] for flat surfaces in $\mathbb{H}^3$ was described in Section 2.22, and we now consider how to write that representation in terms of φ. Using the equations in (2.72),

$$\omega = \hat{\omega}dz = \frac{-1}{\xi^2}dG \ , \quad \theta = \hat{\theta}dz = \frac{\xi^2}{(G-G_*)^2}dG_* \ , \tag{3.12}$$

where

$$\xi = \delta \cdot \exp \int_{z_0}^{z} \frac{dG}{G-G_*} \ .$$

Having curvature line coordinates means, without loss of generality, that

$$\hat{\omega}\hat{\theta} = \tfrac{1}{4} \ .$$

Then, without loss of generality (and with $z = u + iv$),

$$|dx|^2 = |\omega + \bar{\theta}|^2 =$$

$$\left(\hat{\omega}\bar{\hat{\omega}} + \frac{1}{16\hat{\omega}\bar{\hat{\omega}}} + \frac{1}{2}\right) du^2 + \left(\hat{\omega}\bar{\hat{\omega}} + \frac{1}{16\hat{\omega}\bar{\hat{\omega}}} - \frac{1}{2}\right) dv^2 =$$

$$\cosh^2\varphi du^2 + \sinh^2\varphi dv^2 \ .$$

Then

$$\cosh^2\varphi = \hat{\omega}\bar{\hat{\omega}} + \frac{1}{16\hat{\omega}\bar{\hat{\omega}}} + \frac{1}{2}$$

implies

$$\varphi = \pm\frac{1}{2}\log\left(4\hat{\omega}\bar{\hat{\omega}}\right) = \mp\frac{1}{2}\log|\rho| \ ,$$

since

$$|\rho| = \frac{|\hat{\theta}|}{|\hat{\omega}|} = \frac{1}{4\hat{\omega}\bar{\hat{\omega}}} \ .$$

Furthermore, Equation (3.12) gives

$$\varphi = \mp\frac{1}{2}\log\left(\frac{|\xi|^4|G'_*|}{|G-G_*|^2|G'|}\right) .$$

Remark 3.35. In general, $\hat{\omega}$ being holomorphic implies

$$\varphi = \pm\log(2|\hat{\omega}|)$$

is harmonic, and

$$\cosh\varphi = |\hat{\omega}| + 1/(4|\hat{\omega}|) , \quad \sinh\varphi = \pm(|\hat{\omega}| - 1/(4|\hat{\omega}|))$$

implies $|\hat{\omega}| = \frac{1}{2}e^{\pm\varphi}$.

Now we consider some simple explicit examples.

Note that the horosphere has been excluded from the beginning, because the conditions $k_1 = \tanh\varphi$ and $k_2 = \coth\varphi$ exclude the possibility that $k_1 = k_2 = 1$.

Example 3.36. Round cylinder. See Figure 2.21. In this case, k_1 and k_2 are constant, so φ is constant as well.

Example 3.37. Peach fronts. See Figure 2.22. We take

$$G = \frac{i}{2}z + \frac{1}{2} , \quad G_* = \frac{i}{2}z - \frac{1}{2} ,$$

so $\hat{\omega}\hat{\theta} = 1/4$ and

$$\hat{\omega} = -i\tilde{c}e^{-iz} , \quad \tilde{c} > 0$$

and

$$\varphi = \hat{c} \pm v , \quad \hat{c} \in \mathbb{R} .$$

Example 3.38. Surfaces of revolution. See Figure 2.21. We take

$$G = e^{\sqrt{\frac{(1-\mu)^2}{-4\mu}}z} , \quad G_* = \mu e^{\sqrt{\frac{(1-\mu)^2}{-4\mu}}z} , \quad \mu \in (0,1)\cup(1,\infty) ,$$

so $\hat{\omega}\hat{\theta} = 1/4$ and

$$\hat{\omega} = \tilde{c}e^{\sqrt{\frac{(1-\mu)^2}{-4\mu}}\frac{\mu+1}{\mu-1}z} ,$$

and

$$\varphi = \hat{c} \pm i\frac{\mu+1}{\sqrt{-4\mu}}v , \quad \hat{c} \in \mathbb{R} .$$

4. Ω SURFACES

To discuss Ω surfaces, we first need to describe the relevant normal bundles of sphere congruences, which we do in the next two sections.

4.1. **Normal bundle for surfaces in PL^5.** Suppose $[b]$ lies in PL^5, where L^5 is the lightcone in $\mathbb{R}^{4,2}$. There exists, of course, a timelike vector $\vec{v} \in \mathbb{R}^{4,2}$ such that $(b, \vec{v}) \neq 0$. We choose an orthonormal basis

$$\{x_0, x_1, x_2, x_3, x_4, \vec{v}\}$$

of $\mathbb{R}^{4,2}$ of type $(-,+,+,+,+,-)$. Let us assume (without loss of generality) we have chosen the basis elements x_j so that $x_j \perp b$ for $j \leq 3$, and

$$(b, x_4) = -(b, \vec{v}) \;, \quad \text{i.e. } \; b \in \text{span}\{x_4, \vec{v}\} \; .$$

There exists a neighborhood U of $[b]$ in $P\mathbb{R}^{4,2}$ so that $(y, \vec{v}) \neq 0$ for all $[y] \in U$. Define the map

$$\phi : U \to \phi(U) \subseteq \mathbb{R}^{4,2} \;, \quad \phi \left[\begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ 0 \end{pmatrix} \;,$$

where $\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, 1$ are the coordinates (with respect to the basis we have chosen) of the appropriate representative y of $[y] \in U$. Using the natural topology on $P\mathbb{R}^{4,2}$, ϕ is a local homeomorphism, and thus gives a local coordinate chart at b for $P\mathbb{R}^{4,2}$.

Let $[\gamma] : \mathbb{R} \to U$ satisfy $[\gamma(0)] = [b]$, then

$$\phi[\gamma] = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ 0 \end{pmatrix}$$

when taking γ so that

$$\gamma = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ 1 \end{pmatrix} \;, \quad \text{so } \; (\phi[\gamma])' = \begin{pmatrix} \gamma_0' \\ \gamma_1' \\ \gamma_2' \\ \gamma_3' \\ \gamma_4' \\ 0 \end{pmatrix} \;,$$

and thus

$$T_{[b]} P\mathbb{R}^{4,2} \subseteq \text{span}\{\vec{v}\}^{\perp} \; .$$

Then by considering specific curves γ, such as

$$\gamma(t) = \begin{pmatrix} t \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

for example, one sees that

$$T_{[b]} P\mathbb{R}^{4,2} = \text{span}\{\vec{v}\}^{\perp} .$$

Suppose also that $(\gamma, \gamma) = 0$, i.e. γ is lightlike. Then $(\phi[\gamma], \phi[\gamma]) = 1$, and then $((\phi[\gamma])', \phi[\gamma]) = 0$, so in particular $((\phi[\gamma])'(0), \phi[b]) = 0$, and so

$$(\phi[\gamma])'(0) \in \text{span}\{\phi[b], \vec{v}\}^{\perp} .$$

Note that

$$\text{span}\{b\}^{\perp} = \text{span}\{x_0, x_1, x_2, x_3, b\} .$$

If we take the correct scalar multiple of the representative b for the class $[b]$, then also $\phi[b] + \vec{v} = b$, hence $\text{span}\{\phi[b], \vec{v}\} = \text{span}\{b, \vec{v}\}$. Thus

$$(\phi[\gamma])'(0) \in \text{span}\{x_0, x_1, x_2, x_3\} \approx \text{span}\{b\}^{\perp}/\text{span}\{b\} ,$$

and so

$$T_{[b]} PL^5 \subseteq \text{span}\{x_0, x_1, x_2, x_3\} .$$

Again, we can show that $T_{[b]} PL^5$ and $\text{span}\{x_0, x_1, x_2, x_3\}$ are equal by using specific choices for γ, such as

$$\gamma(t) = \frac{2}{1+t^2} \begin{pmatrix} 0 \\ t \\ 0 \\ 0 \\ (1/2)(1-t^2) \\ (1/2)(1+t^2) \end{pmatrix}$$

for t close to 0. We conclude that

$$T_{[b]} PL^5 = \text{span}\{b\}^{\perp}/\text{span}\{b\} .$$

Now let $[s] : M^2 \to PL^5$ be a surface with $b = s(u_0, v_0)$, and so

$$T_{[s]} PL^5 = \text{span}\{s\}^{\perp}/\text{span}\{s\} .$$

The tangent space of $[s]$ at (u_0, v_0) is

$$T_{(u_0,v_0)}[s] = \text{span}\{(\phi[s])_u, (\phi[s])_v\}|_{(u_0,v_0)} \subseteq \text{span}\{s(u_0, v_0)\}^{\perp}/\text{span}\{s(u_0, v_0)\} .$$

The normal space $\mathcal{N}_{(u_0,v_0)}[s]$ at (u_0, v_0) of $[s]$ is then the set of vectors

$$N \perp (\phi[s])_u, (\phi[s])_v|_{(u_0,v_0)}$$

(using the $\mathbb{R}^{4,2}$ metric to define perpendicularity) so that $N \in \text{span}\{x_0, x_1, x_2, x_3\}$, i.e.

$$N + \text{span}\{s(u_0, v_0)\} \in (\text{span}\{(\phi[s])_u, (\phi[s])_v, s\}|_{(u_0,v_0)})^{\perp}/\text{span}\{s(u_0, v_0)\} .$$

To avoid using the map ϕ, we can do the following: If s is normalized so that the last coordinate of s (with respect to the basis chosen) is 1, and if $\tilde{s} = \lambda s$, then

$$(\lambda s)_u = \lambda_u s + \lambda s_u = \lambda_u s + \lambda(\phi[s])_u \ ,$$

so

$$(\phi[s])_u = \lambda^{-1}\tilde{s}_u - \lambda^{-2}\lambda_u \tilde{s} \ ,$$

so

$$\mathrm{span}\{(\phi[s])_u, (\phi[s])_v, s\} = \\ \mathrm{span}\{\lambda^{-1}\tilde{s}_u - \lambda_u\lambda^{-2}\tilde{s}, \lambda^{-1}\tilde{s}_v - \lambda_v\lambda^{-2}\tilde{s}, \tilde{s}\} = \mathrm{span}\{\tilde{s}_u, \tilde{s}_v, \tilde{s}\} \ .$$

Therefore the fibres of the normal bundle to $[s]$ in PL^5 are

$$\mathcal{N}_{(u,v)}[s] = \{N + \mathrm{span}\{\tilde{s}\} \mid N \perp \tilde{s}, \tilde{s}_u, \tilde{s}_v\} \tag{4.1}$$

for any choice of lift $\tilde{s}$. This normal bundle has 2-dimensional fibers.

The conditions for $[s]$ to be isothermic with respect to the coordinates u, v are:

(1) The first fundamental form for s is conformal, i.e.

$$((\phi[s])_u, (\phi[s])_u) = ((\phi[s])_v, (\phi[s])_v) \quad \text{and} \quad ((\phi[s])_u, (\phi[s])_v) = 0 \ .$$

We can rewrite this condition as simply $(s_u, s_u) = (s_v, s_v)$ and $(s_u, s_v) = 0$ for any choice of lift of s, once we note that this conformality property is invariant of choice of that lift, giving a conformal equivalence class.

(2) The second fundamental form is diagonal, i.e. $((\phi[s])_{uv}, N) = 0$ for all $N \in \mathcal{N}_{(u,v)}[s]$, or equivalently, $(s_{uv}, N) = 0$ for all such N.

4.2. **Normal bundle for surfaces in PL^4.** Take p timelike in $\mathbb{R}^{4,2}$, and a surface $[s(u,v)] \in \{p\}^\perp \approx P\mathbb{R}^{4,1}$. We can take $\vec{v}$ as in Section 4.1 so that $b = s(u_0, v_0) \perp p$ and $s(u,v) \not\perp \vec{v}$ for (u,v) close to (u_0, v_0). Now one can argue like in Section 4.1 that

$$T_{[b]}P\mathbb{R}^{4,1} = \mathrm{span}\{\vec{v}, p\}^\perp \ ,$$

$$T_{[b]}PL^4 = \mathrm{span}\{b, p\}^\perp / \mathrm{span}\{b\} \ ,$$

$$T_{(u_0,v_0)}[s] = \mathrm{span}\{(\phi[s])_u, (\phi[s])_v\}|_{(u_0,v_0)} \subseteq \mathrm{span}\{s(u_0,v_0), p\}^\perp / \mathrm{span}\{s(u_0,v_0)\} \ ,$$

$$\mathcal{N}^{p^\perp}_{(u,v)}[s] = \{N + \mathrm{span}\{\tilde{s}\} \mid N \perp \tilde{s}, \tilde{s}_u, \tilde{s}_v, p\} \tag{4.2}$$

for any choice of lift $\tilde{s}$. This normal bundle has 1-dimensional fibers.

Remark 4.1. Again considering the $\mathbb{R}^{4,2}$ Lie sphere geometry setting, when $s = s(u,v)$ is not perpendicular to p, then s gives a sphere congruence with spheres of non-zero radius in resulting spaceforms. Assuming

$$||s_u||^2 > 0 \ , \quad ||s_v||^2 > 0 \ \text{ and } \ s_u \perp s_v \ ,$$

then

$$\mathrm{span}\{s, s_u, s_v\}^\perp \cap \mathrm{span}\{p\}^\perp$$

has signature $(+,-)$ and there exist precisely two independent vectors g, $\hat{g}$ (up to scalar factors) so that

$$g, \hat{g} \in \mathrm{span}\{s, s_u, s_v\}^\perp \cap \mathrm{span}\{p\}^\perp \cap L^5 \ ,$$

since $V := \mathrm{span}\{s, s_u, s_v\}$ has signature $(+,+,0)$, so $V^\perp$ has signature $(+,-,0)$, and then $s \not\perp p$ implies $V^\perp \cap \mathrm{span}\{p\}^\perp$ has signature $(+,-)$.

Thus, when $s \not\perp p$, $\mathcal{N}_{(u_0,v_0)}[s]$ contains two distinct lines (projectivized null planes) $\text{span}\{s,g\}$ and $\text{span}\{s,\hat{g}\}$ in the quadric PL^5, where $g,\hat{g} \in \text{span}\{p\}^\perp$. Each of these two families of lines envelops s. Then the normal bundle of the projection of s to Möbius geometry $p^\perp$ is

$$\text{Proj}_{p^\perp}(\mathcal{N}_{(u,v)}[s]) = \text{span}\{g,\hat{g}\} \ . \tag{4.3}$$

Now the conditions for s to be isothermic with respect to the coordinates u, v can be written in terms of g and $\hat{g}$ as:

(1) The first fundamental is conformal, i.e.

$$(s_u,s_u) = (s_v,s_v) \ \text{ and } \ (s_u,s_v) = 0 \ .$$

(2) The second fundamental form is diagonal, i.e.

$$(s_{uv},s) = (s_{uv},g) = (s_{uv},\hat{g}) = 0 \ .$$

(Should $s \perp p$, the directions of g and $\hat{g}$ would coincide, and in fact would become the direction of s itself.)

4.3. **Isothermic sphere congruences and Ω surfaces.** In the following discussion, any choice of 3-dimensional spaceform M^3 will suffice, and we will consider a surface x in M^3 with unit normal field n in T_xM^3 that lifts to a Legendre immersion Λ as in Section 3.5. We can obtain $\mathbb{R}^3$ by taking

$$p = \begin{pmatrix} 0\\0\\0\\0\\0\\1 \end{pmatrix} , \quad q = \begin{pmatrix} 1\\-1\\0\\0\\0\\0 \end{pmatrix} ,$$

and then the lifts of x and n will be

$$X = \begin{pmatrix} \frac{1}{2}(1+|x|^2)\\ \frac{1}{2}(1-|x|^2)\\ x^t\\ 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} x\cdot n\\ -x\cdot n\\ n^t\\ 1 \end{pmatrix}$$

in $\mathbb{R}^{4,2}$, respectively. We could obtain $\mathbb{S}^3$ by taking

$$p = \begin{pmatrix} 0\\0\\0\\0\\0\\1 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 1\\0\\0\\0\\0\\0 \end{pmatrix} ,$$

and then the lifts of x and n would be

$$X = \begin{pmatrix} 1\\ x^t\\ 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0\\ n^t\\ 1 \end{pmatrix} .$$

We could obtain $\mathbb{H}^3$ by taking

$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} ,$$

and then the lifts of x and n would be

$$X = \begin{pmatrix} x^t \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} n^t \\ 0 \\ 1 \end{pmatrix} .$$

However, to be explicit, let us take the case of $\mathbb{R}^3$.

Assume that x has coordinates u, v, and has first and second fundamental form terms g_{ij} and b_{ij}, respectively, with respect to those coordinates. Assume also that x is an immersion, so

$$g_{11}g_{22} - g_{12}^2 > 0 ,$$

and that x is umbilic free, so the principle curvatures k_1 and k_2 are not equal:

$$k_1 \neq k_2 .$$

Let $b = b(u, v)$ be a free real-valued function. We define

$$s = bX + N ,$$

which is a generic sphere congruence enveloped by span$\{X, N\}$. (See Remark 3.24.) Since changing s by a scalar factor will not change the resulting sphere congruences, we can also take the case "$b = \infty$" to be

$$s = X .$$

Lemma 4.2. *If there exists an isothermic sphere congruence s with isothermic coordinates u, v, then the u, v are curvature line coordinates for x.*

Proof. The trick for this proof is to consider the three fundamental forms I, II, III for the isothermic sphere congruence s of X, with respect to the normal X of s. What was written in Section 2.6 applies to s with normal X as well. By assumption $I = (ds, ds)$ and $II = -(dX, ds)$ are both diagonal, so by Section 2.6, $III = (dX, dX)$ is also diagonal. This completes the proof. □

4.4. **The first fundamental form for an isothermic sphere congruence s.** Now assume that s is an isothermic sphere congruence with isothermic coordinates u, v, and hence $g_{12} = b_{12} = 0$, and $b_{11} = k_1 g_{11}$ and $b_{22} = k_2 g_{22}$, by Lemma 4.2.

A computation gives

$$(ds, ds) = (b - k_1)^2 g_{11} du^2 + (b - k_2)^2 g_{22} dv^2 . \tag{4.4}$$

Note that this metric (4.4) would automatically be conformal if $g_{11} = g_{22}$ and $b = \frac{1}{2}(k_1 + k_2)$ were the mean curvature. Also, in the case $g_{11} = g_{22}$ and "$b = \infty$", i.e. $s = X$, then $(ds, ds) = g_{11}(du^2 + dv^2)$ is conformal as well.

Note also that this metric (4.4) cannot be conformal if b equals one of k_1 or k_2. (Since we are avoiding umbilic points, we have $k_1 \neq k_2$.) Thus

$$b \neq k_1 \quad \text{and} \quad b \neq k_2 .$$

We now consider the case when g_{11} is not equal to g_{22}. By assumption, the metric (4.4) is conformal. Because $(s_u, s_u) = (s_v, s_v)$, we have

$$b = \frac{k_1\sqrt{g_{11}} \mp k_2\sqrt{g_{22}}}{\sqrt{g_{11}} \mp \sqrt{g_{22}}} \ , \tag{4.5}$$

$$b - k_1 = \pm(k_1 - k_2)\frac{\sqrt{g_{22}}}{\sqrt{g_{11}} \mp \sqrt{g_{22}}} \ ,$$

$$b - k_2 = (k_1 - k_2)\frac{\sqrt{g_{11}}}{\sqrt{g_{11}} \mp \sqrt{g_{22}}} \ , \quad \frac{b - k_1}{b - k_2} = \pm\frac{\sqrt{g_{22}}}{\sqrt{g_{11}}} \ ,$$

$$(b - k_1)(b - k_2) = \pm(k_1 - k_2)^2\frac{\sqrt{g_{11}}\sqrt{g_{22}}}{(\sqrt{g_{11}} \mp \sqrt{g_{22}})^2} \ .$$

Using the Codazzi equations (2.22), we can compute that

$$\frac{b_u}{k_1 - b} = (\log(\sqrt{g_{11}} \mp \sqrt{g_{22}}))_u \mp \frac{\sqrt{g_{11}}}{\sqrt{g_{22}}}\frac{k_{1,u}}{k_1 - k_2} \ ,$$

$$\frac{b_v}{k_2 - b} = (\log(\sqrt{g_{11}} \mp \sqrt{g_{22}}))_v \pm \frac{\sqrt{g_{22}}}{\sqrt{g_{11}}}\frac{k_{2,v}}{k_1 - k_2} \ .$$

So

$$\left(\frac{b_u}{k_1 - b}\right)_v - \left(\frac{b_v}{k_2 - b}\right)_u = \mp\left(\left(\frac{\sqrt{g_{11}}}{\sqrt{g_{22}}}\frac{k_{1,u}}{k_1 - k_2}\right)_v + \left(\frac{\sqrt{g_{22}}}{\sqrt{g_{11}}}\frac{k_{2,v}}{k_1 - k_2}\right)_u\right) \ . \tag{4.6}$$

4.5. **The second fundamental form for s (case of b neither 0 nor ∞).** By isothermicity, (s_{uv}, Y) must be zero for any choice of normal field Y in $\mathcal{N}[s]$. The (projectively 2-dimensional) normal bundle $\mathcal{N}[s]$ to $[s]$ is as in (4.1). We have $X + \text{span}\{s\} \in \mathcal{N}[s]$, and

$$(ds, dX) = b(dX, dX) + (dX, dN) = (b - k_1)g_{11}du^2 + (b - k_2)g_{22}dv^2 \ ,$$

which is diagonal with respect to u, v. However, because the normal bundle is 2-dimensional, this does not yet tell us the second fundamental form of s is diagonal, and we need the following more general argument.

For $Y + \text{span}\{s\}$ to be in $\mathcal{N}[s]$, we need that $(Y, s) = (Y, s_u) = (Y, s_v) = 0$, i.e. (using $N_u = -k_1X_u$ and $N_v = -k_2X_v$)

$$(Y, X) = -b^{-1}(Y, N) \ ,$$

$$(Y, X_u) = \frac{b_u}{b(b - k_1)}(Y, N) \ , \tag{4.7}$$

$$(Y, X_v) = \frac{b_v}{b(b - k_2)}(Y, N) \ . \tag{4.8}$$

Since the case of $Y + \text{span}\{s\} = X + \text{span}\{s\}$ was already dealt with in the previous paragraph, we may now assume that $Y + \text{span}\{s\} \in \mathcal{N}[s]$ satisfies

$$Y \notin \text{span}\{X, N\} \ .$$

Then, if Y were perpendicular to N, we would also have $Y \perp X$, and noting that $\text{span}\{s_u, s_v\}$ is spacelike, we would have $Y \in \text{span}\{X, N, s_u, s_v\}^\perp = \text{span}\{X, N\}$, a contradiction. Therefore

$$(Y, N) \neq 0 \ .$$

Then

$$(Y, s_{uv}) = (Y, b_{uv}X + b_u X_v + b_v X_u + bX_{uv} + N_{uv}) = \alpha \cdot (Y, N) \ , \tag{4.9}$$

where the scalar α in (4.9) is zero if and only if

$$\left(b_v - (b-k_2)\frac{k_{1,v}}{k_1-k_2}\right)\frac{b_u}{b-k_1} + \left(b_u - (b-k_1)\frac{k_{2,u}}{k_2-k_1}\right)\frac{b_v}{b-k_2} = b_{uv} \ , \tag{4.10}$$

which can be seen using that $N_u = -k_1 X_u$ and $N_v = -k_2 X_v$ imply

$$N_{uv} = -\tfrac{1}{2}k_{1,v}X_u - \tfrac{1}{2}k_{2,u}X_v - \tfrac{1}{2}(k_1+k_2)X_{uv} \ ,$$

and that

$$X_{uv} = \tfrac{(g_{11})_v}{2g_{11}}X_u + \tfrac{(g_{22})_u}{2g_{22}}X_v = k_{1,v}(k_2-k_1)^{-1}X_u + k_{2,u}(k_1-k_2)^{-1}X_v \ ,$$

coming from equations like those in (2.15), (2.16) and (2.22).

We now rewrite Equation (4.10) in a cleaner form:

Lemma 4.3. *Equation* (4.10) *is equivalent to*

$$d\left(\frac{b_u}{k_1-b}du + \frac{b_v}{k_2-b}dv\right) = 0 \ . \tag{4.11}$$

Proof. Noting that

$$\frac{1}{k_1-k_2}\frac{b-k_2}{b-k_1} + \frac{1}{k_2-k_1}\frac{b-k_1}{b-k_2} = \frac{1}{b-k_1} + \frac{1}{b-k_2} \ ,$$

Equation (4.10) becomes

$$\left(b_v\frac{b-k_2}{k_1-k_2} - (b-k_2)\frac{k_{1,v}}{k_1-k_2}\right)\frac{b_u}{b-k_1} + \left(b_u\frac{b-k_1}{k_2-k_1} - \right.$$

$$\left.(b-k_1)\frac{k_{2,u}}{k_2-k_1}\right)\frac{b_v}{b-k_2} = b_{uv} \ ,$$

so

$$\frac{1}{k_1-k_2}\frac{k_2-b}{k_1-b}(b_v - k_{1,v})b_u - \frac{1}{k_1-k_2}\frac{k_1-b}{k_2-b}(b_u - k_{2,u})b_v = b_{uv} \ ,$$

so

$$\frac{b_{uv}}{k_1-b} + (b-k_1)_v\frac{b_u}{(k_1-b)^2} = \frac{b_{uv}}{k_2-b} + (b-k_2)_u\frac{b_v}{(k_2-b)^2} \ ,$$

which is equivalent to

$$\left(\frac{b_u}{k_1-b}\right)_v = \left(\frac{b_v}{k_2-b}\right)_u \ .$$

This last equation proves the lemma. □

We have now proven the following:

Lemma 4.4. *$s = bX + N$, with $b \in \mathbb{R} \setminus \{0\}$, is an isothermic sphere congruence enveloped by span$\{X, N\}$ if and only if b is as in* (4.5) *and is a solution to Equations* (4.10) *and* (4.11)*.*

Lemma 4.3 implies that both sides of (4.6) being equal to zero is the condition for s to be an isothermic sphere congruence. This provides Demoulin's equation:

$$\mathcal{D} := \left(\frac{\sqrt{g_{11}}}{\sqrt{g_{22}}}\frac{k_{1,u}}{k_1-k_2}\right)_v + \left(\frac{\sqrt{g_{22}}}{\sqrt{g_{11}}}\frac{k_{2,v}}{k_1-k_2}\right)_u = 0 \ .$$

Because of the $\mp$ freedom in choosing b in (4.5), we have the following corollary.

Corollary 4.5. *Whenever we have one isothermic sphere congruence $bX + N$ with $b \in \mathbb{R} \setminus \{0\}$ for a non-umbilic immersion x, then there is a second isothermic sphere congruence. We call these two isothermic sphere congruences an Ω* pair.

4.6. **Definition of Ω surfaces.** Although we have only considered the case $b \in \mathbb{R} \setminus \{0\}$ so far, the following definition includes the cases $b = 0$ and $b = \infty$ as well.

Definition 4.6. *The Legendre immersion $\Lambda = span\{X, N\}$ is an Ω* surface *if there exists an isothermic sphere congruence s enveloped by Λ (congruences with principal curvature spheres are excluded).*

Since span$\{X, N\}$ is determined by x alone, we can also refer to x itself as an Ω surface.

Remark 4.7. Suppose a surface x lifts to a Legendre immersion $\Lambda = \text{span}\{X, N\}$ enveloping an isothermic sphere congruence s. Then the sphere congruence resulting from s is enveloped by another surface $\hat{x}$ (with lift $\hat{X}$ and corresponding Legendre immersion span$\{\hat{X}, s\}$). This defines a transformation between x and $\hat{x}$. This transformation takes curvature line coordinates of x to curvature lines coordinates of $\hat{x}$, so it is a *Ribaucour* transformation. (We referred to X and $\hat{X}$ as g and $\hat{g}$ in Remark 4.1.)

4.7. **The second fundamental form for s ($b = \infty$ case).** In this case, $s = X$, and the normal bundle $\mathcal{N}^{p^\perp}_{(u,v)}[s]$ with 1-dimensional fibres is as in Section 4.2, with fibres determined by the span of the lift N of n. Thus isothermicity of $s = X$ implies isothermicity of the x in $\mathbb{R}^3$. We also conclude, conversely, that the following lemma holds, first proven by Demoulin:

Lemma 4.8. *All isothermic surfaces are Ω surfaces.*

Like in Lemma 4.4 and Corollary 4.5, the only other possible isothermic sphere congruence is given by the mean curvature $b = \frac{1}{2}(k_1 + k_2) = H_0$. When we take that other choice $b = H_0$, and when $H_0 \neq 0$, we do indeed have this second isothermic sphere congruence, by the following Lemma 4.9. Thus Corollary 4.5 holds when $b = \infty$, i.e. $s = X$, and $H_0 \neq 0$, as well.

Lemma 4.9. *When $g_{11} = g_{22}$ and $b = \frac{1}{2}(k_1 + k_2) = H_0 \neq 0$, Demoulin's equation holds, and becomes*

$$k_{1,v}H_{0,u} - k_{2,u}H_{0,v} = H_{0,uv}(k_1 - k_2) \ . \tag{4.12}$$

Proof. The Codazzi equations for curvature line coordinates are as in (2.22).

$$k_{1,uv} = \frac{g_{11,uv}}{2g_{11}}(k_2 - k_1) - \frac{g_{11,u}g_{11,v}}{2g_{11}^2}(k_2 - k_1) + \frac{g_{11,v}}{2g_{11}}(k_{2,u} - k_{1,u}) =$$

$$\frac{g_{11,uv}}{2g_{11}}(k_2 - k_1) - \frac{g_{11,u}g_{11,v}}{2g_{11}^2}(k_2 - k_1) + \frac{k_{1,v}}{k_2 - k_1}(k_{2,u} - k_{1,u}) \ ,$$

$$k_{2,uv} = \frac{g_{22,uv}}{2g_{22}}(k_1 - k_2) - \frac{g_{22,u}g_{22,v}}{2g_{22}^2}(k_1 - k_2) + \frac{k_{2,u}}{k_1 - k_2}(k_{1,v} - k_{2,v}) \ .$$

Now,

$$\tfrac{-1}{2}(k_{1,uv} + k_{2,uv})(k_2 - k_1)^2 = \tfrac{-1}{2}(k_2 - k_1)(-k_{2,u}(k_{1,v} - k_{2,v}) + k_{1,v}(k_{2,u} - k_{1,u})) \ ,$$

and this is exactly the same equation as (4.12), and confirms Demoulin's equation. □

4.8. **The second fundamental form for s ($b = 0$ case).** Because we have divided by b at some places in the above computations, we deal with the only remaining case $b = 0$ here separately. This is the case that

$$s = N$$

and

$$k_1^2 g_{11} = k_2^2 g_{22} \ .$$

(When this property is satisfied, we can call the surface a *Laguerre isothermic surface.*) If k_1 were zero, then the equation immediately above implies k_2 is also zero, so we would have umbilic points, which have been excluded, allowing us to conclude that

$$k_1 \neq k_2 \ , \quad K := k_1 k_2 \neq 0 \ .$$

The conditions for $Y + \text{span}\{s\} \in \mathcal{N}[s]$ are

$$(Y, N) = (Y, N_u) = (Y, N_v) = 0 \ .$$

Then

$$(Y, s_{uv}) = (Y, N_{uv}) = (Y, -\tfrac{1}{2}k_{1,v}X_u - \tfrac{1}{2}k_{2,u}X_v - \tfrac{1}{2}(k_1 + k_2)X_{uv}) =$$

$$\left(Y, \frac{k_{1,v}k_2}{k_1 - k_2}X_u + \frac{k_{2,u}k_1}{k_2 - k_1}X_v\right) = \left(Y, \frac{k_{1,v}k_2}{k_1(k_2 - k_1)}N_u + \frac{k_{2,u}k_1}{k_2(k_1 - k_2)}N_v\right) = 0 \ ,$$

and thus diagonality of the second fundamental form is automatic.

Two possibilities occur:

(1) $H_0 = 0$ and $g_{11} = g_{22}$, and $b = \infty$ gives the second isothermic sphere congruence (i.e. the point sphere congruence).
(2) $H_0 \neq 0$, and the other b as in (4.5) is $KH_0^{-1} \in \mathbb{R} \setminus \{0\}$.

One special case of this second possibility (2) is that x is the parallel surface at distance t of an isothermically parametrized minimal surface, and we find that K/H_0 is then constant, and

$$t = H_0/K \ , \quad \text{and} \quad k_1^2 g_{11} = k_2^2 g_{22} \ .$$

So in this case, the other choice of $b = K/H_0$ in (4.5) is constant, and then clearly, via (4.6), Demoulin's equation holds.

4.9. **Demoulin's equation in general spaceforms M^3.**

Lemma 4.10. *Demoulin's equation is independent of choice of spaceform.*

Proof. In the case of $M_0 = \mathbb{R}^3$, Demoulin's equation is $\mathcal{D} = 0$. When changing to the spaceform M_κ for general κ, we have

$$\hat{k}_j = t \cdot k_j + 2\kappa(x \cdot n_0) \ , \quad \hat{g}_{11} = t^{-2} g_{11} \ , \quad \hat{g}_{22} = t^{-2} g_{22} \ ,$$

with

$$t = 1 + \kappa |x|^2$$

(see the proof of Proposition 2.28). So

$$\left(\frac{\sqrt{\hat{g}_{11}}}{\sqrt{\hat{g}_{22}}} \frac{\hat{k}_{1,u}}{\hat{k}_1 - \hat{k}_2} \right)_v + \left(\frac{\sqrt{\hat{g}_{22}}}{\sqrt{\hat{g}_{11}}} \frac{\hat{k}_{2,v}}{\hat{k}_1 - \hat{k}_2} \right)_u =$$

$$\left(\frac{\sqrt{g_{11}}}{\sqrt{g_{22}}} \frac{2\kappa(x \cdot x_u) k_1 + 2\kappa(x \cdot n_{0,u}) + t k_{1,u}}{t(k_1 - k_2)} \right)_v +$$

$$\left(\frac{\sqrt{g_{22}}}{\sqrt{g_{11}}} \frac{2\kappa(x \cdot x_v) k_2 + 2\kappa(x \cdot n_{0,v}) + t k_{2,v}}{t(k_1 - k_2)} \right)_u =$$

$$\left(\frac{\sqrt{g_{11}}}{\sqrt{g_{22}}} \frac{0 + t k_{1,u}}{t(k_1 - k_2)} \right)_v + \left(\frac{\sqrt{g_{22}}}{\sqrt{g_{11}}} \frac{0 + t k_{2,v}}{t(k_1 - k_2)} \right)_u = \mathcal{D} \ .$$

So Demoulin's equation holds for $M_0 = \mathbb{R}^3$ if and only if it holds for any M_κ. □

Similarly, Equation (4.4) with b as in Equation (4.5) is invariant under choice of spaceform, so we also have the following lemma.

Lemma 4.11. *With x and n given by any choice of spaceform, and with κ_j and g_{ii} determined by that spaceform, the form of b for isothermic $s = bX + N$ in Equation* (4.5) *remains valid.*

Example 4.12. With Lemma 4.11 now available to us, we can check that flat fronts in $\mathbb{H}^3$ are Ω. With φ as in Section 3.10, and setting $C := \cosh \varphi$ and $S := \sinh \varphi$, we have

$$g_{11} = C^2 \ , \quad g_{22} = S^2 \ , \quad k_1 = S/C \ , \quad k_2 = C/S \ .$$

$$\left(\frac{C}{S} \frac{(S/C)_u}{(S/C) - (C/S)} \right)_v + \left(\frac{S}{C} \frac{(C/S)_v}{(S/C) - (C/S)} \right)_u = 0 \ ,$$

so Demoulin's equation holds.

Furthermore, by Lemma 4.11,

$$b = \frac{(S/C)C \mp (C/S)S}{C \mp S} = \pm 1 \ ,$$

so

$$s = N \pm X \ ,$$

and these are exactly the two hyperbolic Gauss maps (so the two Gauss maps lie in the two sphere congruences, respectively). This means that the isothermic sphere congruences are the two horosphere congruences, and the two hyperbolic Gauss maps are conformal maps, i.e. holomorphic.

We have now shown that:

Lemma 4.13. *Flat fronts are Ω surfaces.*

Note that flat fronts are not isothermic in general, because

$$\left(\log \frac{\cosh^2 \varphi}{\sinh^2 \varphi}\right)_{uv} \neq 0$$

in general, so (see Section 2.5):

Corollary 4.14. *Not all Ω surfaces x are isothermic.*

4.10. **Harmonic separation, Moutard lifts of isothermic sphere congruences.** Now we consider the notion of harmonic separation. Four vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ are said to be *harmonically separated* if their cross ratio is -1, i.e.

$$\mathrm{cr}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = -1 \; .$$

Lemma 4.15. *The two isothermic sphere congruences of an Ω surface separate the curvature sphere congruences harmonically.*

Proof. Any sphere congruence, as at the beginning of Section 4.3, is of the form

$$bX + N \; .$$

The four values of b under consideration, for one curvature sphere congruence, then one isothermic sphere congruence, then the other curvature sphere congruence, then the other isothermic sphere congruence, are

$$b_1 = k_1 \; , \quad b_2 = \frac{k_1\sqrt{g_{11}} + k_2\sqrt{g_{22}}}{\sqrt{g_{11}} + \sqrt{g_{22}}} \; , \quad b_3 = k_2 \; , \quad b_4 = \frac{k_1\sqrt{g_{11}} - k_2\sqrt{g_{22}}}{\sqrt{g_{11}} - \sqrt{g_{22}}} \; .$$

Then the cross ratio is

$$(b_2 - b_1)(b_3 - b_2)^{-1}(b_4 - b_3)(b_1 - b_4)^{-1} =$$
$$\left(-k_1 + \frac{k_1\sqrt{g_{11}} + k_2\sqrt{g_{22}}}{\sqrt{g_{11}} + \sqrt{g_{22}}}\right)\left(k_2 - \frac{k_1\sqrt{g_{11}} + k_2\sqrt{g_{22}}}{\sqrt{g_{11}} + \sqrt{g_{22}}}\right)^{-1} \cdot$$
$$\left(\frac{k_1\sqrt{g_{11}} - k_2\sqrt{g_{22}}}{\sqrt{g_{11}} - \sqrt{g_{22}}} - k_2\right)\left(k_1 - \frac{k_1\sqrt{g_{11}} - k_2\sqrt{g_{22}}}{\sqrt{g_{11}} - \sqrt{g_{22}}}\right)^{-1} = -1 \; .$$

□

The next technical lemma is for the purpose of establishing Corollary 4.17.

Lemma 4.16. *$cr(K_1, aK_1 + bK_2, K_2, cK_1 - dK_2) = -1$ implies the 2-dimensional vectors (a, b) and (c, d) are parallel.*

Proof.

$$-1 = \mathrm{cr}(K_1, aK_1 + bK_2, K_2, cK_1 - dK_2) =$$
$$\mathrm{cr}(K_1, K_1 + (b/a)K_2, K_2, K_1 + (-d/c)K_2) =$$
$$\mathrm{cr}(0, b/a, \infty, -d/c) = -bc/(ad) \; .$$

□

Corollary 4.17. *If K_1, K_2 are the principal curvature sphere congruences, and s_1, s_2 are the isothermic sphere congruences, then we can choose the lifts s_j so that*

$$s_1 = aK_1 + bK_2 \quad \textit{and} \quad s_2 = aK_1 - bK_2$$

The next remark will be used in the proof of Lemma 4.19.

Remark 4.18. Theorem 2.2 in [31] is the following: Suppose that $(,)$ is a scalar product on a finite-dimensional real vector space V and that U is a subspace of V.

(1) Then $U^{\perp\perp} = U$ and $\dim U + \dim U^\perp = \dim V$.
(2) The scalar product $(,)$ is nondegenerate on U if and only if it is nondegenerate on $U^\perp$. If the form is nondegenerate on U, then V is the direct sum of U and $U^\perp$.
(3) If V is the orthogonal direct sum of two spaces U and W, then the scalar product $(,)$ is nondegenerate on U and W, and $W = U^\perp$.

The next lemma provides us with a third alternative for determining Ω surfaces, other than just direct application of the definition of Ω surfaces or confirmation of Demoulin's equation. We can instead investigate whether there exists a section s having a Moutard lift.

In this lemma, we consider a section $s = s(u,v)$ of a Legendre immersion $\Lambda = \Lambda(u,v)$. We say that s is a *Moutard lift* if there exist coordinates u, v for which s satisfies

$$s_{uv} || s \ .$$

Nondegeneracy of $s = s(u,v)$ is assumed, which means that s contains no principal curvature spheres. (This is analogous to the assumption of immersedness in Lemma 2.19.) Like in Lemma 3.26, we can then check that neither s_u nor s_v lie in the null plane given by Λ, so in particular

$$(s_u, s_u) > 0 \quad \text{and} \quad (s_v, s_v) > 0 \ .$$

Lemma 4.19. *Let $s = s(u,v)$ be a nondegenerate section of a Legendre immersion $\Lambda(u,v)$. Then s can be scaled to a Moutard lift if and only if it is isothermic.*

Proof. To prove one direction: Assume s is a Moutard lift, so $s_{uv} = \alpha s$ for some scalar α. Then

$$(N_u, s_v) = -(N, s_{uv}) = -\alpha(N, s) = 0$$

for all $N \in \mathcal{N}[s]$. In particular,

$$(s_u, s_v) = 0 \ .$$

This implies we have curvature line coordinates u, v for s.

Then $(s_u, s_u)_v = 2(s_u, s_{uv}) = 0$ because s_{uv} is parallel to s, which implies (s_u, s_u) is independent of v. Similarly, (s_v, s_v) is independent of u. Thus we can scale the diagonal entries of the first fundamental form, by a change of coordinates of the form $u \to \tilde{u}(u)$ and $v \to \tilde{v}(v)$, so that they are equal to each other. In fact, such a change of coordinates can be chosen so that the metric for s becomes $du^2 + dv^2$ (or the identity in matrix form).

To prove the opposite direction: Assume s is isothermic with isothermic coordinates u, v. We can rescale s so that $|ds|^2 = du^2 + dv^2$ without affecting the Moutard equation $s_{uv} = \alpha s$, although the scalar α will change. Taking the envelopes g, $\hat{g}$ as in Remark 4.1, $g, \hat{g} \perp s, ds$. The assumption that we have curvature line coordinates implies, using Remark 4.18,

$$s_{uv} \in (\text{span}\{s, g, \hat{g}\})^\perp = \text{span}\{s, s_u, s_v\} \ .$$

Now

$$0 = \tfrac{1}{2}(1)_v = \tfrac{1}{2}(|s_u|^2)_v = (s_u, s_{uv})$$

implies $s_{uv} \perp s_u$. Similarly, $s_{uv} \perp s_v$. So

$$s_{uv} \in (\text{span}\{s, s_u, s_v\})^\perp = \text{span}\{s, g, \hat{g}\} .$$

Therefore s_{uv} is parallel to s. □

Note that, in Lemma 4.19, the coordinates u, v for which the Moutard equation $s_{uv} = \alpha s$ holds are the isothermic coordinates.

Remark 4.20. Regarding the second half of the above proof, note that once

$$s_{uv} = a s_u + b s_v + c s \tag{4.13}$$

holds for one particular choice of lift s, it holds again for any choice αs (although the coefficients a, b and c will change). Then, as we already know

$$s_{uv} \in \text{span}\{s_u, s_v, s\} ,$$

we have the following way to test whether s is an isothermic sphere congruence: We are given that there exist a, b and c so that (4.13) holds. Now, if αs is the actual Moutard lift, so $(\alpha s)_{uv} || s$, it follows that $a = -(\log \alpha)_v$ and $b = -(\log \alpha)_u$. Thus $a_u = b_v$ is the test equation for s being isothermic. Then, once that holds, we find that

$$\alpha = e^{-\int a dv} = e^{-\int b du} .$$

Before stating more results on Moutard lifts, we prepare the next two lemmas. Regarding the next lemma, see also Lemma 3.26.

Lemma 4.21. *Let u, v be coordinates for a Legendre immersion $\Lambda = \Lambda(u, v)$ that are curvature line coordinates for the projection of Λ to a 3-dimensional spaceform. If K_1, K_2 are the principal curvature spheres with respect to u, v, respectively, then*

$$K_{1,u}, K_{2,v} \in \mathit{span}\{K_1, K_2\} .$$

Proof. $K_1 = r(N + k_1 X)$ for some scalar function r, thus

$$K_{1,u} = r_u(N + k_1 X) + r(N_u + k_{1,u} X + k_1 X_u) =$$

$$r_u(N + k_1 X) + r k_{1,u} X \in \text{span}\{X, N\} = \text{span}\{K_1, K_2\} .$$

The argument is similar for $K_{2,v}$. □

As we saw in Section 3.8,

$$\text{span}\{K_1, K_{1,v}, K_{1,vv}\} \perp \text{span}\{K_2, K_{2,u}, K_{2,uu}\} ,$$

and thus we have:

Lemma 4.22. *$\mathit{span}\{K_{1,v}\}$, $\mathit{span}\{K_{2,u}\}$, $\mathit{span}\{K_1, K_2\}$ are all independent vector subspaces.*

Proof. It suffices to note that K_1, $K_{1,v}$, K_2, $K_{2,u}$ are all perpendicular, and $|K_{1,v}|^2 > 0$, $|K_{2,u}|^2 > 0$. (These last two inequalities were seen in Section 3.8.) □

The next lemma also shows us that, generically, existence of one isothermic sphere congruence implies existence of a second one. This is something we also saw in Corollary 4.5, but now we phrase the result in terms of Moutard lifts.

Lemma 4.23. *If $aK_1 + bK_2$ is Moutard, then so is $aK_1 - bK_2$.*

Proof. By Lemma 4.21, there exist scalars e, f, g and h such that

$$K_{1,u} = eK_1 + fK_2 \ , \quad K_{2,v} = gK_1 + hK_2 \ .$$

By assumption, we have that $aK_1 + bK_2$ and $(aK_1 + bK_2)_{uv}$ are parallel, and we also have

$$\begin{aligned}(aK_1 + bK_2)_{uv} = (a_{uv} + a_v e + b_u g + ae_v + bg_u + afg + bge)K_1 + \\ (b_{uv} + a_v f + b_u h + af_v + bh_u + afh + bgf)K_2 + \\ (ae + a_u)K_{1,v} + (bh + b_v)K_{2,u} \ .\end{aligned}$$

Then Lemma 4.22 implies $ae + a_u = bh + b_v = 0$, so

$$\begin{aligned}(aK_1 + bK_2)||(aK_1 + bK_2)_{uv} = \\ ((bg)_u + g(af + be))K_1 + ((af)_v + f(ah + bg))K_2 \ ,\end{aligned}$$

which gives

$$b(bg)_u + b^2 ge = a(af)_v + a^2 fh \ . \tag{4.14}$$

Now, changing $aK_1 + bK_2$ to $aK_1 - bK_2$, we obtain

$$(aK_1 - bK_2)_{uv} = (-(bg)_u + g(af - be))K_1 + ((af)_v + f(ah - bg))K_2 \ ,$$

and we want this to be parallel to $aK_1 - bK_2$. For this, we need

$$(-b)(-bg)_u + (-b)^2 ge = a(af)_v + a^2 fh \ ,$$

which holds, as it is equivalent to (4.14). □

Remark 4.24. If we rescale K_1 and K_2 so that $a = b = 1$ in Lemma 4.23, then the argument in the proof of Lemma 4.23 shows $e = h = 0$, so

$$K_{1,u}||K_2 \quad \text{and} \quad K_{2,v}||K_1 \ .$$

4.11. **Flat connections and T-transforms.** With the wedge product for $\mathbb{R}^{4,2}$ as described in Remark 2.43, the following lemma can be proven in just the same way as Lemma 2.46 was.

Lemma 4.25. *For an isothermic sphere congruence s with isothermic coordinates u, v, $\Gamma^\lambda = d + \lambda\tau$ is flat for any choice of λ, where the retraction form τ of s is*

$$\tau := \frac{2}{(s_u, s_u)} s \wedge (-s_u du + s_v dv) \ .$$

Like in the proof of Lemma 2.46,

$$[\tau \wedge \tau] = d\tau = 0$$

here as well.

Remark 4.26. Note that τ is invariant of the choice of lift s (see also Remark 2.45).

Definition 4.27. *A* Calapso transform $T \in O_{4,2}$ *(also called a T-*transform*) is a solution of*

$$dT = T \cdot \lambda\tau \ . \tag{4.15}$$

Note that such a solution T exists because $d\tau = [\tau \wedge \tau] = 0$.

Remark 4.28. $\tau \in o_{4,2}$ implies $T \in O_{4,2}$ if the initial condition for (4.15) is chosen in $O_{4,2}$. Then

$$(\vec{v}, \vec{w}) = (T\vec{v}, T\vec{w})$$

for all $\vec{v}, \vec{w} \in \mathbb{R}^{4,2}$. Thus

$$(T\vec{v}, \vec{w}) = (T\vec{v}, T \cdot T^{-1}\vec{w}) = (\vec{v}, T^{-1}\vec{w}) \ .$$

Lemma 4.29. *If s is a Moutard lift, then Ts is also a Moutard lift.*

Proof. Note that $\tau s = 0$, and $(s \wedge s_v)s_u = 0$. Then

$$(Ts)_{uv} = (\lambda T(\tau(\partial_u))(s) + Ts_u)_v = (Ts_u)_v =$$
$$\lambda T(\tau(\partial_v))(s_u) + Ts_{uv} = Ts_{uv}$$

is parallel to Ts. □

Remark 4.30. In fact, in the proof of Lemma 4.29, we have seen that s and Ts even have the same factor function in the Moutard equation.

Corollary 4.31. *Ts is also isothermic.*

Proof. This follows from Lemma 4.19. □

Remark 4.32. Note that the retraction form for Ts becomes $T\tau T^{-1}$. seen by direct computation.

Remark 4.33. If, in addition to being isothermic, s is also a curvature sphere congruence for some Legendre immersion Λ, then Ts is a curvature sphere congruence for $T\Lambda$ as well.

4.12. **Retraction forms for pairs of isothermic sphere congruences.** We now notate the two isothermic sphere congruences of an Ω surface x (or rather, its lift to a Legendre immersion Λ in $\mathbb{R}^{4,2}$, an Ω surface in Lie sphere geometry) by $s^\pm$, with corresponding retraction forms

$$\tau^\pm = \frac{2}{(s_u^\pm, s_u^\pm)} s^\pm \wedge (-s_u^\pm du + s_v^\pm dv)$$

and corresponding solutions $T^\pm$ of

$$dT^\pm = T^\pm \cdot \lambda\tau^\pm \ ,$$

respectively. So

$$\Lambda = \text{span}\{s^+, s^-\} \ .$$

Lemma 4.34. *The connection*

$$\Gamma^{\lambda,t} := d + \lambda(t\tau^+ + (1-t)\tau^-)$$

is flat for any choices of t and λ.

Proof. The proof goes along the same lines as the proof of Lemma 4.25. We need

$$(t\tau^+(\partial_u) + (1-t)\tau^-(\partial_u))_v - (t\tau^+(\partial_v) + (1-t)\tau^-(\partial_v))_u = 0 \ ,$$

but this follows directly from Lemma 4.25 applied to τ^+ and τ^- separately. We also need

$$\{(t\tau^+(\partial_u) + (1-t)\tau^-(\partial_u))(t\tau^+(\partial_v) + (1-t)\tau^-(\partial_v)) -$$
$$(t\tau^+(\partial_v) + (1-t)\tau^-(\partial_v))(t\tau^+(\partial_u) + (1-t)\tau^-(\partial_u))\}Z = 0$$

for all $Z \in \mathbb{R}^{4,2}$. Lemma 4.25 implies it suffices to show

$$\tau^+(\partial_u)\tau^-(\partial_v) + \tau^-(\partial_u)\tau^+(\partial_v) - \tau^-(\partial_v)\tau^+(\partial_u) - \tau^+(\partial_v)\tau^-(\partial_u) = 0 \;,$$

which follows from a computation using

$$(s_u^+, s_v^-) = -(s^+, s_{uv}^-) = (s_v^+, s_u^-) \;.$$

(This computation simplifies somewhat when using Moutard lifts for $s^\pm$.) □

Lemma 4.35. *For the right choices of Moutard lifts $s^\pm$, and the right choices of coordinates u, v,*

$$\tau^+ - \tau^- = d(s^+ \wedge s^-) \;,$$

that is,

$$(\tau^+ - \tau^- - d(s^+ \wedge s^-))Z = 0$$

for all $Z \in \mathbb{R}^{4,2}$.

Proof. By Corollary 4.17, Lemma 4.23 and Remark 4.24, we can normalize the lifts of the principal curvature spheres K_1 and K_2 so that

$$s^+ = K_1 + K_2 \;, \quad s^- = K_1 - K_2 \tag{4.16}$$

are Moutard lifts of the isothermic sphere congruences, and then

$$(K_1)_u = \beta K_2 \quad \text{and} \quad (K_2)_v = \gamma K_1$$

for some functions β and γ, by Remark 4.24. As seen in the proof of Lemma 4.19, $||s_u^+||^2 = ||s_v^+||^2$ can be taken to be constant (i.e. independent of u and v), so we can rescale u, v so that $||s_u^\pm||^2 = ||s_v^\pm||^2 = 2$. Then

$$\begin{gathered}
\tau^+ - \tau^- - d(s^+ \wedge s^-) = \\
s^+ \wedge (-s_u^+ du + s_v^+ dv) - s^- \wedge (-s_u^- du + s_v^- dv) - \\
s^+ \wedge (s_u^- du + s_v^- dv) + s^- \wedge (s_u^+ du + s_v^+ dv) = \\
-s^+ \wedge ((s^+ + s^-)_u du + (s^- - s^+)_v dv) + \\
s^- \wedge ((s^+ + s^-)_u du + (s^+ - s^-)_v dv) = \\
-2(s^+ \wedge (K_{1,u} du - K_{2,v} dv) - s^- \wedge (K_{1,u} du + K_{2,v} dv)) = \\
-2(s^+ \wedge (\beta K_2 du - \gamma K_1 dv) - s^- \wedge (\beta K_2 du + \gamma K_1 dv)) = \\
-2\left(\beta(s^+ - s^-) \wedge K_2 du - \gamma(s^+ + s^-) \wedge K_1 dv\right) = \\
-2\left(\beta(2K_2) \wedge K_2 du - \gamma(2K_1) \wedge K_1 dv\right) = 0 \;.
\end{gathered}$$

□

By Equation (4.16), we also then have:

Corollary 4.36. *For the right choices of lifts $s^\pm$,*

$$s^+ \wedge s^- = -2K_1 \wedge K_2 \;.$$

Corollary 4.37. *For the right choices of Moutard lifts $s^\pm$, and the right choices of coordinates u, v,*

$$\tau^+ - \tau^- = -2d(K_1 \wedge K_2) \;.$$

4.13. **T-transforms of pairs of isothermic sphere congruences.** We now wish to show that, when

$$T^+(u_0, v_0) = T^-(u_0, v_0) \tag{4.17}$$

for some value (u_0, v_0), then, where the $T^\pm$ are defined as in (4.15) with τ replaced by $\tau^\pm$, respectively,

$$T^+(\text{span}\{s^+, s^-\}) = T^-(\text{span}\{s^+, s^-\}) = \text{span}\{T^+s^+, T^-s^-\} \ , \tag{4.18}$$

so that the two deformations, either by T^+ or by T^-, of the sphere congruences (sections of span$\{s^+, s^-\}$) give the same Legendre immersions. Thus this constitutes a 1-dimensional deformation of the original Legendre immersion. In particular, by Corollary 4.31, the deformed Legendre immersions are also Ω.

Lemma 4.38. *Under suitable initial conditions for $T^\pm$,*

$$T^+(1 - \tfrac{1}{2}\lambda s^+ \wedge s^-) = T^-(1 - \tfrac{1}{2}\lambda s^- \wedge s^+) =: \hat{T} \ , \tag{4.19}$$

$$T^+ = T^-(1 - \lambda s^- \wedge s^+) \ , \quad T^- = T^+(1 - \lambda s^+ \wedge s^-) \ . \tag{4.20}$$

Proof.

$$d(T^+(1 - \tfrac{1}{2}\lambda s^+ \wedge s^-)) = T^+(\lambda\tau^+(1 - \frac{1}{2}\lambda s^+ \wedge s^-) - \frac{1}{2}\lambda d(s^+ \wedge s^-)) =$$

$$T^+(1 - \tfrac{1}{2}\lambda s^+ \wedge s^-) \cdot \lambda(\tau^+ - \tfrac{1}{2}d(s^+ \wedge s^-)) \ .$$

A corresponding equation holds for τ^- as well. So

$$T^+(1 - \tfrac{1}{2}\lambda s^+ \wedge s^-)$$

and

$$T^-(1 - \tfrac{1}{2}\lambda s^- \wedge s^+)$$

satisfy the same differential equation, by Lemma 4.35. If we choose the initial conditions of $T^\pm$ appropriately, we have (4.19). Equation (4.20) is immediate from

$$(1 - \tfrac{1}{2}\lambda s^- \wedge s^+)^{-1} = 1 - \tfrac{1}{2}\lambda s^+ \wedge s^- \ .$$

□

If fact, from Equation (4.19), we have this corollary:

Corollary 4.39. *$T^+s^+ = T^-s^+$ and $T^+s^- = T^-s^-$.*

This corollary is actually a stronger statement than the relations in (4.18) we first set out to prove.

Lemma 4.40. *When $T^\pm \in O_{4,2}$, then also $\hat{T} \in O_{4,2}$.*

Proof. For $Z_1, Z_2 \in \mathbb{R}^{4,2}$,

$$((1 - \tfrac{1}{2}\lambda s^+ \wedge s^-)Z_1, (1 - \tfrac{1}{2}\lambda s^+ \wedge s^-)Z_2) =$$

$$(Z_1, Z_2) + \tfrac{1}{4}\lambda^2((s^+ \wedge s^-)Z_1, (s^+ \wedge s^-)Z_2) -$$

$$\tfrac{1}{2}\lambda(((s^+ \wedge s^-)Z_1, Z_2) + (Z_1, (s^+ \wedge s^-)Z_2)) = (Z_1, Z_2) \ .$$

So $1 - \frac{1}{2}\lambda s^+ \wedge s^- \in O_{4,2}$. Similarly, $1 - \frac{1}{2}\lambda s^- \wedge s^+ \in O_{4,2}$. □

Lemma 4.41. *span$\{T^+s^+, T^+s^-\}$ is a Legendre immersion.*

Proof. We need only check the contact condition now:

$$(d(T^+s^+), T^+s^-) = (T^+\lambda\tau^+s^+ + T^+ds^+, T^+s^-) =$$

$$(T^+ds^+, T^+s^-) = (ds^+, s^-) = 0 \ .$$

□

4.14. **The case of flat surfaces in $\mathbb{H}^3$.** In this section, we describe some properties of flat fronts in $\mathbb{H}^3$.

Theorem 4.42. *Suppose Λ is an Ω surface with isothermic sphere congruences s^+ and s^-, and that s^+ and s^- are each enveloped by a constant sphere $\tilde{\mathcal{S}}^+$ and $\tilde{\mathcal{S}}^-$, respectively. Then Λ projects to a flat surface in some $\mathbb{H}^3$. The converse also holds.*

Proof. The spheres $\tilde{\mathcal{S}}^+$ and $\tilde{\mathcal{S}}^-$ are given by vectors $\mathcal{S}^+$ and $\mathcal{S}^-$ in the light cone L^5 of $\mathbb{R}^{4,2}$. Then

$$(s^\pm, \mathcal{S}^\pm) = (s^\pm_u, \mathcal{S}^\pm) = (s^\pm_v, \mathcal{S}^\pm) = 0 \ .$$

Now we need to see that if we project to a correctly chosen $\mathbb{H}^3$, we can have $\mathcal{S}^+$ and $\mathcal{S}^-$ both giving a single sphere $\partial\mathbb{H}^3$ (with opposite orientations), and then we need to see that the resulting surface x is flat in that $\mathbb{H}^3$.

An $O_{4,2}$ isometric motion can take any $\mathcal{S}^+$ and $\mathcal{S}^-$ to any other two lightlike vectors $\hat{\mathcal{S}}^+$ and $\hat{\mathcal{S}}^-$ such that $(\mathcal{S}^+, \mathcal{S}^-) = (\hat{\mathcal{S}}^+, \hat{\mathcal{S}}^-)$, so we can take

$$\hat{\mathcal{S}}^+ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ 1/2 \end{pmatrix} \quad \text{and} \quad \hat{\mathcal{S}}^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (\mathcal{S}^+, \mathcal{S}^-) \\ -(\mathcal{S}^+, \mathcal{S}^-) \end{pmatrix} \ .$$

Then we can rescale $\hat{\mathcal{S}}^\pm$ as we please, as this will not change the spheres they represent. So without loss of generality,

$$\mathcal{S}^+ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \ , \quad \mathcal{S}^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \ . \tag{4.21}$$

We take

$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \ ,$$

so that $\{p\}^\perp$ has signature $(-,+,+,+,+)$, and then

$$q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

will give $\mathbb{H}^3$ (the same $\mathbb{H}^3$ as $M_\kappa = M_{-1}$ in (2.4), but with the fifth coordinate of M_{-1} in (2.4) shifted to the first coordinate here). Then $\mathcal{S}^\pm$ both represent $\partial\mathbb{H}^3$, i.e. the same ideal boundary sphere of $\mathbb{H}^3$, but with opposite orientations.

Now $\Lambda = \text{span}\{s^+, s^-\}$ gives the Legendre lift of the surface x in this $\mathbb{H}^3$, and (ds^+, ds^+) and (ds^-, ds^-) are conformally related. Also, $s^+ \perp \mathcal{S}^+$ and $s^- \perp \mathcal{S}^-$. We can rescale $s^\pm$ without affecting conformality of $(ds^\pm, ds^\pm)$, so without loss of generality

$$s^+ = \begin{pmatrix} s_0^+ \\ s_1^+ \\ s_2^+ \\ s_3^+ \\ 1 \\ 1 \end{pmatrix}, \quad s^- = \begin{pmatrix} s_0^- \\ s_1^- \\ s_2^- \\ s_3^- \\ 1 \\ -1 \end{pmatrix}. \tag{4.22}$$

The vector $(s_0^+, s_1^+, s_2^+, s_3^+)$ represents the hyperbolic Gauss map G, and the vector $(s_0^-, s_1^-, s_2^-, s_3^-)$ represents the other hyperbolic Gauss map G_*, where G and G_* are as in Section 2.22. Conformality of $(ds^\pm, ds^\pm)$ implies

$$-(s_{0,u}^\pm)^2 + (s_{1,u}^\pm)^2 + (s_{2,u}^\pm)^2 + (s_{3,u}^\pm)^2 = -(s_{0,v}^\pm)^2 + (s_{1,v}^\pm)^2 + (s_{2,v}^\pm)^2 + (s_{3,v}^\pm)^2 \,,$$

$$-s_{0,u}^\pm s_{0,v}^\pm + s_{1,u}^\pm s_{1,v}^\pm + s_{2,u}^\pm s_{2,v}^\pm + s_{3,u}^\pm s_{3,v}^\pm = 0 \,.$$

This means that G and G_* are holomorphic maps, and this implies that the surface x, when projected to the $\mathbb{H}^3$ above, is flat (see Section 2.22).

To show the converse, consider a flat surface x in the $\mathbb{H}^3$ determined by

$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

We have already established in Example 4.12 that the surface is Ω, and it remains only to note that, for $s^\pm = N \mp X$ as in Example 4.12, we have $(s^+, \mathcal{S}^+) = (s^-, \mathcal{S}^-) = 0$ with $\mathcal{S}^\pm$ as in (4.21). □

We can now define the notion of polynomial conserved quantities $P^\pm$ for the connections $\Gamma^\pm = d + \lambda\tau^\pm$, respectively, just like in Section 2.16. The next corollary employs notation defined in Lemma 4.34.

Corollary 4.43. *An Ω surface x is a flat surface in some $\mathbb{H}^3$ if and only if x has two order-zero (i.e. constant in λ) conserved quantities $\mathcal{S}^\pm \in L^5$ for $\Gamma^+ = \Gamma^{\lambda,1}$, $\Gamma^- = \Gamma^{\lambda,0}$, respectively.*

Remark 4.44. The $\mathcal{S}^\pm$ will be constant with respect to the coordinates u and v for the surface as well, and thus

$$\tau_+\mathcal{S}^+ = \tau_-\mathcal{S}^- = 0 \ ,$$

as we will see in the proof of Corollary 4.43 just below.

Proof. Suppose x is flat. Then there exist constants $\mathcal{S}^\pm \in L^5$ enveloping the isothermic sphere congruences $s^\pm$, by Theorem 4.42. So

$$d(T^\pm \cdot \mathcal{S}^\pm) = T^\pm \cdot \lambda\tau^\pm\mathcal{S}^\pm = T^\pm \cdot \lambda \cdot 0 \ . \tag{4.23}$$

Thus x has two order-zero conserved quantities $\mathcal{S}^\pm$, because

$$\Gamma^\pm\mathcal{S}^\pm = (d + \lambda\tau^\pm)\mathcal{S}^\pm = 0 \ .$$

Conversely, suppose $\mathcal{S}^\pm \in L^5$ are two order-zero conserved quantities, i.e.

$$\Gamma^\pm\mathcal{S}^\pm = (d + \lambda\tau^\pm)\mathcal{S}^\pm = 0 \ .$$

Because this is true for all real λ, it follows that $\mathcal{S}^\pm$ are constant and $\tau^\pm\mathcal{S}^\pm = 0$, and thus $\mathcal{S}^\pm \perp s^\pm, s^\pm_u, s^\pm_v$. Hence the constants $\mathcal{S}^\pm$ give fixed spheres that envelop $s^\pm$, respectively. Then Theorem 4.42 implies x is a flat surface. □

Theorem 4.45. *$T^\pm$-transforms of a flat surface in $\mathbb{H}^3$ are again flat in $\mathbb{H}^3$.*

Proof. As in Corollary 4.43, there exist constant conserved quantities $\mathcal{S}^\pm$. By (4.23) and the fact that $T^{\pm,\lambda}\mathcal{S}^\pm = \mathcal{S}^\pm$ at one point (because of (4.17)),

$$T^{\pm,\lambda}\mathcal{S}^\pm = \mathcal{S}^\pm$$

at all points, and for all values of λ. Then

$$0 = (s^\pm, \mathcal{S}^\pm) = (T^\pm s^\pm, T^\pm\mathcal{S}^\pm) = (T^\pm s^\pm, \mathcal{S}^\pm)$$

implies T^+s^+ and T^-s^- are each always enveloped by the same constant sphere given by $\mathcal{S}^\pm$, respectively. They are also isothermic, by Corollary 4.31. Then Corollary 4.43 implies all the deformed surfaces (the $T^\pm$-transforms, or equivalently $\hat{T}$-transforms) are flat in $\mathbb{H}^3$. □

4.15. **The gauging principle.** We consider a gauge theoretic approach here, which is useful when one studies discrete analogs of Ω surfaces, as we do in a subsequent text.

Let us first state the gauging principle in vague terms, without any specification of the spaces in which various objects lie: $P = P(\lambda)$ is a conserved quantity for a connection $\nabla = \nabla^\lambda$ if

$$\nabla P = 0 \ .$$

Then for any transformation g, we have the gauging $g\nabla g^{-1}gP = 0$, i.e.

$$(g\nabla g^{-1})(gP) = 0 \ .$$

This means that gP is a conserved quantity of $g\nabla g^{-1}$. We can refer to g simply as a "gauge".

We will illustrate this gauging principle here. Let K_j, $s^\pm$, $\tau^\pm$ be as in previous sections, with the lifts K_j chosen so that Equation (4.16) and Corollaries 4.36 and 4.37 hold, and u, v so that $||s_u||^2 = ||s_v||^2 = 2$.

Lemma 4.46. *$(K_1 \wedge K_2)^2 = 0$ and $d(K_1 \wedge K_2)(K_1 \wedge K_2) = 0$.*

Proof. This result follows from noting that

$$\text{Image}(K_1 \wedge K_2) \subset \text{span}\{K_1, K_2\} ,$$

and $K_1 \wedge K_2$ restricted to $\text{span}\{K_1, K_2\}$ is zero, and, for $A \in \mathbb{R}^{4,2}$ with $\hat{A} := (K_1 \wedge K_2)A \in \text{span}\{K_1, K_2\}$,

$$d(K_1 \wedge K_2)(K_1 \wedge K_2)(A) = (dK_1 \wedge K_2 + K_1 \wedge dK_2)(\hat{A}) = 0 .$$

□

We take, for $\lambda \in \mathbb{R}$,

$$g^\lambda := \exp(\lambda K_1 \wedge K_2) = 1 + \lambda K_1 \wedge K_2 + \sum_{n=2}^{\infty} \tfrac{1}{n!}\lambda^n (K_1 \wedge K_2)^n = 1 + \lambda K_1 \wedge K_2$$

as our gauge.

Lemma 4.47. $g^\lambda \circ d \circ (g^\lambda)^{-1} = d - \lambda d(K_1 \wedge K_2)$. *(”$\circ$” denotes composition.)*

Proof. We have

$$g^\lambda \circ d \circ (g^\lambda)^{-1} = d + g^\lambda d((g^\lambda)^{-1}) = d - (dg^\lambda)(g^\lambda)^{-1} =$$
$$d - \lambda d(K_1 \wedge K_2)(1 - \lambda K_1 \wedge K_2) ,$$

and the result follows from Lemma 4.46. □

Corollary 4.48. $g^\lambda \circ (d + \hat{\lambda}\tau^+) \circ (g^\lambda)^{-1} = d + (\hat{\lambda} + \lambda/2)\tau^+ - (\lambda/2)\tau^-$*, for* $\lambda, \hat{\lambda} \in \mathbb{R}$.

Proof. We have

$$g^\lambda \circ (d + \hat{\lambda}\tau^+) \circ (g^\lambda)^{-1} = d - \lambda d(K_1 \wedge K_2) + \hat{\lambda} g^\lambda \circ \tau^+ \circ (g^\lambda)^{-1} =$$
$$d + \tfrac{1}{2}\lambda(\tau^+ - \tau^-) + \hat{\lambda} g^\lambda \circ \tau^+ \circ (g^\lambda)^{-1} ,$$

by Lemma 4.46 and Corollary 4.37. Now,

$$g^\lambda \circ \tau^+ \circ (g^\lambda)^{-1} = (1 + \lambda K_1 \wedge K_2) \circ \tau^+ \circ (1 - \lambda K_1 \wedge K_2) =$$
$$(1 - \tfrac{\lambda}{2} s^+ \wedge s^-)((-(s^+, \cdot)s^+_u + (s^+_u, \cdot)s^+)du + ((s^+, \cdot)s^+_v - (s^+_v, \cdot)s^+)dv)(1 + \tfrac{\lambda}{2}\lambda s^+ \wedge s^-) =$$
$$(1 - \tfrac{\lambda}{2} s^+ \wedge s^-)((-(s^+, \cdot)s^+_u + (s^+_u, \cdot)s^+)du + ((s^+, \cdot)s^+_v - (s^+_v, \cdot)s^+)dv)(1) =$$
$$(1)((-(s^+, \cdot)s^+_u + (s^+_u, \cdot)s^+)du + (s^+, \cdot)s^+_v - (s^+_v, \cdot)s^+)dv)(1) = \tau^+ ,$$

so

$$g^\lambda \circ (d + \hat{\lambda}\tau^+) \circ (g^\lambda)^{-1} = d + \tfrac{1}{2}\lambda(\tau^+ - \tau^-) + \hat{\lambda}\tau^+ .$$

The result follows. □

Corollary 4.49. *The connections* $d + \lambda\tau^+$ *and* $d + \lambda\tau^-$ *are related via the gauge* g *by*

$$g^{-2\lambda} \circ (d + \lambda\tau^+) \circ (g^{-2\lambda})^{-1} = d + \lambda\tau^- .$$

Remark 4.50. In the case of flat surfaces in $\mathbb{H}^3$, for which we have lightlike constant conserved quantities $\mathcal{S}^\pm$ as in Corollary 4.43,

$$(d + \lambda\tau^+)\mathcal{S}^+ = 0$$

is gauge equivalent to

$$(d + \lambda\tau^-)P_+ = 0 ,$$

where, using Corollary 4.36,

$$P_+ := g^{-2\lambda}\mathcal{S}^+ = (1 - 2\lambda K_1 \wedge K_2)\mathcal{S}^+ = (1 + \lambda s^+ \wedge s^-)\mathcal{S}^+ = \mathcal{S}^+ - \lambda(s^-, \mathcal{S}^+)s^+ ,$$

since $(s^+, \mathcal{S}^+) = 0$. Similarly,

$$(d + \lambda\tau^-)\mathcal{S}^- = 0$$

is gauge equivalent to

$$(d + \lambda\tau^+)P_- = 0 \ ,$$

for some linear conserved quantity P_-. So $\mathcal{S}^+$, P_+, $\mathcal{S}^-$, P_- are all conserved quantities. Note that $\mathcal{S}^\pm$ are lightlike constant conserved quantities, while $P_\pm$ are linear conserved quantities.

Remark 4.51. If an Ω surface Λ has one timelike constant conserved quantity p for the connection $d + \lambda\tau$ coming from some isothermic sphere congruence $s \in \Lambda$, then s projects to an isothermic surface in the Möbius geometry given by $\{p\}^\perp \approx \mathbb{R}^{4,1}$, since $(d + \lambda\tau)p = 0$ implies $(s, p) = 0$.

Remark 4.52. Note that, by (4.19) and Corollary 4.36,

$$(g^{-\lambda})^{-1}T^+g^{-\lambda} = T^- \ ,$$

so $g^{-\lambda}$ acts as a gauge between T^+ and T^-.

In fact, T^+ and T^- themselves are gauge transformations of a different type, so we have interrelated gauge transformations, because $T^\pm$ gauge the trivial connection d to the $\Gamma^\pm = d + \lambda\tau^\pm$ connections, respectively, as follows:

$$T^\pm : (\mathbb{R}^{4,2}, \Gamma^\pm) \to (\mathbb{R}^{4,2}, d)$$

via

$$d(T^\pm y) = T^\pm dy + dT^\pm \cdot y = T^\pm(d + \lambda\tau^\pm)y = T^\pm\Gamma^\pm y \ .$$

4.16. **Guichard surfaces.** Guichard surfaces are Ω, as we are about to see in Lemma 4.54 below, and can be defined as those surfaces x in $\mathbb{R}^3$ that satisfy Calapso's equation

$$cg_{11}g_{22}(k_1 - k_2)^2 = g_{22} - \epsilon^2 g_{11} \ , \tag{4.24}$$

for some choice of curvature line coordinates u, v, where c is a constant, ϵ is 1 or i, and

$$g_{11} = x_u \cdot x_u \ , \quad g_{12} = x_u \cdot x_v = 0 \ , \quad g_{22} = x_v \cdot x_v \ ,$$

and k_1 and k_2 are the principal curvatures. (Equation (4.24) actually includes the case of isothermic surfaces as well, when taking $c = 0$ and $\epsilon = 1$.)

In fact, however, the notion of Guichard surfaces is sensible in Möbius geometry, so one need never reduce to a 3-dimensional spaceform.

Remark 4.53. It was shown by Calapso [30] that Equation (4.24) is equivalent to the existence of a Guichard dual, which is a surface with parallel curvature directions and principal curvatures k_i^* satisfying

$$\frac{1}{k_1k_2^*} + \frac{1}{k_2k_1^*} = \text{constant} \ .$$

Lemma 4.54. *Guichard surfaces are* Ω*.*

Proof. We take a Guichard surface and give it curvature line coordinates so that Calapso's equation (4.24) holds. We need to show that Calapso's equation implies Demoulin's equation

$$\left(\frac{\sqrt{g_{11}}}{\sqrt{g_{22}}}\frac{k_{1,u}}{k_1 - k_2}\right)_v + \epsilon^2\left(\frac{\sqrt{g_{22}}}{\sqrt{g_{11}}}\frac{k_{2,v}}{k_1 - k_2}\right)_u = 0$$

holds, for either $\epsilon = 1$ or $\epsilon = i$. The reason for also considering $\epsilon = i$ will be explained in Section 4.19. We take up just the case $\epsilon = 1$ here. Using the Codazzi equations (2.22), the left-hand side of Demoulin's equation becomes

$$\left(\frac{\sqrt{g_{11}}}{\sqrt{g_{22}}}\left((\log(k_1-k_2))_u+\tfrac{1}{2}(\log g_{22})_u\right)\right)_v-\left(\frac{\sqrt{g_{22}}}{\sqrt{g_{11}}}\left((\log(k_1-k_2))_v+\tfrac{1}{2}(\log g_{11})_v\right)\right)_u ,$$

which, using that the Calapso equation implies

$$\log(k_1-k_2)=\tfrac{1}{2}\log(g_{22}-g_{11})-\tfrac{1}{2}\log c-\tfrac{1}{2}\log g_{11}-\tfrac{1}{2}\log g_{22} ,$$

becomes

$$\tfrac{1}{2}\left(\frac{\sqrt{g_{11}}}{\sqrt{g_{22}}}(\log(g_{22}-g_{11}))_u\right)_v-\tfrac{1}{2}\left(\frac{\sqrt{g_{22}}}{\sqrt{g_{11}}}(\log(g_{22}-g_{11}))_v\right)_u+$$
$$\tfrac{-1}{2}\left(\frac{\sqrt{g_{11}}}{\sqrt{g_{22}}}(\log g_{11})_u\right)_v+\tfrac{1}{2}\left(\frac{\sqrt{g_{22}}}{\sqrt{g_{11}}}(\log g_{22})_v\right)_u=$$
$$\tfrac{1}{2}\left(\frac{\sqrt{g_{11}}}{\sqrt{g_{22}}}\left(\log\left(\frac{g_{22}}{g_{11}}-1\right)\right)_u\right)_v-\tfrac{1}{2}\left(\frac{\sqrt{g_{22}}}{\sqrt{g_{11}}}\left(\log\left(1-\frac{g_{11}}{g_{22}}\right)\right)_v\right)_u ,$$

and then a direct computation shows this is zero. □

Noting that tubular surfaces are defined as those surfaces for which one principal curvature is constant (see the upcoming Definition 4.64), we have the following lemma:

Lemma 4.55. *Nontubular non-CMC linear Weingarten surfaces are Guichard.*

Proof. By assumption, we have the linear Weingarten equation (see the definition of linear Weingarten surfaces at the beginning of Section 4.18)

$$\alpha K-2\beta H+\gamma=0 .$$

Because the surface is not CMC, α is not zero, so without loss of generality we can take $\alpha = 1$, and the linear Weingarten equation becomes

$$(k_1-\beta)(k_2-\beta)=\rho:=\beta^2-\gamma ,$$

and $\rho \neq 0$ because the surface is non-tubular. We give the surface curvature line coordinates u, v.

<u>The case $\rho > 0$</u>: Choose f so that

$$k_1-\beta=\sqrt{\rho}\tanh f ,\quad k_2-\beta=\sqrt{\rho}\coth f .$$

Using the Codazzi equations (2.22), we have

$$k_{1,v}=\sqrt{\rho}\frac{f_v}{\cosh^2 f}=\frac{g_{11,v}}{2g_{11}}(k_2-k_1)=\sqrt{\rho}(\coth f-\tanh f)\cdot\tfrac{1}{2}(\log g_{11})_v ,$$

implying

$$\frac{f_v}{\cosh f}=\frac{(\log g_{11})_v}{2\sinh f} ,$$

and in turn implying

$$\log(\cosh f)=\tfrac{1}{2}\log g_{11}+h(u) ,$$

where $h(u)$ is some function depending only on u. We can rescale the u coordinate so that $h(u) = c_1$ for some constant c_1. Thus

$$c_2\cosh^2 f=g_{11}$$

for some constant c_2. We can then scale u by a constant scalar factor so that

$$g_{11} = \cosh^2 f \ .$$

Similarly, we can arrange that

$$g_{22} = \sinh^2 f \ .$$

Now

$$c g_{11} g_{22} (k_1 - k_2)^2 - g_{22} + \epsilon^2 g_{11} =$$
$$c \cdot \cosh^2 f \cdot \sinh^2 f \cdot (\sqrt{\rho} \tanh f - \sqrt{\rho} \coth f)^2 - \sinh^2 f + \epsilon^2 \cosh^2 f \ ,$$

and this is equal to zero when we set $c = -1/\rho$ and $\epsilon = 1$, thus Calapso's equation holds and the surface is Guichard.

The case $\rho < 0$: In this case we can similarly arrange that

$$k_1 - \beta = \sqrt{-\rho} \tan f \ , \quad k_2 - \beta = -\sqrt{-\rho} \cot f$$

and

$$g_{11} = \cos^2 f \ , \quad g_{22} = \sin^2 f \ .$$

Calapso's equation again holds, now using $\epsilon = i$. □

We have the following immediate corollary:

Corollary 4.56. *All nontubular linear Weingarten surfaces are Ω.*

4.17. **Christoffel duals.** The following lemma is particularly useful for understanding discrete Ω surfaces, as we will see in a subsequent text.

Lemma 4.57. *Consider a Legendre immersion Λ from M^2 to the null planes in L^5, and suppose there exist sections (i.e. particular lifts of sphere congruences) $\sigma^\pm \in \Lambda$ such that*

$$d\sigma^+ \wedge d\sigma^- = 0 \ .$$

Then

(1) *$\sigma^\pm$ are isothermic with the same isothermic coordinates u, v on M^2 (and in particular, Λ is an Ω surface),*
(2) *there exists a scalar function f such that $\sigma^+_u = f\sigma^-_u$ and $\sigma^+_v = -f\sigma^-_v$ (and then $\sigma^\pm$ are called* Christoffel duals*),*
(3) *$\sigma^+_{uv} \in span\{\sigma^+_u, \sigma^+_v\}$, $\sigma^-_{uv} \in span\{\sigma^-_u, \sigma^-_v\}$ (and because of this, σ^+ and σ^- are each called a* Christoffel lift*).*

Proof. Take u, v to be conformal coordinates for σ^+. Because $d\sigma^+ \wedge d\sigma^- = 0$, we have

$$\sigma^+_u \wedge \sigma^-_v - \sigma^+_v \wedge \sigma^-_u = 0$$

and also

$$\sigma^-_u = \alpha\sigma^+_u - \beta\sigma^+_v \ , \quad \sigma^-_v = \gamma\sigma^+_u - \alpha\sigma^+_v$$

for some scalar functions α, β, γ. Then, because

$$(\sigma^-_u, \sigma^+_v) = (\sigma^-, \sigma^+_v)_u - (\sigma^-, \sigma^+_{uv}) = -(\sigma^-, \sigma^+_{uv}) =$$
$$-(\sigma^-, \sigma^+_u)_v + (\sigma^-_v, \sigma^+_u) = (\sigma^-_v, \sigma^+_u) \ ,$$

we have

$$\gamma = -\beta \ .$$

We can also now see that u, v are conformal coordinates for σ^- as well. Set $E := (\sigma_u^+, \sigma_u^+)$. The compatibility condition $\sigma_{uv}^- = \sigma_{vu}^-$ gives $(\sigma_{uv}^- - \sigma_{vu}^-, \sigma_u^+) = (\sigma_{uv}^- - \sigma_{vu}^-, \sigma_v^+) = 0$, and thus

$$(\alpha_v + \beta_u)\sigma_u^+ + (\alpha_u - \beta_v)\sigma_v^+ + \beta(\sigma_{uu}^+ - \sigma_{vv}^+) + 2\alpha\sigma_{uv}^+ = 0 \ , \tag{4.25}$$

implying

$$0 = (\alpha_v + \beta_u)E + \beta E_u + \alpha E_v = (\alpha_u - \beta_v)E - \beta E_v + \alpha E_u \ ,$$

so

$$0 = (\beta E)_u + (\alpha E)_v = (-\beta E)_v + (\alpha E)_u \ ,$$

and so $E(\alpha + i\beta)$ is holomorphic with respect to $u + iv$. For any vector field N lying in the bundle with fibres consisting of the vector spaces $(\text{span}\{\sigma_u^+, \sigma_v^+\})^\perp$ with nondegenerate signature $(+, +, -, -)$, Equation (4.25) implies

$$2\alpha(\sigma_{uv}^+, N) + \beta(\sigma_{uu}^+ - \sigma_{vv}^+, N) = 0 \ . \tag{4.26}$$

Define $a + ib$, $a, b \in \mathbb{R}$, by

$$(a + ib)^2 = E(\alpha + i\beta) \ ,$$

so $a + ib$ is holomorphic as well. It follows that

$$d(bdu + adv) = d(-adu + bdv) = 0 \ ,$$

so we can make a conformal change of coordinates $(u, v) \to (\tilde{u}, \tilde{v})$ so that

$$d\tilde{u} = bdu + adv \ , \quad d\tilde{v} = -adu + bdv \ .$$

Also,

$$\partial_{\tilde{u}} = \frac{b}{a^2 + b^2}\partial_u + \frac{a}{a^2 + b^2}\partial_v \ , \quad \partial_{\tilde{v}} = \frac{-a}{a^2 + b^2}\partial_u + \frac{b}{a^2 + b^2}\partial_v \ .$$

A direct computation gives

$$\sigma_{\tilde{u}\tilde{v}}^+ = \frac{1}{(a^2 + b^2)^2}\left(-ab\sigma_{uu}^+ - a^2\sigma_{uv}^+ + b^2\sigma_{uv}^+ + ab\sigma_{vv}^+\right) - \frac{a}{a^2 + b^2}\left(\frac{b}{a^2 + b^2}\right)_u \sigma_u^+ +$$
$$-\frac{a}{a^2 + b^2}\left(\frac{a}{a^2 + b^2}\right)_u \sigma_v^+ + \frac{b}{a^2 + b^2}\left(\frac{b}{a^2 + b^2}\right)_v \sigma_u^+ + \frac{b}{a^2 + b^2}\left(\frac{a}{a^2 + b^2}\right)_v \sigma_v^+ \ ,$$

and then it follows from (4.26) that $(\sigma_{\tilde{u}\tilde{v}}^+, N) = 0$, so $\tilde{u}, \tilde{v}$ are isothermic coordinates for σ^+. Similarly, $(\sigma_{\tilde{u}\tilde{v}}^-, N) = 0$, and $\tilde{u}, \tilde{v}$ are isothermic coordinates for σ^- as well. Also,

$$\sigma_{\tilde{u}}^- = \frac{b\sigma_u^- + a\sigma_v^-}{a^2 + b^2} = \frac{b(\alpha\sigma_u^+ - \beta\sigma_v^+) - a(\beta\sigma_u^+ + \alpha\sigma_v^+)}{a^2 + b^2} =$$
$$\frac{-b}{E}\sigma_u^+ - \frac{a}{E}\sigma_v^+ = -\frac{a^2 + b^2}{E}\sigma_{\tilde{u}}^+ \ ,$$

and similarly

$$\sigma_{\tilde{v}}^- = \frac{a^2 + b^2}{E}\sigma_{\tilde{v}}^+ \ ,$$

so item 2 of the lemma holds with

$$f = \frac{-E}{a^2 + b^2} \ .$$

Because $\sigma_{\tilde{u}\tilde{v}}^+$ is perpendicular to the entire space $(\text{span}\{\sigma_u^+, \sigma_v^+\})^\perp$, and in particular $(\sigma_{\tilde{u}\tilde{v}}^\pm, N) = 0$ even for fields N in $(\text{span}\{\sigma_u^+, \sigma_v^+\})^\perp$ such that $N \not\perp \sigma^\pm$, it follows that $\sigma_{\tilde{u}\tilde{v}}^+ \in \text{span}\{\sigma_u^+, \sigma_v^+\}$. Similarly, $\sigma_{\tilde{u}\tilde{v}}^- \in \text{span}\{\sigma_u^-, \sigma_v^-\}$. $\square$

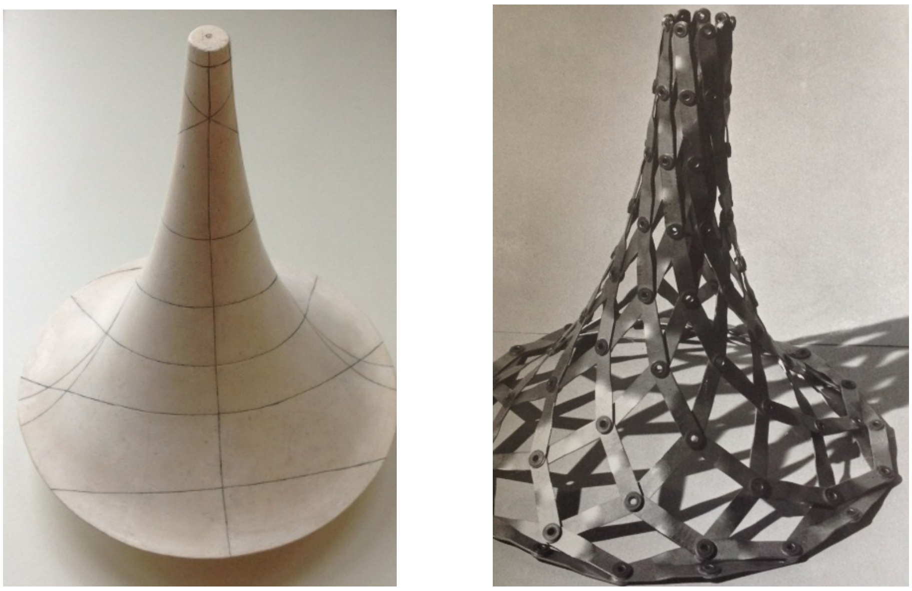

FIGURE 4.1. Physical models of the psuedosphere in $\mathbb{R}^3$, which has constant negative Gaussian curvature, and hence is a linear Weingarten surface (owned by the geometry group at the Technical University of Vienna)

FIGURE 4.2. Physical models of other surfaces of revolution in $\mathbb{R}^3$ with constant Gaussian curvature, which are again linear Weingarten surfaces (owned by the geometry group at the Technical University of Vienna)

4.18. **Linear Weingarten surfaces.** Up to now, we have given some special attention to flat surfaces in $\mathbb{H}^3$, but now we consider a more general class of surfaces called linear Weingarten surfaces. Linear Weingarten surfaces in a spaceform M^3 are those whose Gauss and mean curvatures K and H (with respect to M^3) satisfy

$$aH + bK + c = 0$$

for some constants a, b, c.

Theorem 4.58. *For an Ω surface x with isothermic sphere congruences $s^\pm$ and Legendre lift $\Lambda = span\{s^+, s^-\}$, suppose there exist constant conserved quantities*

$$Q^\pm \in \mathbb{R}^{4,2}\ , \quad i.e.\ \ (s^\pm, Q^\pm) = 0\ ,$$

and suppose

$$(Q^+, Q^-) \neq 0\ , \quad (s^+, Q^-) \neq 0\ , \quad (s^-, Q^+) \neq 0\ ,$$

and the subspace $V = span\{Q^+, Q^-\}$ *has signature* $(+,-)$. *Then* Λ *projects to a linear Weingarten surface in a spaceform.*

Proof. Take an orthonormal pair $p, q \in V$ such that $(p,p) = -1$ and $(q,q) = 1$ and

$$Q^+ = p + \beta q \quad \text{for some} \quad \beta \in \mathbb{R} \setminus \{0\}\ ,$$

and neither p nor q are parallel to Q^-. We can normalize Q^- so that

$$(Q^+, Q^-) = -2\ .$$

We can also scale $s^\pm$ appropriately so that

$$(s^\pm, Q^\mp) = -2\ .$$

Take $\alpha_-, \beta_- \in \mathbb{R} \setminus \{0\}$ so that

$$Q^- = \alpha_- p + \beta_- q\ .$$

Writing

$$s^\pm = \mathfrak{G}^\pm + \wp^\pm \quad \text{for} \quad \mathfrak{G}^\pm \perp V \quad \text{and} \quad \wp^\pm \in V\ ,$$

it follows that

$$\alpha_- = 2 + \beta\beta_-\ . \tag{4.27}$$

There exist $A_\pm, B_\pm$ such that $\wp^\pm = A_\pm p + B_\pm q$, and then

$$\wp^+ = (2 + \beta_- B_+)\alpha_-^{-1} p + B_+ q\ , \quad \wp^- = (2 + \beta B_-)p + B_- q\ .$$

Now, $(s^\pm, Q^\pm) = 0$ implies $B_+ = 2S^{-1}$ and $B_- = -2\alpha_- S^{-1}$ are constant, with $S = \alpha_-\beta - \beta_-$. Since $B_\pm$ exist, we know $S \neq 0$. Then

$$\wp^+ = (2\beta p + 2q)/S\ , \quad \wp^- = (-2\beta_- p - 2\alpha_- q)/S\ .$$

The properties $(X, p) = (N, q) = 0$ and $(X, q) = (N, p) = -1$ give

$$X = \tfrac{1}{2}(\beta_- s^+ + \beta s^-)\ , \quad N = \tfrac{1}{2}(\alpha_- s^+ + s^-)\ .$$

Then by Lemma 4.15,

$$\mathrm{cr}(\kappa_1 X + N, s^+, \kappa_2 X + N, s^-) = -1 \Rightarrow$$

$$\mathrm{cr}(\kappa_1 X + N, -\beta^{-1} X + N, \kappa_2 X + N, -\alpha_-\beta_-^{-1} X + N) = -1 \Rightarrow$$

$$\mathrm{cr}(\kappa_1, -\beta^{-1}, \kappa_2, -\alpha_-\beta_-^{-1}) = -1 \Rightarrow$$

$$(\kappa_1 - (-\beta^{-1}))(-\beta^{-1} - \kappa_2)^{-1}(\kappa_2 - (-\alpha_-\beta_-^{-1}))(-\alpha_-\beta_-^{-1} - \kappa_1)^{-1} = -1 \Rightarrow$$

$$K_{ext}\beta\beta_- + H(\beta_- + \beta(2 + \beta\beta_-)) + \beta\beta_- + 2 = 0\ ,$$

where $K_{ext} = \kappa_1\kappa_2$ is the extrinsic Gaussian curvature. The final implication above uses (4.27). Thus the surface is linear Weingarten. $\square$

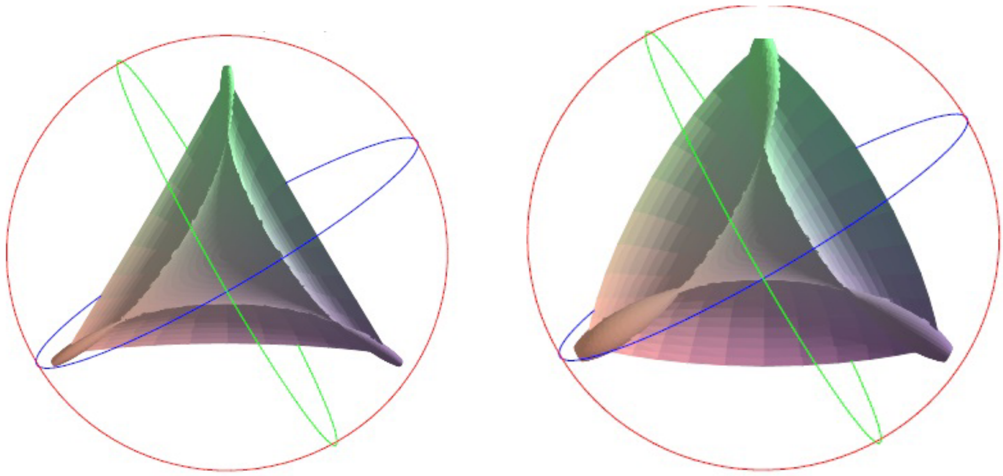

FIGURE 4.3. Two surfaces in a deformation through a one-parameter family of linear Weingarten surfaces of Bryant type in $\mathbb{H}^3$, the first of which is a flat surface (see [65])

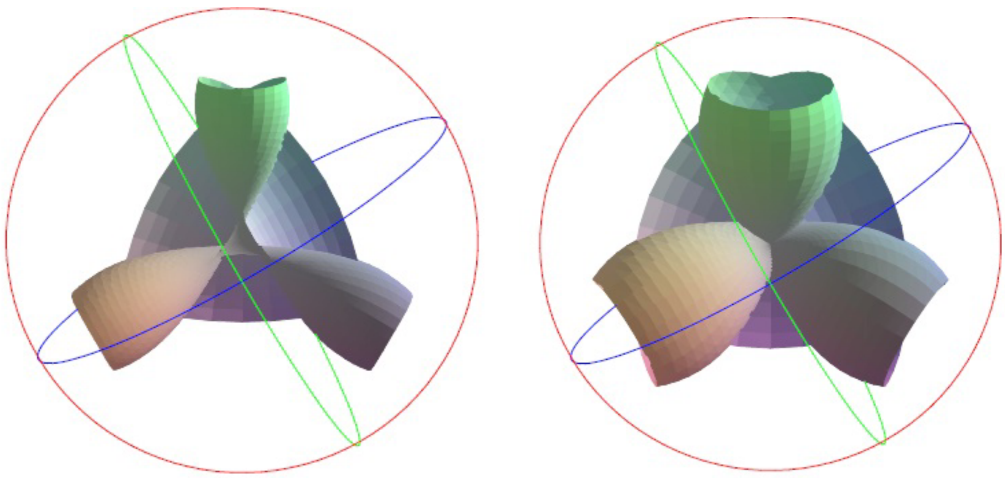

FIGURE 4.4. Two further surfaces in that deformation starting in Figure 4.3 through a one-parameter family of linear Weingarten surfaces of Bryant type in $\mathbb{H}^3$, the second of which is a CMC 1 surface (see [65])

The assumptions in Theorem 4.58 that $(Q^+, Q^-) \neq 0$ and that V has signature $(+,-)$ create restrictions on which types of linear Weingarten surfaces are obtained in that theorem, i.e. not all types of linear Weingarten surfaces are included there. In fact, the only condition that is really needed is that $\text{span}\{Q^+, Q^-\}$ is not a null plane. Once could then restrict to Riemannian geometries by assuming there exists a timelike $p \in \text{span}\{Q^+, Q^-\}$.

Furthermore, we need to consider the case that $s^\pm$ are complex conjugate as well, and also Ω_0 surfaces, to produce all linear Weingarten surfaces. We come back to this in Section 4.19.

Definition 4.59. *Those linear Weingarten surfaces that satisfy an equation of type*

$$\alpha(\pm H + 1) + \beta(K_{ext} - 1) = 0$$

for some $\alpha, \beta \in \mathbb{R}$ ($|\alpha| + |\beta| > 0$) are called linear Weingarten surfaces of Bryant type.

The reason for distinguishing the linear Weingarten surfaces as in Definition 4.59 is that they are the ones with Weierstrass type representations. See [56], [77], for example.

Corollary 4.60. *The linear Weingarten surfaces of Bryant type in Theorem 4.58 correspond to the case that at least one of $Q^{\pm}$ is lightlike.*

Proof. If $(Q^+, Q^+) = 0$, i.e. without loss of generality $\beta = 1$, we have

$$(K_{ext} - 1)\beta_- + 2H(\beta_- + 1) + 2(\beta_- + 1) = 0 \;\Rightarrow$$
$$(K_{ext} - 1)\beta_- + 2(H + 1)(1 + \beta_-) = 0 \;,$$

and this the equation for linear Weingarten surfaces of Bryant type. □

The next corollary was already established in Corollary 4.43.

Corollary 4.61. *Flat surfaces in $\mathbb{H}^3$ correspond to the case that both of Q^+ and Q^- are lightlike.*

Remark 4.62. *The case of CMC surfaces.* For CMC surfaces, $s^+ = X$ is isothermic. By Corollary 4.49

$$g^{-2\lambda} \cdot (d + \lambda\tau^+) \cdot (g^{-2\lambda})^{-1} = d + \lambda\tau^- \;,$$

i.e. $g^{-2\lambda}$ is the gauge for the gauge transformation taking $\Gamma^+ = d + \lambda\tau^+$ to $\Gamma^- = d + \lambda\tau^-$ (and, put more simply, $g^{-2\lambda}$ is the gauge transformation taking Γ^+ to Γ^-).

$$g^{-2\lambda} \cdot (d + \lambda\tau^+) \cdot (g^{-2\lambda})^{-1} Q^- = (d + \lambda\tau^-) Q^- = 0$$

implies

$$(d + \lambda\tau^+)((g^{-2\lambda})^{-1} Q^-) = 0 \;,$$

and now $(g^{-2\lambda})^{-1} q^-$ is a linear conserved quantity for $s^+ = X$. This linear conserved quantity was seen in Theorem 2.54.

4.19. **Complex conjugate $s^{\pm}$, and Ω_0 surfaces.** For the arguments about Ω surfaces in Section 4.3, we could also consider the case that the metric in (4.4) is Lorentz conformal, i.e.

$$(b - k_1)^2 = -(b - k_2)^2 \;.$$

This gives that

$$b = \frac{k_1\sqrt{g_{11}} \mp i k_2 \sqrt{g_{22}}}{\sqrt{g_{11}} \mp i\sqrt{g_{22}}} \;,$$

resulting in $s^{\pm}$ being complex conjugate to each other. Consideration of this case requires us to complexify $\mathbb{R}^{4,2}$. In this case, Demoulin's equation becomes

$$\left(\frac{\sqrt{g_{11}}}{\sqrt{g_{22}}} \frac{k_{1,u}}{k_1 - k_2} \right)_v - \left(\frac{\sqrt{g_{22}}}{\sqrt{g_{11}}} \frac{k_{2,v}}{k_1 - k_2} \right)_u = 0 \;. \tag{4.28}$$

We will call these surfaces Ω as well, and we call the $s^{\pm}$ isothermic sections.

We could also consider the case that the metric in (4.4) is degenerate, i.e.

$$(b - k_1)^2 (b - k_2)^2 = 0 \;,$$

and then without loss of generality that $b = k_1$. We are now considering a principal curvature sphere congruence, of course. To understand what isothermicity is, and what Demoulin's equation is, in this case, one should be careful about what the normal bundle to $s = k_1X + N$ might be. Instead of considering that, let us just take existence of a Moutard lift for $s = k_1X + N$ as a working definition, and then compute the resulting Demoulin equation, as follows:

There exists a scalar factor function α so that

$$(\alpha s)_{uv}||s ,$$

and this implies

$$s_{uv} = hs - (\log\alpha)_u s_v - (\log\alpha)_v s_u$$

for some function $h = h(u, v)$. Now, because $s = k_1X + N$ and $s_u = k_{1,u}X$, h must be zero, and

$$\alpha X + \beta X_v = 0 ,$$

where

$$\gamma = k_{1,uv} + (\log\alpha)_u k_{1,v} + (\log\alpha)_v k_{1,u} , \quad \beta = k_{1,u} + (\log\alpha)_u(k_1 - k_2) ,$$

and so $\gamma = \beta = 0$. This implies

$$(\log\alpha)_u = \frac{-k_{1,u}}{k_1 - k_2} = -(\log(k_1 - k_2))_u - \frac{k_{2,u}}{k_1 - k_2}$$

and

$$k_{1,uv} + \frac{k_{1,u}}{k_2 - k_1} k_{1,v} + (\log\alpha)_v k_{1,u} = 0 .$$

Then the Codazzi equations (2.22) imply

$$(\log\alpha)_v = \frac{-k_{1,uv}}{k_{1,u}} - \tfrac{1}{2}\frac{g_{11,v}}{g_{11}}$$

and

$$(\log\alpha)_u = -(\log(k_1 - k_2))_u - \tfrac{1}{2}(\log g_{22})_u ,$$

so

$$\log\alpha = -\log(k_{1,u}) - \tfrac{1}{2}\log g_{11} + f_1(u) = -\log(k_1 - k_2) - \tfrac{1}{2}\log g_{22} + f_2(v) ,$$

where $f_1(u)$, resp. $f_2(v)$, is some function depending only on u, resp. v. We can reparametrize the v coordinate so that $f_2(v) = c_2$ is constant. Thus, taking the exponential of this equation, we have

$$\frac{k_{1,u}}{k_1 - k_2} \cdot \frac{\sqrt{g_{11}}}{\sqrt{g_{22}}} = f_3(u)$$

for some function $f_3(u)$ depending only on u, so

$$\left(\frac{\sqrt{g_{11}}}{\sqrt{g_{22}}} \cdot \frac{k_{1,u}}{k_1 - k_2}\right)_v = 0 , \tag{4.29}$$

and this is Demoulin's equation for Ω_0 surfaces.

Definition 4.63. *An* Ω_0 surface *is a surface where one of the curvature sphere congruences is isothermic (in the sense that Demoulin's equation* (4.29) *holds, or, equivalently, a Moutard lift exists).*

In regard to Definition 4.63, we can imagine that the two isothermic sphere congruences, as in Corollary 4.5 and Definition 4.6, for Ω surfaces are now coinciding, becoming a single isothermic principal curvature sphere congruence and producing an Ω_0 surface.

Tubular surfaces and channel surfaces are examples of Ω_0 surfaces, and we explain these examples now.

Definition 4.64. *A surface is* tubular *if one principal curvature is constant.*

Thus a surface is tubular if $(k_1-b)(k_2-b) = 0$, where k_j are the principal curvatures of the surface and b is some constant. This amounts to the linear Weingarten condition with the linear Weingarten equation being (H is the mean curvature, and K is the extrinsic Gaussian curvature)

$$0 = K - 2bH + b^2 \ .$$

Remark 4.65. Here we make five comments on tubular and Ω_0 surfaces:

- When a surface is linear Weingarten, the surface generally comes with a pair of isothermic sphere congruences $s^{\pm}$. If the surface is also tubular, then s^+ and s^- coincide, and the surface is Ω_0.
- There are examples of Ω_0 linear Weingarten surfaces that are not tubular. For example, a surface of revolution of constant Gauss curvature 1 in $\mathbb{R}^3$ is multiply Ω, and both Ω and Ω_0, but not tubular. However, this surface being nontubular is possible only because the surface is both Ω and Ω_0, as Corollary 4.68 will show.
- The class of all Lie applicable surfaces is the same as the union of the classes of all Ω surfaces (including the complex conjugate case) and all Ω_0 surfaces (see [92], [43] and volume III of [4]).
- When the ambient space is $\mathbb{R}^3$, and the surface $x(u,v)$ is parametrized by curvature line coordinates (u,v), and $k_1 = b \neq 0$ for the principal curvature associated to the u-direction (b is constant), and n is the unit normal vector to the surface, the Rodrigues equation implies
$$(x + b^{-1}n)_u = 0 \ ,$$
so
$$c(v) = x + b^{-1}n$$
is a curve depending only on v, and is called the *soul* of the tubular surface x.
- Taking the ambient space to be $\mathbb{S}^3$ and taking $x \in \mathbb{S}^3$ with unit normal $n \in \mathbb{S}^3$, setting
$$s := k_1 \begin{pmatrix} 1 \\ x^t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ n^t \\ 1 \end{pmatrix} \ ,$$
we have $s_u = 0$ (because k_1 is constant), and s is a sphere congruence depending on only the one parameter v. So x is a particular case of a channel surface, which we now define.

Definition 4.66. *A surface x is a* channel surface *if it can be enveloped by a 1-parameter family of spheres.*

We are now in a position to consider the converse of Theorem 4.58:

Theorem 4.67. *The converse to Theorem 4.58 also holds. That is, when allowing the case of complex conjugate isothermic sections $s^{\pm}$ and also allowing the case of Ω_0 surfaces, a surface is linear Weingarten if and only if the lift Λ of the surface to Lie sphere geometry has a pair of isothermic sections $s^{\pm}$ with constant conserved quantities $Q^{\pm}$ (or a single isothermic sphere congruence with a single constant conserved quantity in the Ω_0 case).*

Proof. We can extend Theorem 4.58 to the cases of complex conjugate isothermic sections and of Ω_0 surfaces, by computations analogous to those in the proof of that theorem, giving one direction of the theorem being proven here.

To prove the converse, assume

$$\alpha K + 2\beta H + \gamma = 0 \ .$$

(1) If

$$\alpha \neq 0 \quad \text{and} \quad \delta := \sqrt{\beta^2 - \alpha\gamma} \neq 0 \ ,$$

define

$$Q^{\pm} = \alpha q + (\beta \mp \delta) p \ , \quad s^{\pm} = \delta^{-1}((\delta \mp \beta) X \pm \alpha N) \ .$$

When $\beta^2 - \alpha\gamma > 0$, the $s^{\pm}$ will be real, and when $\beta^2 - \alpha\gamma < 0$, the $s^{\pm}$ will be complex conjugate.

(2) If $\alpha = 0$, define

$$Q^{+} = \beta q + \tfrac{1}{2}\gamma p \ , \quad Q^{-} = p \ , \quad s^{+} = HX + N \ , \quad s^{-} = X \ .$$

Then, in both cases (1) and (2), one can check that $(s^{\pm}, Q^{\pm}) = 0$ and $ds^{+} \wedge ds^{-} = 0$, which implies $s^{\pm}$ are a dual pair of isothermic sphere congruences (see Lemma 4.57).

If

$$\alpha \neq 0 \quad \text{and} \quad \delta = 0 \ ,$$

then without loss of generality, we may assume $k_1 = -\beta$ is constant. Defining

$$s = k_1 X + N \ ,$$

it is immediate that Demoulin's equation (4.29) for Ω_0 surfaces holds. In this case the constant conserved quantity will be

$$q - k_1 p \ .$$

This completes the proof of the theorem. □

The above proof provides the following corollary, since in the case (1) that

$$s^{\pm} = \delta^{-1}((\delta \mp \beta) X \pm \alpha N) \quad \text{and} \quad \delta \neq 0 \ ,$$

we cannot have the vector s^{+} parallel to the vector s^{-}.

Corollary 4.68. *If an Ω_0 surface is both linear Weingarten and not Ω, then it is tubular.*

We also have this corollary:

Corollary 4.69. *Calapso transformations of linear Weingarten surfaces are again linear Weingarten.*

Proof. Take a linear Weingarten surface and its lift Λ to an Ω or Ω_0 surface (we know Λ is Ω or Ω_0, by Corollary 4.56). By Theorem 4.67, Λ has a pair of isothermic sections $s^{\pm}$ (or possibly a single section $s = k_1 X + N$ in the Ω_0 case) with constant conserved quantities $Q^{\pm} \in \mathbb{R}^{4,2}$ (or possibly a single conserved quantity for s, in the Ω_0 case). Like as argued in Equation (4.23), $T^{\pm}Q^{\pm}$ are also constant, and satisfy

$$(T^{\pm}s^{\pm}, T^{\pm}Q^{\pm}) = 0 \ ,$$

so $T^{\pm}Q^{\pm}$ are constant conserved quantities for $T^{\pm}s^{\pm}$, respectively. Then, applying Corollary 4.31 and Equation (4.18), we have concluded that the Calapso transformation

$$\text{span}\{T^{+}s^{+}, T^{-}s^{-}\}$$

of Λ has a pair of isothermic sections $T^{\pm}s^{\pm}$ with constant conserved quantities $T^{\pm}Q^{\pm}$. By Theorem 4.67, the result follows. $\square$

5. Closing remark

With all that we have described in this text, it is now possible to consider discrete Ω surfaces. As noted in the introduction, this will be done in a separate text.

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