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Frobenius manifolds and tt^* geometry for singularities

M = a complex manifold with hol. coordinates $t_1, ..., t_m$.

tt^* geometry on a holomorphic vector bundle over M: a generalization of variation of Hodge structure.

E.g. $m = 0$, $M = \{pt\}$.

For $m \geq 1$, M a Frobenius manifold: additional hol. structure on TM.

tt^* geometry on TM
& flat connection
$\Longleftrightarrow$ M Frobenius manifold
& real structure.

tt^* geometry and Frobenius manifolds have a common source (certain meromorphic connections) and arise together.

tt^* geometry in:

S. Cecotti, C. Vafa:
Topological-antitopological fusion (1991).
On classification of $N = 2$ supersymmetric theories (1993).

B. Dubrovin (1992).

A weaker version of tt^* geometry is in the work of
C. Simpson ($\geq$ 1988) on
harmonic bundles,
his nonabelian Hodge theorem,
(mixed) twistor structures.
C. Sabbah (2001): Polarizable twistor $\mathcal{D}$-modules

A distinguished class of examples:

$f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ holomorphic, with an isolated singularity at 0,

Milnor number $\quad \mu := \dim \frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{(\frac{\partial f}{\partial x_i})} < \infty$

Choose $m_1, ..., m_\mu \in \mathcal{O}_{\mathbb{C}^{n+1},0}$, which represent at basis of the Jacobi algebra.

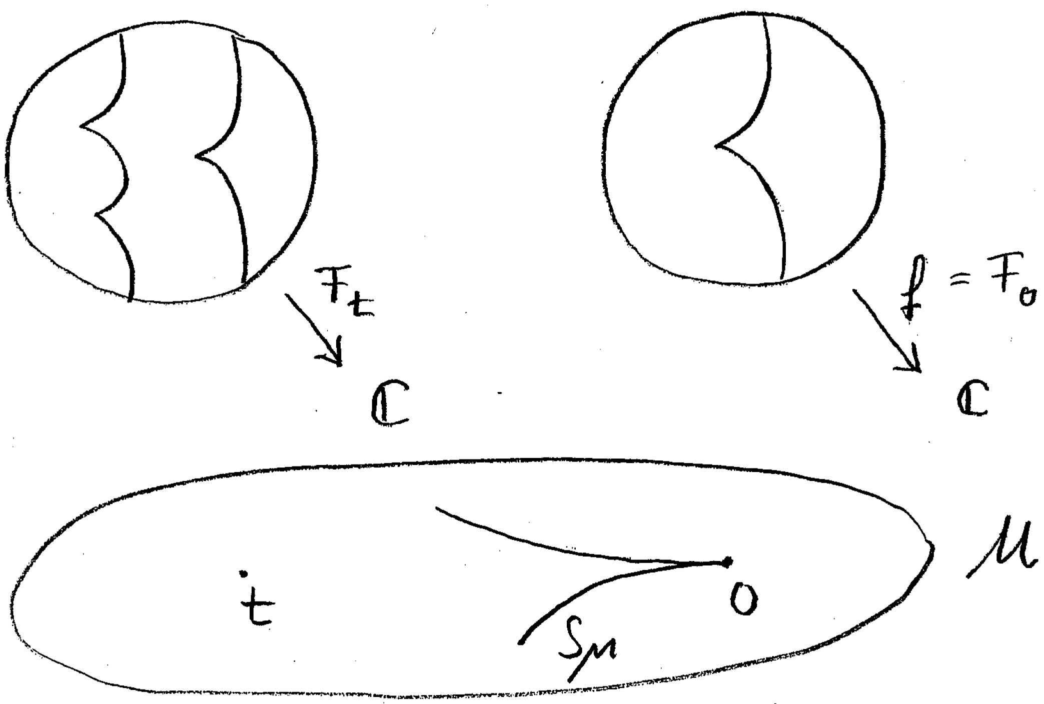
A semiuniversal unfolding F of f:

$$\begin{aligned} F &= F(x,t) = F(x_0, ..., x_n, t_1, ..., t_\mu) \\ &= f(x) + \sum_{i=1}^{\mu} t_i m_i, \end{aligned}$$

$F :$ (nbhd of 0 in $\mathbb{C}^{n+1} \times \mathbb{C}^\mu, 0) \to (\mathbb{C}, 0)$;

$M =$ suitable nbhd of 0 in $\mathbb{C}^\mu$,
M base space of the unfolding.

ball in $\mathbb{C}^{n+1}$ ball in $\mathbb{C}^{n+1}$



μ-constant stratum $S_\mu \subset M$:

$S_\mu = \{t \in M \mid \mathrm{Crit}(F_t) = \{x\}$ with $F_t(x) = 0\}$.

For $t \in S_\mu$, there exists a canonical polarized mixed Hodge structure (Steenbrink '76, Varchenko '80, M. Saito, Scherk-Steenbrink, Pham).

<u>Aim:</u>
(a) tt^* geometry on M, which extends and "explains" this structure on S_μ.
(b) M Frobenius manifold.

(b): K. Saito ('70ies and '80ies),
M. Saito ('83),
C. Sabbah ('96,'02), S. Barannikov ('00),

C. Hertling: tt^* geometry, Frobenius manifolds, their connections, and the construction for singularities.
math.AG/0203054, 81 pages.

<u>Definition 1:</u> Given $H' \to \mathbb{C}^*$ a hol. vector bundle with a flat connection ∇.

(a) An extension of $H' \to \mathbb{C}^*$ to a hol. vector bundle $H \to \mathbb{C}$ has a <u>pole of Poincaré rank $\leq r$</u> ($r \in \mathbb{Z}_{\geq 0}$) at 0 if

$$\nabla_{z\partial_z} : \mathcal{O}(H) \to \frac{1}{z^r}\mathcal{O}(H).$$

(b) A <u>logarithmic pole</u>

:= a pole of Poincaré rank ≤ 0.

(c) An extension $H \to \mathbb{C}$ has a <u>regular singular pole</u> at 0 if its sections are of moderate growth,
i.e. in a sector $\subset \mathbb{C}^*$

$$\text{hol section } = \sum_i \text{coeff}_i(z) \cdot (\text{flat section})_i$$

$$|\text{coeff}_i(z)| \leq b_1 |z|^{b_2} \text{ for some } b_1 > 0, b_2 \in \mathbb{R}.$$

(b)

$\Swarrow$ $\Searrow$

(a) with $r = 1$ $\;\not\Rightarrow\;$ / $\;\not\Leftarrow\;$ (c)

Given $(H' \to \mathbb{C}^*, \nabla)$ a flat vector bundle.

$$\begin{aligned} \pi &: \mathbb{C} \to \mathbb{C}^*, \quad \zeta \mapsto e^{2\pi i \zeta} = z, \\ pr &: \pi^* H' \to H', \\ H^\infty &:= \{\text{global flat manyvalued sections} \\ &\qquad\qquad \text{of } H' \to \mathbb{C}^*\} \\ &= \{pr \circ \sigma \mid \sigma : \mathbb{C} \to \pi^* H' \text{ flat section}\}, \end{aligned}$$

monodrony $M_{mon} : H'_z \to H'_z$,
$M_{mon} : H^\infty \to H^\infty$, $M_{mon} = M_s \cdot M_u$,
$N := \log M_u$,
$H^\infty_\lambda := \ker(M_s - \lambda)$, $\quad H^\infty_{\neq 1} := \bigoplus_{\lambda \neq 1} H^\infty_\lambda$,
$H^\infty = \bigoplus_\lambda H^\infty_\lambda = H^\infty_1 \oplus H^\infty_{\neq 1}$.

Proposition 2: There is a natural 1–1 correspondence between the sets

{extensions $H \to \mathbb{C}$ of $H' \to \mathbb{C}^*$ with logarithmic pole} and

$\{M_{mon}$-invariant (exhaustive) decreasing filtrations $F^\bullet$ of $H^\infty\}$.

Construct an <u>elementary section $es(A,\alpha)$</u>,
a global hol. section of $H' \to \mathbb{C}^*$:

Choose $\alpha \in \mathbb{C}$, $A \in H^{\infty}_{e^{-2\pi i\alpha}}$,
then for $\zeta \in \mathbb{C}$

$$A(\zeta+1) = M_{mon}A(\zeta) \in H'_{e^{2\pi i\zeta}}.$$

$$\begin{aligned} es(A,\alpha) &= [z \mapsto e^{\zeta\alpha}\exp(\zeta(-N))A(\zeta)] \\ &\qquad\qquad \text{for } \zeta \text{ with } e^{2\pi i\zeta} = z \\ &= [z \mapsto {}''z^{\alpha}\exp(\log z\frac{-N}{2\pi i})A''].\end{aligned}$$

M_{mon}-invariant filtration $F^{\bullet} \mapsto$
extension $H \to \mathbb{C}$ with logarithmic pole:

$\mathcal{O}(H)$ is generated by

$$es(A,\alpha) \text{ with } A \in F^{[-\alpha]}H^{\infty}_{e^{-2\pi i\alpha}}.$$

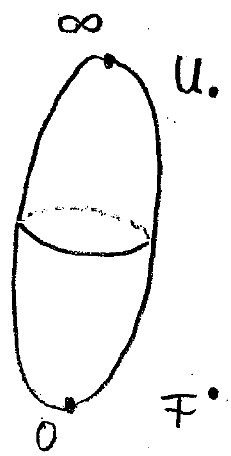
Given $(H' \to \mathbb{C}^*, \nabla)$ a flat vector bundle.

Convention:
decreasing M_{mon}-invariant filtration $F^\bullet$
$\leftrightarrow$ extension with log. pole at 0,
increasing M_{mon}-invariant filtration $U_\bullet$
$\leftrightarrow$ extension with log. pole at ∞.

A decreasing filtration $F^\bullet$ and an increasing filtration $U_\bullet$ are opposite if

$$H^\infty = \bigoplus_p F^p \cap U_p.$$

Proposition 3: An extension $\hat{H} \to \mathbb{P}^1$ of $H' \to \mathbb{C}^*$ with logarithmic poles at 0 and ∞ is the trivial bundle iff the corresponding filtrations satisfy:



$$F^\bullet H^\infty_{\neq 1} \text{ and } U_{\bullet+1} H^\infty_{\neq 1} \text{ are opposite,}$$
$$F^\bullet H^\infty_1 \text{ and } U_\bullet H^\infty_1 \text{ are opposite.}$$

<u>Definition 4:</u> A <u>polarized Hodge structure</u> (PHS) of weight $w \in \mathbb{Z}$ is a tuple $(H^\infty, F^\bullet, H^\infty_{\mathbb{R}}, S)$ with

H^∞ a finite dim. $\mathbb{C}$-vector space;
$F^\bullet$ a decreasing filtration on H^∞;
$H^\infty_{\mathbb{R}} \subset H^\infty$ an $\mathbb{R}$-vector space with

$$H^\infty = H^\infty_{\mathbb{R}} \oplus iH^\infty_{\mathbb{R}};$$

S a $\mathbb{C}$-bilinear $(-1)^w$-symmetric nondegenerate pairing on H^∞ with $S : H^\infty_{\mathbb{R}} \times H^\infty_{\mathbb{R}} \to \mathbb{R}$;

such that

$F^\bullet$ and $\overline{F^{w-\bullet}}$ are opposite,
i.e. $H^\infty = \oplus_p H^{p,w-p}$
with $H^{p,w-p} = F^p \cap \overline{F^{w-p}}$;

S gives a polarization, i.e.

$$S(F^p, F^{w+1-p}) = 0 \qquad \text{and}$$

the form $h_{Hodge} : H^\infty \times H^\infty \to \mathbb{C}$ with

$$h_{Hodge}(a, b) := i^{p-(w-p)} S(a, \overline{b})$$

for $a \in H^{p,w-p}, b \in H^\infty$, is hermitian and positive definite.

Definition 5: (a) A $(TERP)$-structure (Twistor Extension Real Pairing) of weight $w \in \mathbb{Z}$ is a tuple $(H \to \mathbb{C}, \nabla, H_{\mathbb{R}}, P)$ with

$H \to \mathbb{C}$ a hol. vector bundle;

∇ a flat connection on $H|_{\mathbb{C}^*}$ with a pole of Poincaré rank ≤ 1 at 0;

$H_{\mathbb{R}} \to \mathbb{C}^*$ a ∇-flat subbundle of $H|_{\mathbb{C}^*}$ of real vector spaces with $H_z = (H_{\mathbb{R}})_z \oplus i(H_{\mathbb{R}})_z$ for $z \in \mathbb{C}^*$;

P a $\mathbb{C}$-bilinear $(-1)^w$-symmetric nondegenerate ∇-flat pairing

$$P : H_z \times H_{-z} \to \mathbb{C} \quad \text{for } z \in \mathbb{C}^*$$

such that

$$P : (H_{\mathbb{R}})_z \times (H_{\mathbb{R}})_{-z} \to i^w\mathbb{R}$$

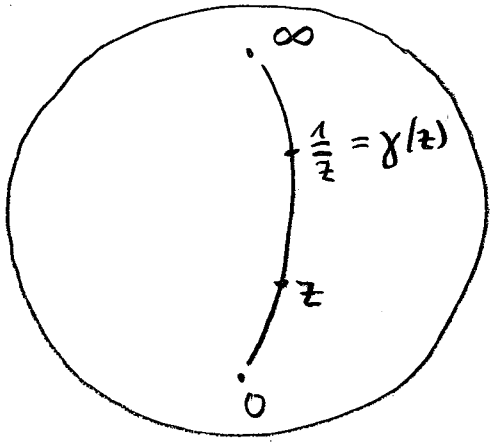
and

$$P : \mathcal{O}(H)_0 \times \mathcal{O}(H)_0 \to z^w\mathcal{O}_{\mathbb{C},0}$$

is nondegenerate.

Given a $(TERP)$-structure $(H \to \mathbb{C}, \nabla, H_{\mathbb{R}}, P)$ of weight w.

$\gamma : \mathbb{P}^1 \to \mathbb{P}^1, z \mapsto \frac{1}{\bar{z}}$.



Define

$\tau \;:\; H_z \to H_{\gamma(z)}$ a $\mathbb{C}$-antilinear isom.

$a \mapsto \nabla$-flat shift to $H_{\gamma(z)}$ of $\overline{z^{-w}a}$.

$\tau^2 = \mathrm{id}$.

Glue $H \to \mathbb{C}$ and $\overline{\gamma^* H} \to \mathbb{P}^1 - \{0\}$ with τ to a bundle

$$\hat{H} \to \mathbb{P}^1.$$

It has a pole of Poincaré rank ≤ 1 at ∞.

Define $K := H_0$ (fiber at 0) and

define the $\mathbb{C}$-bilinear symmetric nondegenerate pairing

$$\begin{aligned} g \ : \ & K \times K \to \mathbb{C} \\ & (a,b) \mapsto \left(z^{-w} P(\tilde{a},\tilde{b})\right)\Big|_{z=0} \end{aligned}$$

for $\tilde{a},\tilde{b} \in \mathcal{O}(H)_0$ with $\tilde{a}(0) = a, \tilde{b}(0) = b$.

If $\widehat{H} \to \mathbb{P}^1$ is the trivial bundle then

$$K \cong \Gamma(\mathbb{P}^1, \mathcal{O}(\widehat{H})),$$

and

$$\tau : \Gamma(\mathbb{P}^1, \mathcal{O}(\widehat{H})) \to \Gamma(\mathbb{P}^1, \mathcal{O}(\widehat{H}))$$

induces a $\mathbb{C}$-antilinear involution

$$\kappa : K \to K.$$

Then define a hermitian nondegenerate pairing

$$\begin{aligned} h \ : \ & K \times K \to \mathbb{C} \\ & (a,b) \mapsto g(a, \kappa(b)). \end{aligned}$$

Definition 5: (b) A $(TERP)$-structure is a $(trTERP)$-structure if $\widehat{H} \to \mathbb{P}^1$ is the trivial bundle.

(c) It is a $(pos.def.trTERP)$-structure if additionally h is positive definite.

(This is a generalization of a PHS (with an automorphism).)

Definition 6: A variation of PHS of weight w over a manifold M is a tuple $(H^\infty \to M, \nabla, F^\bullet, H^\infty_{\mathbb{R}}, S)$ with

$H^\infty \to M$	a hol. vector bundle with flat connection ∇;
$(H^\infty, F^\bullet, H^\infty_{\mathbb{R}}, S)\|_t$	a PHS of weight w for $t \in M$;
$H^\infty_{\mathbb{R}}$ and S	∇-flat;
$F^p \subset H^\infty$	hol. subbundles with

Griffiths transversality

$$\nabla : \mathcal{O}(F^p) \to \Omega^1_M \otimes \mathcal{O}(F^{p-1}).$$

Definition 7: (a) A variation of $(TERP)$-structures (a (VTERP)-structure) over a manifold M is a tuple $(H \to \mathbb{C} \times M, \nabla, H_{\mathbb{R}}, P)$ with

$H \to \mathbb{C} \times M$ a hol. vector bundle;

∇ a flat connection on $H|_{\mathbb{C}^* \times M}$ with a pole of Poincaré rank ≤ 1 along $\{0\} \times M$, i.e.

$$\nabla : \mathcal{O}(H) \to \frac{1}{z} \cdot \Omega^1_{\mathbb{C} \times M}(\log\{0\} \times M) \otimes \mathcal{O}(H).$$

[$\supset$ Griffiths transversality];

$(H \to \mathbb{C} \times M, \nabla, H_{\mathbb{R}}, P)|_{\mathbb{C} \times \{t\}}$ a $(TERP)$-structure of weight w;

$H_{\mathbb{R}}$ and P ∇-flat.

(b) A $(VtrTERP)$-structure ...

(c) A $(Vpos.def.trTERP)$-structure ...

Given a $(VTERP)$-structure, define

$K := H|_{\{0\}\times M}$ vector bundle on M.

Define for $X \in \mathcal{T}_M$ (hol. vector field)

$$\begin{aligned} \mathcal{C}_X &: \mathcal{O}(K) \to \mathcal{O}(K), \\ \mathcal{U} &: \mathcal{O}(K) \to \mathcal{O}(K) \end{aligned}$$

by

$$\begin{aligned} \mathcal{C}_X &= \lim_{z\to 0} z\nabla_X|_{\{z\}\times M}, \\ \mathcal{U} &= \lim_{z\to 0} z\nabla_{z\partial_z}|_{\{z\}\times M}. \end{aligned}$$

$\mathcal{C}$ is a Higgs field,
$[\mathcal{C}_X, \mathcal{C}_Y] = 0$, $[\mathcal{C}_X, \mathcal{U}] = 0$.

Given a $(VtrTERP)$-structure.

On K define g, κ and h as above,
define

$D :=$ Chern connection on K w.r.to h.

Lift $\mathcal{C}, \mathcal{U}, D$ to $\widehat{H} \to \mathbb{P}^1 \times M$, using

$$K_t \cong \Gamma_{hol}(\mathbb{P}^1 \times \{t\}, \widehat{H}|_{\mathbb{P}^1 \times \{t\}}) \quad \cong \widehat{H}_{(z,t)} \quad \forall z \in \mathbb{P}^1$$

Then

$$\nabla = D + \frac{1}{z} \cdot \mathcal{C} + z \cdot \kappa \mathcal{C} \kappa + \left(\frac{1}{z} \cdot \mathcal{U} - \underbrace{\mathcal{Q}}_{+\frac{w}{2} id} - z \cdot \kappa \mathcal{U} \kappa\right) \frac{dz}{z}$$

for some

$$\mathcal{Q} : K_t \to K_t$$

$\mathbb{C}$-linear, real analytic in t.

Cecotti-Fendley-Intriligator-Vafa (1992):
$\mathcal{Q}$ is "A new supersymmetric index".

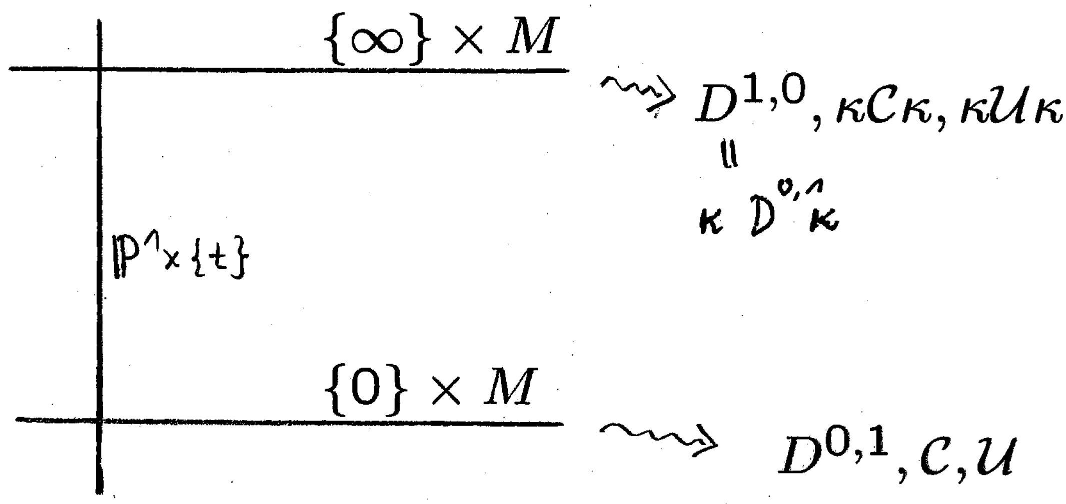
VPHS : then $\quad \mathcal{Q}|_{H^{p,w-p}} = (p - \frac{w}{2}) id$

tr

"Theorem 8:" A $(VTERP)$-structure is equivalent to a structure on a hol. vector bundle $K \to M$ with data $(D, \mathcal{C}, \kappa, g, h, \mathcal{U}, \mathcal{Q})$ and many conditions, e.g. the tt^* equations:
for $X, Y \in \mathcal{T}_M$

$$[D_X, D_{\overline{Y}}] = -[\mathcal{C}_X, (\kappa \mathcal{C} \kappa)_{\overline{Y}}],$$
$$D_X(\mathcal{C}_Y) - D_Y(\mathcal{C}_X) = \mathcal{C}_{[X,Y]}.$$

$(\widehat{H}, \nabla)$ has at $\{\infty\} \times M$ in $\frac{\partial}{\partial t_i}$ no pole, in $\frac{\partial}{\partial \overline{t_i}}$ and $z\partial_z$ a pole of order 1



at $\{0\} \times M$ a pole of Poincaré rank 1.

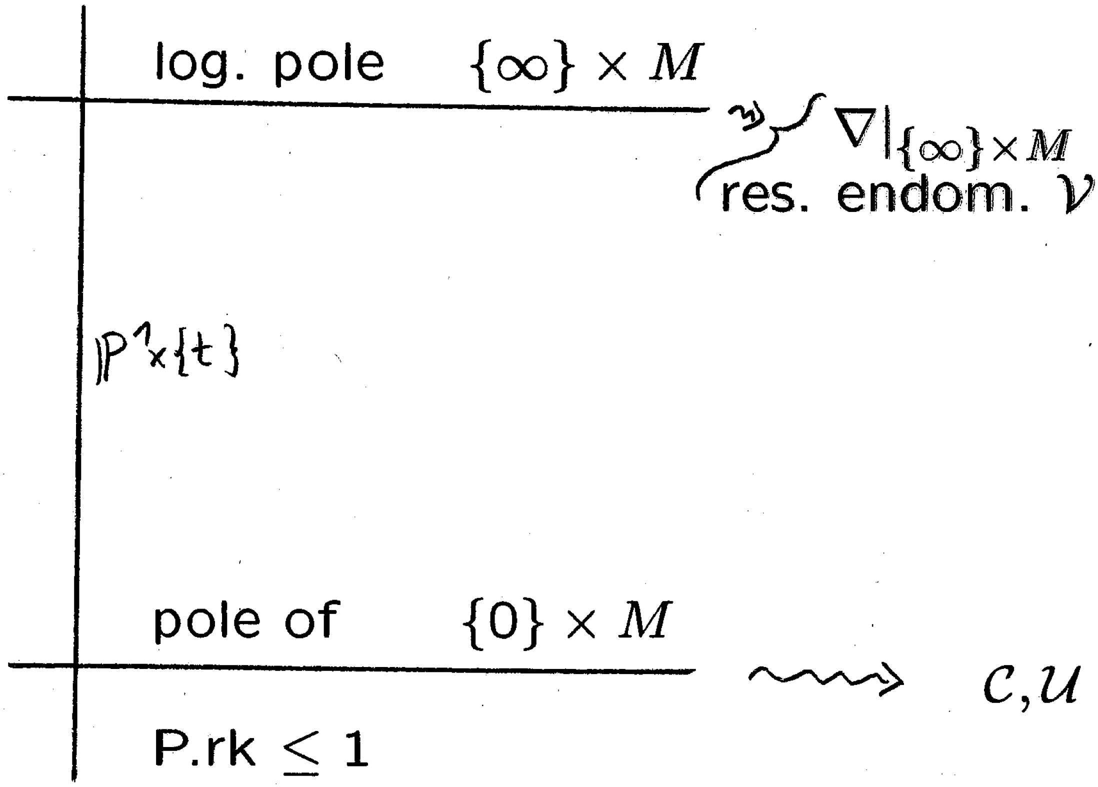
From $(VTERP)$-structures to Frobenius manifolds:

Given a $(VTERP)$-structure $(H \to \mathbb{C} \times M, \nabla, H_{\mathbb{R}}, P)$.

Instead of constructing $\widehat{H} \to \mathbb{P}^1 \times M$ with the real structure, one can try to choose an extension $\widetilde{H} \to \mathbb{P}^1 \times M$

- with a logarithmic pole along $\{\infty\} \times M$,
- and such that $\widetilde{H} \to \mathbb{P}^1 \times M$ is the trivial bundle.

[Birkhoff problem]



$$K := H|_{\{0\}\times M} \cong H|_{\{\infty\}\times M}$$

together with $\mathcal{C}, \mathcal{U}, g, \nabla|_{\{\infty\}\times M}, \mathcal{V}$:

"Frobenius type structure".

Theorem 9: (K. Saito $\leq$ '82, C. Sabbah '96, S. Barannikov '00) (Situation as above.) Suppose that $\mathrm{rk}\, K = \dim M$ and suppose that a section $\zeta \in \Gamma(\mathbb{P}^1 \times M, \mathcal{O}(\widetilde{H}))$ exists with:

α) $\mathcal{C}_\bullet \zeta|_{0\times M} : \mathcal{T}_M \to \mathcal{O}(K),\ X \mapsto \mathcal{C}_X \zeta|_{0\times M}$ is an isomorphism,

β) $\zeta|_{\{\infty\}\times M}$ is $\nabla|_{\{\infty\}\times M}$-flat,

γ) $\zeta|_{\{\infty\}\times M}$ is an eigenvector of $\mathcal{V}$.

Then

$$-\mathcal{C}_\bullet \zeta : \mathcal{T}_M \to \mathcal{O}(K)$$

and $\mathcal{C}, \mathcal{U}, \nabla|_{\{\infty\}\times M}, \mathcal{V}$ induce on TM the structure of a Frobenius manifold.

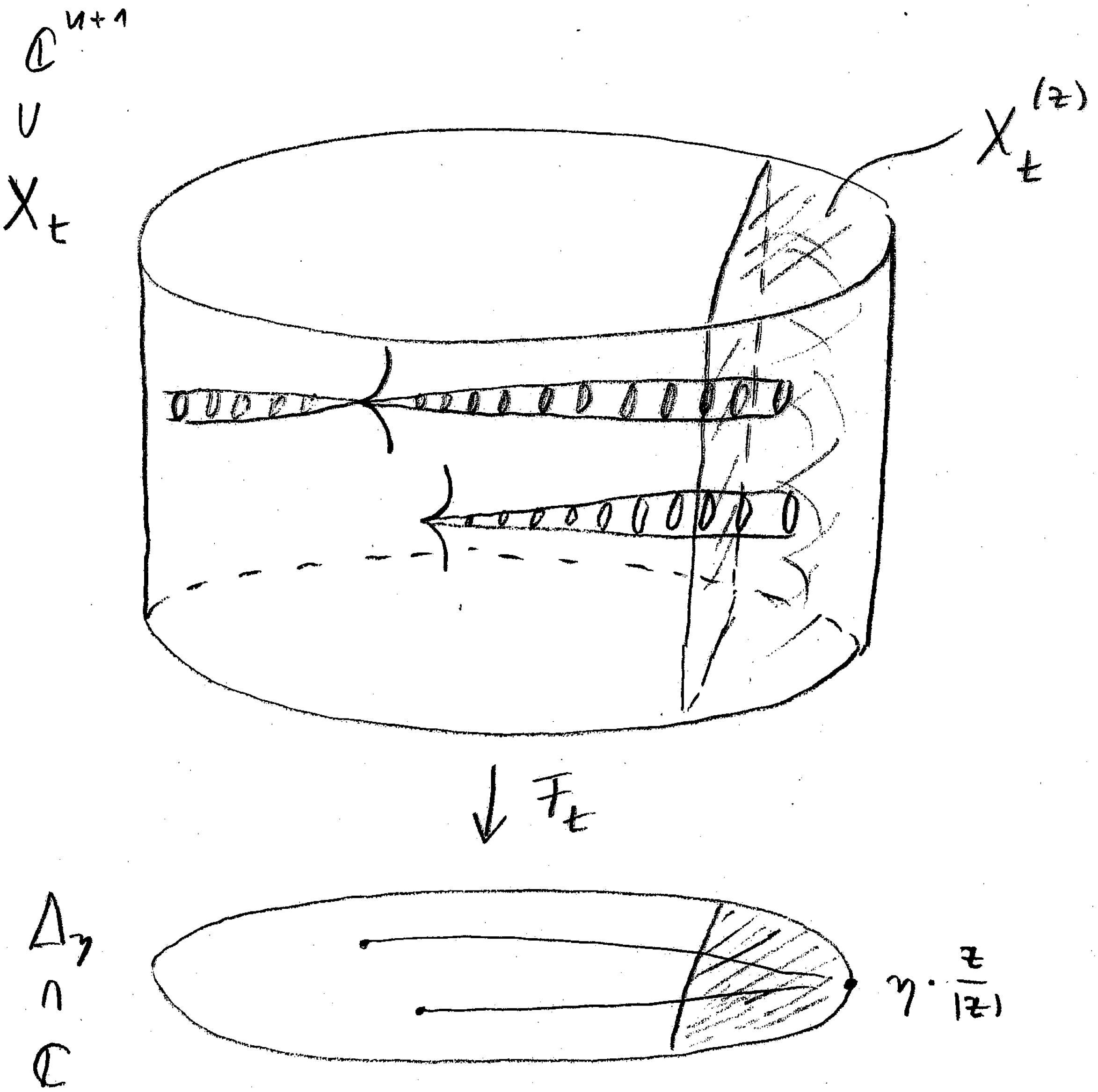
$\zeta \sim$ K. Saito's primitive form.

$f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ holomorphic,
with an isolated singularity at 0,
Milnor number μ, and a semiuniversal unfolding F with base space $M \subset \mathbb{C}^{\mu}$.

<u>Theorem 10:</u> There exists a canonical $(VTERP)$-structure $(H \to \mathbb{C} \times M, \nabla, H_{\mathbb{R}}, P)$ of weight $n+1$, with $\operatorname{rk} H = \mu$.
It can be used to give M the structure of a Frobenius manifold.

For $t \in M$ fixed, the top. part $(H|_{\mathbb{C}^* \times \{t\}}, \nabla, H_{\mathbb{R}}, P)$ is given as follows:

$$\begin{array}{l} F_t : X_t \to \Delta_\eta = \{z \in \mathbb{C} \mid |z| < \eta\} \\ \quad\ \ \cap \\ \quad\ \mathbb{C}^{n+1} \end{array}$$



$$X_t^{(z)} := F_t^{-1}(\text{nbhd in } \Delta_\eta \text{ of } \eta \cdot \frac{z}{|z|}) \qquad \text{for } z \in \mathbb{C}^*$$

$$\Lambda_{(z,t)} := H_{n+1}(X_t, X_t^{(z)}, \mathbb{Z}) \cong \mathbb{Z}^\mu$$

is generated by μ Lefschetz thimbles.

$$H_{(z,t)} := \text{Hom}\,(\Lambda_{(z,t)}, \mathbb{C}) \cong \mathbb{C}^{\mu}, \qquad \text{for } z \in \mathbb{C}^*$$
$$(H_{\mathbb{R}})_{(z,t)} := \text{Hom}\,(\Lambda_{(z,t)}, \mathbb{R}) \cong \mathbb{R}^{\mu}.$$

The intersection form for Lefschetz thimbles,

$$< .,. >: \Lambda_{(z,t)} \times \Lambda_{(-z,t)} \to \mathbb{Z}$$

induces a dual form

$$< .,. >^*: H_{(z,t)} \times H_{(-z,t)} \to \mathbb{C}.$$

Define

$$P := \frac{(-1)^{n(n+1)/2}}{(2\pi i)^{n+1}} < .,. >^* .$$

The extension of $H|_{\mathbb{C}^* \times M}$ to $\{0\} \times M$:

from "oscillating integrals" resp.
from a partial Fourier-Laplace transformation of the Gauss-Manin connection of F.

$P \xleftrightarrow{\text{FL trf.}}$ K. Saito's higher residue pairings

(F. Pham)

The <u>Euler field</u> E on M satisfies:
for any $t \in M$

$$\begin{aligned} & \text{the unfolding } F|(\text{the } E\text{-orbit through } t) \\ \cong\ & \text{the 1-par. unfolding} \\ & (e^\rho F_t \mid \rho \in \text{ nbhd of 0 in } \mathbb{C}) \end{aligned}$$

where $E \cong \frac{\partial}{\partial \rho}$,
and

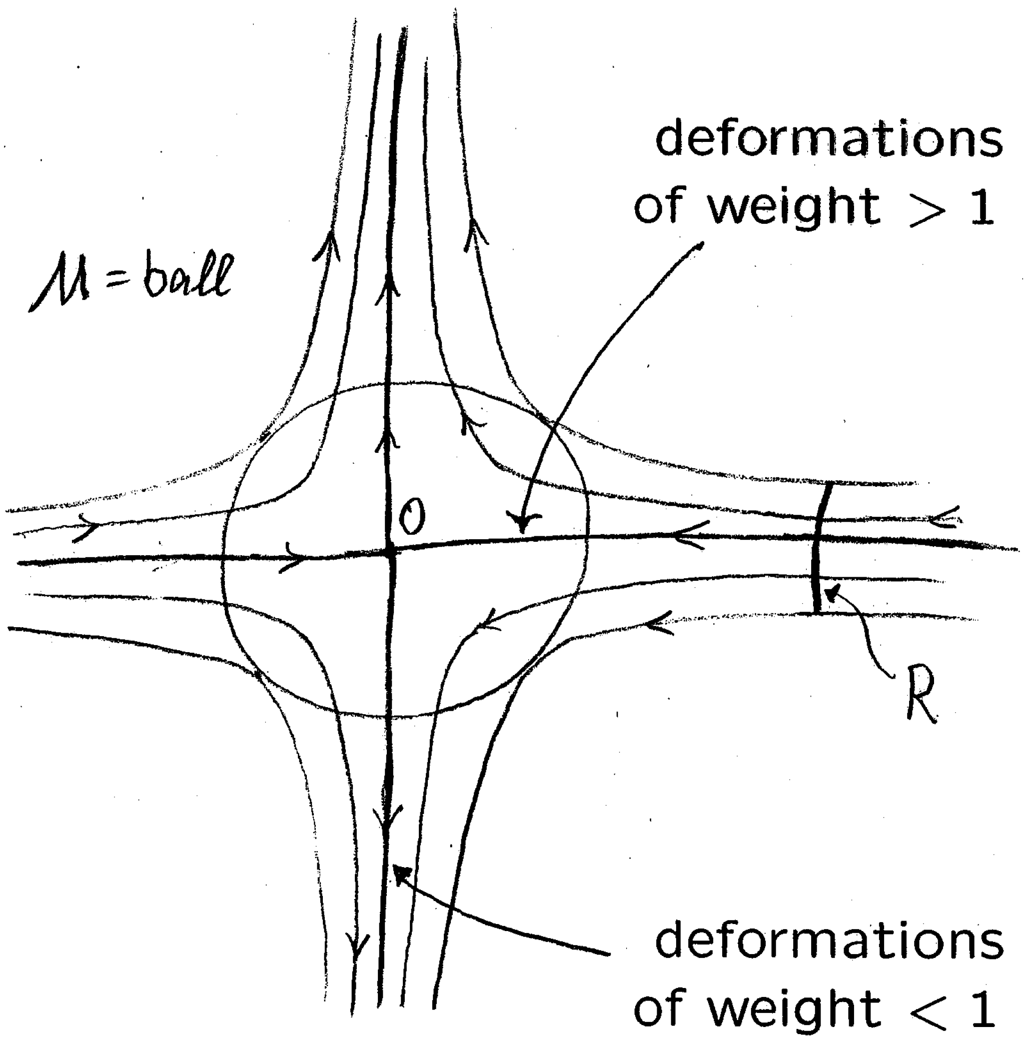
$$(H, \nabla, H_{\mathbb{R}}, P)|\,(\text{the } E\text{-orbit through } t)$$

$$\cong \text{ the 1-par. } (VTERP)\text{-structure}$$

$$\bigcup_{\rho \in (\text{nbhd of 0 in } \mathbb{C})} \pi_\rho^*((H, \nabla, H_{\mathbb{R}}, P)|_{\mathbb{C}\times\{t\}})$$

with $\pi_\rho : \mathbb{C} \to \mathbb{C},\ z \mapsto e^\rho \cdot z.$

Example:
f a quasihomogeneous singularity,



$E \sim$ physicists' renormalization group flow.

<u>Theorem 11:</u> (a) M can be extended uniquely to a manifold M^{ext} with all E-orbits in M^{ext} isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$ or $\{pt\}$.

(b) The canonical $(VTERP)$-structure extends to M^{ext}.

(c) There exists a real analytic subvariety $R \subset M^{ext}$ such that the restrictions of the $(VTERP)$-structure to the components of $M - R$ are $(VtrTERP)$-structures.

For any $t \in M$ the set Crit(F_t) is finite with

$$\sum_{x \in \text{Crit}(F_t)} \mu(F_t, x) = \mu.$$

Associated to $x \in \text{Crit}(F_t)$ is a tuple ("exponents") $\text{Exp}(F_t, x)$: $\mu(F_t, x)$ rational numbers, symmetric around $\frac{n+1}{2}$.

Conjecture 12: If one starts at any $t \in M^{ext}$ and goes sufficiently far along the flow of $\text{Re}\, E$,

- then one does not meet R anymore,
- the $(VTERP)$-structure is a $(Vpos.def.trTERP)$-structure,
- and the eigenvalues of $\mathcal{Q}$ tend to $\bigcup_{x \in \text{Crit}(F_t)} \text{Exp}(F_t, x) - \frac{n+1}{2}$.

<u>Theorem 13:</u> The conjecture is true in the two cases:

(a) F_t has μ A_1-singularities with μ different critical values (eq.: $\mathcal{U}_t$ is semisimple with μ different eigenvalues).

(b) $t \in S_\mu$, i.e. F_t has only 1 singularity x, and $F_t(x) = 0$ (eq.: $\mathcal{U}_t$ is nilpotent).

Proof of (a): a result of Dubrovin ('92); (a) is the semisimple case; $(H, \nabla, H_{\mathbb{R}}, P)|_t$ can be described by Stokes data; using this, Dubrovin's proof is fairly short.

Proof of (b):

<u>Theorem 14</u>
($\Leftarrow$ Schmid's SL_2-orbit theorem '73,
$\Rightarrow$ Cattani-Kaplan-Schmid '86)

Given $(H^\infty, H^\infty_{\mathbb{R}}, S)$,
a classifying space $D = \{F^\bullet \subset H^\infty \mid ...\}$
for PHS, and its compact dual $\check{D} \supset D$.

A pair $(F^\bullet, N)$ with $F^\bullet \in \check{D}$ and
$N : H_{\mathbb{R}} \to H_{\mathbb{R}}$ nilpotent and an infinitesimal isometry of S is part of a
polarized mixed Hodge structure

$\iff$ $\{e^{zN}F^\bullet \mid z \in \mathbb{C}\}$ is a nilpotent orbit,
i.e. $e^{zN}F^\bullet \in D$ for $\operatorname{Im} z$ large.

Proof of (b):
Associated to F_t is a PMHS.
Along the E-orbits of t one obtains a nilpotent orbit.
Some additional estimations and comparisons give Theorem 13 (b).