Classification of operator algebraic conformal field theories by representation categories

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(with Roberto Longo)

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- Representation theory for a single operator algebra (von Neumann algebra)

  ⇝ Almost trivial
  not interesting

- Representation theory for a pair $N \subset M$
  - categorical structure
  - more interesting
    Jones, Ocneanu

- Representation theory for a family of operator algebras $\{M(I)\}$ arsing in QFT.
  categorical structure
  - interesting.
  - useful

Chiral conformal field theory

$\rightsquigarrow$

Axiomatization as a family of operator algebras

$\rightsquigarrow$??

Classification of such families by their representation categories (modular tensor category)

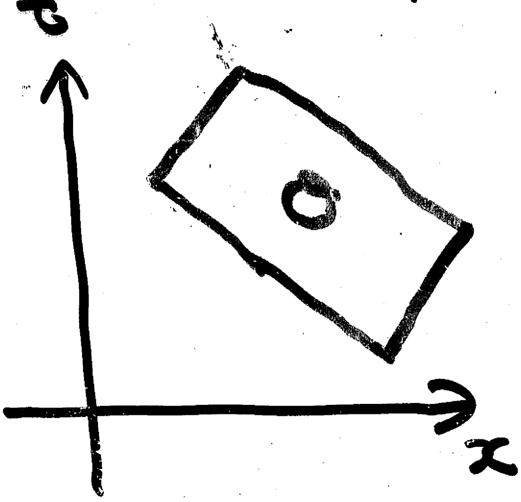
◎ central charge $c < 1$

$\Rightarrow$ such a classification is possible. (K-Longo)

$\begin{cases} \cdot \text{ operator algebras} \\ \cdot \text{ tensor category} \end{cases}$ $\longleftarrow$

- Algebraic Quantum Field Theory

in 1+1 dimensions



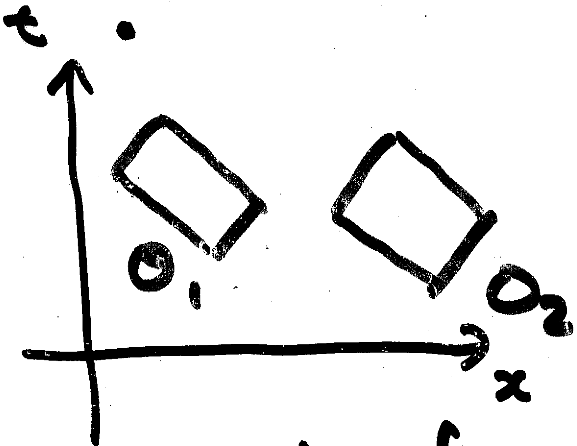
$A(\mathcal{O})$: operator algebra generated by <u>observables</u> in $\mathcal{O}$.

$\mathcal{O}$: "rectangles"

$A(\mathcal{O})$: acts on a fixed Hilbert space H.

$\{A(\mathcal{O})\}$: family of operator algebras on H parametrized by rectangles $\mathcal{O}$

- $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow A(\mathcal{O}_1) \subset A(\mathcal{O}_2)$



speed of light = 1

$\mathcal{O}_1, \mathcal{O}_2$: space-like separated

$\Rightarrow A(\mathcal{O}_1) \subset A(\mathcal{O}_2)'$

$A(\mathcal{O}_2)' = \{ x \text{ on } H : xy = yx \ \forall y \in A(\mathcal{O}_2) \}$

locality

- G : "spacetime symmetry" group (e.g. Poincaré group)

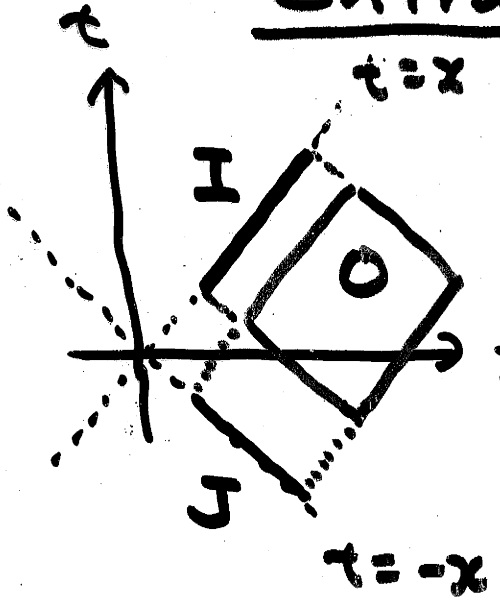
  u_g : projective unitary representation of G on H

$$u_g A(\mathcal{O}) u_g^* = A(g\mathcal{O})$$

"covariance"

- $\Omega \in H$ "vacuum vector"

Chiral theories



$$O = I \times J$$

$$B(O)$$

$\Downarrow$ decomposition

$$A_1(I) \otimes A_2(J)$$

- $I \subset \mathbb{R} \rightsquigarrow A(I)$

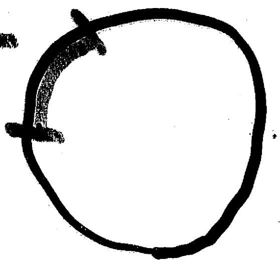
  operator algebra generated by "observables" in I

- $\{A(I)\}$: family of operator algebras parameterized by bounded intervals $I \subset \mathbb{R}$

- Compactify $\mathbb{R}$ to S^1

  higher symmetries

I



$I \subset S^1$

open, connected. $I \neq \phi$.

$\bar{I} \neq S^1$: interval

$A(I)$: operator algebra on H

- $I \subset J \Rightarrow A(I) \subset A(J)$
- $I \cap J = \phi \Rightarrow A(I) \subset A(J)'$ locality
- $G = PSL(2, \mathbb{R})$ Möbius

  or $\mathrm{Diff}(S^1)$

  "spacetime symmetry"

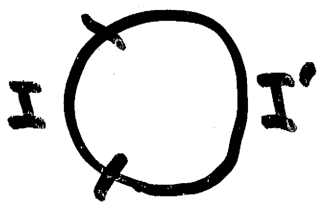
  u_g : projective unitary representation of G on H

  $$u_g A(I) u_g^* = A(gI)$$ covariance
- vacuum vector $\Omega \in H$

$$\overline{\bigcup_I A(I)\Omega} = H$$

- Consequence of axioms

$I \subset S^1$, $I' = \mathrm{int}(S^1 \setminus I)$



$$A(I') = A(I)'$$

Haag duality

- Representation of $\{A(I)\}_{I \subset S^1}$ on another Hilbert space K

$$\pi_I : A(I) \longrightarrow B(K)$$

(no vacuum in K)

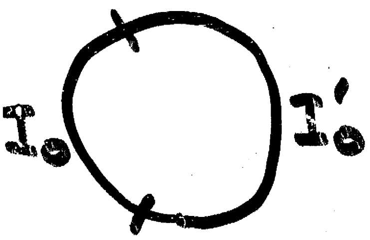
<u>Rem</u> For a fixed I, all representations of $A(I)$ are unitarily equivalent and never irreducible.

- Doplicher - Haag - Roberts theory 1969

Fix $I_0 \subset S^1$

Representation $\pi_{I_0'}$ is unitarily equivalent to $A(I_0') \hookrightarrow B(H)$.

May assume $\pi_{I_0'} = \mathrm{id} : A(I_0') \hookrightarrow B(H)$



$$x \in A(I_0)$$
$$y \in A(I_0')$$
$$xy = yx$$

$$\Rightarrow \pi_{I_0}(x) \underbrace{\pi_{I_0'}(y)}_{y} = \underbrace{\pi_{I_0'}(y)}_{y} \pi_{I_0}(x)$$

$$\Rightarrow \pi_{I_0}(x) \in A(I_0')' = A(I_0)'' = A(I_0)$$

$$\Rightarrow \pi_{I_0} \in \mathrm{End}(A(I_0))$$

$\lambda, \mu \in \mathrm{End}(A(I_0))$

$\Rightarrow \lambda \cdot \mu \in \mathrm{End}(A(I_0))$

"tensor product"

- irreducible decomposition
- direct sum
- dim $\in [1, \infty]$
- Frobenius reciprocity for conjugates

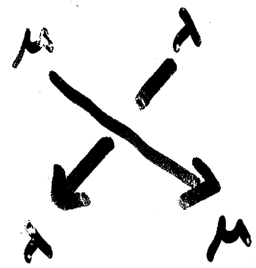
$\rightsquigarrow$ like unitary representations of a compact group

$\rightsquigarrow$ Tensor category of representations of $\{A(I)\}_I$

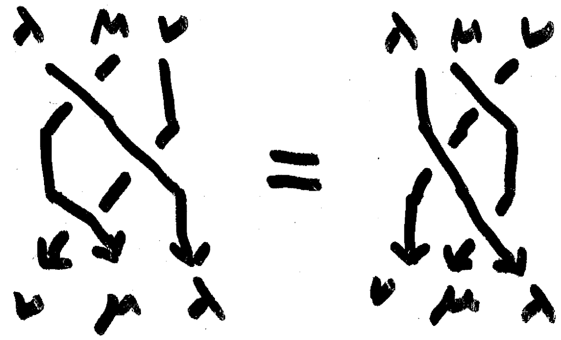
braided tensor category

$$\lambda \cdot \mu = \mathrm{Ad}\, u(\mu, \lambda)\, \mu \cdot \lambda$$

$u(\mu, \lambda)$



braiding



Suppose the number of irreducible representations is finite "rational" theory

$$S_{\lambda\mu} = \lambda$$  $$\mu \in \mathbb{C}$$

$$T_{\lambda\mu} = \delta_{\lambda\mu}$$  $$\lambda \in \mathbb{C}$$

If S is invertible, we get a unitary representation of $SL(2, \mathbb{Z})$

dimension = the number of
irreducible rep.

In such a case, we say
the braided tensor category
is <u>modular</u>.

(• Modular tensor category
⇝ Reshetikhin - Turaev
invariants of 3-manifolds)

• Operator algebraic sufficient
condition for modularity
K - Longo - Müger CMP 2001
"complete rationality"

(use Jones index)

<u>Examples</u> A. Wassermann, Xu

- WZW models $SU(N)_k$, ...
- coset models
  → Virasoro model $c<1$
- orbifold models

Assume $\{A(I)\}$ is diffeomorphism covariant.

⇝ projective rep of $\mathrm{Diff}(S^1)$

⇝ representation of the Virasoro algebra

⇝ central charge $c \in \mathbb{R}$

(numerical invariant of $\{A(I)\}$)

$$c < 1 \Rightarrow c = 1 - \frac{6}{m(m+1)}$$

$m = 3, 4, 5, \ldots$

(Friedan-Qiu-Shenker
Goddard-Kent-Olive)

cf. Jones index < 4

$$\Rightarrow \quad 4\cos^2\frac{\pi}{n} \quad n = 3, 4, 5, \ldots$$

For $c = 1 - \frac{6}{m(m+1)}$, the corresponding "Virasoro" family of operator algebras is constructed by Xu. (coset construction)

- completely rational
- usual fusion rules

$$\{\mathrm{Vir}_c(I)\}_{I \subset S^1}$$

$Vir_c(I)$: "minimal"

General $\{A(I)\}_I$ with

$c < 1$.

$\Rightarrow$ $Vir_c(I) \subset A(I)$

extension of Vir_c

- Representation theory for extensions "α-induction"

$A(I) \subset B(I)$ $\quad I \subset S^1$

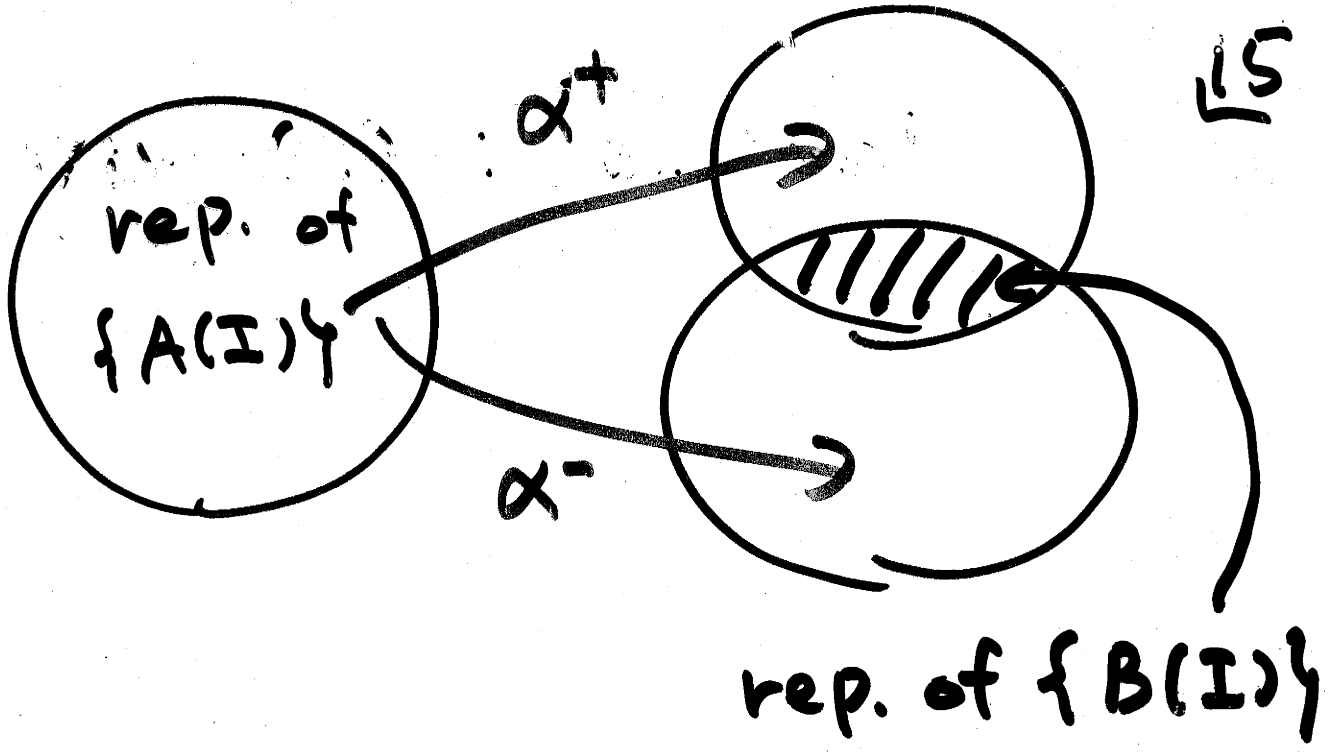
$\lambda \rightsquigarrow \alpha^{\pm}_{\lambda}$

rep. "fake" representation

depends on $\pm$ braiding

Longo-Rehren, Xu

Böckenhauer-Evans-K



$Z_{\lambda\mu} = \dim \operatorname{Hom}(\alpha^+_\lambda, \alpha^-_\mu)$ in $\mathbb{N}$

irreducible rep. of {A(I)}

Böckenhauer - Evans - K CMP 1999

$Z \in (U(SL(2,\mathbb{Z})))'$

"modular invariant"

U : "nearly irreducible" in many cases

rep. of $SL(2,\mathbb{Z})$

The number of such Z is finite.

$$\mathrm{Vir}_c(I) \subset A(I)$$
$$\lambda \qquad\qquad \alpha_\lambda^{\pm}$$

Get a matrix Z from $\{A(I)\}$. invariant of $\{A(I)\}$

- Modular invariants have been classified in many cases.

Virasoro case with $c<1$.

→ Cappelli-Itzykson-Zuber 1987

- at most 3 Z's for each $c<1$

Labelled with pairs of Dynkin diagrams :

type I : (A_{n-1}, A_n)

(A_{4n}, D_{2n+2}), (D_{2n+2}, A_{4n+2})

(A_{10}, E_6) , (E_6, A_{12})

(A_{28}, E_8), (E_8, A_{30})

type II : (A_{4n+2}, D_{2n+3}), (D_{2n+1}, A_{4n})

(A_{16}, E_7), (E_7, A_{18})

Type II modular invariants do not appear now.

(They appear in classification of full conformal field theories.)

Each type I modular invariant corresponds to $\{A(I)\}$ uniquely.

Use Longo's Q-System

"extension of categories"

- Kirillov-Ostrik $SU(2)_k$
  "Quantum subgroup"
- Lam-Lam-Yamauchi (A,D)
  VOA setting. (D,A)

cf. Extensions of $SU(2)_k$

$$A(I) \subset B(I)$$

$SU(2)_k$ classify

Modular invariants for $SU(2)_k$ Cappelli - Itzykson - Zuber

type I : A_n, D_{2n}, E_6, E_8

type II : D_{2n+1}, E_7

A_n : $SU(2)_k$ itself

D_{2n} : simple current extensions

$$\left\{ \begin{array}{l} E_6 : SU(2)_{10} \subset SO(5), \\ E_8 : SU(2)_{28} \subset (G_2). \end{array} \right.$$

conformal embedding

cf Kirillov - Ostrik

Izumi

Main results: K-Longo

$\{A(I)\}_I$ with $c<1$

completely classified by their representation categories.

They are labeled with pairs of Dynkin diagrams as above (type I)

- Virasoro models
- Simple current extensions
- Four exceptionals (Two cosets)