Homological mirror symmetry and the quartic surface

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Kontsevich's conjecture (1994):

$$\boxed{\text{Symplectic geometry on } X} \longleftrightarrow \boxed{\text{Complex geometry on } X^\vee}$$

$(X, X^\vee)$ mirror pair (X symplectic with $c_1 = 0$; $X^\vee$ complex with $K_{X^\vee} \cong \mathcal{O}_{X^\vee}$, actually a family). Noteworthy informal remarks:

- For known $(X, X^\vee)$, conjecture is mathematically precise[1,2]
- Derived from formal analysis of topological field theories (A and B models of Witten) + mathematics intuition

[1] Important correction by Fukaya et al.

[2] Slight correction by P.S.

Simplifications, modifications

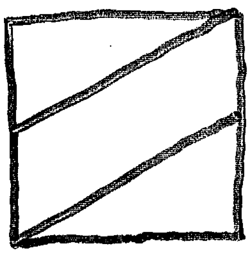
First, a very naive version:

$$\begin{array}{c}\text{Lagrangian submanifolds}^{1} \text{ of } X/\underset{(\text{exact})}{\text{isotopy}} \\ \updownarrow ? \\ \text{Coherent sheaves on } X^\vee/\cong\end{array}$$

This is of interest to symplectic geometers because little is known about the topology of Lagrangian submanifolds.

Basic example (Kontsevich, Polishchuk-Zaslow) 1997

$X = T^2 = X^\vee$ elliptic curve.



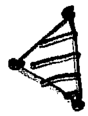
line of slope $\sigma = \frac{d}{r}$ $\longleftrightarrow$ stable vector bundle over $X^\vee$ of rank r, degree d

$\cong$ of moduli spaces

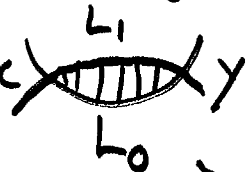
1 carrying flat bundles

To get closer to the spirit of Kontsevich's conjecture, pass from <u>sets</u> to <u>categories</u>.

$$\text{Fukaya}^{2}\text{ category } HF(X)$$
$$\updownarrow ?$$
$$Coh(X^\vee)^{1}$$

<u>Roughly</u>, $HF(X)$ is constructed from Lagrangian submanifolds & (pseudo) holomorphic "bigons"  and triangles , as follows:

<u>objects</u> – Lagrangian submanifolds

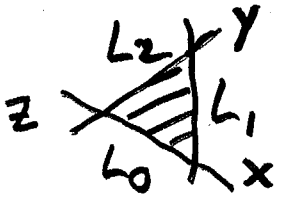
<u>morphisms</u> – $Hom(L_0, L_1) = HF^*(L_0, L_1)$ $= H(CF^*, \partial)$. CF^* freely generated by intersection points, x 

$y \to \partial(y) = x$

weight

weights $e^{-\int \omega}$

<u>composition of morph.</u>



$\longrightarrow y \circ x = z$

compare w/ structure sheaves

1 with all Ext^i as morphisms.

2 Donaldson–Fukaya

- In general, no version of K's conjecture based on $HF(X)$ and $Coh(X^\vee)$ can be appropriate (= true).

The 2 categories are too different, e.g. automorphisms, kernel/cokernel... two ways to proceed from here:

<u>SYZ conjecture</u>

Restrict to **special** Lagrangians/ **stable** sheaves, uses <u>Ricci flat metric</u>...
("quantum" corrections?)

Back to K's original idea: more <u>homological algebra</u>, **derived** categories...

From now on, <u>quartic surfaces</u>. First, the complex side:

$$X^\vee = \{\underbrace{z_0 z_1 z_2 z_3 + q\left(\sum_i z_i^4\right)}_{p_q(z)} = 0\}^{\sim} \Big/ (\mathbb{Z}/4)^2$$

here, $(\mathbb{Z}/4)^2$ acts by $\mathrm{diag}(i,-i,1,1)$ and $\mathrm{diag}(i,1,-i,1)$; $\sim$ is minimal resolution; and q is in the ground field, which is

$$\Lambda = \mathbb{C}[q^{-1}][[q]](q^{1/2}, q^{1/3}, q^{1/4}, \dots)$$

So, $X^\vee$ is a smooth K3 surface over Λ, which is alg. closed. Take <u>derived category</u>

$$D^b(\mathrm{Coh}(X^\vee));$$

by Kapranov-Vasserot, this is $\cong$ to $(\mathbb{Z}/4)^2$-equivariant sheaves on the quartic surface $\{p_q(z)=0\}$. In particular, we get 64 eq. sheaves from $\Omega^i(i)$ on $\mathbb{P}^3_\Lambda$.
$(0 \le i \le 3)$

Formal properties of derived category:

- It is triangulated, i.e. admits <u>mapping cones</u>:

$$
\begin{tikzcd}
 & \mathrm{Cone}(f) \arrow[dl, "{[1]}"'] & \\
E \arrow[rr, "f"'] & & F \arrow[ul]
\end{tikzcd}
$$

As a special case, we have the twisting operation,

$$T_E(F) = \mathrm{Cone}\left(\mathrm{Hom}(E,F)\otimes E \xrightarrow{ev} F\right).$$

- It is idempotent closed (in fact <u>saturated</u>),

$$E \circlearrowleft \pi \quad \pi^2 = \pi \;\rightarrow\; E \cong E_0 \oplus E_1,$$
$$\pi \cong id_{E_0} \oplus 0_{E_1}.$$

Under these 2 operations, $D^b(\mathrm{Coh}(X^\vee))$ can be generated by a finite number of objects; call these <u>split-generators</u>.

The symplectic side: $X = \{\sum_{i=0}^{3} z_i^4 = 0\}$ (or any other smooth quartic) with the F-S symplectic form ω, and $(2,0)$-form η.

$L \subset X$ oriented Lagrangian $\leadsto$ have phase (Lagrangian angle) function $\alpha_L : L \to S^1 \subset \mathbb{C}$,

$$\begin{cases} \alpha_L(x) = \eta_x(e_1 \wedge e_2) \\ TL_x = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \quad (\text{orthonormal}) \end{cases}$$

Def: A Lagrangian "brane" is $L^\# = (L, \alpha^\#, P)$ where $\alpha^\# : L \to \mathbb{R}$ is a lift of α_L - the grading - and $P = TL^{1/2}$ is a Spin structure

Next, recall $\omega = \frac{i}{2\pi} F_A$ for A = connection on $\mathcal{O}_X(1) \leadsto A|L$ flat $U(1)$-bundle.

Def: L is rational (Bohr-Sommerfeld orbit) if the monodromy of $A|L$ is of finite order (roots of unity).

(compare integrable systems)

Finally, let $L \subset M$ be a Lagrangian which admits a grading ($\Leftrightarrow$ zero Maslov class). Then, for generic J, $\nexists$ J-holomorphic discs with boundary on L. Call such J <u>unobstructed</u>.

(the obstructed J form a - possibly dense - set of codim. 1 "walls" in the space of compatible almost cx structures)

<u>Fact</u>: $\boxed{\text{Objects of } \mathcal{F}(X)}$ are pairs

$$(L^{\#}, J)$$

consisting of a <u>rational</u> Lagrangian brane $L^{\#}$ and an unobstructed J.

<u>Note</u>: $L \cong S^2 \Rightarrow$ <u>grading</u> always exists, <u>Spin</u> structure is unique, <u>rationality</u> is trivial, and J is irrelevant!

__Fact__: $\mathcal{F}(X)$ is an $\boxed{A_\infty\text{-category}/\Lambda.}$!

$$\mu^1 = \partial : \hom^*(L_0, L_1) \circlearrowleft [1],$$
$$\mu^2 : \hom^*(L_0, L_1) \otimes_\Lambda \hom^*(L_1, L_2) \xrightarrow[{[0]}]{} \hom^*(L_0, L_2),$$
$$\mu^3 : \hom^*(L_0, L_1) \otimes_\Lambda \hom^*(L_1, L_2) \otimes_\Lambda \hom^*(L_2, L_3) \xrightarrow[{[-1]}]{} \hom^*(L_0, L_3), \ \ldots$$

(but no μ^0). Geometrically

$$\hom^*(L_0, L_1) \cong^1 \Lambda^{|L_0 \cap L_1|\,2}$$

and μ^d counts pseudo-holomorphic $(d+1)$ gons. Formal __closure__ under __cones__ + __idempotent splittings__, and passage to cohomology $\rightsquigarrow$ split-closed derived Fukaya category

$$\boxed{D^\pi \mathcal{F}(X).}$$

[1] Not quite canonical; [2] graded by CZ index.

Change of coordinates: each $\psi(q) \in \mathbb{C}[[q]]$, $\psi(0)=0$, $\psi' \neq 0$, gives rise to a field automorphism of Λ: $\varphi(q) \mapsto \varphi(\psi(q))$, unique up to Galois action of $\hat{\mathbb{Z}}$.

Homological mirror conjecture for the quartic surface (finally!) There exists a ψ (the mirror map) such that

$$\boxed{D^{\pi}\mathcal{F}(X) \cong D^b\,\mathrm{Coh}(X^{\vee})_{q \mapsto \psi(q)}}$$

equivalence of triangulated categories over Λ

Thm (P.S. 2002) It's true.

Note: physics prediction for ψ remains unconfirmed. Also: can do same with finite characteristic ($\neq 2,3$)

Two consequences:

(1) $K_0(D^\pi \mathcal{F}(X)) \cong K_0(Coh(X^\vee))$

$\qquad\qquad\qquad\qquad \cong CH_*(X^\vee)$

mod torsion

∞-dim. by Mumford's thm

Of course, similar implications for higher K-theory, Hochschild and cyclic (co)homology

(2) $$\begin{array}{ccc} Auteq(D^\pi \mathcal{F}(X))^{1} & \cong & Auteq(D^b Coh(X^\vee)) \\ \uparrow & & \vdots \\ \pi_0(Symp(X)) & \xleftarrow{\text{monodromy}} & \pi_1(\mathcal{M}_X) \end{array}$$

$\mathcal{M}_X$ the moduli space of smooth quartic surfaces. In particular, one can use this map to investigate the (∞) kernel of

$$\pi_0(Symp(X)) \to \pi_0(Diff(X)),$$

which (like $\pi_0(Lag(S^2, X))$ is still quite mysterious.

1 up to translation (or extend Symp)

Proof does <u>not</u> use

$\text{SYZ}\left\{\begin{array}{l}\bullet\ \text{Ricci-flat (hyperkähler) metric}\\ \bullet\ \text{Special Lagrangian submanifolds}\\ \bullet\ \text{Elliptic fibration}\end{array}\right.$

- Localization } enumerative mirror symmetry

<u>does</u> use

- Split-generators
- Deformation theory of A_∞ structures
- <u>Symplectic aspects of Picard-Lefschetz theory</u>, in particular one (genus ≥ 2) Lefschetz pencil on the quartic
- Computer (elimination theory, computations in derived cat.)
- Some intuition from physics[1] (linear σ-model)

[1] Thanks to Mike Douglas!

complex	symplectic
64 objects $\mathcal{E}_i$ obtained from $\Omega^k(k)$	64 Lagrangian S^2's L_i obtained as <u>vanishing cycles</u>
Kontsevich relation[1] $\prod_i T_{\mathcal{E}_i} = (-\otimes \mathcal{L})[2]$	$\prod_i \tau_{L_i}$ (large complex limit monodromy) is <u>positive</u>
$\bigoplus_{i,j} \mathrm{Hom}^*(\mathcal{E}_i, \mathcal{E}_j)$ graded algebra	$\bigoplus_{i,j} HF^*(L_i, L_j)$ graded algebra
large complex structure limit $q=0$ $(z_0 \dots z_3 = 0)$	affine limit (makes ω trivial) $z_i \neq 0$, so that $X^{aff} \subset (\mathbb{C}^*)^3$
?	dimensional induction procedure for computations in Fukaya categories (P.S. 2000-2002) based on Picard-Lefschetz theory

[1] In the equivariant sheaf picture; $\mathcal{L} = \mathcal{O}(4)$.

Symplectic topology at a glance

$$\begin{tikzcd}[column sep=small] & \left\{\begin{array}{c}\text{symplectic}\\ \text{manifolds}\end{array}\right\} \arrow[dl, leftrightarrow, "\begin{array}{c}\text{almost}\\ \text{holomorphic}\\ \text{(Donaldson)}\end{array}"'] \arrow[dr, "\begin{array}{c}\text{pseudoholomorphic}\\ \text{(Gromov)}\end{array}"] & \\ \left\{\begin{array}{l}\text{combinatorial}\\ \text{(braid) data}\end{array}\right\} \arrow[rr, dashed, "?"'] & & \left\{\begin{array}{l}\text{(homotopy)}\\ \text{alg. structures}\end{array}\right\} \end{tikzcd}$$

In our situation, given $Y^n \subset \mathbb{C}^N$ generically embedded (+ conditions at ∞) we look at successive linear maps

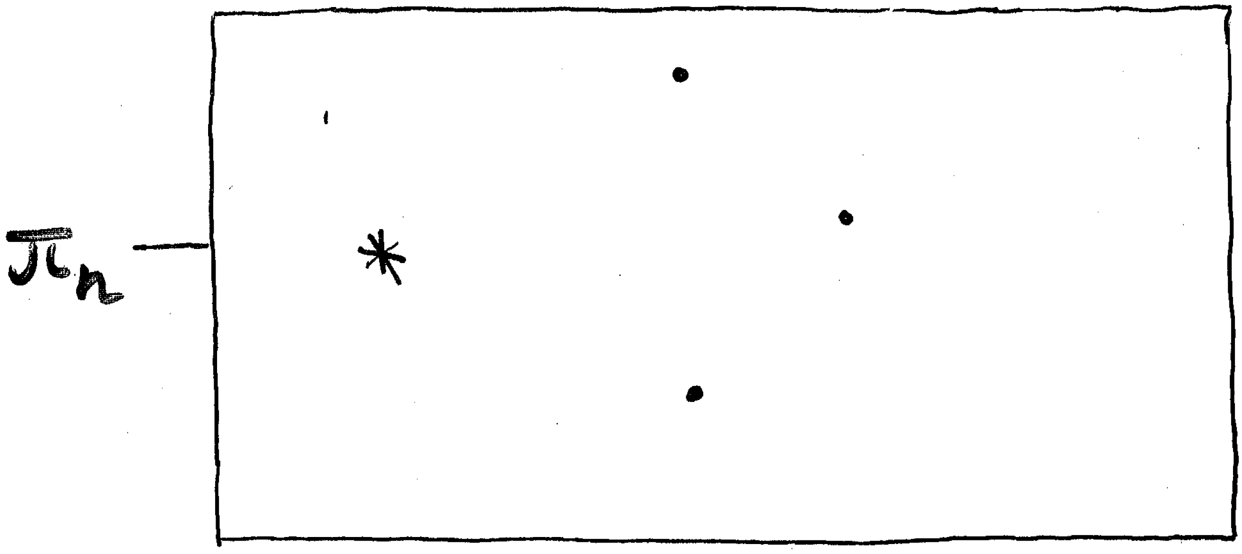
$$\pi_n : Y^n \longrightarrow \mathbb{C},$$
$$Y^{n-1} = \pi_n^{-1}(0)$$
$$\pi_{n-1} : Y^{n-1} \longrightarrow \mathbb{C},$$
$$Y^{n-2} = \pi_{n-1}^{-1}(0)$$

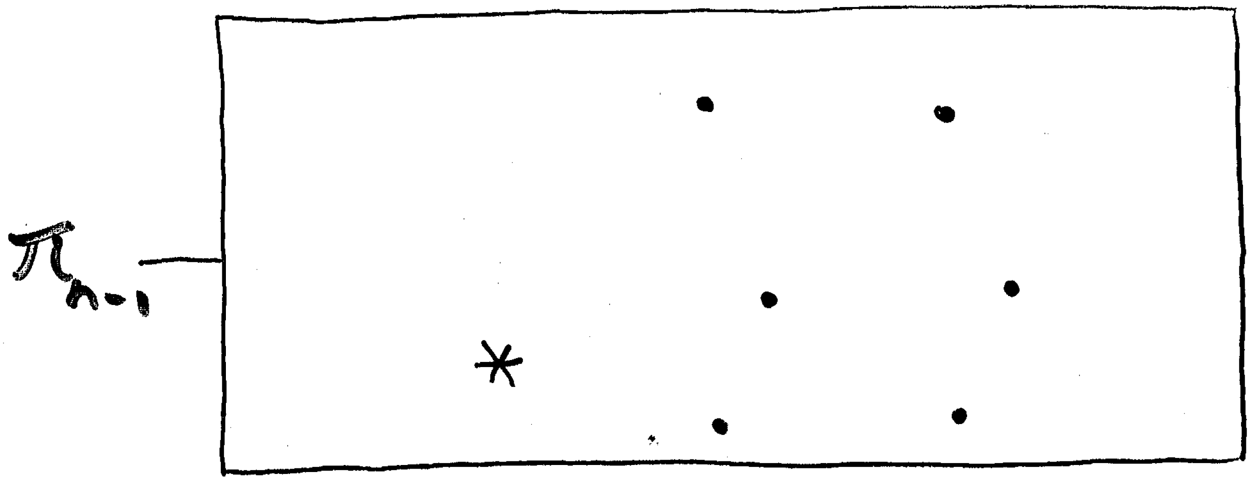
...

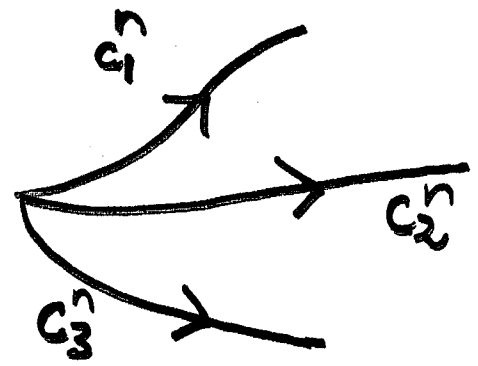
down to the Riemann surface level.

Construct Picard-Lefschetz diagrams:

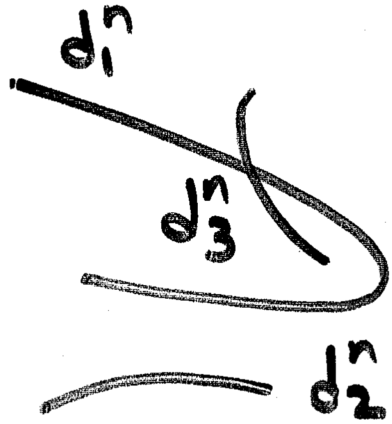
• = Critical points, * = base point





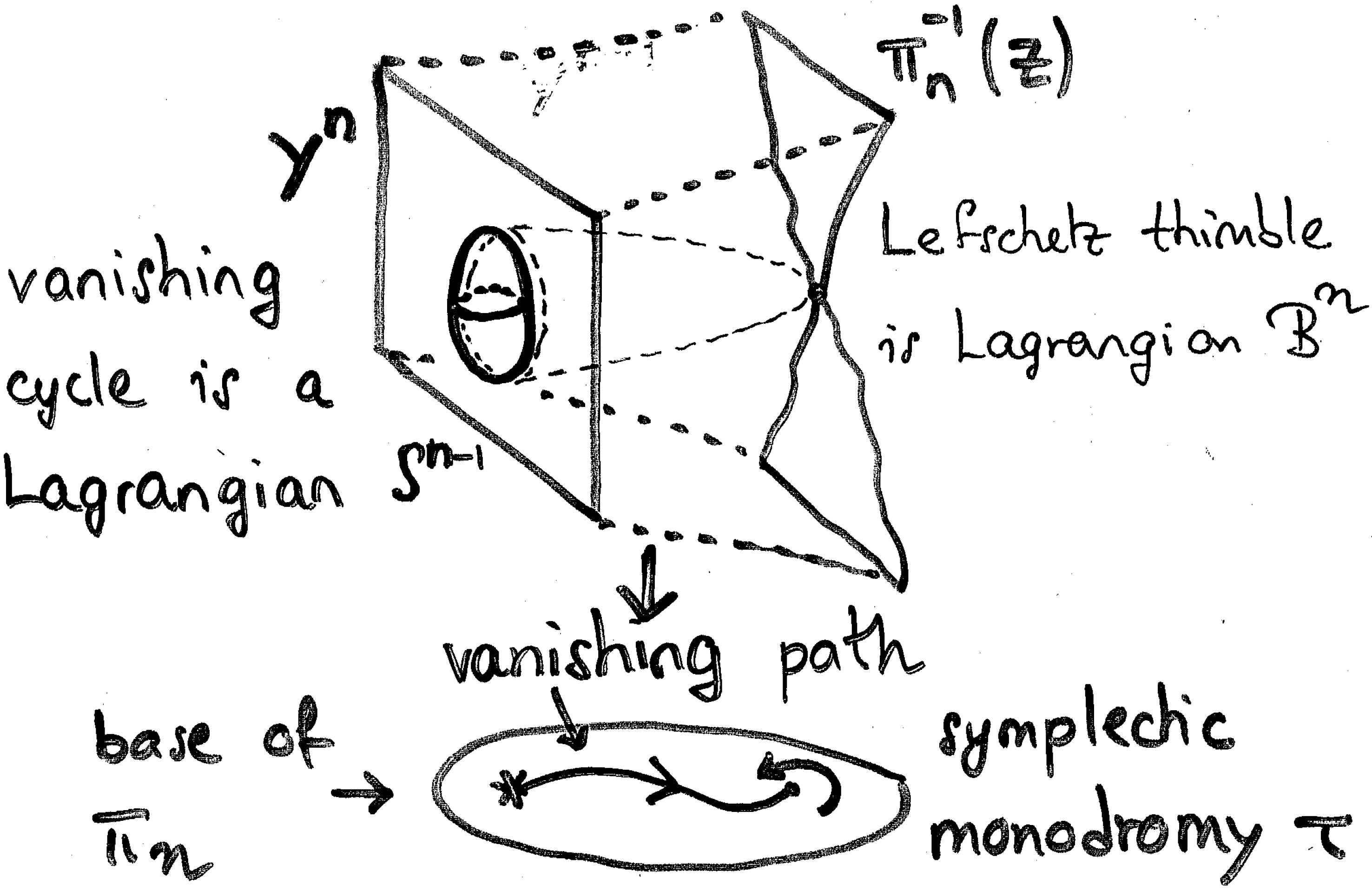


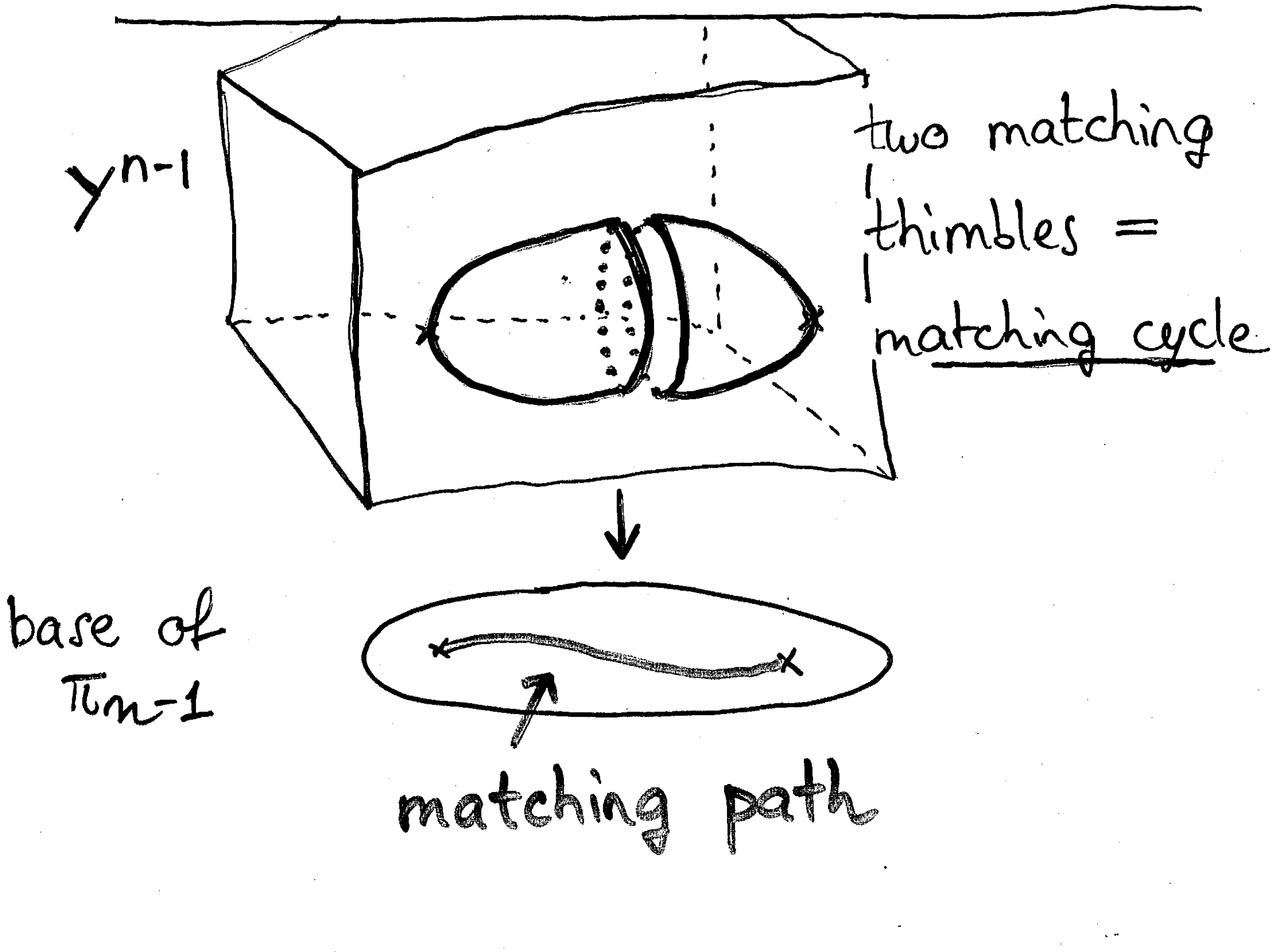
c_k^n - distinguished basis of <u>vanishing paths</u>



d_k^n - corresponding <u>matching paths</u>

Same process for all n (description is highly non-unique). Symplectic geometry interpretation:


Y^n
$\pi_n^{-1}(z)$
vanishing cycle is a Lagrangian S^{n-1}
Lefschetz thimble is Lagrangian B^n
vanishing path
base of π_n
symplectic monodromy τ


Y^{n-1}
two matching thimbles = matching cycle
base of π_{n-1}
matching path

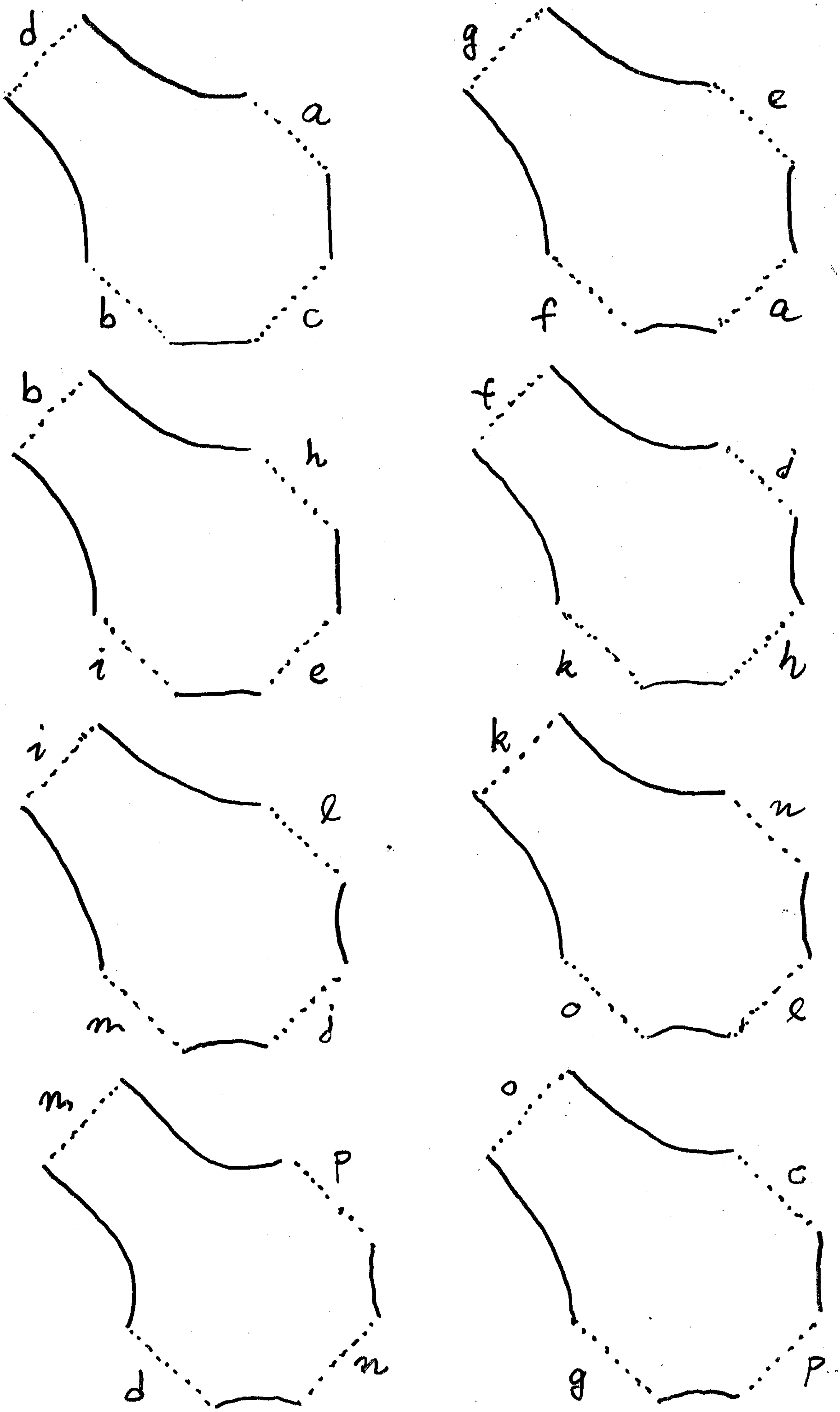
Corresponding hierarchy of (A_∞) categories, leading to $D^\pi \mathcal{F}(Y^n)$:

$$\begin{tikzcd}
D^\pi \mathcal{F}(Y^n) \arrow[r, hook, "\text{full}"] & D^\pi \mathcal{F}(\Pi_n) \arrow[d, phantom, "\cong" rotate=-90] \\
& D^\pi \vec{\mathcal{F}}(Y^{n-1}, \text{vanishing cycles of } \Pi_n) \arrow[dl, hook', "\text{non-full}"'] \\
D^\pi \mathcal{F}(Y^{n-1}) \arrow[r, hook] & D^\pi \mathcal{F}(\Pi_{n-1}) \arrow[d, phantom, "\cong" rotate=-90] \\
& D^\pi \vec{\mathcal{F}}(Y^{n-2}, \text{vanishing cycles of } \Pi_{n-1}) \arrow[dl, hook'] \\
\dots &
\end{tikzcd}$$

down to $D^\pi \vec{\mathcal{F}}(Y^1, \dots)$ which is a purely combinatorial-topological object (built from curves on a surface, intersections and immersed polygons)[1] $\to D^\pi \mathcal{F}(Y^n)$ determined by <u>finitely many $\mathbb{Q}$ numbers</u>.

[1] Actually, can go down to dimension 0...

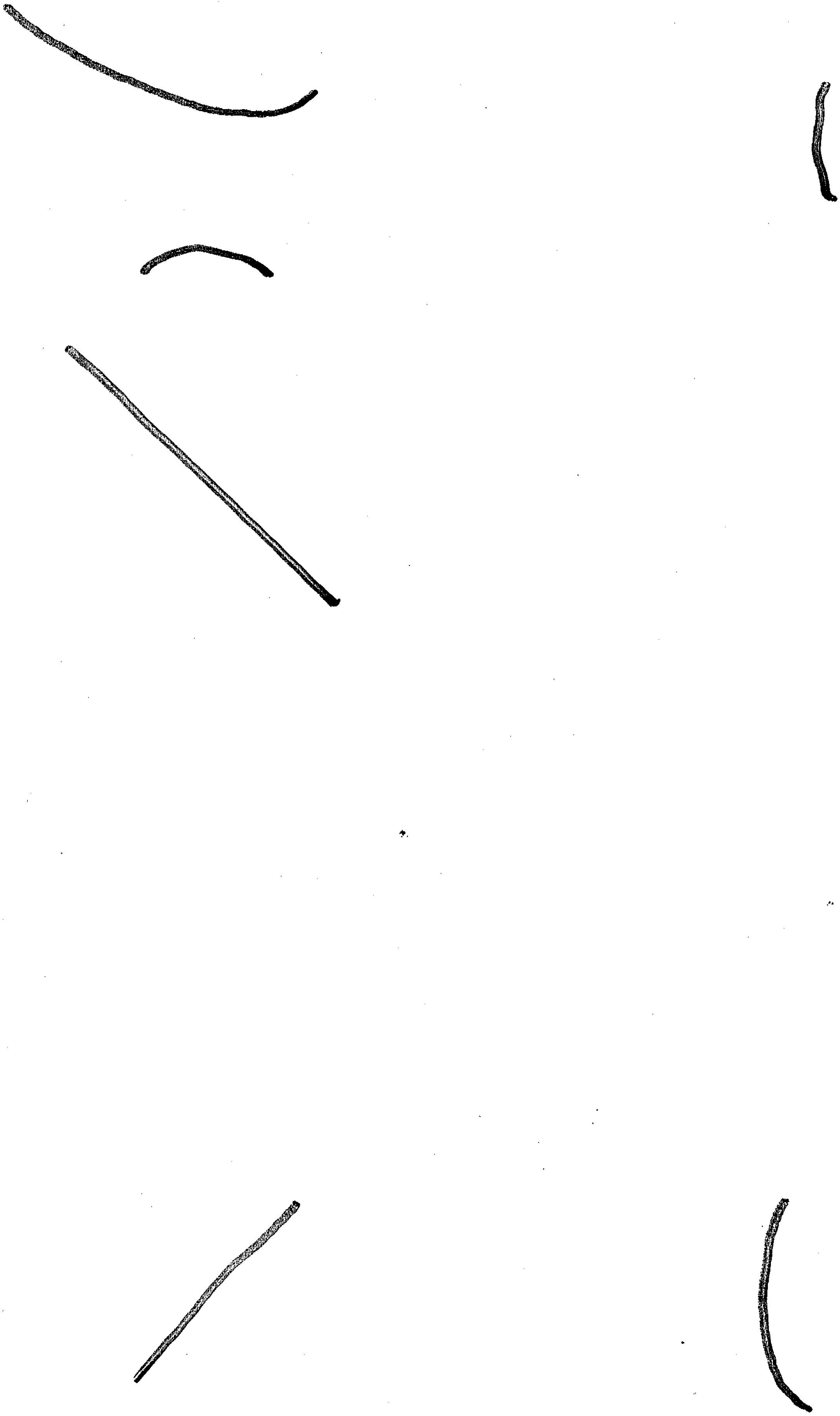
Y^1 = affine genus 3 surface



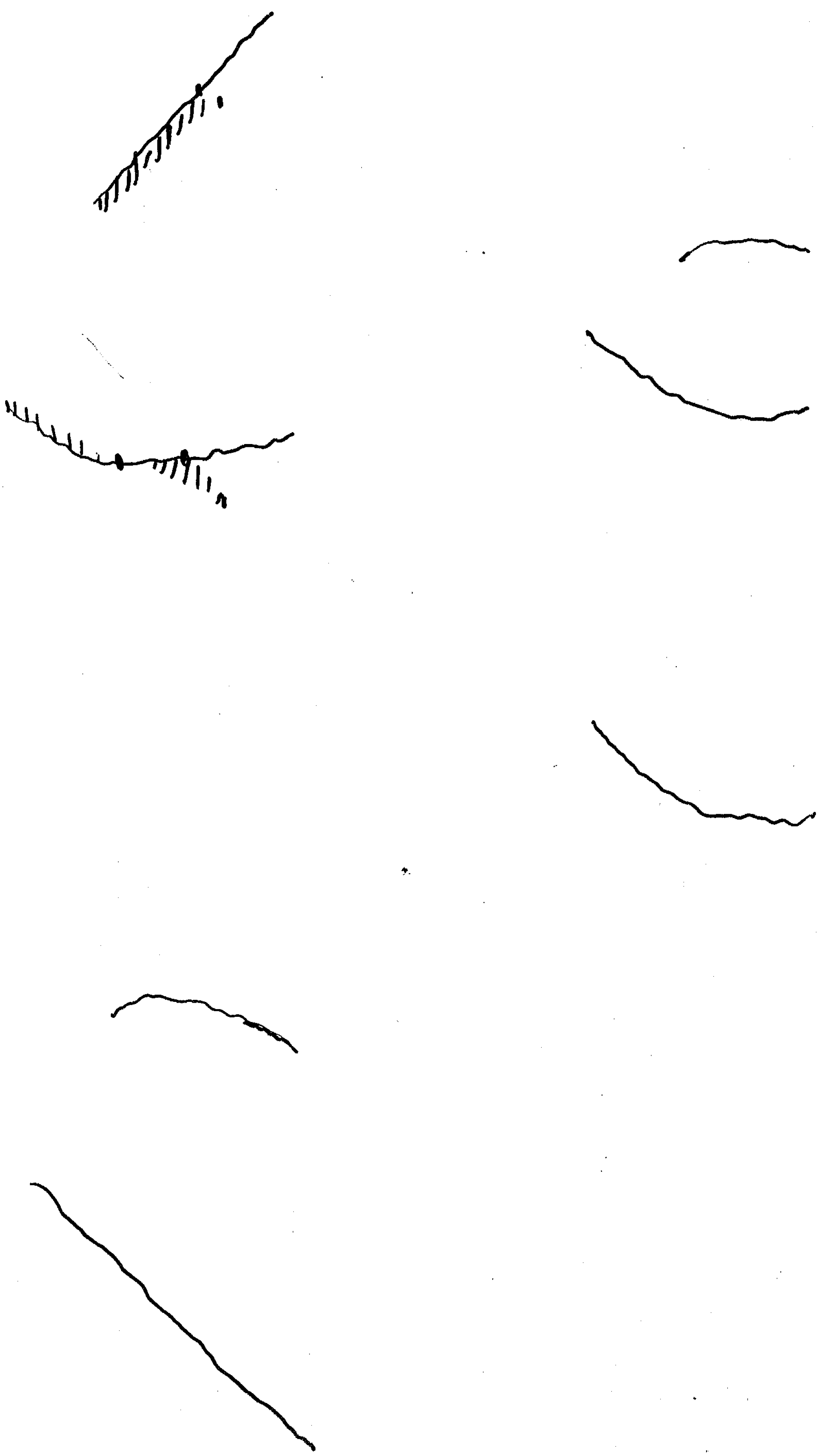
vanishing cycles $\underline{1}$

c

2



3 (one of 89)



For $X = T^2$ this is already quite close to Kontsevich's conjecture, and a version of it can be proved in this way (PZ). Similar partial results for abelian varieties (Fukaya, Kontsevich–Soibelman, ...)
However, in general this kind of "simplification" is bad for 2 reasons:

- It's impossible to understand $HF(X)$ explicitly.

E.g. take X = **quartic surface** $\subset \mathbb{P}^3$.
$Lag(S^2, X)$ = Lagrangian embeddings,
$Emb(S^2, X)$ = all embeddings (C^∞).

$$\begin{tikzcd}
\pi_0(Lag(S^2,X)) \arrow[r, "\infty \text{ to } 1"', "\text{(P.S. 1998)}"'] \arrow[d, "\text{onto (easy)}"'] & \pi_0(Emb(S^2,X)) \\
\{A \in H^2(X) \mid \omega(A) = 0,\ A \cdot A = -2\} & \pi_0(Lag) = ??
\end{tikzcd}$$

In our case, consider (technical simplification)

$$Y^3 = (\mathbb{C}^*)^3/(\mathbb{Z}/4)^2 = \{u_0u_1u_2u_3 = 1\}$$

$$\pi_3(u) = u_0+u_1+u_2+u_3$$

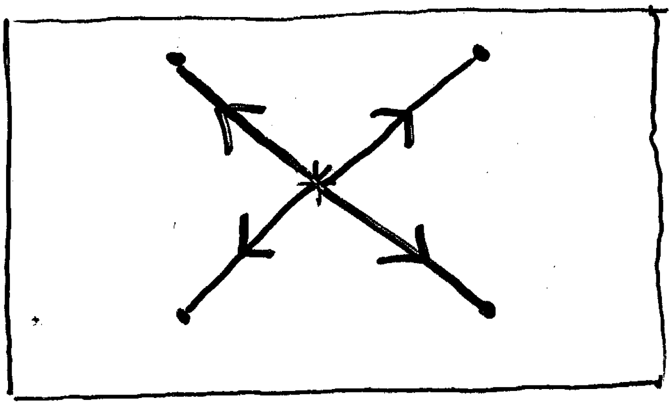
$$Y^2 = \pi_3^{-1}(0) = X^{aff}/(\mathbb{Z}/4)^2 \subset Y^3$$

$$\pi_2(u) = u_0 + iu_1 - u_2 - iu_3$$ perturb slightly

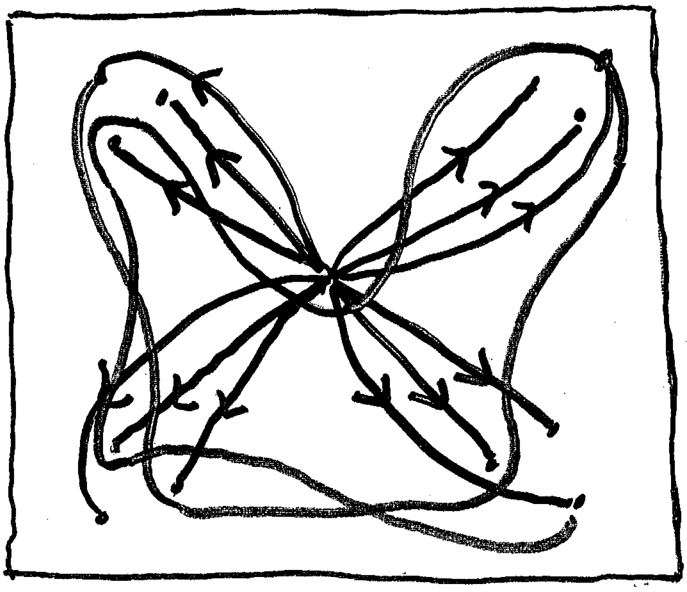
$$Y^1 = \pi_2^{-1}(0) \subset Y^3$$

π_3-vanishing [redacted] paths

64 spheres in X^{aff}



π_2-matching paths and vanishing paths



Possible further developments

Hutchings, Lee

(a) Determining the mirror map

(b) Genus 1 theory (Reidemeister torsion in $HF^* \leftrightarrow$ Ray-Singer torsion)

Consequences: convergence in $\mathcal{F}(X)$, field of definition, enumerative implications

(c) Enriques surfaces (interesting because

Voisin $\rightarrow$ $CH_0(\text{Enriques}) \cong \mathbb{Z}$)

(d) Applications to symplectic topology ($\pi_0(Lag(S^2, X))$, $\pi_0(Symp(X))$, open questions about Lagrangian T^2 ...)

More obvious generalizations:

(e) Other (families of) K3's

(f) Quintic threefold

(g) More systematic approach to homological mirror symmetry !!

toric geometry, linear σ-models