A generalization of Kac's character formula to higher genus Riemann Surface in WZW model

(An abelianization of SU(2) WZW model)

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C : Riemann surface of genus $g\,(\geq 1)$

M_g : the moduli space of rank 2 hol. semi-stable v.b. on C with det $= \mathbb{1}$

$$= \mathrm{Hom}(\pi_1(C), SU(2)) / \mathrm{conj}$$

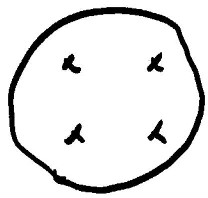
$\mathcal{L}$: det. line bdle on M_g

$\Gamma(M_g, \mathcal{L}^{\otimes k})$: conformal block of level k

Kac's character formula

$g = 1$

$M_1 = \mathbb{P}^1$ with distinguished 4 pts



$\rho : \pi_1(C) \to \{\pm 1\}$

$T \xrightarrow{\pi} \mathbb{P}^1$ 2-fold branched cover

$$\psi_j \in \Gamma(\mathbb{P}^1, \mathcal{L}^k) \xrightarrow{\pi^*} \Gamma(T, \tilde{\mathcal{L}}^{2k}) \ni \tilde{\psi}_j$$

$$\boxed{\tilde{\psi}_j = \frac{\vartheta\begin{bmatrix} \frac{2j+1}{2(k+2)} \\ 0 \end{bmatrix} - \vartheta\begin{bmatrix} -\frac{(2j+1)}{2(k+2)} \\ 0 \end{bmatrix}}{\vartheta\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} - \vartheta\begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix}}}$$

$0 \leq j \leq \frac{k}{2} \qquad j \in \frac{1}{2}\mathbb{Z}$

Abelianization

Hitchin 1985 The self-duality equation on a Riemann surface, Proc. L.M.S.

$$\begin{tikzcd} \tilde{C} \arrow[r, "p"] \arrow[loop below, "\sigma"] & C \end{tikzcd}$$

2-fold branched covering with $4g-4$ branched pts $b = \{x_j\}$

$$g(\tilde{C}) = 4g-3$$

Prym variety P

$$P = \left\{ L \to \tilde{C} \text{ line bdle} \;\middle|\; \begin{array}{l} \sigma^* L \otimes L = [b] \\ \deg L = 2g-2 \end{array} \right\}$$

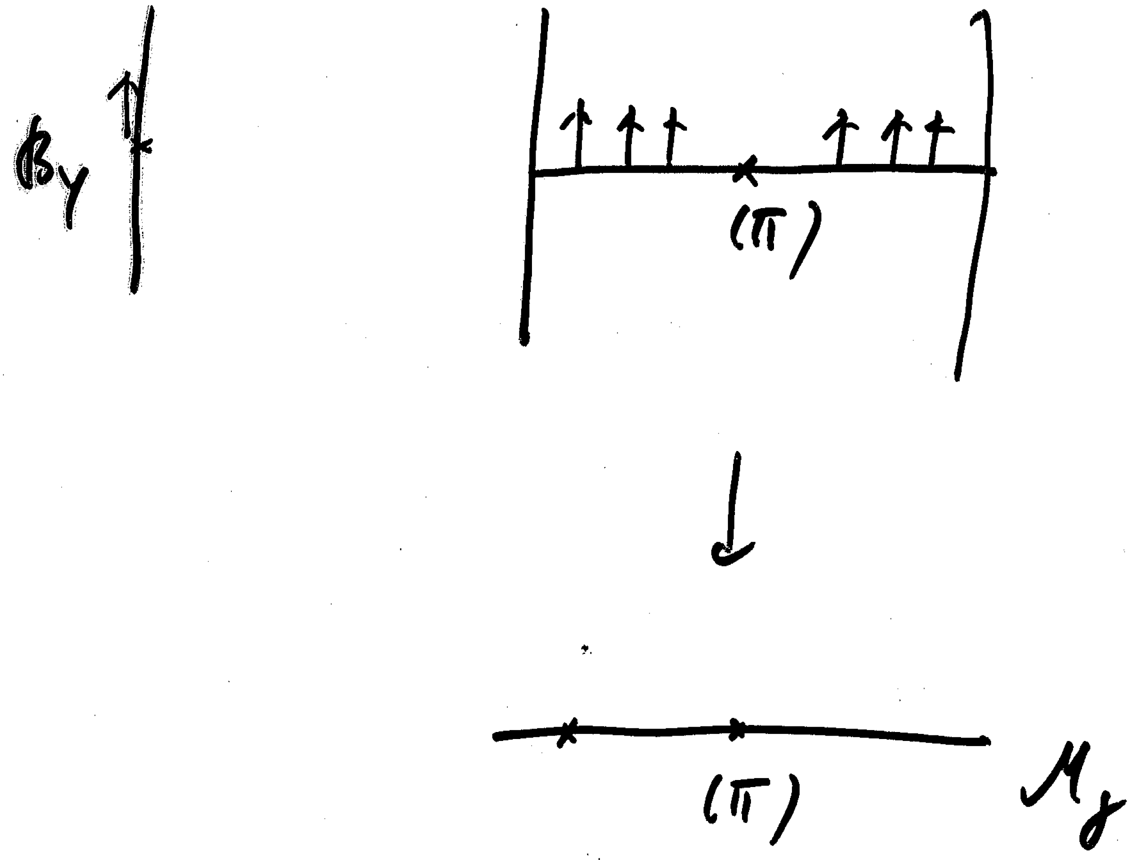
$$\pi : P \to M_g \quad \text{dominant map}$$

$$L \mapsto p_* L \quad \text{direct image}$$

$$\begin{tikzcd} \tilde{\mathcal{L}}^{k} \arrow[r] \arrow[d] & \mathcal{L}^{k} \arrow[d] \\ P \arrow[r, "\pi"] \arrow[u, bend left, "\tilde{\psi}"] & M_g \arrow[u, bend right, "\psi"'] \end{tikzcd}$$

Thm of Hitchin

$\pi : P \longrightarrow M_g$ is biholomorphic isomorphism outside the divisor (Π), where Π is a degree 4 theta function

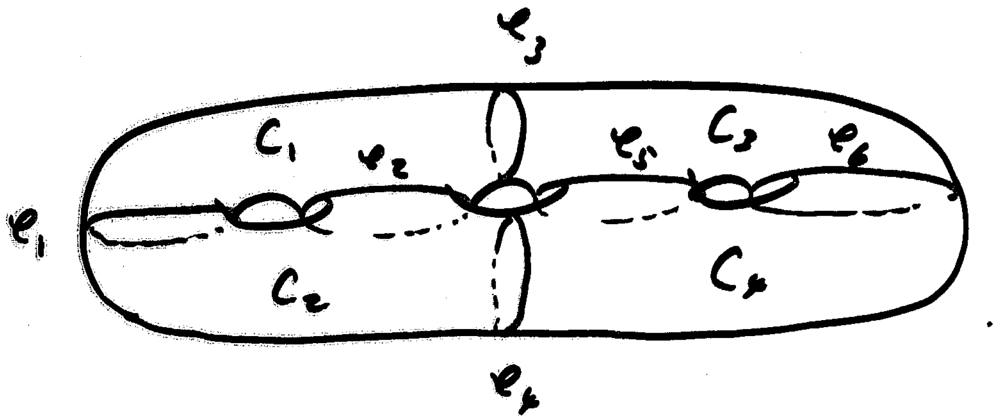


point inverse vector field v

Coordinate on Prym variety

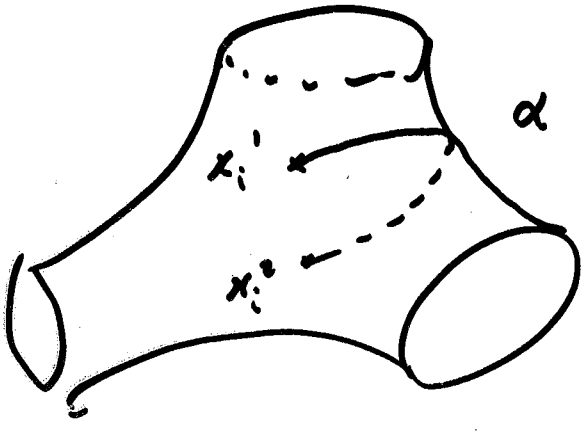
Pant Decomposition of C

$Y = \{ e_l , C_i \}$ $\quad 1 \le l \le 3g-3$, $\quad 1 \le i \le 2g-2$



$B_Y \subset B$ open subset

$B_Y = \{ \bar{b} = (b, \alpha) \mid b \cap C_i = \{x_i^1, x_i^2\}$, $\alpha \cap C_i$ = arc in C_i connecting x_i^1 and x_i^2 $\}$



$\Lambda_0 \subset \Lambda$ in ℓ

$\Lambda_0^* \supset \Lambda^*$ in ℓ^*

$$P = H_1(\tilde{C}, \mathbb{R})_- \big/ \Lambda \oplus \Lambda^*$$

$\Omega = (\Omega_{ij})$ Riemann matrix

$\{w_\ell\}_{1 \le \ell \le 3g-3}$ hol. 1-form on $\tilde{C}$

$\sigma^* w_\ell = -w_\ell$

$$\int_{[\tilde{e}_\ell]-\sigma[\tilde{e}_\ell]} w_{\ell'} = \delta_{\ell\ell'}$$

$$\Omega_{ij} = \int_{\hat{f}_j} w_i$$

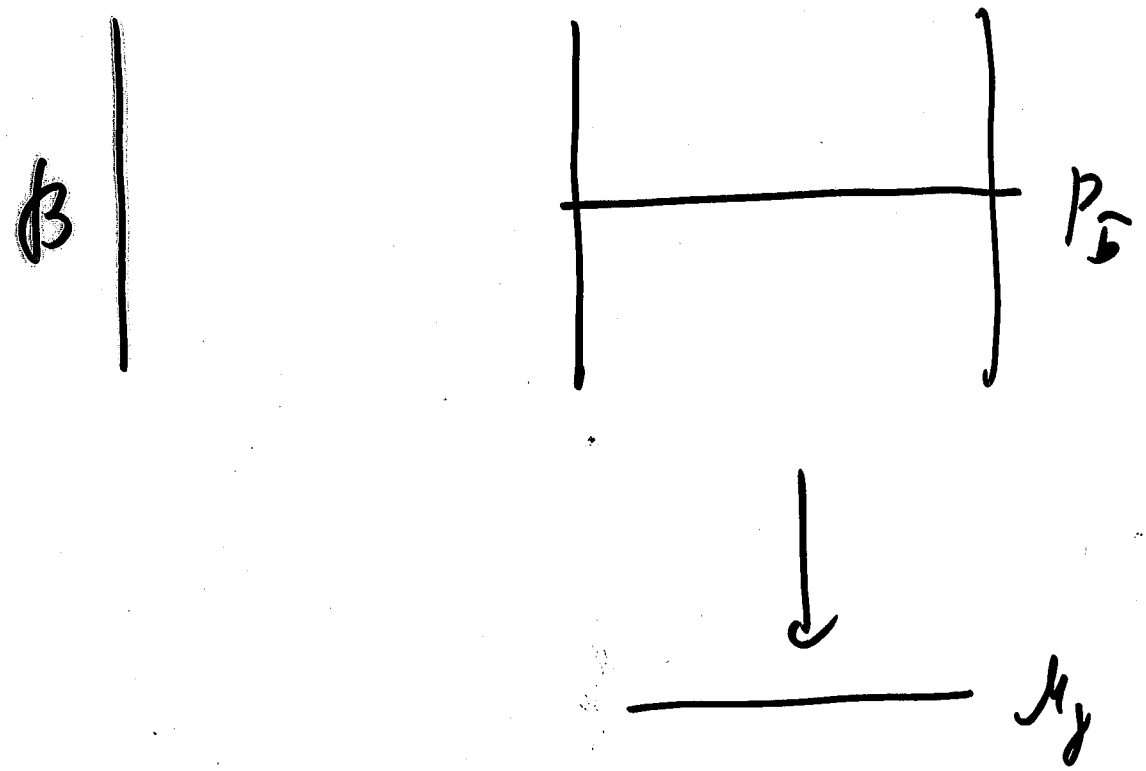
$$P = \mathbb{C}^{3g-3} \big/ \Lambda \oplus \Omega\Lambda^*$$

$$\mathcal{B} = \{ (b,\alpha) = \tilde{b} \mid b = \{x_j\} \in C^{4g-4} - \Delta, \ \alpha \ \text{cut locus} \}$$

$$\tilde{b} \in \mathcal{B} \longrightarrow \tilde{C} = \tilde{C}_{\tilde{b}}$$

$$P_{\tilde{b}}$$

$$\pi = \pi_{\tilde{b}} : P_{\tilde{b}} \to \mathcal{M}_g$$



$$\begin{array}{ccc} \Gamma(\mathcal{M}_g, \mathcal{L}^k) & \longrightarrow & \Gamma(P_{\tilde{b}}, \tilde{\mathcal{L}}^{2k}) \\ \psi & \longrightarrow & \tilde{\Psi} \end{array}$$

$\tilde{b} = (b, \alpha) \in \mathcal{B}_Y$

base pt $\eta \in P_{\tilde{b}}$ $\qquad \eta = \frac{1}{2} \sum_{x_j \in b} [x_j]$

$L \in P_{\tilde{b}} \qquad L = L_0 \eta$

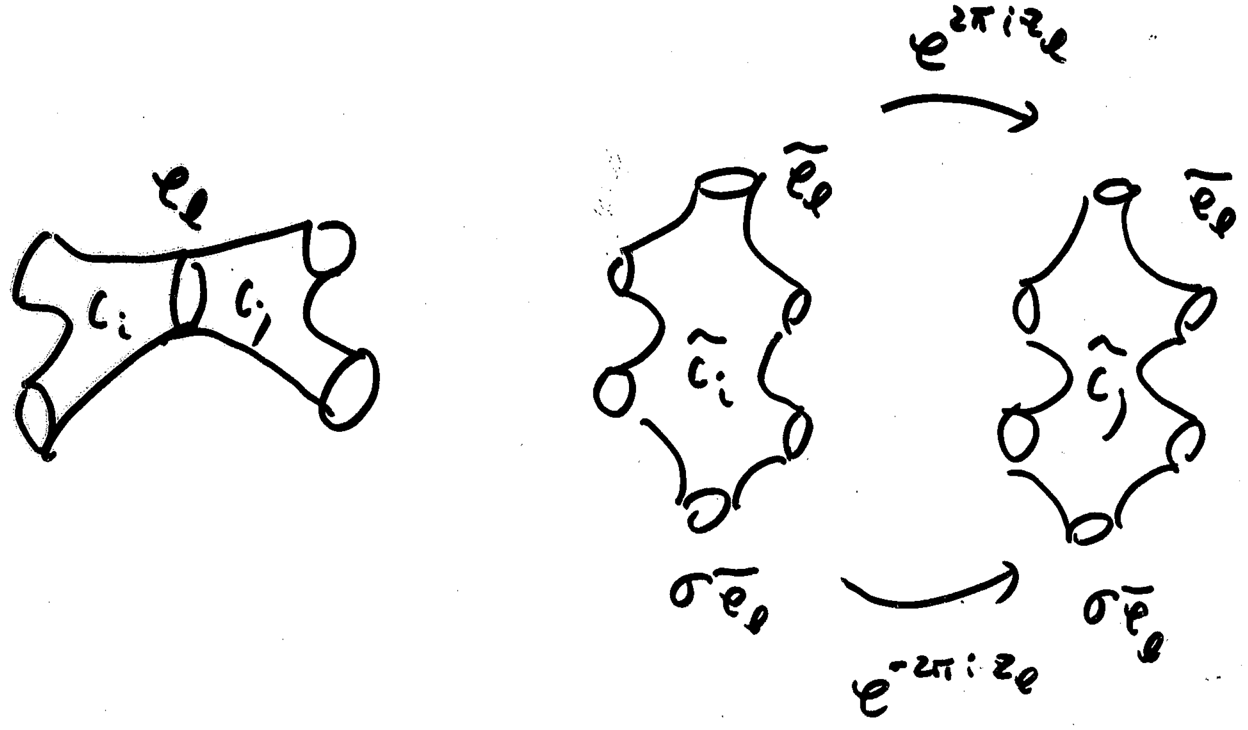
$\sigma^* L_0 \otimes L_0 = \mathbb{I}$

$$P_{\tilde{b}} = \{ L_0 \mid \sigma^* L_0 = L_0^{-1} \}$$

$$= \{ \vec{z} = (z_\ell) \in \mathbb{C}^{3g-3} \}$$

$L_0 \leftrightarrow \vec{z} = (z_\ell)$

$L_0 = \bigcup \tilde{C}_i \times \mathbb{C} \Big/$ transition



Description of $P = P_{\tilde{b}}$ as a symplectic variety

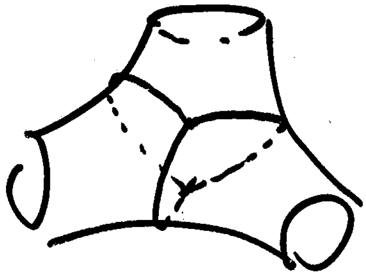
$$H_1(\widetilde{C}, \mathbb{R})_- = \{ x \in H_1(\widetilde{C}, \mathbb{R}) \mid \sigma_* x = -x \}$$

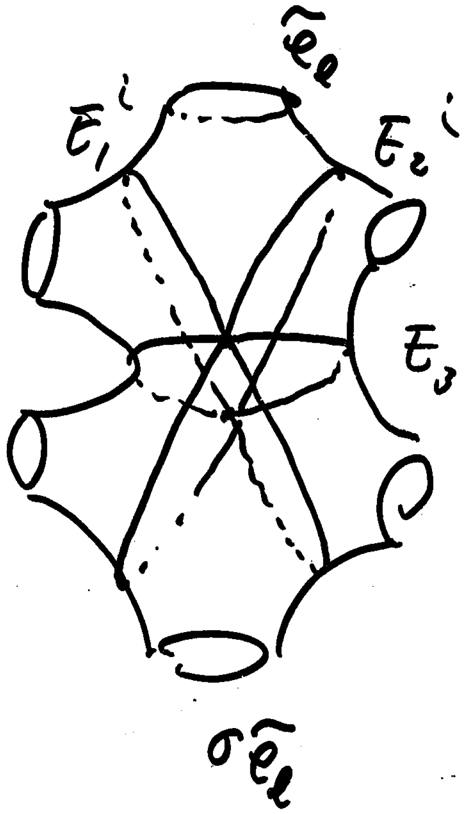
$$\ell = \{ [\tilde{e}_\ell] - \sigma_* [\tilde{e}_\ell] \}_{1 \le \ell \le 3g-3} \quad \text{over } \mathbb{R}$$

$$\ell^* = \{ \tilde{f}_\ell \}$$ complementary Lagrangian subspace

$$\Lambda_0 = \{ [\tilde{e}_\ell] - \sigma_* [\tilde{e}_\ell] \} \quad \text{over } \mathbb{Z}$$

$$\Lambda = \Lambda_0 \cup \{ E_1^i, E_2^i, E_3^i \}_{1 \le i \le 2g-2}$$





Prym Variety

$$P = H_1(\tilde{C}, \mathbf{R})_- / (\Lambda + \Lambda^*).$$

$$\begin{tikzcd} \tilde{\mathcal{L}}^{2k} \arrow[r] \arrow[d] & \mathcal{L}^{k} \arrow[d] \\ P \arrow[r] & \mathcal{M}_g \end{tikzcd}$$

Theta Function of degree $2k$

$$\begin{aligned} &\vartheta \begin{bmatrix} \vec{a} \\ \vec{0} \end{bmatrix} (2k\vec{z}, 2k\Omega) \\ &= \sum_{\vec{n} \in \Lambda^*} \exp\left\{\pi i (\vec{n} + \vec{a})^t 2k\Omega (\vec{n} + \vec{a})\right\} \\ &\qquad \times \exp\left\{2\pi i (\vec{n} + \vec{a})^t 2k\vec{z})\right\} \end{aligned}$$

where

$$\vec{a} \in \frac{1}{2k}\Lambda^* \quad , \quad \vec{n} \in \Lambda^*$$

$H_1(\tilde{C}, \mathbf{R})_-$: σ-anti-invariant homology group of $\tilde{C}$.

$\tilde{\ell} = \{[\tilde{e}_l - \sigma\tilde{e}_l]\}_{1 \le l \le 3g-3}$:Lagrangian subspace

$$E_1^i = \frac{1}{2}\left[-(\tilde{e}_{l_1^i} - \sigma\tilde{e}_{l_1^i}) + (\tilde{e}_{l_2^i} - \sigma\tilde{e}_{l_2^i}) + (\tilde{e}_{l_3^i} - \sigma\tilde{e}_{l_3^i})\right]$$
$$E_2^i = \frac{1}{2}\left[(\tilde{e}_{l_1^i} - \sigma\tilde{e}_{l_1^i}) - (\tilde{e}_{l_2^i} - \sigma\tilde{e}_{l_2^i}) + (\tilde{e}_{l_3^i} - \sigma\tilde{e}_{l_3^i})\right]$$
$$E_3^i = \frac{1}{2}\left[(\tilde{e}_{l_1^i} - \sigma\tilde{e}_{l_1^i}) + (\tilde{e}_{l_2^i} - \sigma\tilde{e}_{l_2^i}) - (\tilde{e}_{l_3^i} - \sigma\tilde{e}_{l_3^i})\right]$$

Integral Lattices in $\tilde{\ell}, \tilde{\ell}^*$

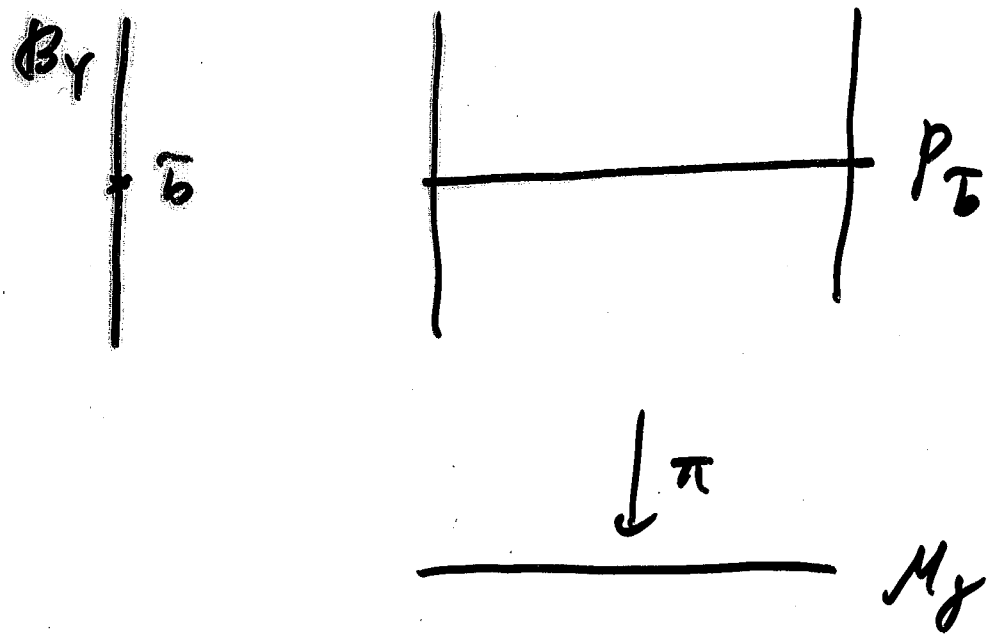
Λ_0 is generated by $\{[\tilde{e}_l - \sigma\tilde{e}_l]\}$.

Λ_0^* is the symplectic dual of Λ_0.

Λ is generated by $\{E_1^i, E_2^i, E_3^i\}_{1 \le i \le 2g-2}$.

Λ^* is the symplectic dual of Λ.

Symmetry which characterize the pull-back section



① Global Symmetry

② Local Symmetry

Global Symmetry

$\mathbf{Z}_2^{3g-3}$-action

$\vec{\epsilon} = (\epsilon_l) = (\epsilon_1, \cdots, \epsilon_{3g-3})\ \epsilon_l = \pm 1.$

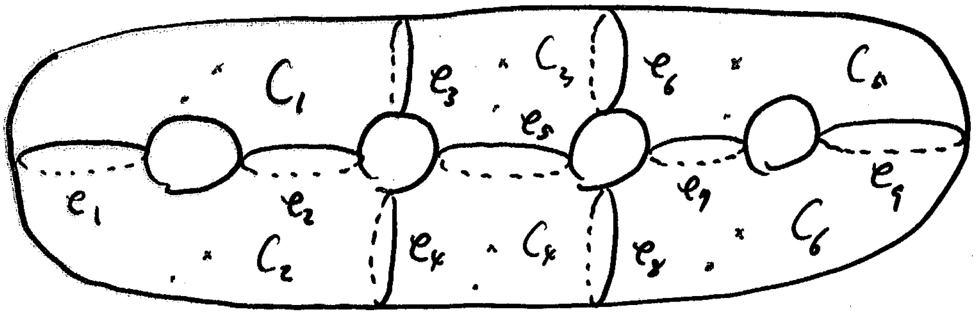
$$\boxed{\begin{gathered} q(\Omega) \to q(\vec{\epsilon}\cdot\Omega) \\ \vartheta\begin{bmatrix}\frac{\vec{j}}{k} \\ \vec{0}\end{bmatrix}(2k\vec{z}, 2k\Omega) \to \vartheta\begin{bmatrix}\vec{\epsilon}\cdot\frac{\vec{j}}{k} \\ \vec{0}\end{bmatrix}(2k\vec{z}, 2k\Omega)\end{gathered}}$$

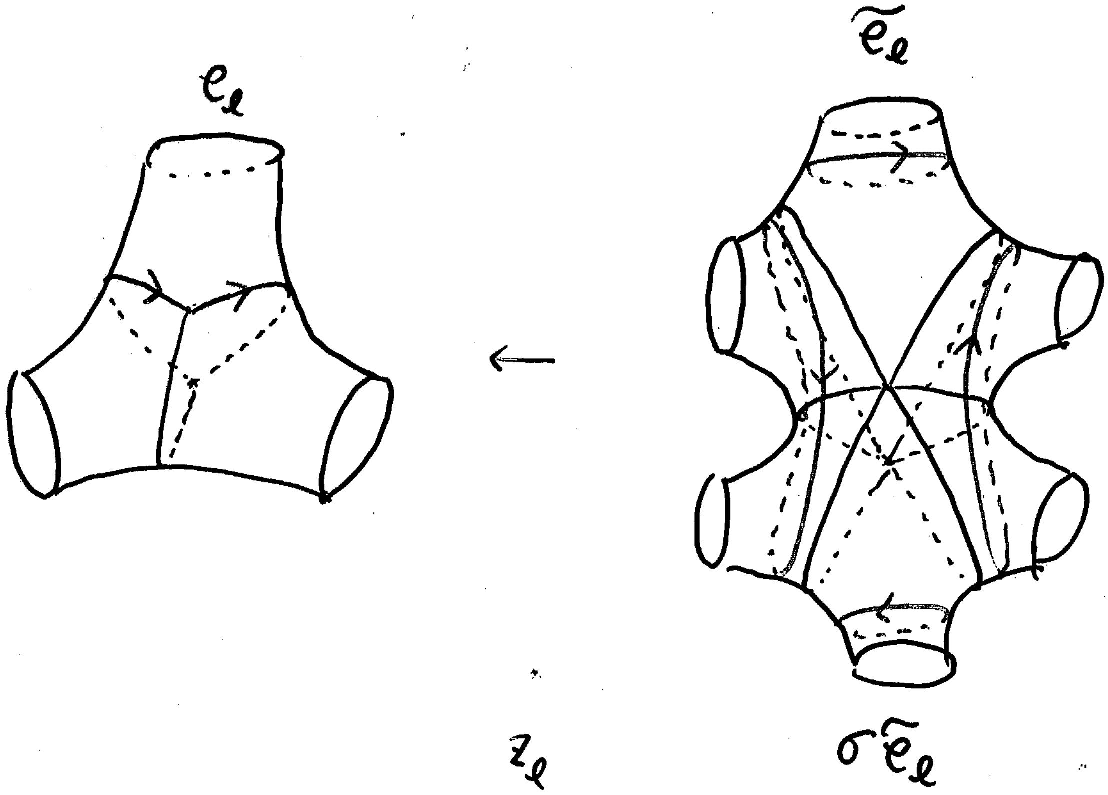
$$\Pi\ \tilde{\psi} = \sum_{\vec{\epsilon}\in\mathbf{Z}_2^{3g-3}} w(\vec{\epsilon})q(\vec{\epsilon}\cdot\Omega) \times \vartheta\begin{bmatrix}\vec{\epsilon}\cdot\frac{(2\vec{j}+\vec{1})}{2(k+2)} \\ \vec{0}\end{bmatrix}$$

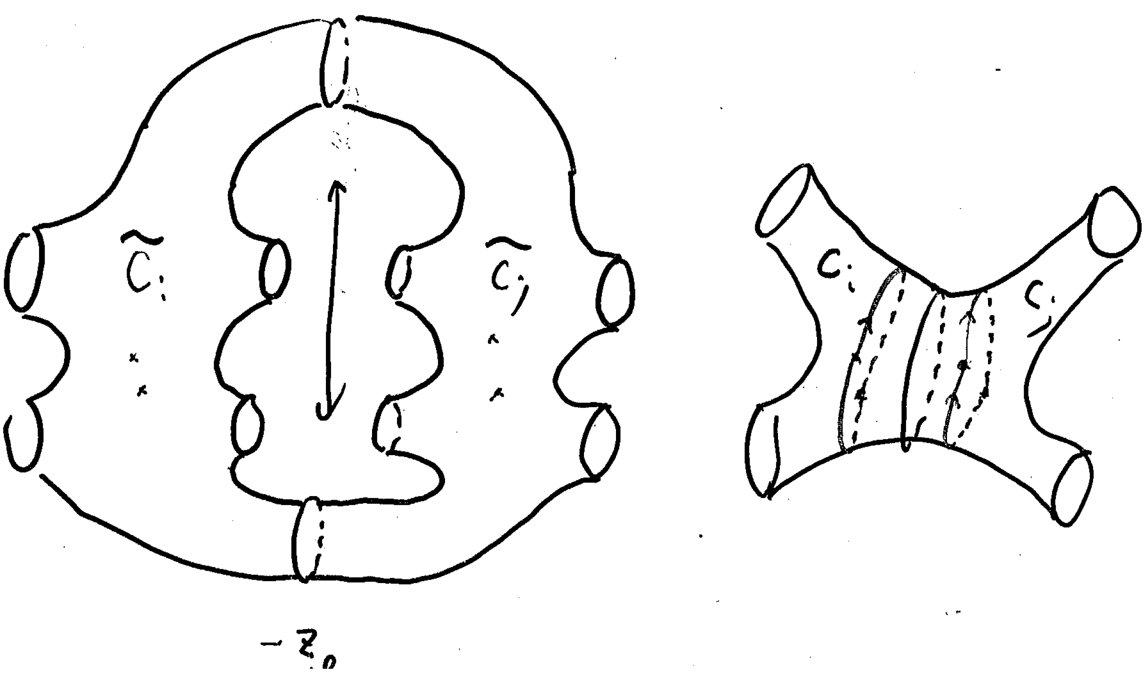
$$w(\vec{\epsilon}) = \epsilon_1 \cdots \epsilon_{3g-3}$$

$$\vec{\epsilon}\cdot\vec{a} = (\epsilon_l a_l)$$

$Y = \{ e_\ell, C_i \}$







Local Symmetry

Vector Field

$$\mathcal{V}_b = \delta + \frac{1}{8}(\delta J\omega^{-1})_{ij}\Pi^{-1}\partial_i\Pi\frac{\partial}{\partial z_j}$$

Differential Operator

$$D = \Pi^{-1}\left(\delta + \frac{1}{8(k+2)}(\delta J\omega^{-1})_{ij}\partial_i\partial_j\right)\Pi$$
$$-\Pi^{-1}\delta P_\Pi \Pi$$
$$+\left(1\otimes\bar{\nabla}^{\mathcal{K}}\right)$$

$$P_\Pi : \Theta_{2(k+2)} \to \Pi\Theta_{2k}$$

$\Theta_{2k}^{\dagger} \to \Theta_{2k+4}$

$$D\tilde{\psi} = 0$$

$Q \quad \frac{\widetilde{\Pi\psi}}{\Pi}$

Automorphic Forms

$$q_{\vec{\epsilon}\cdot\vec{j}} = \sum_{g \in \mathcal{E}_\Upsilon} \vartheta^g_{\vec{\epsilon}\cdot\vec{j}}(\Omega^g_1)\vartheta^g_{\vec{\epsilon}\cdot\vec{j}}(\Omega^g_2)\vartheta^g_{\vec{\epsilon}\cdot\vec{j}}(\Omega^g_3)$$

where

$$\vartheta_{\vec{\epsilon}\cdot\vec{j}}(\Omega^g_m) = \sum_{\vec{n} \in \Lambda} \exp\left\{2(k+2)\pi i N(\vec{n})^t \Omega^g_m N(\vec{n})\right\}$$

$$\left(N(\vec{n}) = \vec{n} + \vec{\epsilon} \cdot \frac{\vec{\delta}^g_m \cdot \left(2\vec{j} + \vec{1}\right)}{2(k+2)} \right)$$

$$(\Pi) = \{ L \in P \mid H^0(\tilde{C}, \sigma^* L \otimes L^{-1} \otimes p^* K_C) \neq 0 \}$$

$$\boxed{\begin{aligned} \Pi(\vec{z}, \Omega) \; = \; \sum_{\vec{n} \in \Lambda^*} \exp \pi i \left(\vec{n} + \frac{\vec{1}}{2} \right)^t \Omega \left(\vec{n} + \frac{\vec{1}}{2} \right) \\ \cdot \exp 2\pi i \left(\vec{n} + \frac{\vec{1}}{2} \right)^t \left(2\vec{z} + \frac{\vec{1}}{4} \right) \end{aligned}}$$

(g=1)

$$\Pi(z, \Omega) = \vartheta \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} (z, 4\Omega) - \vartheta \begin{bmatrix} -1/4 \\ 0 \end{bmatrix} (z, 4\Omega)$$

Kac Character Formula

$$\tilde{\psi}_j = \frac{\vartheta \begin{bmatrix} \frac{(2j+1)}{2(k+2)} \\ 0 \end{bmatrix} - \vartheta \begin{bmatrix} -\frac{(2j+1)}{2(k+2)} \\ 0 \end{bmatrix}}{\Pi}$$

$\vec{\epsilon} = (\epsilon_l) \in \mathbb{Z}_2^{3g-3}$

$w(\vec{\epsilon}) = \epsilon_1 \cdots \epsilon_{3g-3}$

Pant Decomposition

$$\Upsilon = \{e_l, C_i\}$$

Weight System of Level k

$$\vec{j} = (j_l) : \{e_l\} \to \left\{0, \frac{1}{2}, \cdots, \frac{k}{2}\right\} \qquad (1)$$

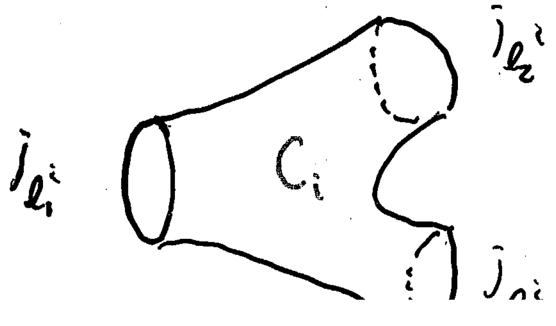
Form of $\tilde{\psi}$

$$\boxed{\tilde{\psi}_{\vec{j}} = \Pi^{-1} \sum_{\vec{\epsilon}} w(\vec{\epsilon})\, q_{\vec{\epsilon}\cdot\vec{j}} \cdot \vartheta \begin{bmatrix} \vec{\epsilon} \cdot \frac{2\vec{j}+\vec{1}}{2(k+2)} \\ \vec{0} \end{bmatrix}}$$

Quantum-Clebsch-Gordan Condition:

$C_i \in \Upsilon (1 \le i \le 2g-2)$

$$\begin{cases} j_{l_1^i} + j_{l_2^i} + j_{l_3^i} \in \mathbf{Z} \\ |j_{l_1^i} - j_{l_2^i}| \le j_{l_3^i} \le j_{l_1^i} + j_{l_2^i} \\ j_{l_1^i} + j_{l_2^i} + j_{l_3^i} \le k, \end{cases}$$



$Y = \{e_j, C_i\}$

Grouping $\mathcal{G}$

$\mathcal{E} = \{e_j\}_{1 \leq j \leq 3g-3}$

$\mathcal{E} = \mathcal{E}_1^g \cup \mathcal{E}_2^g \cup \mathcal{E}_3^g$

decomposition of $\mathcal{E}$ into 3 disjoint subsets each of which containes $(g-1)$- curves

$\# \mathcal{G} = 2^{g-1}$

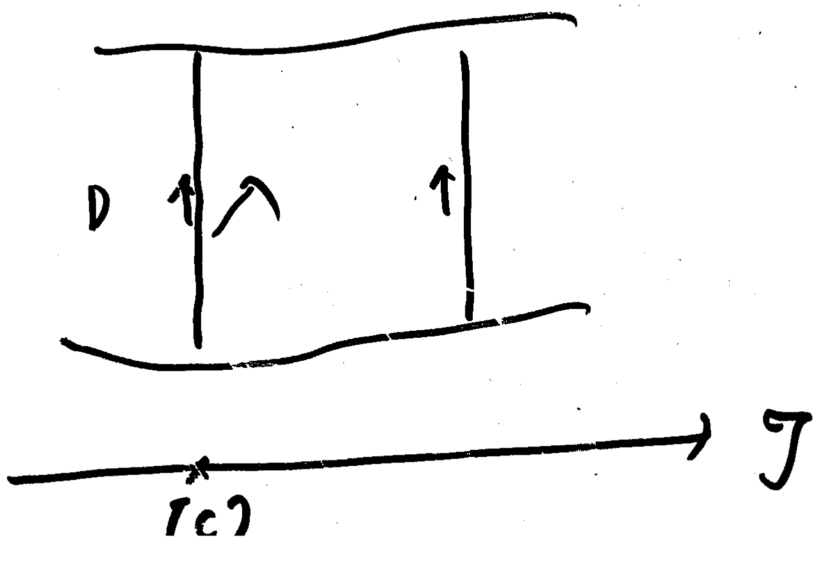
Main Theorem

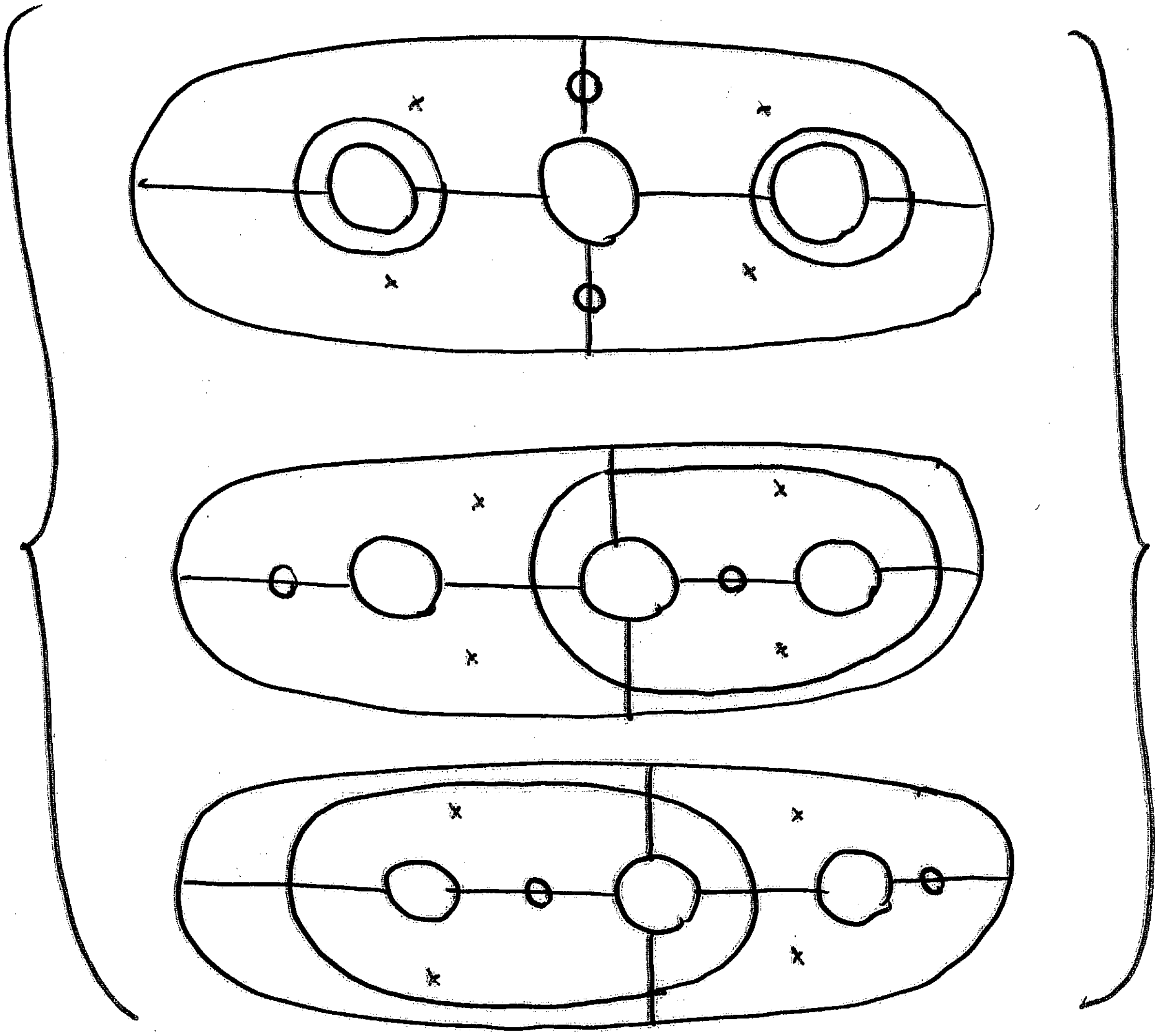
$\{\tilde{\Psi}_{\vec{j}}\}$ forms a basis of the conformal block of level k of $SU(2)$ WZW model.

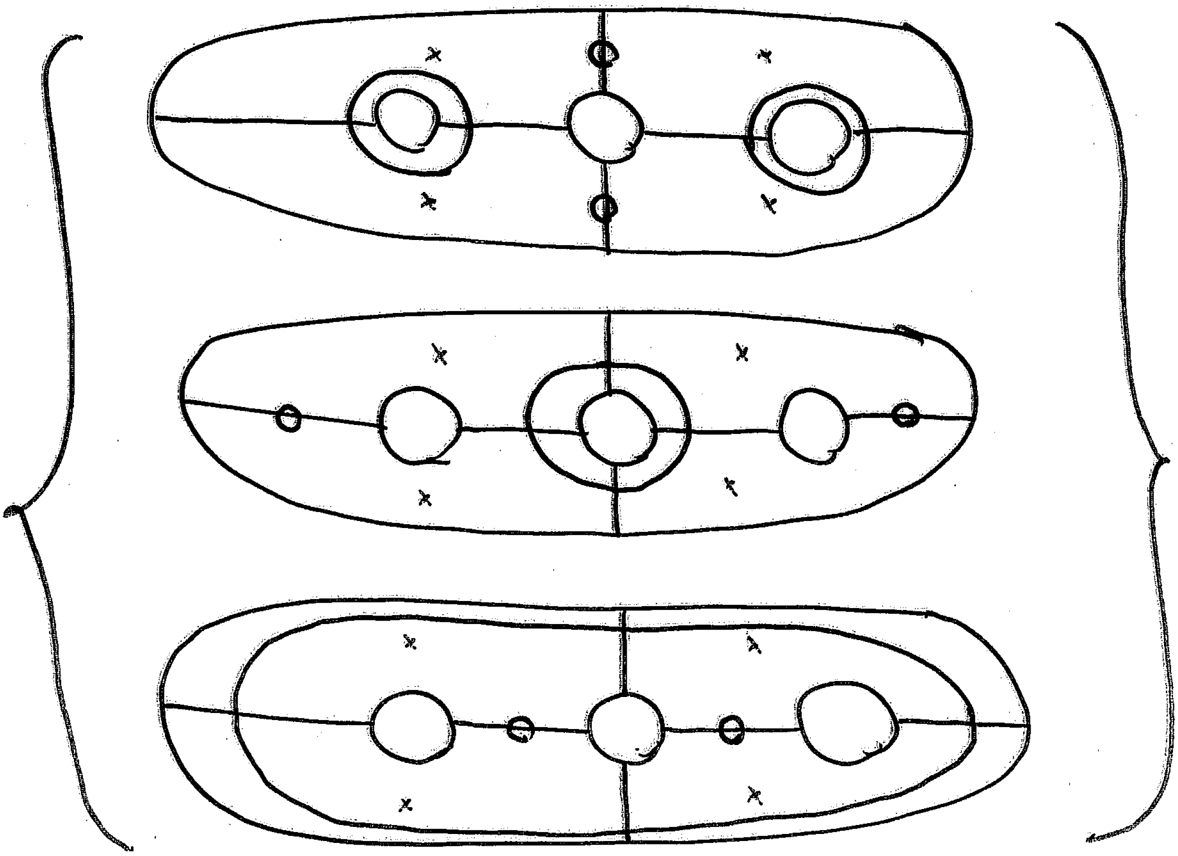
D gives a projectively flat connection and $\tilde{\Psi}_{\vec{j}}$ is parallel.

Corollary

There exist a hermitian product of the conformal block preserved by connection







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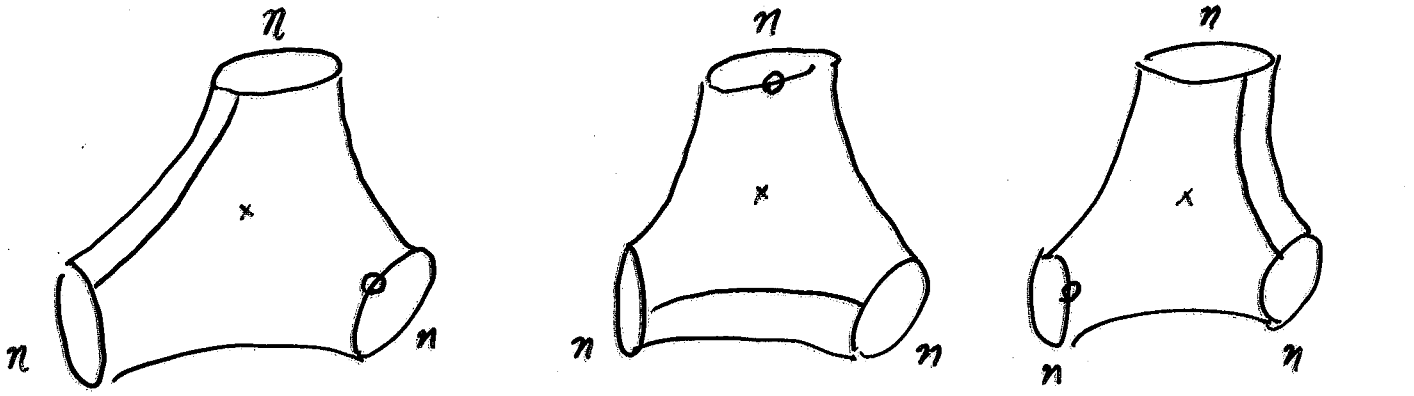

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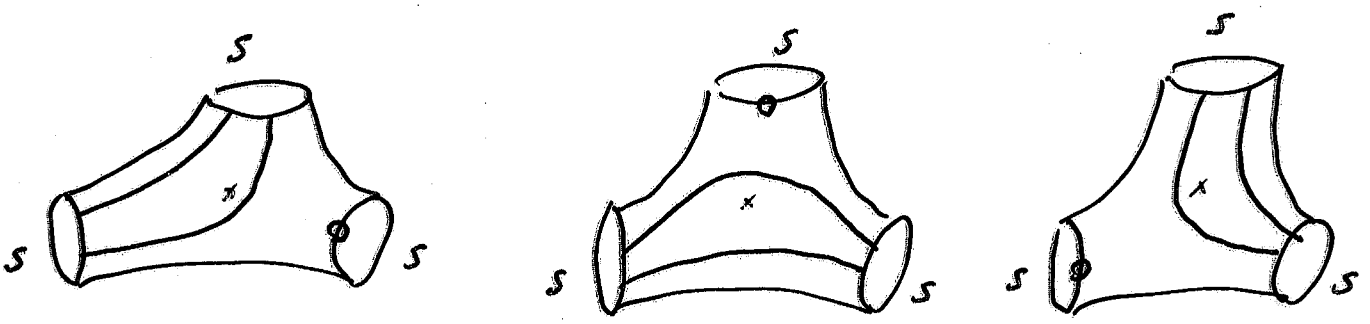

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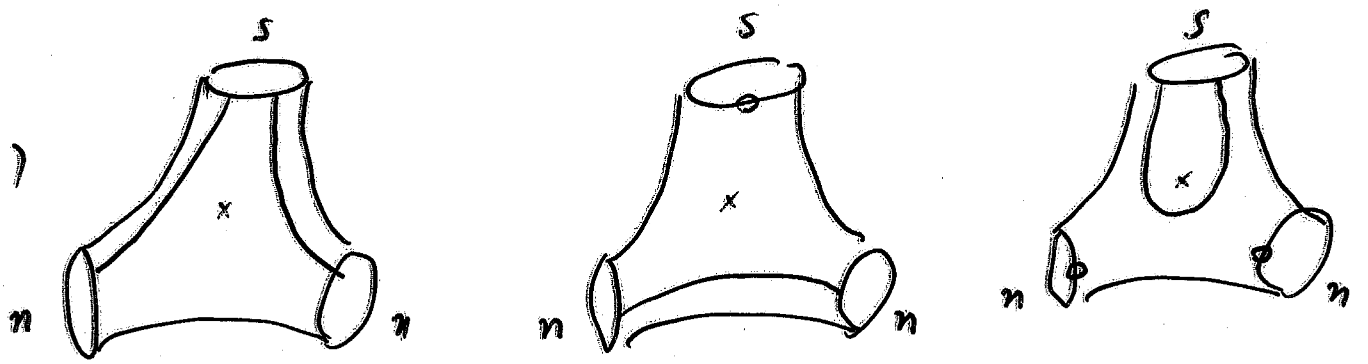

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