SURFACE SINGULARITIES APPEARED IN THE HYPERBOLIC SCHWARZ MAP FOR THE HYPERGEOMETRIC EQUATION

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Abstract. Surface singularities, swallowtail and cuspidal edge, appear in the hyperbolic Schwarz map for the hypergeometric differential equation. Such singularities are studied in detail. After an overview of classical staffs, the hypergeometric equation and the Schwarz map, the hyperbolic Schwarz map is introduced. We study the singularities of this map, whose target is the hyperbolic 3-space, and visualize its image when the monodromy group is a finite group or a typical Fuchsian group. Several confluences of swallowtails are also observed.

Contents

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1. Introduction

Surface singularities $\boldsymbol{R}^3$ such as cuspidal edge and swallowtail are interesting objects. One can visualize them and can easily make paper-models. However, there are still many things to study about them (see for example [AGV], [LLR]). In this note, we show that they appear also in the hypergeometric world.

We first present the swallowtail as the discriminant of the quartic polynomial. The description is very elementary, but highly non-trivial.

We next recall the classical hypergeometric staffs: The *hypergeometric differential equation*

$$E(a,b,c) \qquad x(1-x)u'' + \{c-(a+b+1)x\}u' - abu = 0,$$

and its *Schwarz map* by

$$s : X = \boldsymbol{C} - \{0,1\} \ni x \longmapsto u_0(x) : u_1(x) \in Z \cong \boldsymbol{P}^1, \tag{1.1}$$

where u_0 and u_1 are linearly independent solutions of $E(a, b, c)$ and $\boldsymbol{P}^1$ is the complex projective line. The Schwarz map of the hypergeometric differential equation was studied by Schwarz, around 1870, when the parameters (a, b, c) are real.

We then proceed to the *hyperbolic Schwarz map*, which is introduced in [SYY1]. It is defined as follows: Change the equation $E(a, b, c)$ into the so-called SL-form:

$$E^{\mathrm{SL}} : \quad u'' - q(x)u = 0, \tag{1.2}$$

and transform it to the matrix equation

$$\frac{d}{dx}(u, u') = (u, u')\Omega, \quad \Omega = \begin{pmatrix} 0 & q(x) \\ 1 & 0 \end{pmatrix}. \tag{1.3}$$

We now define the hyperbolic Schwarz map, denoted by $\mathscr{S}$, as the composition of the (multi-valued) map

$$X \ni x \longmapsto H = U(x)\ {}^t\overline{U}(x) \in \mathrm{Her}^+(2) \tag{1.4}$$

and the natural projection $\mathrm{Her}^+(2) \to \boldsymbol{H}^3 := \mathrm{Her}^+(2)/\boldsymbol{R}^+$, where $U(x)$ is a fundamental solution of the system, $\mathrm{Her}^+(2)$ the space of positive-definite Hermitian matrices of size 2, and $\boldsymbol{R}^+$ the multiplicative group of positive real numbers; the space $\boldsymbol{H}^3$ is called the *hyperbolic* 3*-space*.

Note that the target of the hyperbolic Schwarz map is $\boldsymbol{H}^3$, whose boundary is $\boldsymbol{P}^1$, which is the target of the Schwarz map. In this sense, our hyperbolic Schwarz map is a lift-to-the-air of the Schwarz map. The image surface (of X under $\mathscr{S}$) has the following geometrically nice property: The classical Schwarz map s is recovered as one of the two *hyperbolic Gauss maps* of the image surface. The image surface admits singularities; generically they are cuspidal edges and swallowtails.

In this note, we study the hyperbolic Schwarz map $\mathscr{S}$ of the equation $E(a,b,c)$ mainly when the parameters (a,b,c) are real, especially when its monodromy group is a finite (polyhedral) group or a Fuchsian group. We made our best to visualize the image surfaces.

In a computational aspect of this visualization, we use the composition of the hyperbolic Schwarz map $\mathscr{S}$ and the inverse of the Schwarz map s, $\Phi = \mathscr{S} \circ s^{-1}$, especially when the inverse of the Schwarz map is single-valued globally; refer to Section 6.2. This choice is very useful, because the inverse map is often given explicitly as an automorphic function for the monodromy group acting properly discontinuously on the image of the Schwarz map. Moreover, in one of the cases where we treat the lambda function for drawing pictures, it is indispensable, because we have a series that converges very fast.

In the end, we inroduce the derived Schwarz map $ds : X \to \boldsymbol{P}^1$, present an associated (parallel) 1-parameter family of surfaces in $\boldsymbol{H}^3$ having $\mathscr{S}$ as a generic member, and having s and ds as the two extremes. We also study confluence of swallowtail singularities; Refer to §7.3 and §7.4.

Basic ingredients of the hypergeometric function and its Schwarz map can be found in [IKSY] and [Yo1]. For this article only, the first half of [Yo2] is enough.

2. Swallowtail as a discriminant

In this section, we describe surface singularities: cuspidal edges and a swallowtail. We start from the well-known discriminant of a quadratic polynomial.

2.1. **Discriminant of a quadratic polynomial.** Consider a quadratic polynomial

$$F := t^2 + at + b$$

in t with real coefficients (a,b). A polynomial in the coefficients which vanishes only when F has a multiple root is called a discriminant; it is unique up to constant and is given as

$$D := a^2 - 4b.$$

When $D > 0$, F has two distinct real roots, and when $D < 0$, F has a pair of complex conjugate roots. There are at least three ways to derive D.

2.1.1. *Difference of two roots.* Let α and β be two roots of F. Since

$$(t-\alpha)(t-\beta) = t^2 - (\alpha+\beta)t + \alpha\beta,$$

we have

$$\alpha+\beta = -a, \quad \alpha\beta = b.$$

Thus we have

$$(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = a^2 - 4b = D.$$

2.1.2. *Parallel displacement.* A change of variable $t \to t + c$ does not affect the status of roots. Choose c so that the resulting polynomial has no t-term:

$$t^2 + at + b = \left(t + \frac{a}{2}\right)^2 - \frac{a^2-4b}{4} = \left(t + \frac{a}{2}\right)^2 - \frac{D}{4}.$$

2.1.3. *Usage of differentiation.* The polynomial F has a multiple root at a point c if and only if

$$R: \quad F(c) = 0 \quad \text{and} \quad F'(c) = 0.$$

So the point is whether the system $F(c) = F'(c) = 0$ is solvable. Eliminating c from

$$R: \quad c^2 + ac + b = 0, \quad 2c + a = 0,$$

we have

$$0 = \left(-\frac{a}{2}\right)^2 + a\left(-\frac{a}{2}\right) + b = -\frac{D}{4}.$$

This method does not tell us the difference of the status when $D > 0$ and that when $D < 0$; this is the weak point.

2.2. **Discriminant of a cubic plynomial.** We consider

$$F := t^3 + pt^2 + qt + r;$$

by performing a suitable parallel displacement, we (can) assume $p = 0$.

2.2.1. *Difference of three roots.* Let α, β, γ be the three roots of $F = 0$. Since

$$(t-\alpha)(t-\beta)(t-\gamma) = t^3 - (\alpha+\beta+\gamma)t^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)t - \alpha\beta\gamma,$$

we have

$$\alpha+\beta+\gamma = -p, \quad \alpha\beta+\beta\gamma+\gamma\alpha = q, \quad \alpha\beta\gamma = -r.$$

The discriminant should be given as a constant multiple of

$$D = \{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\}^2,$$

which is not so easy to write in terms of q and r.

2.2.2. *Usage of differentiation.* We eliminate t from

$$F = t^3 + qt + r = 0, \quad F' = 3t^2 + q = 0.$$

(The second equation gives an expression of t^2. Substitute this the first, you get an expression of t. Substitute this to the second.) Anyway, we get the expression $4q^3 + 27r^2$, which is now straightforward to see

$$D = -(4q^3 + 27r^2).$$

2.2.3. *Visualization.* Draw in the (q, r)-plane the curve $D = 4q^3 + 27r^2 = 0$. Since

$$F = (t-\alpha)^2(t-\beta), \quad 2\alpha + \beta = 0,$$

the curve can be parameterized as

$$q = -3\alpha^2, \quad r = -2\alpha^3, \quad r \in \boldsymbol{R};$$

by this expression, you can draw its image (see Figure 1).

In this occasion, let us recall that if a polynomial with real coefficients has a complex root α, then its complex conjugate $\bar{\alpha}$ is also a root, where the complex conjugate of $z = x + iy$ is defined as $\bar{z} = x - iy$; this can be proved by repeated use of

$$\overline{z+w} = \bar{z} + \bar{w}, \quad \overline{zw} = \bar{z}\bar{w}.$$

Convention:

- $1+1+1$: polynomial with three distinct real roots
- $1+(1+\bar{1})$: polynomial with a real root and a pair of conjugate imaginary roots
- $2+1$: polynomial with double root on the left of a simple root
- $1+2$: polynomial with double root on the right of a simple root
- 3: polynomial with triple root

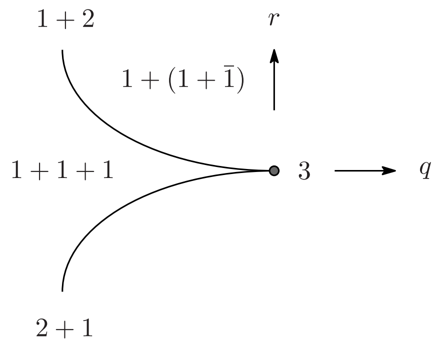

FIGURE 1. The curve $D = 0$ in the qr-plane

The (q, r)-space, the totality of the cubic polynomials F, is divided by the curve $D = 0$ into two parts: the smaller one consisits of the polynomials of type $1+1+1$, and the bigger one $1+(1+\bar{1})$; the curve $D = 0$ consists of two parts $1+2$ and $2+1$ with the common point 3. See the diagram in Figure 2.

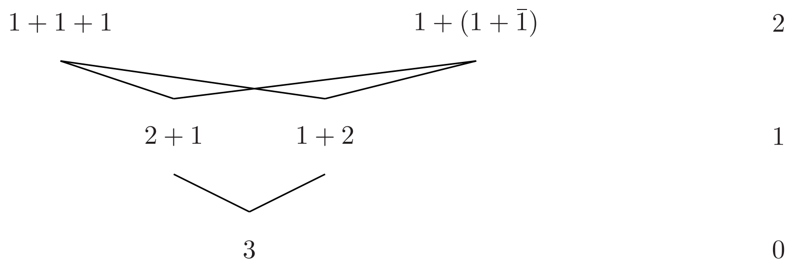

FIGURE 2. Degeneration diagram of the status of three roots

When two are connected by a segment, it means that the lower one is a specialization of the above one. The numerals on the right denote the freedom/dimension. From this diagram, you can draw Figure 3; which is topologically equivalent to Figure 1.

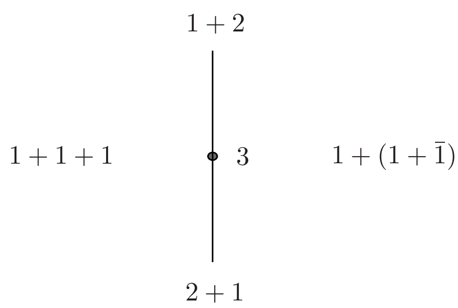

FIGURE 3. The curve $D = 0$ in qr-space, simplified

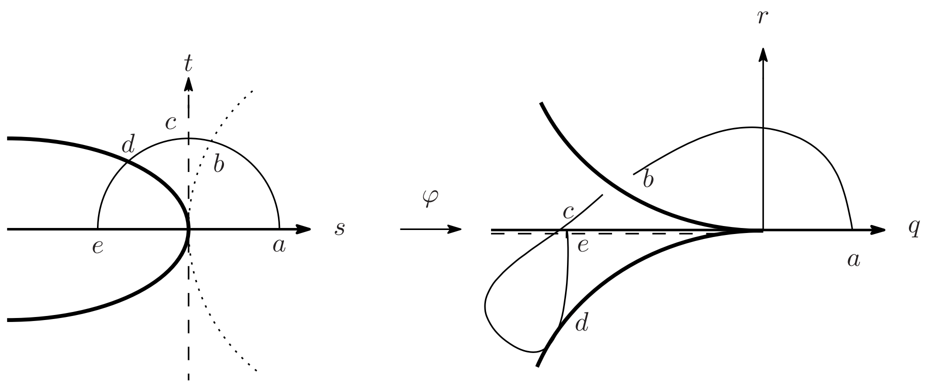

FIGURE 4. Image under the map φ

2.2.4. *Relation between roots and coefficients.* Recall that the coefficients q, r of the polynomial $F = t^3 + qt + r$ and the roots $\alpha, \beta, \gamma = -\alpha - \beta$ are related as

$$\begin{aligned} q &= \alpha\beta + \beta\gamma + \gamma\alpha = \alpha\beta - (\alpha+\beta)^2, \\ r &= -\alpha\beta\gamma = \alpha\beta(\alpha+\beta). \end{aligned}$$

Since these expressions are symmetric in α and β, we put

$$t = \alpha + \beta, \quad s = \alpha\beta;$$

then the expressions above become

$$q = s - t^2, \quad r = st.$$

It is interesting to study the map

$$\varphi : (s,t) \mapsto (q,r) = (s - t^2, st).$$

For generic (q,r), i.e. $D(q,r) = 4q^3 + 27r^2 \neq 0$, its preimage consists of three points, since the polynomial F has three distinct roots. The preimage of the discriminant curve $D = 0$ consists of two parabolas; indeed we have

$$4q^3 + 27r^2 = (s + 2t^2)^2(4s - t^2).$$

Since we have

$$\frac{\partial(q,r)}{\partial(s,t)} = \begin{vmatrix} 1 & -2t \\ t & s \end{vmatrix} = s + 2t^2,$$

the map is degenerate along the parabola $s + 2t^2 = 0$. These two parabolas are shown in Figure 4. To help understanding this map, the image of a hemicircle centered at the origin in the (s,t)-space is also shown.

2.3. **Discriminant of a quartic polynomial.** We consider a quartic plynomial

$$F = t^4 + xt^2 + yt + z.$$

2.3.1. *Discriminant.* Let $\alpha, \beta, \gamma, \delta$ be the four roots of F. By the relations

$$\alpha + \beta + \gamma + \delta = 0, \quad \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = x,$$
$$\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma = -y, \quad \alpha\beta\gamma\delta = z,$$

we can express

$$D = \{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta) \cdot (\beta - \gamma)(\beta - \delta) \cdot (\gamma - \delta)\}^2$$

in terms of x, y, z. But no one would like to carry out the computation.

We eliminate t from $F(t) = F'(t) = 0$:

$$t^4 + xt^2 + yt + z = 0, \quad 4t^3 + 2xt + y = 0.$$

(The second equation gives an expression of t^3. Substitute this the first, you get an expression of t^2. Substitute this to the second, you get an expression of $t, \dots$) Eventually, we get the expression

$$256z^3 - 128x^2z^2 + (144y^2x + 16x^4)z - 4y^2x^3 - 27y^4;$$

you can easily check that it coincides with D.

For cosmetic reason, we change z into $z/4$:

$$F = t^4 + xt^2 + yt + \frac{z}{4},$$

$$D := 4z^3 - 8x^2z^2 + (36y^2x + 4x^4)z - (27y^2 + 4x^3)y^2.$$

Can you guess how the set $D = 0$ looks like?

2.3.2. *Various status of the four roots.* Three-dimensional happenings are difficult to understand. The set $D = 0$ will divide the xyz-space into several parts. In some part, the corresponding quartic polynomial $t^4 + xt^2 + yt + z/4 = 0$ has four distinct real roots, in some other parts two real roots and a pair of complex conjugate roots, . . . :

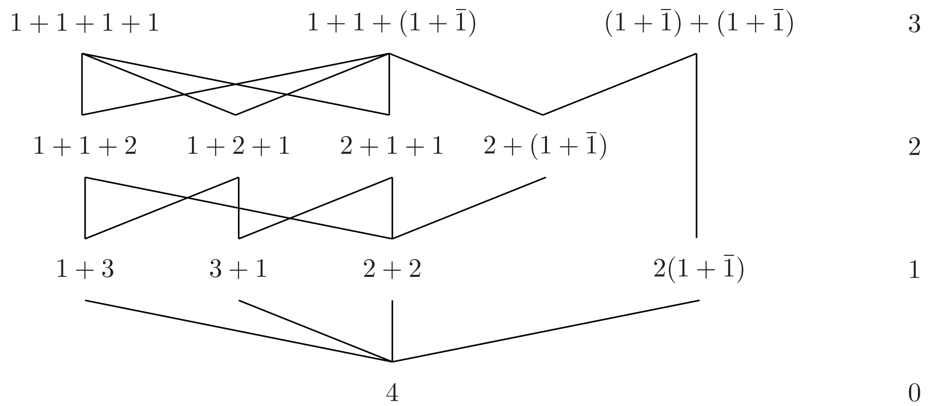

FIGURE 5. Degeneration diagram of the status of four roots

When two are connected by a segment, it means that the lower one is a specialization of the above one. The numerals on the right denote the freedom/dimension. Convention:

- $1+1+1+1$: 4 distinct real roots
- $1+1+(1+\bar{1})$: 2 distinct real, and a pair of conjugate roots
- $(1+\bar{1})+(1+\bar{1})$: 2 distinct pairs of conjugate roots
- $1+1+2$: double real root on the right of 2 distinct real roots
- $1+2+1$: double real root at the middle
- $2+1+1$: double real root on the left of 2 distinct real roots
- $2+(1+\bar{1})$: double real root and a pair of conjugate roots
- $1+3$: triple root on the right of a real root
- $3+1$: triple root on the left of a real root
- $2+2$: distinct 2 double roots
- $2(1+\bar{1})$: a pair of conjugate double roots
- 4: 4-ple root

Only from these data, we can draw a (topological) picture of the surface S consisting of polynomials of types

$$1+1+2, \qquad 1+2+1, \qquad 2+1+1, \qquad 2+(1+\bar{1}).$$

This surface divides the space into three parts:

$$1+1+1+1, \qquad 1+1+(1+\bar{1}), \qquad (1+\bar{1})+(1+\bar{1}),$$

which are the inside of the triangular cone, the inside of the circular cone, and the complement of the two cones, respectively. Note that the set $D=0$ consists of S and a curve corresponding to $2(1+\bar{1})$, which lies in the circular cone $(1+\bar{1})+(1+\bar{1})$.

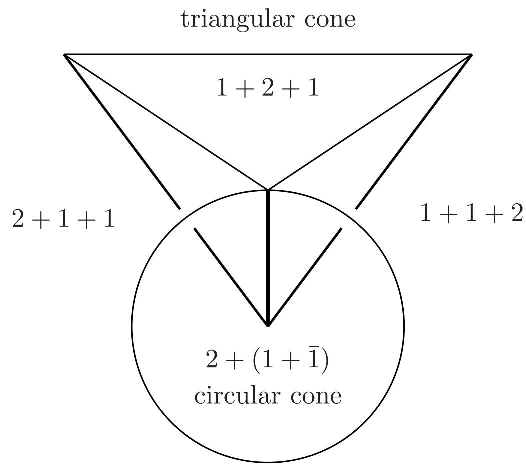

FIGURE 6. A surface homeomorphic to the surface S in xyz-space

2.4. **Parametrization of the surface S.** Let γ be a double root. Then we have

$$t^4+xt^2+yt+z/4=(t-\alpha)(t-\beta)(t-\gamma)^2, \quad \alpha+\beta+2\gamma=0,$$

and so

$$x=-\frac{3}{4}\alpha^2-\frac{1}{2}\alpha\beta-\frac{3}{4}\beta^2, \quad y=-\frac{1}{4}(\alpha+\beta)(-\beta+\alpha)^2, \quad z=(\alpha+\beta)^2\alpha\beta,$$

and

$$x=-\frac{3}{4}a^2+b, \quad y=\frac{1}{4}a(a^2-4b), \quad z=a^2b,$$

where $\alpha+\beta=-a,\ \alpha\beta=b$. The surface S can be parametrized by $a,b\in\boldsymbol{R}$ (see Figure 7); note that

$$\alpha=\beta \Leftrightarrow a^2-4b=0, \qquad \alpha=\gamma\Leftrightarrow 3a^2+4b=0.$$

Now it is not difficult for a machine to draw S (see Figure8). It is not difficult also for human beings to see that the surface S has cuspidal edges along the curves $3+1$ and $1+3$, and has normal-crossings along the curve $2+2$; we make calculus with the expression of x,y,z in (a,b). We get a better picture: see Figure 9.

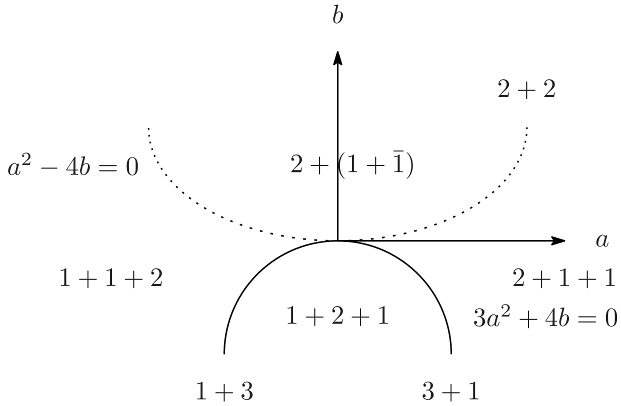

FIGURE 7. (a, b)-space parameterizing S

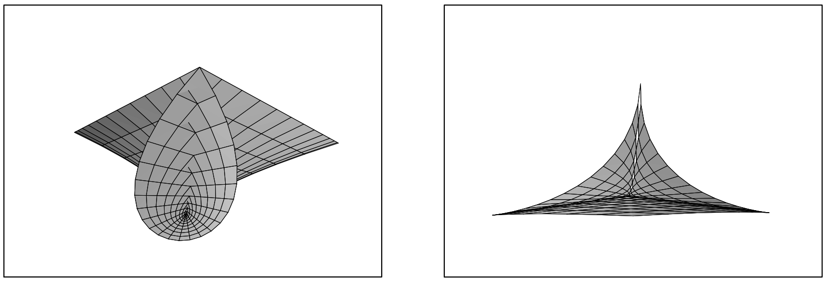

FIGURE 8. Maple figures

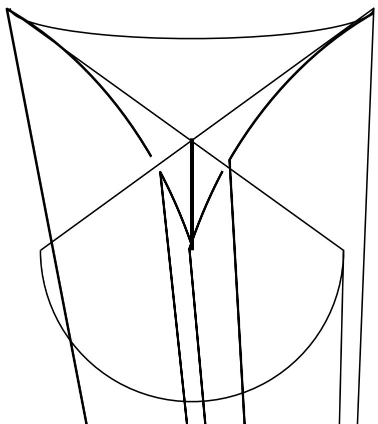

FIGURE 9. A better picture of the surface S with a swallowtail

3. Hypergeometric differential equation

The *hypergeometric differential equation*

$$E(a, b, c) : x(1-x)u'' + \{c - (a + b + 1)x\}u' - abu = 0$$

and its solutions, the hypergeometric functions, are found and studied by Euler and Gauss more than two hundred years ago. It is linear, of second order with singularities at $x = 0, 1$ and ∞. It is the simplest differential equation not solvable by highschool mathematics. The fundamental theorem of Cauchy asserts that

- at any point $x_0 \neq 0, 1, \infty$, the local solutions at x_0 (they are holomorphic) form a 2-dimensional linear space over $\boldsymbol{C}$, and
- any solution at x_0 can be continued holomorphically along any curve starting x_0 not passing through $0, 1$ or ∞.

3.1. **Local properties around the singular points.** By substituting a local expression $u = x^\alpha(1 + c_1x + c_2x^2 + \cdots)$ into the equation, we find two solutions around the origin of the form

$$1 + O(x) \quad \text{and} \quad x^{1-c}(1 + O(x)).$$

In the same way we get two solutions around $x = 1$ of the form

$$1 + O(1 - x) \quad \text{and} \quad (1 - x)^{c-a-b}(1 + O(1 - x)),$$

and around $x = \infty$,

$$\xi^a(1 + O(\xi)) \quad \text{and} \quad \xi^b(1 + O(\xi)), \qquad \xi = 1/x.$$

The pairs

$$\{0, 1 - c\}, \quad \{0, c - a - b\}, \quad \{a, b\}$$

are called the (local) exponents around $0, 1$ and ∞, respectively. Set the exponent differences as

$$\mu_0 = 1 - c, \quad \mu_1 = c - a - b, \quad \mu_\infty = a - b.$$

3.2. **Monodromy group.** The fundamental group of

$$X := \boldsymbol{C} - \{0, 1\}$$

with a base point, say $x_0 \in X$ can be generated by a loop ρ_0 around 0, and a loop ρ_1 around 1. If you continue analytically the pair (u_0, u_1) of linear independent solutions at x_0 along the loop ρ_0, then it changes linearly as

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \longrightarrow M_0 \begin{pmatrix} u_0 \\ u_1 \end{pmatrix},$$

for a matrix $M_0 \in GL_2(\boldsymbol{C})$. Along ρ_1, we get M_1. The group generated by these two matices is called the *monodromy group* of the differential equation $E(a, b, c)$ (with respect to a pair of linearly independent solutions (u_0, u_1) with base x_0). If you make use of another pair of solutions at another base point, the resulting group is conjugate in $GL_2(\boldsymbol{C})$ to this group. So the differential equation determines a conjugacy class represented by this group. Any representative, or its conjugacy class, is called the monodromy group of the equation.

There are many important subgroups of $GL_2(\boldsymbol{C})$ generated by two elements. Almost all such groups (including all the irreducible ones) are realized as the monodromy group of the hypergeometric equation $E(a, b, c)$ for some (a, b, c).

4. The Schwarz map of the hypergeometric differential equation

The Schwarz map is defined as

$$s : X \ni x \mapsto z = u_0(x) : u_1(x) \in \boldsymbol{P}^1_z \ (\cong \boldsymbol{C} \cup \{\infty\}),$$

where u_0 and u_1 are linearly independent solutions of the hypergeometric equation $E(a, b, c)$.

4.1. **Schwarz triangle when exponents are real.** In this section, we assume that the parameters a, b and c are real; so the coefficients of the equation $E(a, b, c)$ are real, if x is real. Along each of the three intervals

$$(-\infty, 0), \quad (0, 1), \quad (1, +\infty),$$

there are two linearly independent solutions which are real-valued on the interval. Note that a real-valued solution along an interval may not be real valued along another interval. Since the Schwarz map is in general multi-valued, we restrict it on the upper half-plane X_+, and study the shape of its image in the target space $\boldsymbol{P}^1_z$, the projective line with coordinate z. The Schwarz map depends on the choice of the two linearly independent solutions; if we choose other two such solutions then the new and the old Schwarz maps relate with a linear fractional transformation, which is an automorphism of $\boldsymbol{P}^1_z$.

Recall the following fundamental fact: A linear fractional transformation takes a circle to a circle, here a line is considered to be a circle which passes through ∞.

Around the singular points $x = 0, 1$ and ∞, the Schwarz map s is, after performing suitable linear fractional transformations, near to the power functions $x \to x^{|\mu_0|}, (1-x)^{|\mu_1|}$ and $x^{|\mu_\infty|}$. Thus small hemi-disks with center $x = 0, 1$ and ∞ are mapped to horns with angle $\pi|\mu_0|, \pi|\mu_1|$ and $\pi|\mu_\infty|$. See Figure 10

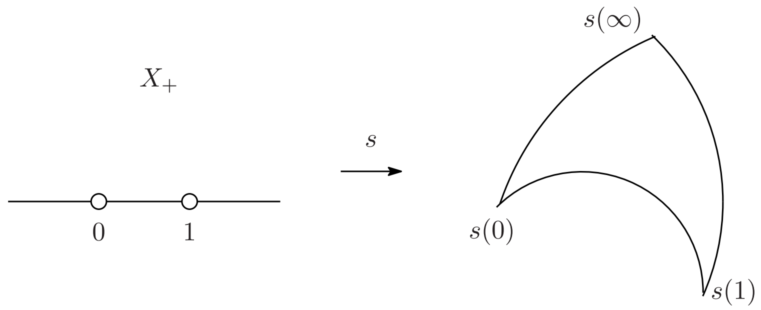

FIGURE 10. Schwarz triangle

Summing up, the image of X_+ under s is bounded by three arcs (part of circles), and the three arcs meet with angle $\pi|\mu_0|, \pi|\mu_1|$ and $\pi|\mu_\infty|$. If

$$|\mu_0| < 1, \quad |\mu_1| < 1, \quad |\mu_\infty| < 1,$$

then the image is an arc-triangle, which is called the *Schwarz triangle.* In this case, the Schwarz map gives a conformal equivalence between X_+ and the Schwarz triangle.

4.2. **Schwarz map when exponents are real.** The global behavior of the Schwarz map can be seen by applying repeatedly the Schwarz reflection principle to the three sides of the Schwarz triangle. For generic parameters, the picture of the reflected triangles gradually become chaotic. For special parameters, it can remain neat. For example, if

$$|\mu_0| = \frac{1}{p}, \quad |\mu_1| = \frac{1}{q}, \quad |\mu_\infty| = \frac{1}{r}, \quad p, q, r \in \{2, 3, \ldots, \infty\},$$

then the whole image is nice; note that this is not a *necessary* condition. Paint the original triangle black (since it is called the Schwarz triangle), adjacent ones white,

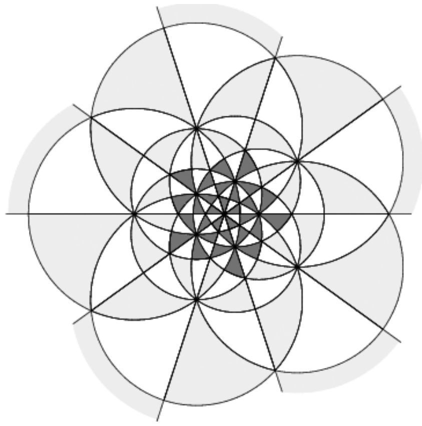

FIGURE 11. 120 icosahedral triangles and 60 central ones

and so on. The picture of these black and white triangles thus obtained can be roughly classified into three cases depending on whether

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$$

is bigger than or equal to or less than 1.

- In the first case, the number of the triangles is finite, and those cover the whole sphere. The possible triples ($p \le q \le r$) are

$$(2,2,r), \quad (2,3,3), \quad (2,3,4), \quad (2,3,5).$$

  The monodromy groups are the dihedral, tetrahedral, octahedral and icosahedral groups, respectively. Figure 11 shows 120 icosahedral triangles.
- In the second case, the three circles generated by the three sides of the triangles pass through a common point. If you send this point to infinity, then the triangles are bounded by (straight) lines, and they cover the whole plane. The possible triples ($p \le q \le r$) are

$$(2,2,\infty), \quad (3,3,3), \quad (2,4,4), \quad (2,3,6).$$

- In the third case, there is a unique circle which is perpendicular to all of the circles generated by the sides of the triangles, and these triangles fill the disc bounded by this circle. In particular, when

$$(p,q,r)=(2,3,\infty), \quad (\infty,\infty,\infty), \quad (2,3,7),$$

  you can find in many books nice pictures of the tessellation of these triangles; See Figure 12.

4.3. **Schwarz map when the exponents are not real.**

4.3.1. *Schwarz maps with purely imaginary exponents.* In this case, the monodromy group is the so-called Schottky group. See [Yo2].

4.3.2. *An example with complex exponents.* If, for example, we take the parameters as

$$a=\frac{\arccos\frac{1+i}{2}}{2\pi}, \quad b=1-a, \quad c=1,$$

then the monodromy group is generated by

$$\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1+i & 1 \end{pmatrix},$$

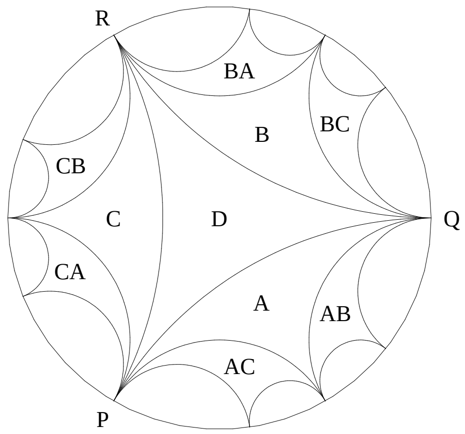

FIGURE 12. (∞, ∞, ∞)

which is the Whitehead-link-complement-group. Though the image of the Schwarz map is chaotic, the image surface of our hyperbolic Schwarz map is a closed surface in hyperbolic 3-space.

5. HYPERBOLIC SPACES

5.1. **2-dimensional models.** The upper-half plane

$$\boldsymbol{H}^2 = \{\tau \in \boldsymbol{C} \mid \Im(\tau) > 0\}$$

is called the hyperbolic 2-space. The group of (orientation-preserving) isometric transformations are given by

$$GL_2(\boldsymbol{R}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \boldsymbol{H}^2 \ni \tau \mapsto \frac{a\tau + b}{c\tau + d} \in \boldsymbol{H}^2.$$

A 'line' is a hemi-circle or a line perpendicular to the boundary line $\Im\tau = 0$. The upper-half plane can be also identified with a Poincaré disc $\{z \in \boldsymbol{C} \mid |z| < 1\}$, by

$$z = \frac{\tau - i}{\tau + i}, \qquad \tau = i\frac{1+z}{1-z}.$$

A 'line' is a hemi-circle perpendicular to the boundary circle $|z| = 1$. See Figure 12.

5.2. **3-dimensional models.** The space $\boldsymbol{H}^3 = \mathrm{Her}^+(2)/\boldsymbol{R}^+$ is called the hyperbolic 3-space. The group of (orientation-preserving) isometric transformations are given by

$$GL_2(\boldsymbol{C}) \ni P : \mathrm{Her}^+(2) \ni H \mapsto PH\ {}^t\overline{P} \in \mathrm{Her}^+(2).$$

This space $\boldsymbol{H}^3$ can be identified with the upper half-space $\boldsymbol{C} \times \boldsymbol{R}^+$ as

$$\boldsymbol{C} \times \boldsymbol{R}^+ \ni (z, t) \longmapsto \begin{pmatrix} t^2 + |z|^2 & \bar{z} \\ z & 1 \end{pmatrix} \in \mathrm{Her}^+(2),$$

$$\begin{pmatrix} h & \bar{w} \\ w & k \end{pmatrix} \in \mathrm{Her}^+(2) \longmapsto \boldsymbol{C} \times \boldsymbol{R}^+ \ni \frac{1}{k}\left(w, \sqrt{hk - |w|^2}\right).$$

A 'line' is a hemi-circle or a line perpendicular to the boundary plane $t = 0$. A 'plane' is a hemi-sphere or a plane perpendicular to the boundary plane $t = 0$. The upper half-space can be also identified with a subvariety

$$L_1 = \{x_0^2 - x_1^2 - x_2^2 - x_3^2 = 1\}$$

of the Lorentz-Minkowski 4-space

$$L(+,-,-,-) = \left\{(x_0, x_1, x_2, x_3) \in \boldsymbol{R}^4 \mid x_0^2 - x_1^2 - x_2^2 - x_3^2 > 0, x_0 > 0\right\}$$

by

$$\mathrm{Her}^+(2) \ni \begin{pmatrix} h & \bar{w} \\ w & k \end{pmatrix} \longmapsto \frac{1}{2\sqrt{hk-|w|^2}}\left(h+k, w+\bar{w}, \frac{w-\bar{w}}{i}, h-k\right) \in L_1$$

and with the Poincaré ball

$$B_3 = \{(x_1,x_2,x_3) \in \boldsymbol{R}^3 \mid x_1^2+x_2^2+x_3^2 < 1\},$$

by

$$L_1 \ni (x_0,x_1,x_2,x_3) \longmapsto \frac{1}{1+x_0}(x_1,x_2,x_3) \in B_3.$$

A 'line' is a hemi-circle perpendicular to the boundary sphere. A 'plane' is a hemisphere perpendicular to the boundary sphere. The Poincaré ball is a space form of constant curvature -1. Through identifications above, each model has also the induced metric that is invariant under respective automorphisms. Its form on the space $\boldsymbol{C} \times \boldsymbol{R}^+$ is $(dzd\bar{z}+dt^2)/t^2$.

We use these models according to convenience.

6. Hyperbolic Schwarz map – fundamental properties

We first transform our equation $E(a,b,c)$ a little. Consider in general an equation of the form

$$u'' + pu' + qu = 0.$$

If we make a change from u to v by $u = fv$, then we have

$$v'' + \left(p + 2\frac{f'}{f}\right)v' + \left(q + p\frac{f'}{f} + \frac{f''}{f}\right)v = 0.$$

Choose f solving $p + 2f'/f = 0$. Then the coefficient of v is given as

$$q - \frac{1}{2}p' - \frac{1}{4}p^2.$$

Perform this transformation to $E(a,b,c)$: By the change of the unknown

$$u \longrightarrow \sqrt{x^c(1-x)^{a+b+1-c}}\ u,$$

we get the SL-form $E^{\mathrm{SL}}(a,b,c)$

$$E^{\mathrm{SL}} : u'' - q(x)u = 0, \quad q = -\frac{1}{4}\left\{\frac{1-\mu_0^2}{x^2} + \frac{1-\mu_1^2}{(1-x)^2} + \frac{1+\mu_\infty^2-\mu_0^2-\mu_1^2}{x(1-x)}\right\},$$

where $\mu_0 = 1-c, \mu_1 = c-a-b, \mu_2 = a-b$.

Unless otherwise stated, we always take a pair (u_0,u_1) of linearly independent solutions of E^{SL} satisfying $u_0u_1' - u_0'u_1 = 1$, and set

$$U = \begin{pmatrix} u_0 & u_0' \\ u_1 & u_1' \end{pmatrix}.$$

The hyperbolic Schwarz map is defined as

$$\mathscr{S} \colon X \ni x \longmapsto H(x) = U(x)\ {}^t\overline{U}(x) = \begin{pmatrix} |u_0|^2+|u_0'|^2 & u_0\bar{u}_1+u_0'\bar{u}_1' \\ \bar{u}_0u_1+\bar{u}_0'u_1' & |u_1|^2+|u_1'|^2 \end{pmatrix} \in \boldsymbol{H}^3.$$

6.1. **Monodromy group.** Let $\{v_0, v_1\}$ be another pair of linearly independent solutions of E^{SL}. Then there is a non-singular matrix, say P, such that $U = PV$ and so that

$$U\ {}^t\overline{U} = PV\ {}^t\overline{V}\ {}^t\overline{P}, \quad \text{where} \quad V = \begin{pmatrix} v_0 & v_0' \\ v_1 & v_1' \end{pmatrix}.$$

Thus the hyperbolic Schwarz map is determined by the system up to orientation preserving automorphisms. The monodromy group $\mathrm{Mon}(a, b, c)$ with respect to U acts naturally on $\boldsymbol{H}^3$ by

$$H \longmapsto MH\ {}^t\overline{M}, \quad M \in \mathrm{Mon}(a, b, c).$$

Note that the hyperbolic Schwarz map to the upper half-space model is given by

$$X \ni x \longmapsto \frac{\Big(\bar{u}_0(x)u_1(x) + \bar{u}_0'(x)u_1'(x),\ 1\Big)}{|u_1(x)|^2 + |u_1'(x)|^2} \in \boldsymbol{C} \times \boldsymbol{R}^+.$$

6.2. **Use of the Schwarz map.** Let u and v be solutions of the equation E^{SL} such that $uv' - vu' = 1$. The Schwarz map is defined as $X \ni x \mapsto z = u(x)/v(x) \in Z$; it is convenient to study the hyperbolic Schwarz map by regarding z as variable.

Especially when the inverse of the Schwarz map is single-valued globally, this choice of variable is very useful, because the inverse map is often given explicitly as an automorphic function for the monodromy group acting properly discontinuously on the image of the Schwarz map. The solution U is written in terms of the Schwarz map z and their derivatives:

$$U = i\frac{1}{\sqrt{\dot{x}}} \begin{pmatrix} z\dot{x} & 1 + \frac{z}{2}\frac{\ddot{x}}{\dot{x}} \\ \dot{x} & \frac{1}{2}\frac{\ddot{x}}{\dot{x}} \end{pmatrix}, \tag{6.1}$$

where $\dot{} = d/dz$. Here, we summarize the way to show the formula: Since $z'(:= dz/dx) = -1/v^2$ and $\ddot{x} = d^2x/dz^2$, we have

$$v = i\sqrt{\frac{1}{z'}} = i\sqrt{\dot{x}}, \quad u = vz,$$

and

$$v' = \frac{dv}{dx} = \frac{dv}{dz}\frac{dz}{dx} = \frac{i}{2}(\dot{x})^{-3/2}\ddot{x}, \qquad u' = i\frac{1}{\sqrt{\dot{x}}} + z\frac{i}{2}(\dot{x})^{-3/2}\ddot{x}.$$

6.3. **Singularities of hyperbolic Schwarz maps.**

6.3.1. *Geometry of surfaces defined by the hyperbolic Schwarz map.* A map f from a 2-manifold M into a 3-manifold N is called a front if it is the projection of a smooth map $L : M \to T_1N$ so that $df_p(X)$ is perpendicular to $L(p)$, where T_1N is the tangent sphere bundle of N. A front may have singularity. However, the parallel immersion f_t, $t \in \boldsymbol{R}$, is well-defined, as the set of points that are equidistant from f. It is also a front. A front is said to be flat, if the Gaussian curvature vanishes at any nonsingular point and if, at a singular point, any parallel front f_t is nonsingular for any small value of t. It is known that the flat front f is written as $U\ {}^t\overline{U}$, where $U : M \to SL(2, \boldsymbol{C})$ is an immersion. Thus, in our case where $N = \boldsymbol{H}^3$, the hyperbolic Schwarz map defines a flat front.

6.3.2. *Singularities of $\mathscr{S} : X \to \boldsymbol{H}^3$.* Since the equation E^{SL} has singularities at 0, 1 and ∞, the corresponding hyperbolic Schwarz map $\mathscr{S}$ has singularities at these points. In terms of flat fronts in $\boldsymbol{H}^3$, they are considered as *ends* of the surface. On the other hand, the map $\mathscr{S}$ may not be an immersion at $x \in X$, even if x is not a singular point of E^{SL}.

In this subsection, we analyze properties of these singular points of the hyperbolic Schwarz maps.

The following criterion is known.

Lemma 6.1 ([KRSUY])**.** (1) *A point $p \in X$ is a singular point of the hyperbolic Schwarz map H if and only if $|q(p)| = 1$,*
(2) *a singular point $x \in X$ of H is equivalent to the cuspidal edge if and only if*

$$q'(x) \neq 0 \quad \textit{and} \quad q^3(x)\bar{q}'(x) - q'(x) \neq 0,$$

(3) *and singular point $x \in X$ of H is equivalent to the swallowtail if and only if $q'(x) \neq 0$,*

$$q^3(x)\bar{q}'(x) - q'(x) = 0, \quad \textit{and} \quad \Re\left\{\frac{1}{q}\left(\left(\frac{q'(x)}{q(x)}\right)' - \frac{1}{2}\left(\frac{q'(x)}{q(x)}\right)^2\right)\right\} \neq 0.$$

We apply Lemma 6.1 to the hypergeometric equation. The coefficient q of the equation E^{SL} is written as

$$q =: \frac{-Q}{4x^2(1-x)^2}, \quad Q = 1 - \mu_0^2 + (\mu_\infty^2 + \mu_0^2 - \mu_1^2 - 1)x + (1 - \mu_\infty^2)x^2. \tag{6.2}$$

Hence $x \in X$ is a singular point if and only if

$$|Q| = 4|x^2(1-x)^2|. \tag{6.3}$$

The condition $q^3(x)\bar{q}'(x) - q'(x) = 0$ is equivalent to

$$Q^3\bar{R}^2 \text{ is real non-positive,} \tag{6.4}$$

where R is given by

$$q' = -\frac{Q'x(1-x) - 2Q(1-2x)}{4x^3(1-x)^3} =: \frac{-R}{4x^3(1-x)^3}.$$

Thus cuspidal edges and swallowtails can be found by solving a system of *algebraic* equations.

6.3.3. *At a singular point of the equation $E(a,b,c)$.* By using local expressions of solutions around the singularities 0, 1 and ∞, we can prove that if the parameters a, b and c are real, the hyperbolic Schwarz map $\mathscr{S}$ extends to these singular points and to the boundary of $\boldsymbol{H}^3$. Its image is nonsingular and is tangent to the boundary at these point.

7. Hyperbolic Schwarz maps – Examples

When the monodromy group of the equation $E(a,b,c)$ is a finite group or a typical Fuchsian group, we study the singularities of the hyperbolic Schwarz map, and visualize the image surface.

7.1. **Finite (polyhedral) monodromy groups.** We first recall fundamental facts about the polyhedral groups and their invariants.

7.1.1. *Basic data.* Let the triple (k_0, k_1, k_∞) be one of

$$(2,2,n)\ (n=1,2,\dots),\quad (2,3,3),\quad (2,3,4),\quad (2,3,5),$$

in which case, the projective monodromy group is of finite order N:

$$N = \quad 2n, \quad 12, \quad 24, \quad 60,$$

respectively. Note that

$$\frac{2}{N} = \frac{1}{k_0} + \frac{1}{k_1} + \frac{1}{k_\infty} - 1.$$

For each case, there is a triplet $\{R_1, R_2, R_3\}$ of reflections whose mirrors bound a Schwarz triangle (cf. [SYY1]). In the dihedral case $(2,2,n)$, for example, we can take

$$R_1 : z \mapsto \bar{z}, \quad R_2 : z \mapsto e^{2\pi i/n}\bar{z}, \quad R_3 : z \mapsto \frac{1}{\bar{z}}.$$

The monodromy group Mon (a polyhedral group) is the group of even words of these three reflections.
The (single-valued) inverse map

$$s^{-1} : Z \ni z \longmapsto x \in \bar{X} \cong \boldsymbol{P}^1,$$

invariant under the action of Mon, is given as follows. Let $f_0(z), f_1(z)$ and $f_\infty(z)$ be the monic polynomials in z with simple zeros exactly at the images $s(0), s(1)$ and $s(\infty)$, respectively. If $\infty \in Z$ is not in these images, then the degrees of these polynomials are $N/k_0, N/k_1$ and N/k_∞, respectively; if for instance $\infty \in s(0)$, then the degree of f_0 is $N/k_0 - 1$. Now the inverse map s^{-1} is given by

$$x = A_0\ \frac{f_0(z)^{k_0}}{f_\infty(z)^{k_\infty}},$$

where A_0 is a constant; we also have

$$1 - x = A_1\ \frac{f_1(z)^{k_1}}{f_\infty(z)^{k_\infty}}, \quad \frac{dx}{dz} = A\frac{f_0(z)^{k_0-1}f_1(z)^{k_1-1}}{f_\infty(z)^{k_\infty+1}},$$

for some constants A_1 and A. Their explicit forms can be found for example in [SYY1]. In the dihedral case $(k_0, k_1, k_\infty) = (2,2,n)$, for example, we have

$$A_0 = \frac{1}{4}, \quad A_1 = -\frac{1}{4}, \quad A = \frac{n}{4}, f_0 = z^n + 1, \quad f_1 = z^n - 1, \quad f_\infty = z.$$

Note that f_∞ is of degree $1 = 2n/n - 1$, since $\infty \in s(\infty)$, that is, $x(\infty) = \infty$.

7.1.2. *Dihedral cases.* We consider a dihedral case: $(k_0, k_1, k_\infty) = (2,2,n)$, $n = 3$. The curve C in the x-plane defined by (6.3): $|Q| = 4|x(1-x)|^2$ is symmetric with respect to the line $\Re(x) = 1/2$ and has a shape of a cocoon (see Figure 13 (left)). We next study the conditions (6.3) and (6.4). We can prove that, on the upper half x-plane, there is a unique point (the intersection P of the curve C and the line $\Re x = 1/2$) that the image surface has a swallowtail at this point, and has cuspidal edges along $\mathscr{S}(C)$ outside $\mathscr{S}(P)$. We omit the computation.

The curve which gives the self-intersection is tangent to C at P, and crosses the real axis perpendicularly; this is the dotted curve in Figure 13 (left), and is made as follows. Since the curve is symmetric with respect to the line $\Re x = 1/2$, on each level line $\Im x = t$, we take two points x_1 and x_2 ($\Re(x_1 + x_2) = 1$), compute the distance between their images $\mathscr{S}(x_1)$ and $\mathscr{S}(x_2)$, and find the points that the two image points coincide.

We substitute the inverse of the Schwarz map:

$$x = \frac{1}{4}\ \frac{(z^n+1)^2}{z^n}, \qquad n = 3$$

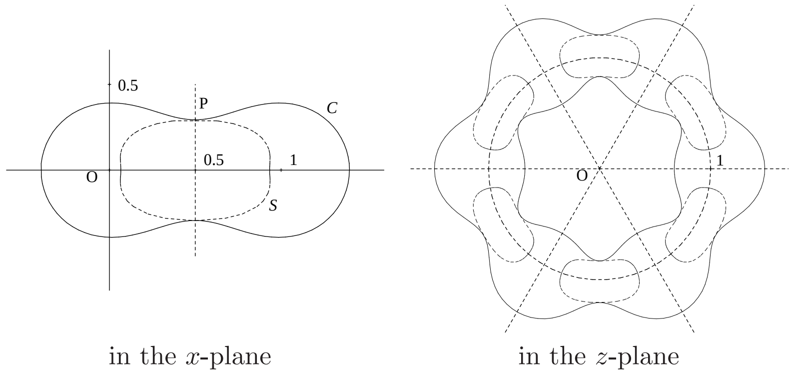

FIGURE 13. The curve $C : |Q| = 4|x(1-x)|^2$, when $(k_0, k_1, k_\infty) = (2, 2, 3)$

into the expression (6.1) of the hyperbolic Schwarz map, and visualize the image surface in the Poincaré ball model explained in Section 5.2. The upper half x-space corresponds to a fan in the z-plane bounded by the lines with argument $0, \pi/3, 2\pi/3$, and the unit circle (see Figure 13(right).). The image $s(C)$ consists of two curves; the dotted curves in the figures form the pre-image of the self-intersection.

Let Φ denote the hyperbolic Schwarz map in z-variable:

$$\Phi := \mathscr{S} \circ s^{-1} : Z \ni z \longmapsto H(z) \in \boldsymbol{H}^3.$$

We visualize the image of the hyperbolic Schwarz map when $n = 3$. Figure 14(upper left) is a view of the image of one fan in the z-plane under Φ (equivalently, the image of upper/lower half x-plane under $\mathscr{S}$). The cuspidal edge traverses the figure from left to right and one swallowtail is visible in the center. The upper right figure is the antipode of the left. Figure 14(below) is a view of the image of six fans dividing the unit z-disk. To draw the images of fans with the same accuracy, we make use of the invariance of the function $x(z)$ under the monodromy groups.

The image of the sixty icosahedral triangles (Figure 11) are shown in Figure 15.

7.2. **A Fuchsian monodromy group.** We study only the case $(k_0, k_1, k_\infty) = (\infty, \infty, \infty)$.

7.2.1. *Singular locus.* We find the singular locus of the image when $\mu_0 = \mu_1 = \mu_\infty = 0$. We have

$$Q = 1 - x + x^2, \quad R = (-1 + 2\,x)\left(x^2 - x + 2\right).$$

The singularities lie on the image of the curve

$$C : f := 16|x(1-x)|^4 - |Q|^2 = 0.$$

Note that this curve is symmetric with respect to the line $\Re x = 1/2$.

We can prove that, on the upper half x-plane, there is a unique point P – the intersection of the curve C and the line $\Re(x) = 1/2$ – that the image surface has a swallowtail singularity at P, and has cuspidal edges along $\mathscr{S}(C)$ outside $\mathscr{S}(P)$.

We omit the actual computation.

7.2.2. *Lambda function.* The inverse of the Schwarz map is a modular function known as the lambda function

$$\lambda : \boldsymbol{H}^2 = \{z \in \boldsymbol{C} \mid \Im z > 0\} \longrightarrow X.$$

The hyperbolic Schwarz map is expressed in terms of its derivatives. The lambda function can be expressed in terms of the theta functions. Since these functions

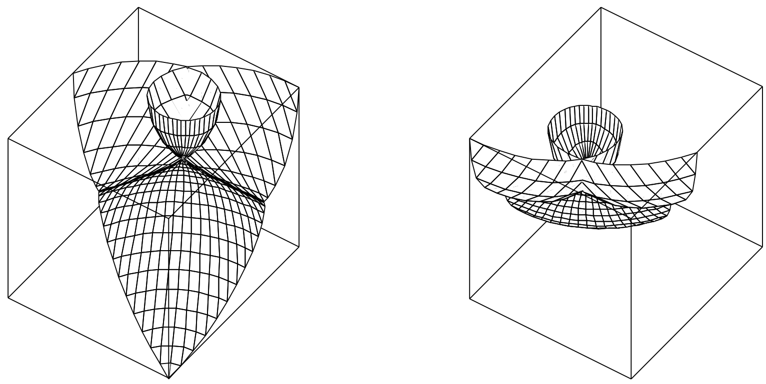

Image of a fan under Φ; a swallowtail can be seen

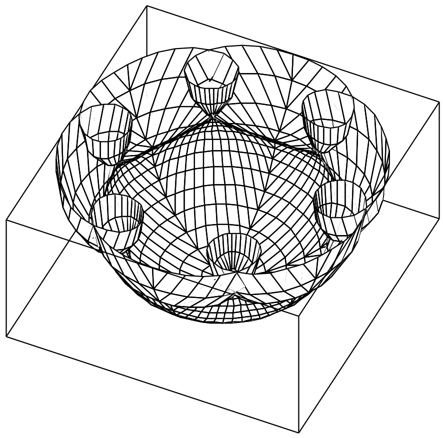

Image of six fans; six swallowtails can be seen

FIGURE 14. Dihedral case

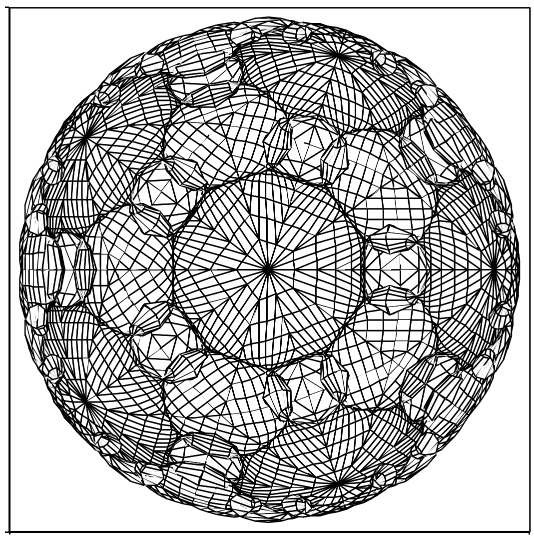

FIGURE 15. Sixty swallowtails can be seen

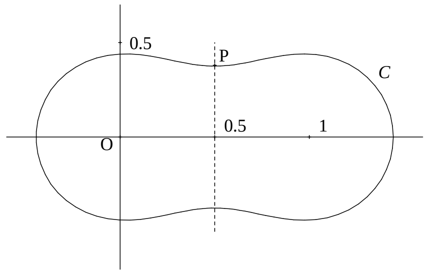

FIGURE 16. The curve $C : |Q| = 4|x(1-x)|^2$, when $(k_0, k_1, k_\infty) = (\infty.\infty, \infty)$

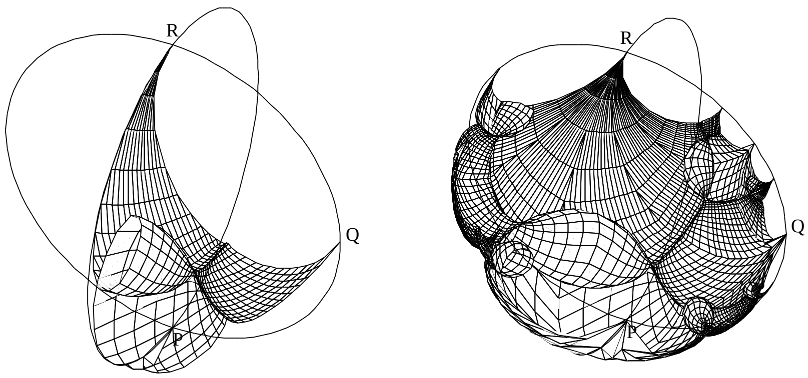

Image of one triangle D Image of $\{D, A, B, C, AB, BA, AC, CA, BC, CB\}$

FIGURE 17. Images of the hyperbolic Schwarz map when $k_0 = k_1 = k_\infty = \infty$

have q-series expansions, which converge very fast, we can accurately compute the image of the hyperbolic Schwarz map.

7.2.3. *Visualizing the image surface.* The image of the hyperbolic Schwarz map is shown in Figure 17. The first one is the image of the triangle $\{D\}$, and the second one the ten triangles $\{D, A, B, C, AB, BA, AC, CA, BC, CB\}$; these triangles are labeled in Figure 12.

7.3. **Parallel family of flat fronts connecting Schwarz and derived Schwarz maps.** The parallel front f_t of distance $t \in \boldsymbol{R}$ of the flat front f is defined as

$$\varphi_t(x) = \exp_{\varphi(x)} t\nu(x) = (\cosh t)\varphi(x) + (\sinh t)\nu(x),$$

where ν is the unit normal vector field and exp denotes the exponential map of $\boldsymbol{H}^3$. It has another expression as

$$\varphi_t = U_t {}^t\overline{U_t}, \qquad U_t = U \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

As t tends to infinity, the parallel front approaches to the hyperbolic Schwarz map in one direction and to the derived hyperbolic Schwarz map in the other direction.

The family of such flat fronts is seen in the Figure 18.

7.4. **Confluence of swallowtail singularities.** The generic singularities of the hyperbolic Schwarz map are cuspidal edges and swallowtail singularities. Their location depends obviously on the parameters (a, b, c). It is interesting to trace

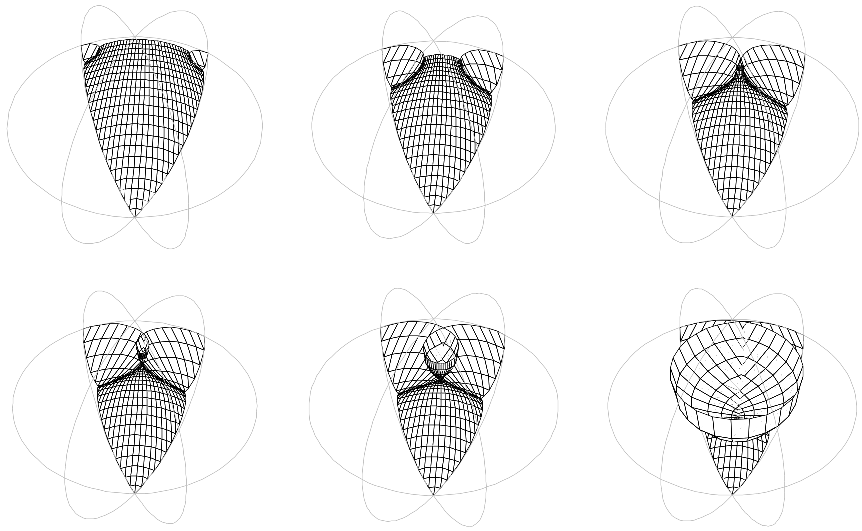

FIGURE 18. Parallel family of the images when $(a, b, c) = (1/6, -1/6, 1/2)$

the shapes of the cuspidal edges and to see how creation/elimination of swallowtail singularities occur depending on the parameters.

We here give one picture when $(a, b, c) = (1/2, 1/2, c)$ which shows change of the surfaces in the upper row and the cuspidal edges in the lower row. The value of parameter c is written in the Figure 19.

When $c > 0.2929$, there is a pair of swallowtail singularities on the image surface carried by a pair of cuspidal components of the cuspidal edge curve. When c tends to 0.2929, two singularities come together and kiss, and when $c < 0.2929$, the cuspidal edge curve becomes a pair of nonsingular curves.

We note that the confluence of swallowtail singularities was studied by Arnold [AGV]; five types $(1, \ldots, 5)$ of confluence (bifurcation) as in the Figure 20 are known. The confluence shown in Figure 19 corresponds to Type 2. Types 3 and 5 also appear in hyperbolic Schwarz maps, see [NSYY].

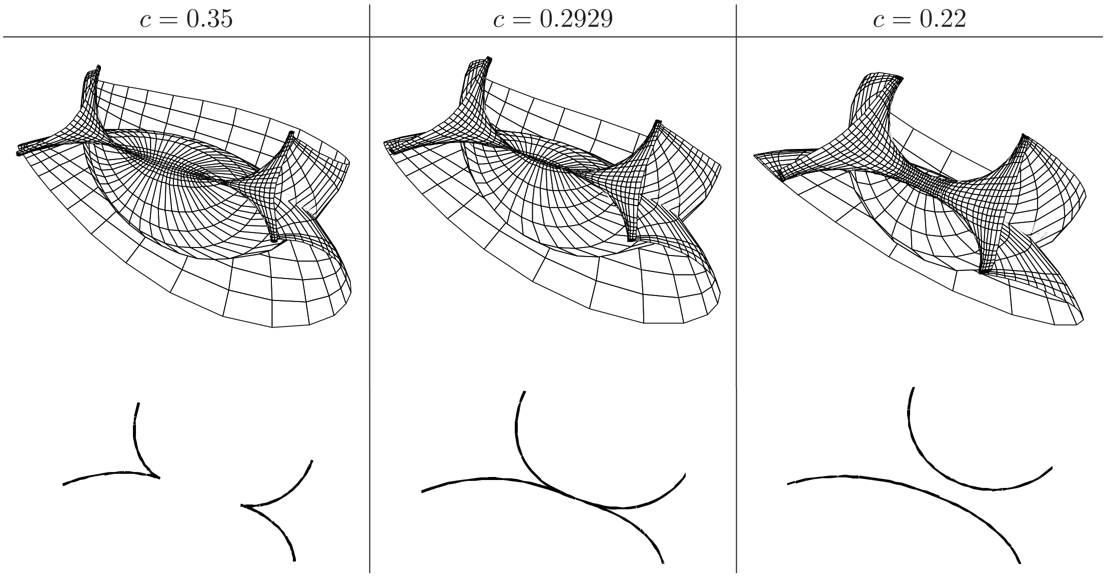

FIGURE 19. Dependence of surfaces and cuspidal edges on parameter c

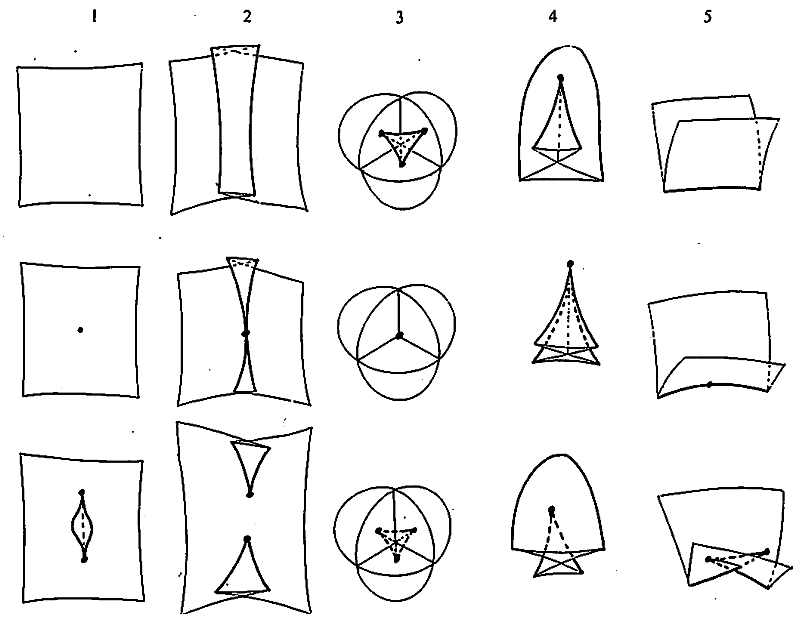

FIGURE 20. Confluence of swallowtail singularities ([LLR])

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