Analytic Gröbner fan and its applications

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Notions in polyhedral geometry (1)

References: G.Ziegler, Lectures on polytopes. R.Thomas, Lectures in geometric combinatorics.

1. Polyhedron is the intersection of finite closed half spaces in $\mathbf{R}^d$.

2. Polytope is a bounded polyhedron.

3. Closed cone is the intersection of finite closed half spaces of the form $\sum_{j=1}^{d} c_{ij}x_j \geq 0$.

4. Closed cone C can be expressed as

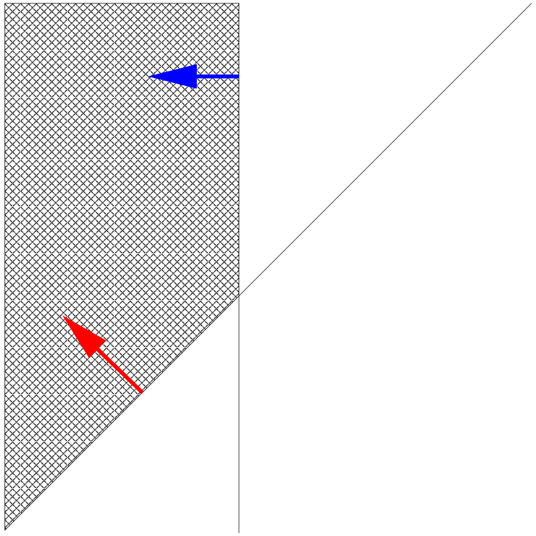
$$C = \mathbf{R}_{\geq 0}\, a_1 + \cdots + \mathbf{R}_{\geq 0}\, a_n$$

where $a_i \in \mathbf{R}^d$ and $\mathbf{R}_{\geq 0} = \{x \in \mathbf{R} \,|\, x \geq 0\}$.

Notions in polyhedral geometry (1) in picture

Closed cone is the intersection of finite closed half spaces of the form $\sum_{j=1}^{d} c_{ij}x_j \geq 0$.

The intersection of $-x \geq 0$ and $-x+y \geq 0$ is



and can be expressed as

$$\mathbf{R}_{\geq 0}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathbf{R}_{\geq 0}\begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Notions in polyhedral geometry (2)

1. Let P be a polyhedron and $w \in \mathbf{R}^d$ a weight vector.

$$\begin{aligned} \mathrm{face}_w(P) &:= \{m \in P \,|\, \langle m, w \rangle \leq \langle x, w \rangle \text{ for all } x \text{ in } P\} \\ &= \text{The lowest points of } P \text{ with respect to } w. \end{aligned}$$

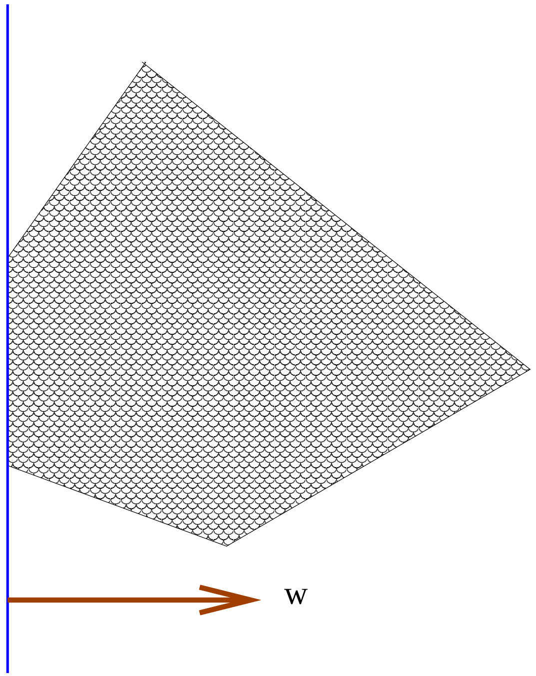
2. $\mathrm{face}_w(P)$ is called the face of P determined by w.

3. Although choices for w are infinite, there are only finite faces.

4. The face is an analogous notion of the initial term for a polynomial f. In fact, the Newton polygon of $\mathrm{in}_w(f)$ agrees with the $\mathrm{face}_w(\mathrm{New}(f))$.

Notions in polyhedral geometry (2) in picuture

$$\begin{aligned} \text{face}_w(P) &:= \{m \in P \,|\, \langle m, w\rangle \le \langle x, w\rangle \text{ for all } x \text{ in } P\} \\ &= \text{The lowest points of } P \text{ with respect to } w. \end{aligned}$$



In this example, $w = (1, 0)$.

For $f=\sum_{\alpha\in E} c_\alpha x^\alpha,\ c_\alpha\neq 0, \alpha\in E,\ x^\alpha = x_1^{\alpha_1}\cdots x_d^{\alpha_d}$ and $w\in\mathbf{R}^d$ we define w-order

$$\mathrm{ord}_w(f)=\min_{\alpha\in E}\langle w,\alpha\rangle$$

and the initial form of f with respect to w

$$\begin{aligned}\mathrm{in}_w(f) &:= \sum_{\alpha\in E,\langle\alpha,w\rangle=\mathrm{ord}_w(f)} c_\alpha x^\alpha \\ &= \text{The lowest order terms of } f \text{ with respect to } w.\end{aligned}$$

Example: $\mathrm{in}_{(2,1)}(x_1x_2+3x_2^3+x_1^2)=x_1x_2+3x_2^3$.
The initial ideal $\mathrm{in}_w(I)$ is the ideal generated by initial forms $\mathrm{in}_w(f)$, $f\in I$.

We may use the following alternative definition

$$\begin{aligned}\mathrm{face}_w(P) &:= \{m\in P\,|\,\langle m,w\rangle\geq\langle x,w\rangle \text{ for all } x \text{ in } P\} \\ &= \text{The highest points of } P \text{ with respect to } w.\end{aligned}$$

In this case, we change the definition of the initial form as

$$\mathrm{in}_w(f) = \text{The highest order terms of } f \text{ with respect to } w.$$

Notions in polyhedral geometry (3)

1. Let e be a face of P. $N(e) = \{w \in \mathbf{R}^d \,|\, \mathrm{face}_w(P) = e\}$ (normal cone).

2. Let F be the set of all faces of P. The collection of $\bar{N}(e)$, $e \in F$ is called the normal fan of P and denoted by $\bar{\mathcal{N}}(P)$.

3. A finite set Δ of closed cones are called a (polyhedral) fan when

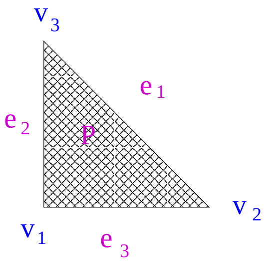
   (a) If $\sigma \in \Delta$, then any face of σ is in Δ.

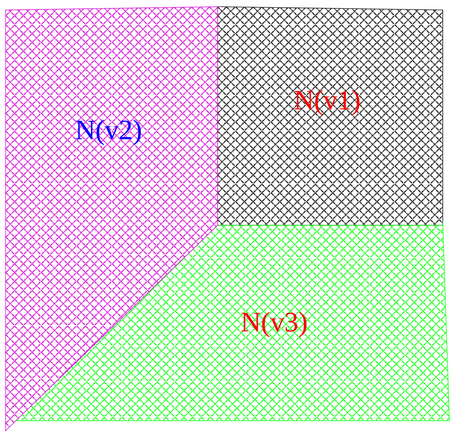
   (b) If $\sigma \in \Delta$ and $\tau \in \Delta$, then $\sigma \cap \tau$ is a face of both of σ and τ.

4. **Theorem**. The normal fan is a fan.

Notions in polyhedral geometry (3) in picuture

Let e be a face of P. $N(e) = \{w \in \mathbf{R}^d \,|\, \text{face}_w(P) = e\}$.



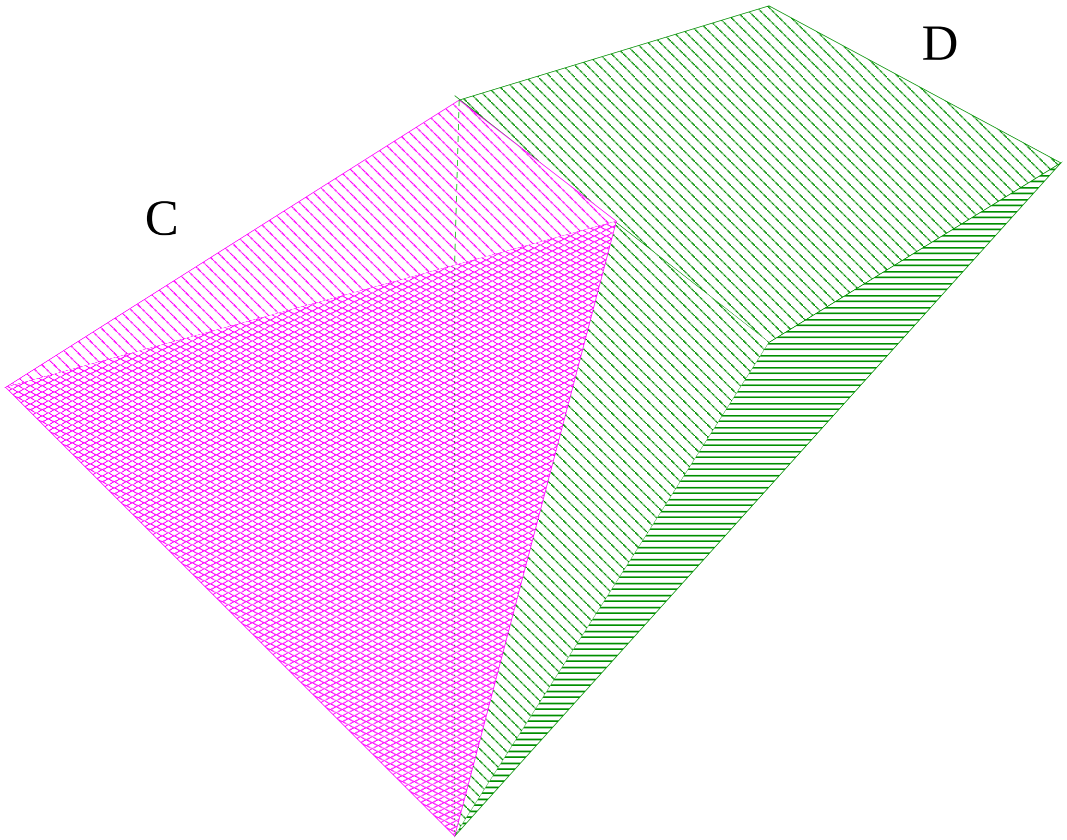


$$
\begin{aligned}
N(v_1) &= \mathbf{R}_{>0}\begin{pmatrix}1\\0\end{pmatrix} + \mathbf{R}_{>0}\begin{pmatrix}0\\1\end{pmatrix}\\
N(v_2) &= \mathbf{R}_{>0}\begin{pmatrix}0\\1\end{pmatrix} + \mathbf{R}_{>0}\begin{pmatrix}-1\\-1\end{pmatrix}\\
N(v_3) &= \mathbf{R}_{>0}\begin{pmatrix}-1\\-1\end{pmatrix} + \mathbf{R}_{>0}\begin{pmatrix}1\\0\end{pmatrix}\\
N(e_1) &= \mathbf{R}_{>0}\begin{pmatrix}-1\\-1\end{pmatrix}\\
N(e_2) &= \mathbf{R}_{>0}\begin{pmatrix}1\\0\end{pmatrix}\\
N(e_3) &= \mathbf{R}_{>0}\begin{pmatrix}0\\1\end{pmatrix}, \quad N(P) = (0,0)
\end{aligned}
$$

What is a fan? What is not a fan?

A finite set Δ of closed cones are called a fan when

1. If $\sigma \in \Delta$, then any face of σ is in Δ.
2. If $\sigma \in \Delta$ and $\tau \in \Delta$, then $\sigma \cap \tau$ is a face of both of σ and τ.



$\Delta = \{C, D, \text{faces of } C \text{ and } D\}$ is not a fan. Because, $C \cap D$ is not a face of D.

A brief history on Gröbner fan (1).

Let I be a homogeneous ideal in $\mathbf{K}[x_1, \ldots, x_d]$ and w a weight vector. Let $e = \mathrm{in}_w(I)$ be the initial ideal of I with respect to w.

$$C[e] = C[w] := \{w \in \mathbf{R}^d \,|\, \mathrm{in}_w(I) = e\}$$

is called the open (global) Gröbner cone for e (for w). the normal cone $N(e) = \{w \in \mathbf{R}^d \,|\, \mathrm{face}_w(P) = e\}$

1. There exist only finite distinct initial ideals. Finite faces.
2. $\bar{C}[e]$ is a rational polyhedral cone. (T.Mora, L.Robbiano, 1988)
3. "$\bar{\mathcal{E}}(I) :=$ the collection of closed Gröbner cones" is called a Gröbner fan.
4. The Gröbner fan is a fan (B.Sturmfels, Gröbner basis and convex polytopes, 1995). The normal fan is a fan.

Example: The Gröbner fan of $\langle f \rangle$ is the normal fan of the Newton polygon of f. $f = 1 + x_1 + x_2$

Theorem (B.Sturmfels, 1990): The Gröbner fan of homogeneous affine toric ideal is a refinement of the secondary fan standing for the regular triangulations.

A brief history on Gröbner fan (2).

1. K.Fukuda, A.Jensen, R.Thomas, Computing Gröbner fans, `math.AC/0509544`.
   A definition of (global) Gröbner fan for inhomogeneous ideal.
   Theorem: The Gröbner fan is a fan.

2. Collection of $\bar{C}[\mathrm{in}_w(I)]$ such that $\mathrm{in}_w(I)$ does not contain a monomial is called the tropical zero set associated to I, e.g., T.Bogart, A.Jensen, D.Speyer, B.Sturmfels, R.Thomas, Computing tropical varieties, `math.AG/0507563`.
   Theorem: The tropical zero set can be regarded as a subfan of the Gröbner fan.

"global" : Ideals are in the polynomial ring $K[x_1, \ldots, x_d]$ or in the Weyl algebra. The initial forms are also in $K[x_1, \ldots, x_d]$.

$$\begin{aligned} \hat{\mathcal{O}}_0 &= \mathbf{C}[[x_1, \ldots, x_n]], \quad \mathcal{O}_0^{an} = \text{ germ of convergent power series} \\ \mathcal{O}_0^{alg} &= \{f/g \mid f, g \in \mathbf{C}[x], g(0) \neq 0\} \\ h_{(0,1)}(\hat{\mathcal{D}})_0 &= \hat{\mathcal{O}}_0[h]\langle \partial_1, \ldots, \partial_n \rangle \\ & \quad \text{where } \partial_i a(x) = a(x)\partial_i + h\frac{\partial a}{\partial x_i} \end{aligned}$$

$$W_{loc} = \{(w_1, \ldots, w_n) \mid w_i \geq 0\}$$

Assi, Castro, Granger (2001) suggested a definition of local Gröbner fan (analytic standard fan) and proved the finiteness and convexity of the local Gröbner cones (analytic standard cones).

Theorem: the set of the $\overline{\text{local Gröbner cones}}$ (= local Gröbner fan) is a polyhedral fan.

Take $w = (0, \ldots, 0, w_k, \ldots, w_n) \in W_{loc}$, $w_k > 0, \ldots,$.

$$\mathrm{gr}^w(\hat{\mathcal{O}}) = \mathbf{C}[[x_1, \ldots, x_{k-1}]][x_k, \ldots, x_n]$$

Fix an ideal I in $\hat{\mathcal{O}}$. $\mathrm{in}_w(I) \subseteq \mathrm{gr}^w(\hat{\mathcal{O}})$.

$$w \sim w' \Leftrightarrow \mathrm{gr}^w(\hat{\mathcal{O}}) = \mathrm{gr}^{w'}(\hat{\mathcal{O}}) \text{ and } \mathrm{in}_w(I) = \mathrm{in}_{w'}(I)$$

$$C[w] = \{w' \mid w \sim w'\} \quad \text{(local or analytic Gröbner cone)}$$

$$\bar{\mathcal{E}}(I) = \{\bar{C}[w]\} \quad \text{(local or analytic Gröbner fan)}$$

Q. Why do we need to prove that $\bar{\mathcal{E}}$ is a polyhedral fan?

(1) It is a natural property. It should be proved.
(2) Since $\bar{\mathcal{E}}$ is a polyhedral fan, we may only enumerate the maximal dimensional cones. $\Rightarrow$ Effective enumeration.

(3) Corollary. Let L be a linear subspace of $\mathbf{R}^n$. Then, $\bar{\mathcal{E}}(I) \cap L$ is a polyhedral fan.

Comparison (1)

1. For homogeneous I, local and global cones agree.

2. For polynomial generators, local and analytic agree.

Theorem: Local (analytic) Gröbner fan is computable if the ideal is generated by polynomials.

1. (Double) homogenization.

2. Tangent cone algorithms in $\mathcal{O}^{alg}$ (Mora, Gräbe, Greuel, Pfister) and $\mathcal{D}^{alg}$ (Grangner-Oaku-T).

Example. Global vs. local

$$
\begin{aligned}
I^{global} &= K[x_1,x_2,x_3](x_1+x_2) + K[x_1,x_2,x_3](1-x_3) \\
I^{an} &= \hat{\mathcal{O}}\cdot(x_1+x_2) + \hat{\mathcal{O}}\cdot(1-x_3) = \langle 1\rangle
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{in}_{(2,1,0)}(x_1+x_2) &= x_2 \\
\mathrm{in}_{(2,1,0)}(1-x_3) &= 1-x_3
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{in}_{(2,1,0)}(I^{global}) &= K[x_1,x_2,x_3]x_2 + K[x_1,x_2,x_3](1-x_3) \\
\mathrm{in}_{(2,1,0)}(I^{an}) &= K[x_1,x_2][[x_3]]\cdot 1
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{in}_{(1,2,0)}(I^{global}) &= K[x_1,x_2,x_3]x_1 + K[x_1,x_2,x_3](1-x_3) \\
\mathrm{in}_{(1,2,0)}(I^{an}) &= K[x_1,x_2][[x_3]]\cdot 1
\end{aligned}
$$

Q. Why do we need $h_{(0,1)}(\hat{\mathcal{D}}) = \hat{\mathcal{O}}[h]\langle \partial_1, \ldots, \partial_n \rangle$?

Informally, it is like that we need a homogeneous input in $\mathbf{C}[[x_1, \ldots, x_n]][x_{n+1}, \ldots, x_{2n}]$ to get the reduced Gröbner basis for local orders.

There will be no polyhedral Gröbner fan for $\mathcal{D}$, because the division theorem does not hold.

Example (T.Oaku):
$n = 1$, $u + v \geq 0$, $u < 0$. u is the weight for x and v is the weight for ∂.
Put $f = x$ and $g = x + x^k \partial$ where k is taken so that $x > x^k \partial$.
Suppose that we can perform a division of the form

$$f = qg + r = q(\underline{x} + x^k \partial) + r(\partial)$$

where the remainder r does not contain the variable x since the leading term of g is x. Take the initial of $f - qg$ with the weight vector $(0, 1)$, then it is of the form $x^k \xi$. It is a contradiction.

References:

1. R.Bahloul and N.Takayama, Local Gröbner fan: polyhedral and computational approach, math.AG/0412044

2. R.Bahloul and N.Takayama, Local Gröbner fan: to appear in C.R. Academy Sci. Paris.
http://www.math.kobe-u.ac.jp/~taka

Application 1: A new algorithm to get local Bernstein-Sato polynomials (by R.Bahloul. computational details are by H.Nakayama(中山) and T)

Rouchdi Bahloul gives a constructive proof of the existence of local Bernstein-Sato polynomial in terms of the local Gröbner fan. (R. Bahloul, *Démonstration constructive de l'existence de polynômes de Bernstein-Sato pour plusieurs fonctions analytiques*, Compositio Math. **141** (2005), 175–191.) Our construction algorithm of local Gröbner fan gives a method to compute local Bernstein-Sato polynomials in two variable case.

Switch "max" and "min" ($w \Leftrightarrow -w$) in the sequel.
Given $f_1, f_2 \in \mathbf{C}[x_1, x_2]$ and $v \in \mathbf{N}_0^2$. v is usually $(1,1)$ or $(1,0)$ or $(0,1)$.
Polynomials b satisfying

$$b(s_1, s_2) f_1^{s_1} f_2^{s_2} \in \mathcal{D}[s] f_1^{s_1+v_1} f_2^{s_2+v_2}$$

are called Bernstein-Sato polynomials for f_1, f_2, v. The set of all BS polynomials is an ideal and is denoted by $\mathcal{B}^v(f_1, f_2)$.

Problem: Compute non-zero $b(s)$.

$$I = \langle t_1 - f_1, t_2 - f_2, \partial_{x_i} + \frac{\partial f_1}{\partial x_i}\partial_{t_1} + \frac{\partial f_2}{\partial x_i}\partial_{t_2}, i = 1, 2\rangle$$

$$\bar{B}(I) = \bar{\mathcal{E}}(I) \cap \{(\overbrace{-w_1}^{t_1}, \overbrace{-w_2}^{t_2}, \overbrace{0}^{x_1}, \overbrace{0}^{x_2}, \overbrace{w_1}^{\partial_{t_1}}, \overbrace{w_2}^{\partial_{t_2}}, \overbrace{0}^{\partial_{x_1}}, \overbrace{0}^{\partial_{x_2}}) \mid w_i \in \mathbf{R}_{\geq 0}\}$$

We call $\bar{B}(I)$, the local BSS Gröbner fan.

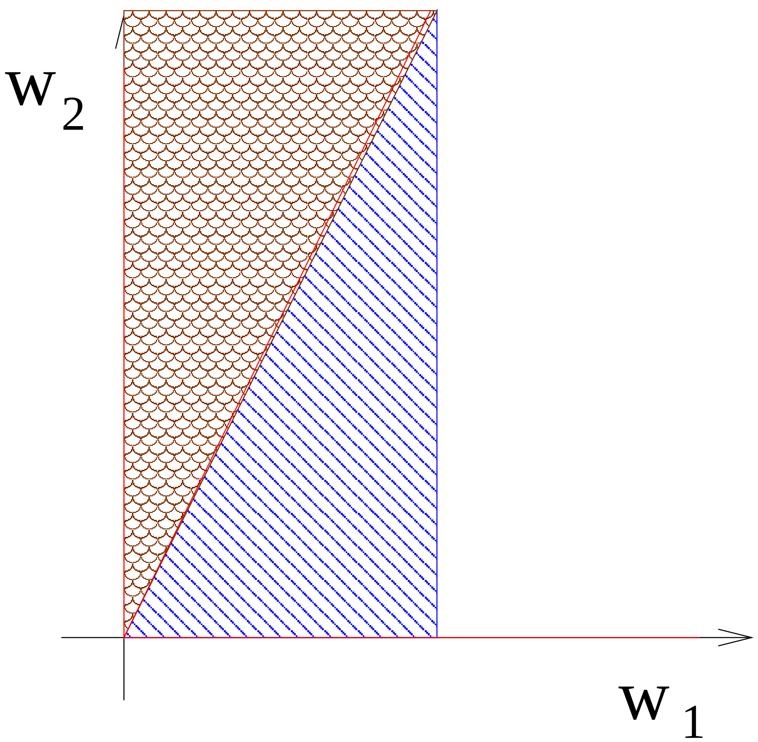
Theorem-Algorithm

$$\prod_{L \in \text{ray in} \bar{B}(I),} \quad \prod_{-\langle L, v+\kappa\rangle < k \leq 0} b_L(L_1 s_1 + L_2 s_2 - k)$$

lies in $\mathcal{B}^v(f_1, f_2)$. Here κ is a vector computed from the local BSS fan and b_L is a local b-function along L.

Example: $f_1 = x_1 + x_2$, $f_2 = x_1^2 + x_2^2$. $\bar{B}(I)$ consists of two maximal dimensional cones $(1,0),(1,2)$ and $(1,2),(0,1)$.

For $L = (1,2)$, $b_L(s)$ is $2s^2 + s$.
For $L = (1,0)$, $b_L(s)$ is s.
For $L = (0,1)$, $b_L(s)$ is s^2.



Application 2: local tropical variety. N.Touda(富田), Local Tropical Variety, math.AG/0511486

$\mathrm{Trop}(I) = \{w \in \bar{\mathcal{E}}(I) \mid \mathrm{in}_w(I) \in \mathrm{gr}^w(\hat{\mathcal{O}})$ does not contain a monomial$\}$

$\mathrm{Trop}(I)$ is called the local tropical zero set.
Theorem (N.Touda): $\mathrm{Trop}(I)$ can be regarded as a sub fan of the local Gröbner fan.

Algorithm(conjectural). $\mathrm{Trop}(I)$ can be computed if I is generated by polynomials.

Open problems. Natural geometric definition. ...

Application 3: Slopes

$$V_1 = (\overbrace{-1}^{x_1}, \overbrace{0}^{x_2}, \ldots, \overbrace{0}^{x_n}; \overbrace{1}^{\partial x_1}, \overbrace{0}^{\partial x_2}, \ldots, \overbrace{0}^{\partial x_n}),\ F = (\overbrace{0}^{x_1}, \overbrace{0}^{x_2}, \ldots, \overbrace{0}^{x_n}; \overbrace{1}^{\partial x_1}, \overbrace{1}^{\partial x_2}, \ldots, \overbrace{1}^{\partial x_n})$$

$$\mathcal{I}_1(I) = \bar{\mathcal{E}}(I) \cap \{pF + qV_1 \mid p, q \geq 0\}$$

We call $\mathcal{I}_1(I)$ the irregularity local Gröbner fan along $x_1 = 0$.

The irregularity global Gröbner fan is a refinement of the irregularity local Gröbner fan. The irregularity local Gröbner fan is a refinement of the fan by the slopes.

Slopes are introduced by Y.Laurent and Z.Mebkhout. Constructive method to compute slopes is given by Assi-Castro-Granger (How to compute slopes). U.Walther's talk.

Application 4: Analytic GKZ compactification and series solutions

Theorem ([SST; Chapter2]) Global small Gröbner fan of regular holonomic I gives a natural compactification to construct series solutions of I.

$\bar{\mathcal{E}}(I) \cap \{(-w, w) \mid w \in \mathbf{R}_{\geq 0}\}$ is called the analytic small Gröbner fan.

Theorem (conjectural) Analytic small Gröbner fan of regular holonomic I gives a natural compactification to construct series solutions of I.

[SST] Saito-Sturmfels-T, Gröbner deformations of hypergeometric differential equations.

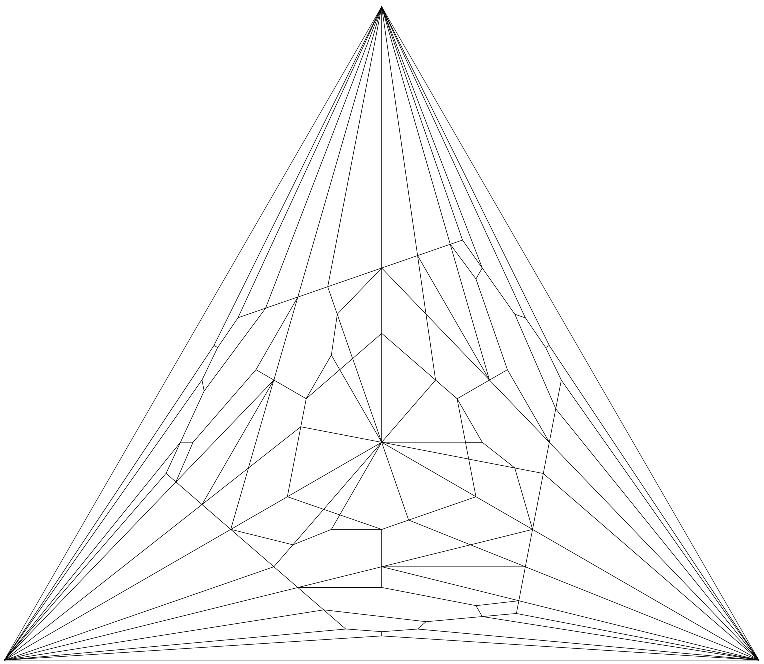
Q. Implementation?

Computation is split into a system for algebra (Kan/sm1) and a system for geometry (Polymake). They are connected by the OpenXM-RFC 104 protocol (OoHG, OpenXM on HTTP/GET). This design gives us a robust system, since the polymake is a strong, flexible, and robust system for polytopes developed by E. Gawrilow and M. Joswig linked with GMP and cdd by K. Fukuda. We use polymake properties `DIM`, `FACETS`, `INEQUALITIES`.

1. Fastest: gfan by Anders Jensen.

2. Broardest scope: gfan.sm1 on kan/sm1

KNOPPIX/Math DVD 2006 for icms2006



Gröbner fan of $\langle a^2b - c - a, b^2c - a - b, c^2a - b - c\rangle$ computed by `gfan`

Input example.

```
% BS poly. x+y, x^2+y^2
/cone.loaded boundp { }
{
  [(parse) (cohom.sm1) pushfile] extension
  [(parse) (dhecart.sm1) pushfile] extension
  [(parse) (gfan.sm1) pushfile] extension
  /cone.loaded 1 def
} ifelse

%------------------Globals----------------------------------------
% 0 : x,y,Dx,Dy
% 1 : x,y,Dx,Dy,h,H
% 2 : x,y,Dx,Dy,h
/cone.type 2 def

/cone.local 1 def

% cone.h0:  1 ==> weight(h)=0
/cone.h0 1 def

% cone.input : input polynomials
/cone.input
  [
   (t1-x-y) (t2-x^2-y^2) (2*x*Dt2+Dt1+Dx) (2*y*Dt2+Dt1+Dy)
  ]
```

```
def

/cone.vlist [(t1) (t2) (x) (y) (Dt1) (Dt2) (Dx) (Dy) (h)] def
/cone.vv (t1,t2,x,y) def

/cone.parametrizeWeightSpace {
  4 2 parametrizeSmallFan
} def

% cone.w_start : starting weight
/cone.w_start
  [1 3]
def
%homogenize
  [cone.vv ring_of_differential_operators
   [[(t1) -1 (t2) -1 (Dt1) 1 (Dt2) 1]] ecart.weight_vector
  0] define_ring
  dh.begin
  cone.input { . homogenize toString } map /cone.input set
  dh.end

/cone.DhH  1 def
/cone.gb {
  cone.gb_DhH
} def
% ----------------  end of configure
```

```
getGrobnerFan
printGrobnerFan

dhCones_h         % dehomogenize the fan
dhcone.rtable
dhcone.printGrobnerFan
```

```
The number of cones = 2

0 : begin dhcone ---------------------------------------
facets=
 [
   [    -2 , 1 ]    % normal vectors for 0th Grobner cone.
   [    1 , 0 ]
 ]
nextcid=
  [    1 , -2 ]
cones=
  [    0 ]
--------------------------------------- end dhcone
1 : begin dhcone ---------------------------------------
facets=
 [
   [    0 , 1 ]   % normal vectors for 1th Grobner cone.
   [    2 , -1 ]
 ]
nextcid=
  [    -2 , 0 ]
cones=
  [    1 ]
--------------------------------------- end dhcone
0 : begin gbasis ---------------------------------------
initial=
 [
```

```
  [    -x-y , 2*y*Dt2 , -2*y^2 , 2*t1*Dt2 , t1*Dt1+2*t2*Dt2-y*Dx+y*Dy+2*h , -2*y*Dt1 ,
  [    t1,t2,x,y, , [    t1 , t2 ]  , [   ]  ]
 ]
weight=
  [    t1 , -1 , t2 , -3 , Dt1 , 1 , Dt2 , 3 ]
---------------------------------------- end gbasis
1 : begin gbasis ----------------------------------------
initial=
 [
  [    -x-y , 2*Dt1 , -2*y^2 , 4*y*Dt2 , -4*t2*Dt2+2*y*Dx-2*y*Dy-4*h , -4*t1^2*Dt1*Dt2-
  [    t1,t2,x,y, , [    t1 , t2 ]  , [   ]  ]
 ]
weight=
  [    t1 , -12 , t2 , -19 , Dt1 , 12 , Dt2 , 19 ]
---------------------------------------- end gbasis
```