Algorithms for D-modules

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1. Restriction and integration complex in one dimensional case (pages 2–16).
2. References of introductory texts, recent topics. Open questions (pages 17–, briefly)

Theory of D-modules (1960's–1970–1990–now).
Algorithms for it (1980's–1997–2003–now)

$\mathbb{C}[x, 1/f]$ is finitely generated left D-module. Restriction algorithm. de Rham cohomology.

`https://www.math.kobe-u.ac.jp/HOME/taka/2025/2025-09-11-rims/`

Weyl algebra of one variable

$K = \mathbb{C}$.

$$D = K\langle x, \partial \rangle \tag{1}$$

$\partial x = x\partial + 1$. Let f be an analytic function or a distribution.

$$x^\alpha \partial^\beta \bullet f = x^\alpha \frac{d^\beta f}{dx^\beta}$$

A left ideal = linear differential equations.

$L_1 \bullet f = 0$ and $L_2 \bullet f = 0$, Ł$_i \in D$. Then, $(\ell_1 L_1 + \ell_2 L_2) \bullet f = 0$ for all $\ell_i \in D$.

Example 1

$I = \langle x\partial - 1, \partial^2 \rangle$

(0-th) restriction

$D/xD \otimes_D M$ is called *the (0-th) restriction* to $x = 0$ of $M = D/I$.
$D/xD \otimes_D M = D/(I + xD)$

Theorem 2

If $I! = 0$, *then* $D/xD \otimes_D M$ *is a finite dimensional* K*-vector space*[1].

Theorem 3 (K-1970[2], ...)

$D/xD \otimes_D M \simeq \mathrm{Hom}_D(M, K[[x]]) = \{$*formal power series solutions*$\}$

[1] 1-dim case of I.N.Bernstein, The analytic continuation of generalized functions with respect to a parameter. Functional Analysis and its Applications 6 (1972), 273–285.

[2] M.Kashiwara, Algebraic study of systems of partial differential equations, 1970. `https://www.kurims.kyoto-u.ac.jp/~kenkyubu/kashiwara/master_thesis.pdf`

Example 4

When $I = D(x\partial - 1)$, $M = D/I$, then

$$D/xD \otimes_D M \simeq D/(I + xD) \simeq K\partial.$$

$J = xD + D(x\partial - 1)$. $\theta_x = x\partial$.

$$\partial^k \theta_x = \theta_x \partial^k + k\partial^k$$

$\partial^k(x\partial - 1) = \theta_x\partial^k + k\partial^k - \partial^k$, $\theta_x\partial^k \in xD$, then

$$(k-1)\partial^k \in J.$$

$1, \partial^2, \partial^3, \ldots \in J$.
Excercise: Show $\partial \notin J$.

> How to compute the restriction?

1. Approximation by Macaulay type matrix.
2. Approximation bound to give the exact answer (Oaku's b-function criterion)

Macaulay type matrix

$K = \mathbb{C}$. $f = \sum_{j=0}^{m} f_j x^j$.
$K[x]_k$: the K vector space of the polynomials of which degree is less than or equal to k.

f induces a linear map $M_k(f)$.

$x^i f(x) = \sum_{j=0}^{m} f_j x^{j+i}$. A K-linear map (by the correspondence $e_i \Leftrightarrow x^i$) as

$$K[x]_{k-m} \simeq K^{k-m+1} \ni e_i \mapsto \sum_{j=0}^{m} f_j e_{j+i} \in K^{k+1} \simeq K[x]_k.$$

The matrix representation of this map: $M_k(f)$ *Macaulay type matrix of the degree k*.

Example 5

$f(x) = x^2 + 1$.

$$e_0 \mapsto 1 \cdot (x^2 + 1) = e_2 + e_0, e_1 \mapsto x \cdot (x^2 + 1) = e_3 + e_1$$

$$M_3(f) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{Columns are indexed by } e_0,\ e_1,\ \ldots$$

Macaulay type matrix in the Weyl algebra D.

Put $w = 1$. We define $(-w, w)$ degree or order by $\mathrm{ord}_{(-w,w)}(x^i\partial^j) = -wi + wj = j - i$.
V_k: the K vector space spanned by $x^i\partial^j$, $j - i \leq k$. V_k is the set of the elements whose $(-w, w)$ order is less than or equal to k.
Note that $V_{-1} \subseteq xD$.
For $f \in D$, the expression as $\sum c_{ij}x^i\partial^j$ (∂'s are collected to the right) is called the normally ordered expression and is denoted by $:f:$. For example, we have $:\partial x := x\partial + 1$.
Fix a natural number k. For $g \in D$ such that $\mathrm{ord}_{(-w,w)}(g) = m$,

the operator g induces a K-linear map

$$K[\partial]_{k-m} \ni \partial^i \mapsto :\partial^i g: |_{x=0} \in K[\partial]_k$$

The matrix representation of this map is called the *Macaulay type matrix for restriction of degree* k and is denoted by $M_k(g)$.

Example 6

$g = x\partial^2 + x\partial$, $\mathrm{ord}_{(-\mathrm{w},\mathrm{w})}(g) = \max(2-1, 1-1) = 1$, $k = 2$.

$$\begin{aligned} 1 &\mapsto \; : x\partial^2 + x\partial : |_{x=0} = 0 \\ \partial &\mapsto \; : \partial g : |_{x=0} = x\partial^3 + \partial^2 + x\partial^2 + \partial \,|_{x=0} = \partial^2 + \partial \end{aligned}$$

$$M_2(g) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Here, the K vector space $K[\partial]_{k-1}$ is regarded as a set of row vectors.

Complex and cohomology groups.

The diagram

$$C^\bullet : 0 \stackrel{\varphi_{m+1}}{\to} K^{b_m} \stackrel{\varphi_m}{\to} K^{b_{m-1}} \stackrel{\varphi_{m-1}}{\to} \cdots \to K^{b_1} \stackrel{\varphi_1}{\to} K^{b_0} \stackrel{\varphi_0}{\to} 0$$

where φ_i's are K-linear maps is called a *complex of vector spaces* when $\varphi_i \circ \varphi_{i+1} = 0$ holds. Define the $-i$-th cohomology group of $C^\bullet$ as

$$H^{-i}(C^\bullet) = \frac{\mathrm{Ker}\,\varphi_i}{\mathrm{Im}\,\varphi_{i+1}}$$

which is a K-vector space.
When $H^{-i}(C^\bullet) = 0$ for all i, the complex is called *exact*.

Example 7

For $g = x\partial^2 + x\partial$, consider the Macaulay type matrix for the restriction of degree 2. The cohomology groups of the complex

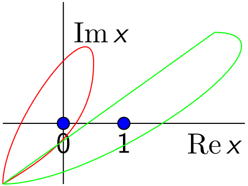
$$C^\bullet : 0 \to K^2 \xrightarrow{M_2(g)} K^3 \to 0$$

are $H^0(C^\bullet) \simeq K^2$, $H^{-1}(C^\bullet) \simeq K$. Here,

$$M_2(g) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$f = x(1-x)$. Consider the left D-module $\mathbb{C}[x, 1/f]$. It is isomorphic to D/I, $I = \mathrm{Ann}\,\frac{1}{f}$ is generated by $x(1-x)\partial - 2x + 1$. Apply $x \mapsto -\partial, \partial \mapsto x$.

$$(-\partial - \partial^2)x + 2\partial + 1 = -(x\partial^2 + x\partial) = -g$$



$$H^1(\mathbb{C} \setminus V(f), \mathbb{C})^* \simeq \mathbb{C}(\text{red loop}) + \mathbb{C}(\text{green loop})$$

It was $\dim_{\mathbb{C}} H^0(C^\bullet) = 2$. By accident?

By the Grothendieck-Deligne comparison theorem and the restriction algorithm, $H^i(\mathbb{C} \setminus V(f), \mathbb{C}) \simeq H^{i-1}(C^\bullet)$.

b-function for restriction

$g \in D$. The b-function (for restriction) [3] [4] $b(\theta)$ of g is defined by

$$D\,\mathrm{in}_{(-1,1)}(g) \cap \mathbb{C}[\theta] = \langle b(\theta) \rangle, \quad \theta = x\partial$$

Example 8

$g = x\partial^2 + x\partial$.

$$\mathrm{in}_{(-1,1)}(g) = x\partial^2$$

$$xx\partial^2 = \theta(\theta - 1) = b(\theta)$$

The maximal integral root of $b(s) = 0$ is $s = 1$.

[3] It is the indicial polynomial in the theory of ODE.

[4] M.Noro, An Efficient Modular Algorithm for Computing the Global b-Function, 2002. `generic_bfct`

Theorem 9

(1-dim case of [Oaku, 1997]) *Suppose that $g \neq 0$ is a given element of D. Define a complex that is called, restriction complex of $M = D/(Dg)$ to $x = 0$,*

$$G^\bullet : \qquad 0 \longrightarrow D/xD \otimes_D D \xrightarrow{(\mathrm{id}, \cdot g)} D/xD \otimes_D D \longrightarrow 0\ .$$

Let k_0 be the maximal integral root of the b-function of g[5]*. For $k \geq k_0$, define a complex of K-vector spaces by*

$$C^\bullet\ :\ 0 \to K^{k+1-\mathrm{ord}_{(-w,w)}(g)} \xrightarrow{M_k(g)} K^{k+1} \to 0$$

Then, we have $H^{-i}(G^\bullet) = H^{-i}(C^\bullet)$. [6]

[5] No integral root case $\Rightarrow$ 0-complex. If $k < 0$, $K^{k+1} := 0$

[6] In other words, $C^\bullet$ is a subcomplex of $G^\bullet$ and they are quasi isomorphic for $k \geq k_0$.

Put $M = D/I$, $I = Dg$. The restriction complex $G^\bullet$ of M to $x = 0$ is

$$D/(xD) \otimes_D M^\bullet$$

where $M^\bullet$ is

$$0 \longrightarrow D \stackrel{g}{\longrightarrow} D \longrightarrow 0.$$

Note that $D \ni \ell \stackrel{\cdot g}{\mapsto} \ell g \in D$ and

$$0 \longrightarrow D \stackrel{g}{\longrightarrow} D \stackrel{id}{\longrightarrow} D/I \longrightarrow 0 \quad \text{(exact)}.$$

The cohomology $H^0(G^\bullet)$ is the 0-th restriction $D/(xD) \otimes_D M$. $D/(xD) \otimes_D M^\bullet$ is called the restriction (in the derived category) of M and is denoted by $Lj^* M$.
The discussion so far is one dimensional specialization of T.Oaku's important paper [7] (Oaku's b-function criterion).

[7] T.Oaku, Algorithms for b-functions, restrictions, and algebraic local cohomology groups of D-modules, 1997, https://doi.org/10.1006/aama.1997.0527. $D_n/(x_n D)$, $(-1, 1)$-Gröbner basis.

Integration functor

Put $M = D/I$. By the Fourier transformation

$$\mathcal{F} : x \mapsto -\partial, \quad \partial \mapsto x,$$

we have

$$D/(\partial D) \otimes_D M^\bullet \simeq_{q.i.s} D/(xD) \otimes_D \mathcal{F}(M)^\bullet$$

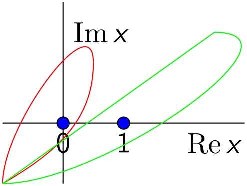
that is called *the integration* of M. Note that 0-th cohomology is $D/(I + \partial D)$ and $D/(\mathcal{F}(I) + xD)$ respectively. These are called the (0-th) integration and the (0-th) restriction respectively.

Example 10

$f = x(1-x)$. Consider the left D-module $\mathbb{C}[x, 1/f]$. It is isomorphic to D/I, $I = \operatorname{Ann} \frac{1}{f}$ is generated by $x(1-x)\partial - 2x + 1$.

$$(-\partial - \partial^2)x + 2\partial + 1 = -(x\partial^2 + x\partial) = -g$$

By the Grothendieck-Deligne comparison theorem, $H^i(\mathbb{C} \setminus V(f), \mathbb{C}) \simeq H^{i-1}(G^\bullet)$.



$$H^1(\mathbb{C} \setminus V(f), \mathbb{C})^* \simeq \mathbb{C}(\text{red loop}) + \mathbb{C}(\text{green loop})$$

It was $\dim_{\mathbb{C}} H^0(G^\bullet) = 2$.

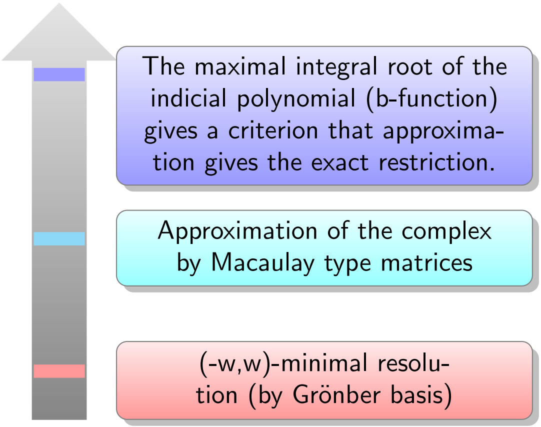
Introductory texts on Algorithms for *D*-modules

1. 大阿久, *D* 加群と計算数学, 2002, (in Japanese). ISBN : 978-4-254-11555-0.

SST M.Saito, B.Sturmfels, N.Takayama, Gröbner deformations of hypergeometric differential equations, 2000. Chap. 1 and chap. 5, ISBN: 978-3540660651.

2. T.Hibi et al, Gröbner bases: statistics and software systems, 2013. Chap. 6 and chap. 7. ISBN 443156215X.
3. T.Oaku, Algorithms for *D*-modules, integration, and generalized functions with applications to statistics, 2018, `https://doi.org/10.2969/aspm/07710253`
4. Ed. by K.Iohara et al, Two Algebraic Byways from Differential Equations: Gröbner Bases and Quivers, 2020. ISBN: 3030264564.
5. 高山, 小松, 松原, モデリングと計算代数, to appear, (in Japanese).

How to compute the restriction and integration complex?



1. T.Oaku, N.Takayama, Algorithms for D-modules — restriction, tensor product, localization, and local cohomology groups (2001). `https://doi.org/10.1016/S0022-4049(00)00004-9`
2. T.Oaku, N.Takayama, Minimal free resolutions of homogenized D-modules (2001). `https://doi.org/10.1006/jsco.2001.0484`

Hom functor

For holonomic D-module M and N, the K vector space $\mathrm{RHom}_D(M, N)$ can be computed[8]. In particular it answers if two D-modules are isomorphic or not.

Idea:

Theorem 11

(well-known in the theory of D-modules) *Let p be a projection from X to a point. Then,*

$$\mathrm{RHom}_D(M^\bullet, N^\bullet)[dx] = \int_p \mathrm{D}M^\bullet \otimes^L_{\mathcal{O}} N^\bullet$$

where D *is the dualizing functor and dx is the dimension of X.*

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[8] H.Tsai, U.Walther, Computing homomorphisms between holonomic D-modules, 2001, `https://doi.org/10.1006/jsco.2001.0485`

Computation of $\mathrm{RHom}_D(M^\bullet, \hat{\mathcal{O}})$ [9]. Note that $\hat{\mathcal{O}}$ is holonomic $\hat{\mathcal{D}}^{an}$ module and not holonomic over the Weyl algebra. The restriction gives dimensions, but does not give bases of vector spaces.

Open question

Efficient algorithm to check if two holonomic modules are isomorphic or not.
Isomorphism between two hypergeometric system $H_A(\beta)$ and $H_A(\beta')$.

1. M.Saito, Isomorphism Classes of A-Hypergeometric Systems, (2001). https://doi.org/10.1023/A:1011877515447
2. H.Nakayama, N.Takayama, Comprehensive restriction algorithm for hypergeometric systems, 2025. Todo: arxiv

[9] Nobuki Takayama, An Algorithm of Constructing Cohomological Series Solutions of Holonomic Systems (2004) https://doi.org/10.48550/arXiv.math/0309378

Application to de Rham cohomology groups $H^i(\mathbb{C}^n \setminus V(f))$

$= H^{i-n}\left(\int_p \mathbb{C}[1/f]\right)$ where p is the projection to the origin.

1. T.Oaku, N.Takayama, An algorithm for de Rham cohomology groups of the complement of an affine variety via D-module computation (1998). https://doi.org/10.1016/S0022-4049(99)00012-2
2. U.Walther, Algorithmic Determination of the Rational Cohomology of Complex Varieties via Differential Forms (2001), http://dx.doi.org/10.48550/arXiv.math/0103013

Works by S.Tajima, K.Nabeshima, K.Ohara, Y.Nakamura, V.Levandovskyy, ... on $D[1/f]$ where f is a polynomial. See Katsusuke Nabeshima's page https://www.rs.tus.ac.jp/~nabeshima/nabe_ja.html and references therein and Viktor Levandovskyy's page https://www.math.rwth-aachen.de/homes/Viktor.Levandovskyy/en/ and references therein.

Implementations

Macaulay2: BernsteinSato, ConnectionMatrices, Dmodules, HolonomicSystems, MultiplierIdeals, MultiplierIdealsDim2, WeylAlgebras: `https://macaulay2.com/doc/Macaulay2/share/doc/Macaulay2/Macaulay2Doc/html/_packages_spprovided_spwith_sp__Macaulay2.html`

Singular: `bfun_lib`, `dmod_lib`, `dmodapp_lib`, `dmodvar_lib`, `dmodloc_lib`, `gmssing_lib`
`https://www.singular.uni-kl.de/Manual/4-0-3/sing_545.htm#SEC596`
in Non-commutative libraries:

Risa/Asir: `bfct`, `generic_bfct`, `ann`, `nk_restriction.restriction`, `nk_restriction.integration`,

kan/k0: `https://www.math.kobe-u.ac.jp/OpenXM/Current/doc/other-docs/k0-man-en.html`, `https://www.openxm.org`

Type E_6, E_7, E_8. $\mathbb{C}^4 \setminus V(f)$

type	f	$\dim [H^4, H^3, H^2, H^1, H^0]$	Time (s)[10]
E_6	$x^4 + y^3 + z^2 + u^2$	[0,0,0,1,1]	1305
E_7	$x^3 y + y^3 + z^2 + u^2$	[1,1,0,1,1]	957
E_8	$x^5 + y^3 + z^2 + u^2$	[0,0,0,1,1]	880

Type D_k, $f = x(x^{k-2} + y^2) + z^2$, $k \geq 4$.

f	$\dim [H^3, H^2, H^1, H^0]$	Time (s)
$x(x^2 + y^2) + z^2$	[2,0,1,1]	43
$x(x^3 + y^2) + z^2$	[2,0,1,1]	121
$x(x^4 + y^2) + z^2$	NA	$\geq$ a week

```
openxm fep k0
 load["lib/all.k"];;
 k=4;
 RingD("x,y,z");
 f = x*(x^(k-2)+y^2)+y^2+z^2;
 fstr = ToString(f);
 Ans=DeRham3(fstr);
```

[10] AMD EPYC 7552 48-Core Processor of 1.5GHz with 1T bytes memory.

$f = x^p + y^q + xy^{q-1}$, $p + 1 \leq q$. $\mathbb{C}^2 \setminus V(f)$,
F.J.Castro, J.M.Ucha's benchmark.

(p, q)	$\dim [H^2, H^1, H^0]$	Time (s)
(9,10)	[1,1,1]	110
(9,11)	[2,1,1]	159
(9,12)	[3,1,1]	132
(9,13)	[4,1,1]	129
(9,14)	[5,1,1]	181
(9,15)	[6,1,1]	224
(9,16)	[7,1,1]	809
(9,17)	[8,1,1]	1174
(9,18)	[9,1,1]	1115
(9,19)	[10,1,1]	1458
(9,20)	[11,1,1]	3428
(9,21)	[12,1,1]	3863
(9,22)	[13,1,1]	5925
(9,23)	[14,1,1]	5644
(9,24)	[15,1,1]	14089
(9,25)	[16,1,1]	15640

Applications to definite integrals with parameters

two variable case: put $x = x_1, y = x_2$. A smooth function $f(x, y)$ of the variables x, y. Let $I = \mathrm{Ann}\, f \subset D_2$. The integration ideal is

$$\mathbb{C}\langle y, \partial_y\rangle \cap (I + \partial_x D_2) \ni \ell. \tag{2}$$

Theorem 12

(well-known, motivation of integration functor)

$$\ell \bullet \int_a^b f(x, y)dx + [\ell_1 \bullet f]_a^b = 0.$$

where $\ell + \ell_1 D_2 \in \mathrm{Ann}\, f$, $\ell_1 \in D_2$.

Proof. $\int_a^b [\ell(y, \partial_y) + \partial_x \ell_1(x, y, \partial_x, \partial_y)] \bullet f\, dx = 0$,
$\ell(y, \partial_y) \bullet \int_a^b f(x, y)dx + \int_a^b \partial_x \bullet (\ell_1 \bullet f)dx = 0$.

Computing integration ideal and restriction ideal.

1. Creative telescoping since D.Zeilberger (1989). Analogous fact holds for discrete case. There are applications to prove combinatorial identities.
   1.1 S.Chen, M.Kauers, C.Koutschan, Creative Telescoping, `https://arxiv.org/abs/2505.05345` (survey)
   1.2 C.Koutschan, HolonomicFunctions (implementation), `https://www3.risc.jku.at/research/combinat/software/ergosum/RISC/HolonomicFunctions.html`
2. Holonomic gradient methods to evaluate normalizing constants in probability distributions.
   2.1 `https://www.math.kobe-u.ac.jp/OpenXM/math/ref-hgm.html`
   2.2 A.Sakoda, N.Takayama, An Application of the Holonomic Gradient Method to the Neural Tangent Kernel, `https://arxiv.org/abs/2410.23626`
3. Differential equations for Feynman amplitude integrals.
   3.1 V.Chestnov, S.J.Matsubara-Heo, H.J.Munch, N.Takayama, Restrictions of Pfaffian Systems for Feynman Integrals, `https://doi.org/10.1007/JHEP11(2023)202"`, 2023.
   3.2 V.Chestnov, F.Gasparotto, M.K.Mandal, P.Mastrolia, S.J.Matsubara-Heo, H.J.Munch, N.Takayama, Macaulay Matrix for Feynman Integrals: Linear Relations and Intersection Numbers, `https://doi.org/10.1007/JHEP09(2022)187"`, 2022.

See also references therein.

Open Problem 1

Restriction to a subvariety

V is smooth variety in $\mathbb{C}^n$.

$$j \ : \ V(f) \longrightarrow \mathbb{C}^n \tag{3}$$

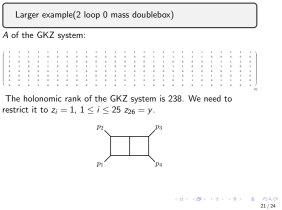
For given holonomic D-module M, give an efficient algorithm to compute Lj^*M. [OT] gives an algorithm for $V(x_{r+1}, \ldots, x_n)$ and Lj^*M is expressed by a local cohomology (slow).

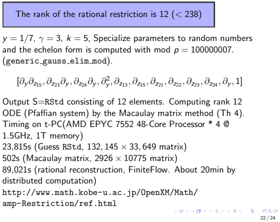
When V is a hypersurface defined by a polynomial f, an construction algorithm of $R_n \otimes_D \frac{M}{fM}$ is given by [CMMT][11] where R_n is the rational Weyl algebra.
Example: Restriction of Horn HG systems to singular locus.
https://www.math.kobe-u.ac.jp/HOME/taka/2023/Horn-n1/

[11] V.Chestnov, S.J.Matsubara-Heo, H.J.Munch, N.Takayama, Restrictions of Pfaffian Systems for Feynman Integrals, https://doi.org/10.1007/JHEP11(2023)202, 2023.

Open Problem 2

More efficient restriction and integration alg.





Some recent works related to the Griffith reduction.

1. V.Chestnov, W.Flieger, P.Mastrolia, S.J.Matsubara-Heo, N.Takayama, W.J.Torres-Bobadilla, Differential Space of Feynman Integrals: Annihilators and D-module, `https://arxiv.org/abs/2506.10456`
2. H.Brochet, F.Chyzak, P.Lairez, Faster multivariate integration in D-modules, `https://arxiv.org/abs/2504.12724`.