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An algorithm of constructing the integral of a module
— an infinite dimensional analog of Gröbner basis

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§1. Introduction

Let $\mathfrak{A}$ be a left ideal of Weyl algebra:

$$A_n = K\langle x_1, \cdots, x_n, \partial_1, \cdots, \partial_n \rangle.$$

Put $M = A_n/\mathfrak{A}$. M is a left A_n module. The purpose of this paper is an explicit construction of the left A_{n-1} module:

$$\int M dx_n := M/\partial_n M$$

by introducing an analog of Gröbner basis of a submodule of a kind of infinite dimensional free module. We call $M/\partial_n M$ the *integral* of the module M. The non-commutativity of A_n prevents us from using the usual Buchberger algorithm to construct $M/\partial_n M$. (If A_n is commutative, then $M/\partial_n M \simeq A_n/(\partial_n, \mathfrak{A})$. There is no problem.) We must consider a sum of left and right ideal of A_n. We overcome this difficulty by using an infinite dimensional analog of Gröbner basis.

The algorithm of constructing the integral of a module is not only important to mathematicians, but also has many impacts on the classical fields of computer algebra. It plays central roles in mathematical formula verification [Zeil], [Tak2], computation of a definite integral [AZ], [Tak2] and an asymptotic expansion of a definite integral with respect to parameters. However, a complete algorithm of obtaining $M/\partial_n M$ has not been known. We give a complete algorithm in this paper. The algorithm is an answer to the research problem of the paper [AZ].

We refer to [Buch1], [Buch2], [MM], [FSK], [Bay] for the Gröbner basis of a polynomial ideal and free module, to [Gal], [Cas], [Tak1], [Nou] , [UT] for the Gröbner basis of the ideal of Weyl algebra, to [Ber], [Bjo] for holonomic system and Weyl algebra. We remark that [Berg] also considered infinite set of reduction systems.

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§2. Gröbner basis for submodule of $R_\infty = \varinjlim R^m$

Put $R = K\langle x_1, \cdots, x_{n-1}, \partial_1, \cdots, \partial_n \rangle = A_{n-1}[\partial_n]$. We define a left R module structure to R^m in the following way. Given

$$R^m \ni \vec{f} = (f(0), \cdots, f(m-1)), \; a \in A_{n-1},$$

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we put

$$a\vec{f} = (af(0), \cdots, af(m-1)) \tag{2.1}$$

and for $a = \partial_n$,

$$a\vec{f} = (af(0) + f(1), \cdots, af(k) + (k+1)f(k+1), \cdots, af(m-1)). \tag{2.2}$$

The Weyl algebra A_n has a left R module structure in the standard way. The map

$$\varphi : R^m \ni \vec{f} \longmapsto \sum_{k=0}^{m-1} x_n^k f(k) \in A_n$$

is homomorphism of left R module.

We can define the notions of addmissible order, reducible, S-polynomial(sp) and Gröbner basis of the ring R in a similar way to the case of the polynomial ring. Let us explain some of them for clarity. We define an order $\prec_1$ between monomials of R by

$$x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} \prec_1 x_1^{\gamma_1} \cdots x_{n-1}^{\gamma_{n-1}} \partial_1^{\delta_1} \cdots \partial_n^{\delta_n} \tag{2.3}$$

$\Longleftrightarrow$

$$(\alpha_1, \cdots, \alpha_{n-1}, \beta_1, \cdots, \beta_n) \prec_2 (\gamma_1, \cdots, \gamma_{n-1}, \delta_1, \cdots, \delta_n)$$

where $\prec_2$ is the total degree order in $\mathbb{N}_0^{2n-1}$. We use the order in the sequel. Let r and s be elements of R. We put

$$\text{head}(r) = \text{ leading term of } r \text{ by the order (2.3).}$$

We assume $\text{head}(r) = cx^\alpha \partial^\beta$ and $\text{head}(s) = dx^\gamma \partial^\delta$, $c, d \in K$. We define

$$\text{lcm}(\alpha, \gamma) = (\max\{\alpha_1, \gamma_1\}, \cdots, \max\{\alpha_{n-1}, \gamma_{n-1}\}) \text{ and } \text{lcm}(\beta, \delta) = (\max\{\beta_1, \delta_1\}, \cdots, \max\{\beta_n, \delta_n\}).$$

Iff $\text{lcm}(\alpha, \gamma) = \alpha$ and $\text{lcm}(\beta, \delta) = \beta$, r is reducible by s. Put $\xi = \text{lcm}(\alpha, \gamma)$ and $\eta = \text{lcm}(\gamma, \delta)$. We define

$$\text{sp}(r, s) = x^{\xi-\alpha} \partial^{\eta-\beta} r - \frac{c}{d} x^{\xi-\gamma} \partial^{\eta-\delta} s.$$

Let r be reducible by s and $t = \text{sp}(r, s)$, then the situation is denoted by "$r \longrightarrow t$ by s". Let $\rightarrow^*$ be a transitive closure of $\rightarrow$. A finite subset G of R is called Gröbner basis of an ideal $\mathfrak{A}$ if $\forall r_i, r_j \in G, \text{sp}(r_i, r_j) \longrightarrow^* 0$ by G and $\mathfrak{A} = RG$. It is well known that every left ideal of R has a Gröbner basis [Gal], [Cas], [Tak1], [Nou], [UT].

Consider R^m. [Bay], [MM] and [FSK] extended the notion of Gröbner basis to free modules. We can apply their extension to R^m. Let us review their extension (See [Tak1] for proofs in the case of a free module over a non-commutative ring). Given an element $\vec{f}$ of R^m that satisfies $f(i) = 0$ $(k < i \le m-1)$ and $f(k) \neq 0$, we put $\text{topIndex}(\vec{f}) = k$. Let $\vec{f}$ and $\vec{g}$ be elements of R^m. The element $\vec{f}$ is reducible by $\vec{g}$ iff $k = \text{topIndex}(\vec{f}) = \text{topIndex}(\vec{g})$ and $f(k)$ is reducible by $g(k)$ in R. We put

$$\text{sp}(\vec{f}, \vec{g}) := \begin{cases} 0, & \text{if } \text{topIndex}(\vec{f}) \neq \text{topIndex}(\vec{g}) \\ c_1 \vec{f} - c_2 \vec{g}, & \text{if } k = \text{topIndex}(\vec{f}) = \text{topIndex}(\vec{g}), \end{cases}$$

where c_1 and c_2 are determined by the relation $\text{sp}(f(k), g(k)) = c_1 f(k) - c_2 g(k)$. We define an order $\succ$ in R^m in the following way.

$$\vec{f} \succ \vec{g} \Longleftrightarrow \begin{array}{l} \text{topIndex}(\vec{f}) > \text{topIndex}(\vec{g}) \\ \text{or } \left(\text{topIndex}(\vec{f}) = \text{topIndex}(\vec{g}) = k \text{ and } f(k) \succ_1 g(k)\right) \end{array} \tag{2.4}$$

We use the order (2.4) of R^m in the sequel. We remark that other order in R^m can be used in our theory. The use of good orders leads us to a fast termination of Buchberger algorithm.

Put

$$\mathcal{G} = \{\vec{g}_1, \cdots, \vec{g}_p\}.$$

The set $\mathcal{G}$ is a Gröbner basis of a left R submodule $\mathcal{M}$ of R^m iff (1) $\forall i, j$, $\mathrm{sp}(\vec{g}_i, \vec{g}_j) \longrightarrow^* 0$ by $\mathcal{G}$ and (2) $\mathcal{G}$ generates $\mathcal{M}$ over R.

Any left submodule $\mathcal{M}$ of R^m has a Gröbner basis ([Bay],[MM], [FSK], [Tak1]).

The i-th unit vector is denoted by $\vec{e}_i$, i.e.,

$$\vec{e}_0 = (1, 0, \cdots, 0),\ \vec{e}_1 = (0, 1, 0, \cdots, 0), \cdots.$$

Any vector $\vec{g}_i \in R^m$ can be decomposed into a sum of (monomial of R) $\times$ (unit vector) which is written as

$$\vec{g}_i = \sum_j c_i^j \vec{e}_{k_j}, \quad c_i^j \text{ is a monomial of } R.$$

The set $\mathcal{G}$ is a reduced Gröbner basis of a left submodule $\mathcal{M}$ iff $\mathcal{G}$ is a Gröbner basis of $\mathcal{M}$,

$$\forall i, j \ \ c_i^j \vec{e}_{k_j} \longrightarrow^* c_i^j \vec{e}_{k_j} \ \text{ by } \mathcal{G} \setminus \{\vec{g}_i\}$$

and the leading coefficient of $\vec{g}_i$ is 1.

We define these notions on

$$R_\infty = \lim_{\rightarrow} R^m \simeq R \text{ module } A_n.$$

Any element $\vec{f}$ of R_∞ can be written as

$$\vec{f} = (f(0), f(1), \cdots), \quad \exists k, i > k \Rightarrow f(i) = 0 \text{ and } f(k) \neq 0.$$

The number k is denoted by $\mathrm{topIndex}(\vec{f})$. Therefore we can consider $\vec{f}$ as the element of R^m, $m \geq k$. We define the notions of reducibility, s-polynomial and order $\prec$ identifying the element $\vec{f}$ of R_∞ with the element $(f(0), \cdots, f(m-1))$ of R^m, $(m \geq k)$.

Put

$$\mathcal{G} = \{\vec{g}_1, \vec{g}_2, \cdots\}, \quad \vec{g}_i \in R_\infty.$$

We do not assume that $\mathcal{G}$ is finite set. Put

$$\mathcal{G}(k) = \{\vec{g} \in \mathcal{G} | \mathrm{topIndex}(\vec{g}) \leq k\}.$$

Assumption 2.1

$$\forall k, \ \#\mathcal{G}(k) < +\infty.$$

We consider the existence of a Gröbner basis under Assumption 2.1 in the sequal.

Definition 2.1

$$\vec{f} \longrightarrow \vec{h} \text{ by } \mathcal{G} \Longleftrightarrow \exists i, \exists m, \vec{g}_i \in \mathcal{G}, \mathrm{topIndex}(g_i) \leq m, \mathrm{topIndex}(f) \leq m \text{ and } \vec{f} \longrightarrow \vec{h} \text{ by } \vec{g}_i \text{ in } R^m.$$

The rewriting $\vec{f} \longrightarrow \vec{h}$ is called reduction of $\vec{f}$.

Proposition 2.1 *For any element $\vec{f} \in R_\infty$, any sequence of reduction of $\vec{f}$ by $\mathcal{G}$ terminates in finite steps.*

Proof. Put $m = \text{topIndex}(\vec{f})$. Note that any sequence of reduction of $\vec{f}$ uses the elements of $\mathcal{G}(m)$. Since $\mathcal{G}(m)$ is the finite set, the sequence terminates in finite steps. ∎

It follows from Proposition 2.1 that we can take a transitive closure of $\longrightarrow$ in finite steps. The transitive closure is denoted by $\longrightarrow^*$.

DEFINITION 2.2 The set $\mathcal{G}$ is a Gröbner basis of a left R submodule $\mathcal{M}$ of R_∞ iff
(1) $\forall i, j,\ \text{sp}(\vec{g}_i, \vec{g}_j) \longrightarrow^* 0$ by $\mathcal{G}$.
(2) $\mathcal{G}$ generates $\mathcal{M}$ over R, i.e., $\forall \vec{f} \in \mathcal{M}, \exists I \subset \mathbb{N}, \exists a_i \in R$ such that $\#I < \infty$ and

$$\vec{f} = \sum_{i \in I} a_i \vec{g}_i.$$

(3) (local finiteness)

$$\forall m,\ \#\mathcal{G}(m) < +\infty.$$

PROPOSITION 2.2 *If $\mathcal{G}$ is a Gröbner basis of an R submodule $\mathcal{M} \subset R_\infty$, then*

$$\forall \vec{g}_i, \vec{g}_j \in \mathcal{G}(m),\ \text{sp}(\vec{g}_i, \vec{g}_j) \longrightarrow^* 0 \text{ by } \mathcal{G}(m).$$

Proof. We have $\text{sp}(\vec{g}_i, \vec{g}_j) \longrightarrow^* 0$ by $\mathcal{G}$. Since $\text{topIndex}(\text{sp}(\vec{g}_i, \vec{g}_j)) \leq m$, we have $\text{sp}(\vec{g}_i, \vec{g}_j) \longrightarrow^* 0$ by $\mathcal{G}(m)$. ∎

THEOREM 2.1 *Let $\mathcal{M}$ be a left R submodule of R_∞ and $\mathcal{G}$ be a Gröbner basis of $\mathcal{M}$. If $\vec{f} \in \mathcal{M}$, then $\vec{f} \longrightarrow^* 0$ by $\mathcal{G}$.*

Proof. Since $\mathcal{G}$ is a set of generators of $\mathcal{M}$, there exist an index set I and elements $a_i \in R,\ i \in I$ such that $\#I < +\infty$ and $\vec{f} = \sum_{i \in I} a_i \vec{g}_i$. Put $m = \max_{i \in I}\{\text{topIndex}(\vec{g}_i)\}$. We can consider $\vec{f}$ as an element of R^m. It follows from Proposition 2.2 that $\mathcal{G}(m)$ is a Gröbner basis of $R\mathcal{G}(m)$ in R^m. Since $\vec{f} \in R\mathcal{G}(m)$, we have $\vec{f} \longrightarrow^* 0$ by $\mathcal{G}(m)$ for any sequence of reduction. ∎

Let $\mathcal{H}_m,\ m = 0, 1, 2, \cdots$ be subsets of R_∞ that satisfy the conditions:

$$\begin{aligned} &\cdots \subseteq \mathcal{H}_m \subseteq \mathcal{H}_{m+1} \subseteq \cdots \\ &\#\mathcal{H}_m < +\infty \quad \text{and} \quad \text{topIndex}(\vec{f}) \leq m \quad \text{for all } \vec{f} \in \mathcal{H}_m. \end{aligned} \tag{2.5}$$

Suppose that $\mathcal{M}_\infty$ is the left R submodule generated by $\bigcup_{m=0}^{\infty} \mathcal{H}_m$. We have $\mathcal{M}_\infty = \bigcup_{m=0}^{\infty} R\mathcal{H}_m$.

THEOREM 2.2 *Let $\mathcal{G}_m$ be the reduced Gröbner basis of $R\mathcal{H}_m$ in R^m. The set*

$$\mathcal{G}_\infty = \bigcup_{m=0}^{\infty} \mathcal{G}_m$$

is a Gröbner basis of $\mathcal{M}_\infty$.

Proof. We prove the local finiteness condition: $\#\mathcal{G}_\infty(m) < +\infty$. We remark that $\mathcal{G}_\infty(m) \neq \mathcal{G}_m$ in general. Put

$$\mathcal{G}_k(m) = \{\vec{f} \in \mathcal{G}_k | \text{topIndex}(\vec{f}) \leq m\}.$$

$\mathcal{G}_k(m)$ is a Gröbner basis of $R\mathcal{G}_k(m)$ in R^m. Since $\cdots \subseteq R\mathcal{G}_k(m) \subseteq R\mathcal{G}_{k+1}(m) \subseteq \cdots$ in R^m, there exists k_0 such that $\forall k \geq k_0,\ R\mathcal{G}_k(m) = R\mathcal{G}_{k_0}(m)$. $\mathcal{G}_k$ is the reduced Gröbner basis, then we have $\forall k \geq k_0,\ \mathcal{G}_k(m) = \mathcal{G}_{k_0}(m)$. Hence $\#\mathcal{G}_\infty(m) < +\infty$.

Other conditions are easily verified. ∎

§3. Computation of the integral of A_n module

Let $\mathfrak{A}$ be a left ideal of A_n and M be $A_n/\mathfrak{A}$. We have

$$M/\partial_n M \simeq A_n/(\partial_n A_n + \mathfrak{A}) \quad \text{as } A_{n-1} \text{ module.}$$

The set $\partial_n A_n + A_n\mathfrak{A} = \partial_n A_n + \mathfrak{A}$ is not left A_n module. Let us note that R is a subalgebra which is commutative to ∂_n. Therefore $\partial_n A_n + \mathfrak{A}$ has a left R module structure. We will show that $\partial_n A_n + \mathfrak{A}$ is the left R submodule of R_∞, prove the existence of a Gröbner basis (with the local finiteness property) of the module and present a construction algorithm of the basis.

Let

$$G = \{g_1 \cdots, g_p\} \tag{3.1}$$

be generators of the left ideal $\mathfrak{A}$ of A_n. Any element g_k can be written as

$$g_k = \sum_{j=0}^{s_k} x_n^j g_{kj}, \quad g_{kj} \in R.$$

We put

$$\psi(\partial_n x_n^k) = (0, \cdots, 0, k, \partial_n, 0, \cdots, 0) \in R^m,$$

and

$$\psi(x_n^i g_k) = (0, \cdots, 0, g_{k0}, g_{k1}, \cdots, g_{ks_k}, 0, \cdots, 0) \in R^m.$$

Let $\mathcal{H}_m \subset R^m$ be

$$\left(\bigcup_{k=0}^{m-1} \{\psi(\partial_n x_n^k)\}\right) \bigcup \left(\bigcup_{k=1}^{p} \bigcup_{i=0}^{m-s_k-1} \{\psi(x_n^i g_k)\}\right). \tag{3.2}$$

We have $\cdots \subseteq \mathcal{H}_m \subseteq \mathcal{H}_{m+1} \subseteq \cdots$ and $\#\mathcal{H}_m < +\infty$. $\mathcal{M}_\infty = \bigcup_{m=0}^{\infty} R\mathcal{H}_m$ is the left R submodule of R_∞. It follows from Theorem 2.2 that $\mathcal{M}_\infty$ has a Gröbner basis $\mathcal{G}_\infty$.

THEOREM 3.1.

$$R_\infty/\mathcal{M}_\infty \simeq A_n/(\partial_n A_n + \mathfrak{A}) = \int M dx_n$$

as left A_{n-1} module.

Proof. We define a map:

$$\theta : R_\infty \ni \vec{f} = (f(0), f(1), \cdots, f(m), 0, \cdots) \mapsto f(0) + x_n f(1) + \cdots + (x_n)^m f(m) \in A_n$$

where $m = \text{topIndex}(\vec{f})$.

We prove if $\vec{f} \in \mathcal{M}_\infty$, then $\theta(\vec{f}) \in \partial_n A_n + \mathfrak{A}$. Since $\vec{f} \in \mathcal{M}_\infty$, there exists $a_j, b_j \in R$ such that

$$\vec{f} = \sum_j a_j \psi(\partial_n x_n^{k_j}) + \sum_j b_j \psi(x_n^{i_j} g_{k_j})$$

where $\sum_j$ is a finite sum. Then we have

$$\begin{aligned}\theta(\vec{f}) &= \sum_j a_j \partial_n x_n^{k_j} + \sum_j b_j x_n^{i_j} g_{k_j} \\ &= \sum_j \partial_n (a_j x_n^{k_j}) + \sum_j (b_j x_n^{i_j}) g_{k_j} \in \partial_n A_n + \mathfrak{A}.\end{aligned}$$

Therefore we can define a map:

$$\hat{\theta} : R_\infty/\mathcal{M}_\infty \longrightarrow A_n/(\partial_n A_n + \mathfrak{A}),$$

by $\hat{\theta}([\vec{f}]) = [\theta(\vec{f})]$.

It is easily verified that $\hat{\theta}$ is A_{n-1} homomorphism and surjective.

We will show that $\hat{\theta}$ is injective. We assume that $\theta(\vec{f}) = \partial_n h + g \in \partial_n A_n + \mathfrak{A}$, $h \in A_n, g \in \mathfrak{A}$. h can be written as $h = \sum_k h_k x_n^k$, $h_k \in R$. Then we have $\partial_n h = \sum_k h_k(\partial_n x_n^k)$. g can be written as $g = \sum_k c_k g_k$. c_k has an expression of the form $c_k = \sum_j b_{kj} x_n^j$, $b_{kj} \in R$. Then we have $g = \sum_{k,j} b_{kj} x_n^j g_k$. Since θ is injective, then we have

$$\vec{f} = \sum_k h_k \psi(\partial_n x_n^k) + \sum_{k,j} b_{kj} \psi(x_n^j g_k) \in \mathcal{M}_\infty.$$

Therefore $\hat{\theta}$ is injective. ∎

COROLLARY 3.1 *If M is holonomic, then there exists a number m such that*

$$R^m/R\mathcal{G}_\infty(m) \simeq \int M dx_n$$

as A_{n-1} module where $\mathcal{G}_\infty$ is a Gröbner basis of $\mathcal{M}_\infty = \bigcup R\mathcal{H}_m$ of (3.2).

We will show an application of our theory to the zero recognition problem [Zeil] [Tak2] and the computation of a definite integral with parameters [AZ] [Tak2]. Algorithm 3.1 can be used in Algorithm 1.2 of [Tak2] and is "correct " algorithm in the sense of [Tak2].

ALGORITHM 3.1 (Computation of differential equations for a definite integral with parameters)
INPUT: $G = \{g_k\}$, generators (3.1) of a left ideal $\mathfrak{A}$ of A_n. We assume that $M = A_n/\mathfrak{A}$ is holonomic.
OUTPUT: $\mathcal{G}(0)$, a Gröbner basis in R such that $R/R\mathcal{G}(0)$ is holonomic A_{n-1} module, i.e. , $\mathcal{G}(0)$ is a very large system of differential equations such that $\mathcal{G}(0) \subseteq \partial_n A_n + \mathfrak{A}$.

(1) $m := \max\{s_k + 1\}$; $\mathcal{G} := \emptyset$;
(2) repeat
(3) $\mathcal{H}_m :=$(3.2);
(4) $\mathcal{G} := \mathcal{G} \cup \{$ reduced Gröbner basis of $R\mathcal{H}_m$ in R^m by the order (2.4) $\}$;
(5) $m := m + 1$;
(6) until ($R/R\mathcal{G}(0)$ is holonomic)

THEOREM 3.2 *Algorithm 3.1 stops.*

THEOREM 3.3 *Assume a function f of $x_1, \cdots, x_n$ is rapidly decreasing with respect to x_n. Let $\mathfrak{A}$ be an ideal of A_n such that $\mathfrak{A} f = 0$. If $A_n/\mathfrak{A}$ is holonomic, then the integral*

$$\int_{-\infty}^{\infty} f dx_n$$

is annihilated by differential operators $\mathcal{G}(0)$ where $\mathfrak{A}$ is the input of Algorithm 3.1 and $\mathcal{G}(0)$ is the output. $R/R\mathcal{G}(0)$ is holonomic A_{n-1} module.

Examples and timing data appear in [NT].

References

[NT] Takayama,N., Gröbner basis, integration and transcendental functions, *in this proceedings*.

See the references of [NT] to consult the cited papers.