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Gröbner Basis and the Problem of Contiguous Relations

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It is a classical problem to find contiguous relations of hypergeometric functions of several variables. Recently Kametaka [11] and Okamoto [15] have developed the theory of hypergeometric solutions of the Toda equation. We need to find the explicit formulas of contiguous relations (or ladders) to construct the hypergeometric solutions of the Toda equation explicitly. We present an algorithm to obtain contiguous relations of hypergeometric functions of several variables. The algorithm is based on Buchberger's algorithm [3] on the Gröbner basis.

Key words: hypergeometric function of several variables, Toda equation, contiguous relation, Gröbner basis, computer algebra

§ 0. Introduction

In this paper we answer the following problem.

PROBLEM ([12], 54–60). *Find a systematic method to obtain contiguous relations (or ladders) of hypergeometric functions of several variables.*

The problem is classical, but we need to answer the problem in view of the recent studies of hypergeometric solutions of the Toda equation [11], [15]. Contiguous relations are also used to make correspondence between Lie algebra and special functions. The correspondence yields formulas of special functions [13].

We present a new algorithm to obtain contiguous relations of hypergeometric functions of several variables. The author implemented the algorithm on the computer algebra system REDUCE3.2.

Our algorithm is based on Buchberger's algorithm that constructs a Gröbner basis ([3]). But we need to generalize the notion of Gröbner basis to the following rings.

Let k be a field of characteristic 0. A ring of differential operators with rational function coefficients

$$k(x_1, \cdots, x_n)\left[\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right]$$

is denoted by $\mathscr{A}$. A product in $\mathscr{A}$ is defined by the relation

$$\frac{\partial}{\partial x_i} x_j = x_j \frac{\partial}{\partial x_i} + \delta_{ij},$$

where δ_{ij} is Kronecker's delta.

Let Δ_i be a difference operator defined by

$$\Delta_i f(\lambda_1, \cdots, \lambda_i, \cdots, \lambda_m) = f(\lambda_1, \cdots, \lambda_i + 1, \cdots, \lambda_m).$$

A ring of difference-differential operators with rational function coefficients

$$k(\lambda_1, \cdots, \lambda_m, x_1, \cdots, x_n)\left[\Delta_1, \cdots, \Delta_m, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right]$$

is denoted by $\mathscr{A}(m, n)$. Note that $\mathscr{A}(0, n) = \mathscr{A}$.

Buchberger [3] found Buchberger's algorithm that constructs a Gröbner basis of an ideal of a polynomial ring. His algorithm has been extended in many fields. Zacharias [17] found the efficient algorithm that solves a linear indefinite equation in a polynomial ring and is based on the Gröbner basis. These algorithms are extended to modules by [1], [14], [8]. Galligo [9] also extended them to modules over the rings of differential operators.

In § 1 we generalize Buchberger's algorithm and the algorithm to solve a linear indefinite equaion to a class of modules that include $\mathscr{A}(m, n)$. There is no published Buchberger's algorithm for $\mathscr{A}(m, n)$, but we can generalize the algorithm by the same idea with Buchberger's original work. We remark that Bergman [2] essentially suggested these algorithms.

In § 2 we state the algorithm to obtain contiguous relations. The notion of a Gröbner basis for $\mathscr{A}$, $\mathscr{A}(m, n)$ plays a crucial role. We present the explicit formula of the contiguous relation of Appell's F_4 with respect to the parameter α (see [7] 5.7 on the Appell's functions). It is a new formula. The first motivation of the paper was to answer the question "Is F_4 a hypergeometric solution of the Toda equation?". The answer is negative by the formula.

§ 1. Gröbner Basis

We define $G := \{0, 1, 2, \cdots\}$ and $G_\omega := G \cup \{\omega\}$ where ω is a symbol that is not an element of G. G is a commutative semigroup with respect to '+'. We define $\omega + k = \omega$, $k \in G_\omega$. It is a natural extension of '+' to G_ω. The action of G^q on $(G_\omega)^q$ is defined by

$$G^q \times G^q_\omega \ni ((k_1, \cdots, k_q), (i_1, \cdots, i_q)) \longmapsto (k_1 + i_1, \cdots, k_q + i_q) \in G^q_\omega.$$

Let I be a subset of $(G_\omega)^q$. I is a *monoideal* iff $G^q + I \subseteq I$. Let k_i be elements of G^q_ω. $\langle k_1, \cdots, k_l \rangle$ is $\bigcup_{1 \le i \le l}(k_i + G^q)$. Any set of monoideals I_i $(i = 1, 2, \cdots)$ satisfies the ascending chain condition, i.e. if $I_i \subseteq I_{i+1}$, then there exists i_0 such that $I_{i_0} = I_i$ for all $i \ge i_0$.

Let R be an associative ring (with unit) and M be a left R module. Suppose that 'deg' is a map from M to $(G_\omega)^q$ where q is a natural number and fixed. Let $\succ$ be a linear order on the $\deg(M)$. We suppose that 'deg' and $\succ$ satisfy the following conditions (1.1)–(1.8) in the sequel.

(1.1) $F=0$ iff $\deg(F)=(\omega, \cdots, \omega)$.

(1.2) $\forall k\in\deg(M), k\succeq(\omega, \cdots, \omega)$.

(1.3) If $\deg(F)\succ\deg(G)$, then $\deg(F\pm G)=\deg(F)$ and $\deg(cF)\succ\deg(cG)$ for $\forall c\in R\backslash\{0\}$.

(1.4) If $\deg(F)=\deg(G)$, then $\deg(F\pm G)\preceq\deg(F)$ and $\deg(cF)=\deg(cG)$ for $\forall c\in R\backslash\{0\}$.

(1.5) $\forall c\in R\backslash\{0\}, \langle\deg(cF)\rangle\subseteq\langle\deg(F)\rangle$.

(1.6) If $\langle\deg(G)\rangle\subseteq\langle\deg(F)\rangle$, then $\deg(G)\succeq\deg(F)$.

(1.7) If $\langle\deg(G)\rangle\subseteq\langle\deg(F)\rangle$ and $F\neq 0$, then $\exists h\in R$ such that

$$\deg(G-hF)\prec\deg(G).$$

Let $\mathrm{cm}(F, G)$ be a set

$$\{\deg(cF)\mid c\in R\}\cap\{\deg(dG)\mid d\in R\}.$$

If $\mathrm{cm}(F, G)=\{(\omega, \cdots, \omega)\}$, then

$$\mathrm{lcm}(F, G):=(\omega, \cdots, \omega)$$

else

$$\mathrm{lcm}(F, G):=\text{minimum of } \mathrm{cm}(F, G)\backslash\{(\omega, \cdots, \omega)\}.$$

(1.8) $$\langle\mathrm{cm}(F, G)\rangle\subseteq\langle\mathrm{lcm}(F, G)\rangle.$$

The existence of the minimum follows on the fact that $(\deg(M), \prec)$ is a well-founded set, i.e.

PROPOSITION 1-1. *If* $\deg(F_i)\succeq\deg(F_{i+1})$ $(i\geq 1)$, *then there exists a number* i_0 *such that* $\deg(F_i)=\deg(F_{i_0})$ $(i\geq i_0)$.

Proof. Suppose that

$$\deg(F_i)\succ\deg(F_{i+1})$$

for all i. Let S_k be $\langle\deg(F_1), \cdots, \deg(F_k)\rangle$. Since S_k is a mono-ideal, there exists a number k such that $S_k=S_{k+1}$. Hence it follows that there exists a number i_0 such that $\deg(F_{k+1})\in\langle\deg(F_{i_0})\rangle$ $(i_0\leq k)$. We have $\deg(F_{k+1})\succeq\deg(F_{i_0})$ by (1.6). It is a contradiction. ■

Let G_i $(i=1, \cdots, m)$ be elements of the module M and $\mathscr{G}$ be $\{G_1, \cdots, G_m\}$.

DEFINITION 1-1. Let F be an element of M. F is *weakly reducible* by $\mathscr{G}$ iff $F\neq 0$ and $\langle\deg(F)\rangle\subseteq\langle\deg(G_1), \cdots, \deg(G_m)\rangle$. F is *weakly irreducible* by $\mathscr{G}$ iff F is not weakly reducible by $\mathscr{G}$.

If F is weakly reducible by $\mathscr{G}$, then there exist $h\in R$ and G_i such that $\deg(F-hG_i)\prec\deg(F)$ by the (1.7). We say that F can be rewritten to $F-hG_i$ in the case. We call the rewriting procedure the *weak reduction*. By the Proposition 1-1, we can verify that a weak reduction by $\mathscr{G}$ terminates in finite steps.

Let F, G be elements of M. If $\mathrm{lcm}(F, G)=(\omega, \cdots, \omega)$, we define the *critical pair* of F and G as

$$\mathrm{sp}(F, G):=0 .$$

If $\mathrm{lcm}(F, G)\neq(\omega, \cdots, \omega)$, there exists $c, d\in R$ such that $\deg(cF)=\deg(dG)=\mathrm{lcm}(F, G)$. We have $\langle\deg(dG)\rangle\subseteq\langle\deg(G)\rangle$ by (1.5). The condition (1.7) says that there exists $h\in R$ such that $\deg(cF-hG)\prec\deg(cF)=\mathrm{lcm}(F, G)$ and $\deg(cF)=\deg(hG)$. We define the *critical pair* of F and G as

$$\mathrm{sp}(F, G):=cF-hG .$$

There is ambiguity in our definition of the critical pair $\mathrm{sp}(F, G)$. We choose one of the elements that satisfies the definition of the critical pair and fix it.

Example 1-1. A left ideal $\mathfrak{N}$ of the ring $\mathscr{A}(m, n)$ is left $\mathscr{A}(m, n)$ submodule of $\mathscr{A}(m, n)$. Let an order $\succ_1$ on G^m be a lexicographic order, i.e.

$(p_1, \cdots, p_m)\succ_1(q_1, \cdots, q_m)$ iff $p_m>q_m$ or $(p_m=q_m$ and $(p_1, \cdots, p_{m-1})\succ_1$ $(q_1, \cdots, q_{m-1}))$,

and an order $\succ_2$ on G^n be a total degree order, i.e.

$(p_1, \cdots, p_n)\succ_2(q_1, \cdots, q_n)$ iff $(p_1+\cdots+p_n>q_1+\cdots+q_n)$ or $(p_1+\cdots+p_n$ $=q_1+\cdots+q_n$ and $(p_1>q_1$ or $(p_1=q_1$ and $(p_2, \cdots, p_n)\succ_2(q_2, \cdots, q_n))))$.

We define an order $\succ$ on $G^m\times G^n=G^{m+n}$ as

$$(v_1, v_2)\succ(w_1, w_2) \quad \text{iff} \quad v_2\succ_2 w_2 \quad \text{or} \quad (v_2=w_2 \text{ and } v_1\succ_1 w_1),$$

where $v_1, w_1\in G^m$, $v_2, w_2\in G^n$. Put

$$\deg\left(\sum_{k\preceq\alpha} a_k\Delta_1^{k_1}\cdots\Delta_m^{k_m}\left(\frac{\partial}{\partial x_1}\right)^{k_{m+1}}\cdots\left(\frac{\partial}{\partial x_n}\right)^{k_{m+n}}\right):=\alpha, \qquad (a_\alpha\neq 0)$$

and $\deg(0):=(\omega, \cdots, \omega)$, where $k=(k_1, \cdots, k_{m+n})$ and $\alpha=(\alpha_1, \cdots, \alpha_{m+n})$. 'deg' and $\succ$ satisfy the conditions (1.1)–(1.8).

Example 1-2 (cf. [1], [8], [14], [9]). Let R be an associative ring (with unit). Suppose that there exists a map

$$\deg_1: R\to(G_\omega)^n$$

and an order $\succ_1$ that satisfies the conditions (1.1)–(1.8). R^r is a left R module. We define a 'deg' as

$$\deg: R^r\ni(F^{(1)}, \cdots, F^{(r)})\longmapsto(\Omega_1, \cdots, \Omega_{i-1}, \deg_1(F^{(i)}), \cdots, \Omega_r)\in(G_\omega)^{rn} ,$$

where $\Omega_k=(\omega, \cdots, \omega)$ (n-tuple), $\forall j \deg_1(F^{(i)})\succeq_1 \deg_1(F^{(j)})$ and if $\deg_1(F^{(i)})=$

$\deg_1(F^{(j)})$, then $j \geq i$.

We define an order $\succ$ as

$$(\Omega_1, \cdots, \Omega_{i-1}, \deg_1(F^{(i)}), \cdots, \Omega_r) \succ (\Omega_1, \cdots, \Omega_{j-1}, \deg_1(F^{(j)}), \cdots, \Omega_r)$$

iff $\deg_1(F^{(i)}) \succ_1 \deg_1(F^{(j)})$ or $(\deg_1(F^{(i)}) = \deg_1(F^{(j)})$ and $i < j)$. It satisfies the conditions (1.1)–(1.8).

Let $(L^{(1)}, \cdots, L^{(p)})$ be an R submodule of M generated by $L^{(i)} \in M$ $(i=1, \cdots, p)$.

Algorithm 1-1 (*Buchberger's algorithm*, [3]).

input: $\{L^{(1)}, \cdots, L^{(p)}\}$: generator of the submodule $(L^{(1)}, \cdots, L^{(p)})$.

output: $\mathcal{G}$: Gröbner basis of $(L^{(1)}, \cdots, L^{(p)})$.

$\mathcal{G} := \emptyset$; $\mathcal{S} := \{L^{(1)}, \cdots, L^{(p)}\}$;

while $\mathcal{S} \neq \emptyset$ **do**
  begin $\mathcal{G} := \mathcal{G} \cup \mathcal{S}$; $\mathcal{S} := \emptyset$;
    while *there is a weakly reducible elements in* $\mathcal{G}$ **do**
    begin
      $L_0 :=$ *one of the weakly reducible elements of* $\mathcal{G}$;
      $\mathcal{G} := \mathcal{G} \setminus \{L_0\}$;
      $L := L_0$;
      repeat *weak reduction of* L
      until L *becomes weakly irreducible by* $\mathcal{G}$;
      if $L \neq 0$ **then** $\mathcal{G} := \mathcal{G} \cup \{L\}$;
    end;
    for
      all combinations (P, Q) $(P \neq Q)$ *of the elements of* $\mathcal{G}$
    do begin
      $T := \mathrm{sp}(P, Q)$;
      repeat *weak reduction of* T
      until T *becomes weakly irreducible by* $\mathcal{G}$;
      if $T \neq 0$ **then** $\mathcal{S} := \mathcal{S} \cup \{T\}$;
    end;
  end;

The chain of the mono-ideals generated by $\deg(d)$, $d \in \mathcal{S}$ satisfies the ascending chain condition. Therefore we can verify that the algorithm 1-1 terminates in finite steps ([3]).

DEFINITION 1-2. The output $\mathcal{G}$ of the algorithm 1-1 is called the *Gröbner basis* of the left R submodule $(L^{(1)}, \cdots, L^{(p)})$.

Let $\mathcal{G} = \{G_1, \cdots, G_m\}$ be a Gröbner basis of the finitely generated left R submodule $\mathfrak{N}$ of the left R module M. We fix the Gröbner basis. A *representation of the element* D *of* $\mathfrak{N}$ *on* $\mathcal{G}$ is an element $\vec{a}$ of R^m such that

$$D=\sum_{i=1}^{m} a_i G_i\,, \qquad \text{where} \quad \vec{a}=(a_1, \cdots, a_m)\,.$$

The element D of $\mathfrak{N}$ may have more than one representation on $\mathscr{G}$. The following proposition is an immediate consequence of the definition of the Gröbner basis.

PROPOSITION 1-2. $\mathrm{sp}(G_i, G_j)=\hat{u}_i^{(ij)}G_i-\hat{u}_j^{(ij)}G_j$ $(i\neq j)$ *has a representation* $\vec{s}^{(ij)}=(s_1^{(ij)}, \cdots, s_m^{(ij)})$ *such that* $\deg(s_k^{(ij)}G_k)\preceq\deg(\mathrm{sp}(G_i, G_j))$ *for all* k.

THEOREM 1-1 (*cf.* [3], [2]). *Suppose that* $\mathscr{G}=\{G_1, \cdots, G_m\}$ *is a Gröbner basis of a submodule* $\mathfrak{N}$, *then*

$$\bigcup_{L\in\mathfrak{N}\setminus\{0\}} \langle\deg(L)\rangle=\langle\deg(G_1), \cdots, \deg(G_m)\rangle\,.$$

That is to say, $N\in M$ *is weakly reducible by* $\mathscr{G}$ *or equal to* 0 *if* $N\in\mathfrak{N}$.

Proof. Let $\vec{h}=(h_1, \cdots, h_m)$ be an element of R^m. We set

$$\deg(\vec{h}):=\mathrm{Max}_{i=1,\cdots,m}[\deg(h_iG_i)]\,,$$

$$M(\vec{h}):=\sharp\{h_i \mid \deg(h_iG_i)=\deg(\vec{h}),\ 1\leq i\leq m\}\,.$$

Suppose the $L\in\mathfrak{N}$ and $L\neq 0$. If we prove that L has a representation $\vec{h}$ such that $\deg(L)=\deg(\vec{h})$, the proof is completed by (1.5). So the proof of the theorem is reduced to proving that if $L=\sum_{i=1}^{m} h_iG_i$ and $\deg(\vec{h})\succ\deg(L)$, then we can construct a representation $\vec{j}$ of L such that

$$\deg(\vec{j})\prec\deg(\vec{h}) \qquad \text{or} \qquad M(\vec{j})<M(\vec{h})\,.$$

If $\deg(\vec{h})\succ\deg(L)$, then we have $M(\vec{h})\geq 2$ by (1.3). We can suppose that $\deg(\vec{h})=\deg(h_1G_1)=\deg(h_2G_2)$ (renumber the indexes of G_i, if necessary). We have $\mathrm{sp}(G_1, G_2)=c_1G_1-c_2G_2$, $\deg(c_1G_1)=\deg(c_2G_2)$ and $\langle\deg(h_1G_1)\rangle\subseteq\langle\deg(c_1G_1)\rangle$ by (1.8). Therefore there exists $q\in R$ such that

$$\deg(h_1G_1-qc_1G_1)\prec\deg(h_1G_1)$$

by (1.7). We have

$$\begin{aligned}
L&=h_1G_1-qc_1G_1+qc_1G_1+\sum_{i\geq 2}^{m} h_iG_i\\
&=(h_1-qc_1)G_1+q\,\mathrm{sp}(G_1, G_2)+qc_2G_2+\sum_{i\geq 2}^{m} h_iG_i\\
&=(h_1-qc_1)G_1+q\sum_{k=1}^{m} s_k^{(12)}G_k+qc_2G_2+\sum_{i\geq 2}^{m} h_iG_i\,.
\end{aligned}$$

Put

$$j_1 := h_1 - qc_1 + qs_1^{(12)},$$
$$j_2 := qs_2^{(12)} + qc_2 + h_2,$$
$$j_i := h_i + qs_i^{(12)}, \ (i \neq 1, 2).$$

$\vec{j}$ satisfies the conclusion. ■

Once we construct the Gröbner basis, we can obtain a special solution of a linear indefinite equation. We will describe the procedure. It is the same as the well known procedure for polynomial ring (see [17], [1], [8], [14]).

Let C_i $(i=1, \cdots, l)$ and D be elements of M, and $\mathfrak{R}$ be a left R submodule of M generated by C_i $(i=1, \cdots, l)$. A linear indefinite equation

$$\sum_{i=1}^{l} x_i C_i = D, \qquad x_i \in R \tag{1.9}$$

has a solution $(x_1, \cdots, x_l)$ iff $D \in \mathfrak{R}$. We can construct a Gröbner basis $\mathscr{G} = \{G_k \mid k = 1, \cdots, m\}$ of $\mathfrak{R}$ by the algorithm 1-1. Hence it follows that we can express G_i by $\{C_i \mid i=1, \cdots, l\}$ explicitly

$$G_k = \sum_{i=1}^{l} b_k^i C_i.$$

Therefore we have

$$\sum_{k=1}^{m} y_k G_k = \sum_{i=1}^{l} \left(\sum_{k=1}^{m} y_k b_k^i \right) C_i$$
$$= \sum_{i=1}^{l} x_i C_i = D.$$

So we may solve

$$\sum_{k=1}^{m} y_k G_k = D,$$

to solve the (1.9). If $D \in \mathfrak{R}$, then there exists a sequence of weak reductions of D by $\mathscr{G}$ such that

$$F_{i_k} - s_{i_{k+1}} G_{i_{k+1}} = F_{i_{k+1}}, \ (0 \leq k \leq q-1, \ F_{i_0} = D),$$
$$F_{i_q} = 0.$$

Eliminating F_{i_k} from the above sequence, we obtain one special solution of $\sum_{k=1}^{m} y_k G_k = D$.

§ 2. Answer to the Problem

Let $f_\lambda(x_1, \cdots, x_n)$ be a hypergeometric function with a parameter λ. A differential operator H_λ that satisfies

$$H_\lambda f_\lambda = f_{\lambda+1} \tag{2.1}$$

is an *step-up operator*, and a differential operator B_λ that satisfies

$$B_\lambda f_\lambda = f_{\lambda-1} \tag{2.2}$$

is a *step-down operator.*

Example 2-1. Put

$$f(\alpha, \beta, \gamma; x) = \sum_{m=0}^{\infty} \frac{(\alpha, m)(\beta, m)}{(1, m)(\gamma, m)} x^m ,$$

$$H_\alpha = \frac{1}{\alpha}\left(x\frac{d}{dx} + \alpha\right),$$

$$B_\alpha = \frac{1}{\gamma - \alpha}\left\{x(1-x)\frac{d}{dx} + (\gamma - \alpha - \beta x)\right\}.$$

We have

$$H_\alpha f(\alpha, \beta, \gamma; x) = f(\alpha+1, \beta, \gamma; x) ,$$

$$B_\alpha f(\alpha, \beta, \gamma; x) = f(\alpha-1, \beta, \gamma; x) .$$

The pair of identities (2.1) and (2.2) is called a *contiguous relation* (*or ladder*). The problem is to "Find an algorithm to obtain a step-up operator and a step-down operator".

It is well known ([7], [12]) that f_λ is a solution of a system of partial differential equations

$$D_i^{(\lambda)} f_\lambda = 0 , \quad D_i^{(\lambda)} \in \mathscr{A}, \qquad (i = 1, \cdots, l) .$$

$$f_\lambda(0, \cdots, 0) = 1 .$$

Let $\mathfrak{N}_\lambda$ be the left ideal of the ring of differential operators $\mathscr{A}$ generated by $D_i^{(\lambda)}$ ($i = 1, \cdots, l$) and $\mathscr{G} = \{G_i^{(\lambda)} \mid i = 1, \cdots, m\}$ be the Gröbner basis of $\mathfrak{N}_\lambda$.

PROPOSITION 2-1. *If we have a step-up operator H_λ (resp. a step-down operator B_λ), then a step-down operator $B_{\lambda+1}$ (resp. a step-up operator $H_{\lambda-1}$) is a solution of a linear indefinite equation in $\mathscr{A}$*

$$\sum_{i=1}^{m} X_i G_i^{(\lambda)} + B_{\lambda+1} H_\lambda = 1 , \tag{2.3}$$

(*resp.*

$$\sum_{i=1}^{m} X_i G_i^{(\lambda)} + H_{\lambda-1} B_\lambda = 1 ,)$$

where X_i, $B_{\lambda+1}$ (resp. $H_{\lambda-1}$) are unknown elements.

Proof. We prove the first case. Let $(X_1, \cdots, X_m, B_{\lambda+1})$ be a solution of (2.3). Since $G_i^{(\lambda)} f_\lambda = 0$ and $H_\lambda f_\lambda = f_{\lambda+1}$, we have

$$\sum_{i=1}^{m} X_i G_i^{(\lambda)} f_\lambda + B_{\lambda+1} H_\lambda f_\lambda = 1 \cdot f_\lambda$$

$$B_{\lambda+1} H_\lambda f_\lambda = f_\lambda$$

$$B_{\lambda+1} f_{\lambda+1} = f_\lambda . \quad \blacksquare$$

The equation (2.3) has a solution iff the left ideal generated by $\mathscr{G} \cup \{H_\lambda\}$ is equal to $\mathscr{A}$. The condition holds if $\mathfrak{R}_\lambda$ is a left maximal ideal and $H_\lambda \notin \mathfrak{R}_\lambda$.

PROPOSITION 2-2. *If $\mathfrak{R}_\lambda$ is left maximal and $f_\lambda \not\equiv 0$, then H_λ and B_λ are unique by modulo $\mathfrak{R}_\lambda$.*

Proof. We prove the uniqueness of H_λ. Suppose that

$$H_\lambda f_\lambda = f_{\lambda+1}, \quad \tilde{H}_\lambda f_\lambda = f_{\lambda+1} \quad \text{and} \quad H_\lambda \neq \tilde{H}_\lambda \bmod \mathfrak{R}_\lambda .$$

We have $(H_\lambda - \tilde{H}_\lambda) f_\lambda = 0$, $H_\lambda - \tilde{H}_\lambda \notin \mathfrak{R}_\lambda$ and $\mathfrak{R}_\lambda f_\lambda = 0$. Since $\mathfrak{R}_\lambda$ is left maximal, $H_\lambda - \tilde{H}_\lambda$ and $\mathfrak{R}_\lambda$ generates $\mathscr{A}$. Therefore we have $1 \cdot f_\lambda = 0$. It is a contradiction. $\blacksquare$

PROPOSITION 2-3. (a) *If the system of differential equations $\mathfrak{R}_\lambda f = 0$ is irreducible, then $\mathfrak{R}_\lambda$ is left maximal.*

(b) *Suppose that any solution f of $\mathfrak{R}_\lambda f = 0$ has regular singularities on the n-dimensional projective space and that the dimension of the solution space of the system of differential equations $\mathfrak{R}_\lambda f = 0$ is finite. $\mathfrak{R}_\lambda$ is irreducible iff the monodromy group of the solution of $\mathfrak{R}_\lambda f = 0$ is irreducible.*

Proof of (a). *Suppose that* $\mathfrak{R}_\lambda$ is not left maximal, we have the operator P such that

$$(\mathfrak{R}_\lambda, P) \subset \mathscr{A}, \quad (\mathfrak{R}_\lambda, P) \neq \mathscr{A} .$$

A solution space of equations $(\mathfrak{R}_\lambda, P) f = 0$ is a proper subspace of the solution space of the equations $\mathfrak{R}_\lambda f = 0$. It means that $\mathfrak{R}_\lambda f = 0$ is reducible. $\blacksquare$

The fact (b) is well known, so we omit the proof.

PROPOSITION 2-4.

$$x_k{}^r \left(\sum_{i=1}^{n} a_i \delta_{xi} + a_0 + r a_k \right) = \left(\sum_{i=1}^{n} a_i \delta_{xi} + a_0 \right) x_k{}^r ,$$

where a_i $(i = 0, \cdots, n)$ and r are complex numbers and $\delta_{xi} = x_i(\partial/\partial x_i)$.

Proof. By a calculation. $\blacksquare$

Let

$$L_k(\{a_i^j\}, \{b_i^j\}) := \prod_{j=1}^{p} \left(\sum_{i=1}^{n} a_i^j \delta_{xi} + a_0^j \right) - (x_s)^r \prod_{j=1}^{q} \left(\sum_{i=1}^{n} b_i^j \delta_{xi} + b_0^j \right), \tag{2.4}$$

where $\{a_i^j\}$, $\{b_i^j\}$ are complex numbers and r is an integer. By the Proposition 2-4, we have

$$L_k(\{a_i^j\}, \{\tilde{b}_i^j\}) \left(\sum_{i=1}^{n} b_i^k \delta_{xi} + b_0^k \right) = \left(\sum_{i=1}^{n} b_i^k \delta_{xi} + b_0^k \right) L_k(\{a_i^j\}, \{b_i^j\}),$$

where

$$\tilde{b}_i^j = b_i^j \ (i \neq 0 \text{ or } j \neq k),$$

$$\tilde{b}_0^k = b_0^k + r b_s^k .$$

Hence it follows that if the function $f(\{a_i^j\}, \{b_i^j\}; x_1, \cdots, x_n)$ is a solution of the partial differential equation $L_k(\{a_i^j\}, \{b_i^j\}) f = 0$, then

$$\left(\sum_{i=1}^{n} b_i^k \delta_{xi} + b_0^k \right) f(\{a_i^j\}, \{b_i^j\}; x_1, \cdots, x_n)$$

is a solution of the partial differential equation $L_k(\{a_i^j\}, \{\tilde{b}_i^j\}) \tilde{f} = 0$.

The differential operators that define the hypergeometric functions of several variables consist of the operators of the form (2.4). So either a step-up or a step-down operator is of the form

$$c \left(\sum_{i=1}^{n} b_i^k \delta_{xi} + b_0^k \right),$$

where c is a constant for a normalization.

Algorithm 2-1.
input: A system of partial differential equations

$$D_i^{(\lambda)} f_\lambda = 0 \ (i = 1, \cdots, l),$$

that defines a hypergeometric function of several variables.
output: Step-up operator H_λ and step-down operator B_λ.

begin
Construct a Gröbner basis $\{G_i^{(\lambda)}\}$
of the left ideal generated by $\{D_i^{(\lambda)}\}$;
Find H_λ *(resp.* B_λ*) by the Proposition* 2-4;
Solve the linear indefinite equation (2.3);
Do weak reduction of $B_{\lambda+1}$ *(resp.* $H_{\lambda-1}$*),*
then we obtain the output;
end;

We remark that the contiguous relation of the holonomic solution of the Euler-Poisson-Darboux equation or harmonic equation of Darboux with respect to the parameters that are contained in these equations can be obtained by the algorithm 2-1 if the equation (2.3) has a solution. See [10] and [16] on these equations.

Example 2-2. Appell's F_4 is

$$F_4(\alpha, \beta, \gamma, \gamma'; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)(\beta, m+n)}{(1, m)(1, n)(\gamma, m)(\gamma', n)} x^m y^n .$$

Put

$$\begin{cases} L_1^{(\alpha)} = \delta_x(\delta_x + \gamma - 1) - x(\delta_x + \delta_y + \alpha)(\delta_x + \delta_y + \beta) , \\ L_2^{(\alpha)} = \delta_y(\delta_y + \gamma' - 1) - y(\delta_x + \delta_y + \alpha)(\delta_x + \delta_y + \beta) , \end{cases}$$

then F_4 is the solution of

$$L_1^{(\alpha)} f = L_2^{(\alpha)} f = 0 , \qquad f(0, 0) = 1 .$$

Put $k := \boldsymbol{C}(\beta, \gamma, \gamma')$ and $\mathscr{A} := k(x, y)[\partial/\partial x, \partial/\partial y]$. Let $\mathfrak{R}_\alpha$ be the left ideal of $\mathscr{A}$ generated by $L_1^{(\alpha)}$ and $L_2^{(\alpha)}$. Put

$$H_\alpha := \frac{1}{\alpha}(\delta_x + \delta_y + \alpha) , \tag{2.5}$$

then

$$H_\alpha F_4(\alpha, \beta, \gamma, \gamma'; x, y) = F_4(\alpha + 1, \beta, \gamma, \gamma'; x, y) .$$

We use the 'deg' and $\succ$ of Example 1-1. The Gröbner basis of $\mathfrak{R}_\alpha$ is

$$x\frac{\partial^2}{\partial x^2} + \gamma\frac{\partial}{\partial x} - y\frac{\partial^2}{\partial y^2} - \gamma'\frac{\partial}{\partial y} ,$$

$$2xy\frac{\partial^2}{\partial x \partial y} + (y^2 - y(1-x))\frac{\partial^2}{\partial y^2} + (\alpha + \beta + 1 - \gamma)x\frac{\partial}{\partial x} + ((\alpha + \beta + 1)y - \gamma'(1-x))\frac{\partial}{\partial y} + \alpha\beta ,$$

$$2y^2(x^2 - 2xy - 2x + y^2 - 2y + 1)\frac{\partial^3}{\partial y^3} + \text{lower order terms} .$$

Solve the linear indefinite equation (2.3). We have

$$B_{\alpha+1} = \frac{1}{c}\left(c_0 + c_1\frac{\partial}{\partial x} + c_2\frac{\partial}{\partial y} + c_3\frac{\partial^2}{\partial y^2}\right), \tag{2.6}$$

where

$$c = -2(-\alpha + \gamma' - 1)(-\alpha + \gamma + \gamma' - 2)(-\alpha + \gamma - 1) ,$$

$$\begin{aligned} c_0 = {} & 2\alpha^3 + 4\alpha^2\beta x + 4\alpha^2\beta y - 2\alpha^2\beta - 4\alpha^2\gamma - 4\alpha^2\gamma' + 8\alpha^2 - 3\alpha\beta\gamma x - 5\alpha\beta\gamma y + \\ & \alpha\beta\gamma - 5\alpha\beta\gamma' x - 3\alpha\beta\gamma' y + \alpha\beta\gamma' + 10\alpha\beta x + 10\alpha\beta y - 4\alpha\beta + 2\alpha\gamma^2 + 6\alpha\gamma\gamma' - 10\alpha\gamma + \\ & 2\alpha\gamma'^2 - 10\alpha\gamma' + 10\alpha + 2\beta\gamma^2 y + 2\beta\gamma\gamma' x + 2\beta\gamma\gamma' y - 3\beta\gamma x - 7\beta\gamma y + \beta\gamma + 2\beta\gamma'^2 x - \\ & 7\beta\gamma' x - 3\beta\gamma' y + \beta\gamma' + 6\beta x + 6\beta y - 2\beta - 2\gamma^2\gamma' + 2\gamma^2 - 2\gamma\gamma'^2 + 8\gamma\gamma' - 6\gamma + 2\gamma'^2 - 6\gamma' + 4, \end{aligned}$$

$$c_1 = x(4\alpha^2 x + 4\alpha^2 y - 4\alpha^2 + 2\alpha\beta x - 2\alpha\beta y - 2\alpha\beta - 5\alpha\gamma x - 3\alpha\gamma y + 5\alpha\gamma - 5\alpha\gamma' x - 3\alpha\gamma' y + 5\alpha\gamma' + 12\alpha x + 8\alpha y - 12\alpha - \beta\gamma x + \beta\gamma y + \beta\gamma - \beta\gamma' x + \beta\gamma' y + \beta\gamma' + 2\beta x - 2\beta y - 2\beta + \gamma^2 x + \gamma^2 y - \gamma^2 + 3\gamma\gamma' x + \gamma\gamma' y - 3\gamma\gamma' - 6\gamma x - 4\gamma y + 6\gamma + 2\gamma'^2 x - 2\gamma'^2 - 8\gamma' x - 2\gamma' y + 8\gamma' + 8x + 4y - 8),$$

$$c_2 = 4\alpha^2 xy + 4\alpha^2 y^2 - 4\alpha^2 y - 2\alpha\beta xy + 2\alpha\beta y^2 - 2\alpha\beta y - 3\alpha\gamma xy - 5\alpha\gamma y^2 + 5\alpha\gamma y + 2\alpha\gamma' x^2 - 7\alpha\gamma' xy - 4\alpha\gamma' x - 3\alpha\gamma' y^2 + \alpha\gamma' y + 2\alpha\gamma' + 8\alpha xy + 12\alpha y^2 - 12\alpha y + \beta\gamma xy - \beta\gamma y^2 + \beta\gamma y + \beta\gamma' xy - \beta\gamma' y^2 + \beta\gamma' y - 2\beta xy + 2\beta y^2 - 2\beta y + 2\gamma^2 y^2 - 2\gamma^2 y - \gamma\gamma' x^2 + 3\gamma\gamma' xy + 2\gamma\gamma' x + 2\gamma\gamma' y^2 - \gamma\gamma' y - \gamma\gamma' - 2\gamma xy - 8\gamma y^2 + 8\gamma y - \gamma'^2 x^2 + 3\gamma'^2 xy + 2\gamma'^2 x + \gamma'^2 y - \gamma'^2 + 2\gamma' x^2 - 8\gamma' xy - 4\gamma' x - 4\gamma' y^2 + 2\gamma' y + 2\gamma' + 4xy + 8y^2 - 8y,$$

$$c_3 = 3(2\alpha - \gamma - \gamma' + 2)y(x^2 - 2xy - 2x + y^2 - 2y + 1).$$

This is a new formula.

PROPOSITION 2-5. *Suppose that* $\alpha, \gamma, \gamma', \gamma+\gamma' \notin \boldsymbol{Z}$ *and* $2\alpha - (\gamma+\gamma') \neq 0$. *If the monodromy group of* $F_4(\alpha, \beta, \gamma, \gamma'; x, y)$ *is irreducible, then any ladder of* F_4 *with respect to the parameters* α

$$(H_{\alpha+n}, B_{\alpha+n}), (n \in \boldsymbol{Z})$$

is not a ladder of Laplace.

Proof. Since the monodromy group is irreducible, then H_α and B_α are unique by modulo $\mathfrak{R}_\alpha$ by the Propositions 2-2 and 2-3. The step-down operator B_α (2.6) is weakly irreducible by the Gröbner basis of the ideal $\mathfrak{R}_\lambda$. Hence it follows that (2.6) is the lowest degree expression of the step-down operator by the order $\succ$ of the Example 1-1. Therefore we cannot construct a ladder which consists of first order operators (see [15] on the ladder of Laplace). ■

We conclude that $F_4(\alpha+n, \beta, \gamma, \gamma'; x, y)$ is not a hypergeometric solution of the Toda equation in the context of [11], [15]. A complete list of hypergeometric solutions of the Toda equation will be presented in a future paper.

We use (deg, $\succ$) defined in Example 1-1 in the sequel. We can obtain a difference contiguous relation by constructing a Gröbner basis in the ring $\mathscr{A}(1, n)$ by (deg, $\succ$). We write λ for λ_1 in the sequel. Let $D_1^{(\lambda)}, \cdots, D_l^{(\lambda)}$ be differential opertors that define hypergeometric functions of several variables and H_λ be a step-up operator.

PROPOSITION 2-6. *Put*

$$\mathfrak{R} := (D_1^{(\lambda)}, \cdots, D_l^{(\lambda)}, H_\lambda - \Delta_1).$$

If

$$\dim_{k(\lambda, x_1, \cdots, x_n)} \mathscr{A}(1, n)/\mathfrak{R} < +\infty,$$

then the Gröbner basis of the ideal $\mathfrak{R}$ *by* (deg, $\succ$) *contains an element*

$$L \in k(\lambda, x_1, \cdots, x_n)[\Delta_1] .$$

Proof. If there exists no such element in the Gröbner basis, $\Delta_1{}^i - \Delta_1{}^j$ $(i \neq j)$ is weakly irreducible by the Gröbner basis of the ideal $\mathfrak{R}$. Therefore we have

$$\Delta_1{}^i - \Delta_1{}^j \notin \mathfrak{R} \ (i \neq j)$$

by Theorem 1-1. It means

$$\dim_{k(\lambda, x_1, \cdots, x_n)} \mathscr{A}(1, n)/\mathfrak{R} = +\infty .$$

It is a contradiction. ■

Example 2-3. Let

$$H_\lambda = \frac{1}{\lambda}\left(x_1 \frac{\partial}{\partial x_1} + \lambda\right)$$

be the step-up operator of the Gauss hypergeometric function, and $D_0^{(\lambda)}$ be

$$x_1(1-x_1)\frac{\partial^2}{\partial x_1^2} + [\gamma - (\lambda + \beta + 1)x_1]\frac{\partial}{\partial x_1} - \lambda\beta .$$

Put $k := \boldsymbol{C}(\beta, \gamma)$. The Gröbner basis of the ideal $(D_0^{(\lambda)}, H_\lambda - \Delta_1)$ of the ring $\mathscr{A}(1, 1)$ by $(\deg, \succ)$ is

$$\begin{cases} L = (\lambda+1)(1-x_1)\Delta_1{}^2 + [\gamma - 2(\lambda+1) + (\lambda+1-\beta)x_1]\Delta_1 + (\lambda+1-\gamma) , \\ \dfrac{\partial}{\partial x_1} - \lambda\Delta_1 + \lambda , \end{cases}$$

we have

$$LF(\lambda, \beta, \gamma; x_1) = 0 .$$

It is the well known difference contiguous relation of the Gauss hypergeometric function.

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