ALGEBRAIC ALGORITHMS FOR D-MODULES AND NUMERICAL ANALYSIS

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Algorithmic methods in D modules have been used in mathematical study of hypergeometric functions and in computational algebraic geometry. In this paper, we show that these algorithms give correct algorithms to perform several operations for holonomic functions and also generates substantial information for numerical evaluation of holonomic functions.

1. Introduction

As was observed by Castro and Galligo [3] , [5], the Buchberger algorithm for computing Gröbner bases of ideals of the polynomial ring applies also to the Weyl algebra, i.e., the ring of differential operators with polynomial coefficients. This generalization of the Buchberger algorithm has turned out to be very fruitful in the computational approach to the theory of D-modules, which aims at an algebraic treatment of systems of linear partial (or ordinary) differential equations and the theoretical foundation of which was laid by Bernstein, Kashiwara, M. Sato, and many others.

The aim of this paper is to show that such an algorithmic approach to the D-module theory, which essentially depends on the Buchberger algorithm, enables us to solve some fundamental problems in symbolic computation, namely to perform computation with so-called holonomic functions. Our motivation of studying computation with holonomic functions comes from signal processing and numerical analysis. We sketch some applications of computation of holonomic functions to these areas.

A function u is called holonomic, roughly speaking, if u satisfies a system of linear differential equations $P_1 u = \ldots = P_r u = 0$ whose solutions form a finite dimensional vector space; here $P_1, \ldots, P_r$ are elements of the Weyl algebra $D_n = \mathbf{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$ over the field of the complex numbers with $\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i}$. Such a system is called holonomic and plays a key role in the theory of D-modules.

Since a linear ordinary differential equation is always holonomic, special functions of one variable, such as the Gauss hypergeometric function and the Bessel function are holonomic by definition. Moreover, the rational functions of arbitrary number of variables and their exponentials are simple examples of holonomic functions. As nontrivial examples, the expression f^λ for an arbitrary polynomial f and an arbitrary complex number λ and GKZ-hypergeometric systems (see, e.g., [18]) are holonomic.

We can expect to obtain substantial information on a holonomic function by studying the differential equations which it satisfies, rather than dealing with the function itself. This holonomic approach to special function identities was initiated by Zeilberger et al. ([1], [22], [16], [23]).

We are concerned with the following computational issues on holonomic functions: (1) Given two holonomic functions f, g and two differential operators P, Q, find a holonomic system which the function $Pf + Qg$ satisfies; (2) Given two holonomic functions f, g, find a holonomic system which the function fg satisfies; (3) Given a holonomic function $f(t,x)$, find a holonomic system which the integral $\int_C f(t,x)\,dt$ satisfies.

We give answers to the three problems (under a technical condition for the third one) by using the Buchberger algorithm applied to the Weyl algebra. The class of holonomic functions are stable under these three operations (addition, multiplication, integration) and two more operations of restriction and localization [13]. We give explicit algorithms for these constructions. Partial answers to the above three problems were given in [1], [19], [22], [23].

2. Holonomic functions

Definition 2.1. A multi-valued analytic function f defined on (the universal covering of) $\mathbf{C}^n \setminus S$ with an algebraic set S of $\mathbf{C}^n$ is called a holonomic function if there exists a left ideal I of D_n so that $M = D_n/I$ is a holonomic system and $Pf = 0$ holds on $\mathbf{C}^n \setminus S$ for any $P \in I$.

We set $\mathrm{Ann}(f) := \{P \in D_n \mid Pf = 0 \text{ on } \mathbf{C}^n \setminus S\}$. Then f is holonomic if and only if $D_n/\mathrm{Ann}(f)$ is holonomic.

Proposition 2.2. *[2] Let $f \in \mathbf{C}[x]$ be a nonzero polynomial and let λ be an arbitrary complex number. Then f^λ is holonomic.*

Algorithms to compute a holonomic system which f^λ satisfies are given in [10] and [8].

Proposition 2.3. *Let f and g be holonomic functions and $P, Q \in D_n$. Then $Pf + Qg$ and fg are holonomic.*

We shall give an algorithmic proof to this proposition in Section 3.3. The class of the holonomic functions is not closed under the division [23].

Proposition 2.4. *Let $f \in \mathbf{C}(x)$ be a rational function. Then f, $\exp(f)$, and $\log f$ are holonomic.*

Proof. Suppose $f = p/q$ with $p, q \in \mathbf{C}[x]$. Then by Proposition 2.2, p and q^{-1} are holonomic. Hence f is holonomic by Proposition 2.3. The holonomicity of $\exp(f)$ is a special case of the proposition below. To prove that $u := \log f$ is holonomic, we may assume that f is a polynomial. Then u satisfies $f\partial_i f = f_i$ with $f_i := \partial_i(f)$. Let f_i be of degree n_i with respect to x_i. Then we have $\partial_i^{n_i} f \partial_i u = 0 \quad (i = 1, \ldots, n)$. Then, this system is identified with the left D_n-module $M = D_n/(D_n P_1 + \ldots + D_n P_n)$ with $P_i := \partial_i^{n_i} f \partial_i$ and M is a holonomic system on $\{x \in \mathbf{C}^n \mid f(x) \neq 0\}$ since $\mathrm{Char}(M) \subset \{(x, \xi) \mid \xi_i f(x) = 0 \ (i = 1, \ldots, n)\}$. In view of Theorem 3.1 of Kashiwara [7], the localization $M[1/f]$ is holonomic. Since $M[1/f]$ is isomorphic to M outside $f = 0$, we are done. □

We note that an algorithmic method for the localization is given in [13].

Proposition 2.5. *[1] Let f be a multi-valued analytic function and assume that $(\partial f/\partial x_i)/f$ is a rational function for every $i = 1, \ldots, n$. Then f is holonomic.*

Proof. Put $a_i := (\partial f/\partial x_i)/f = p_i/q_i$ with $p_i, q_i \in \mathbf{C}[x]$. Then f satisfies $M \quad : \quad (q_i\partial_i - p_i)f = 0 \quad (i = 1, \ldots, n)$. Let q be the least common multiple of $q_1, \ldots, q_n$. Then M is holonomic outside the hypersurface defined by $q = 0$. This implies that f is holonomic in the same way as the proof of the preceding proposition. □

Example 2.6. For two polynomials $f_1(x)$, $f_2(x)$ in $\mathbf{C}[x_1, \ldots, x_n]$, put $f(x) = \exp(f_1(x)/f_2(x))$. The system of differential equations M above is not holonomic in general (consider, e.g., $\exp(1/(x_1^3 - x_2^2 x_3^2))$). A holonomic system for $f(x)$ can be found by the method in [13].

Let f be a holonomic function. By definition, it is a multi-valued analytic function defined on $\mathbf{C}^n \setminus S$. The algebraic set S is contained in the singular locus of the annihilating ideal I of f. The singular locus is the zero set of $\left(\mathrm{in}_{(\mathbf{0},\mathbf{1})}(I) : (\xi_1, \ldots, \xi_n)^\infty\right) \cap \mathbf{C}[x_1, \ldots, x_n]$, generators of which are computable by the Buchberger algorithm in D from generators of I. See [9] and [18, §1.4].

3. Four operations on holonomic functions

3.1. *Restriction to $x_{m+1} = \cdots = x_n = 0$*

Let $u(x)$ be a holonomic function and suppose that a left ideal I of D_n is explicitly given so that $M := D_n/I$ is a holonomic system. Then $M_Y = M/x_n M$ is a holonomic system. This holonomic system is called the *restriction* of M to $x_n = 0$. As a left D_{n-1}-module, M_Y is generated by the residue classes of $1, \partial_n, \ldots, \partial_n^{k_0}$. Hence, there exists a submodule J such that $D_{n-1}^{k_0+1}/J \simeq M_Y$; J is a system of equations for $u(x', 0), (\partial_n u)(x', 0), \ldots, (\partial_n^{k_0} u)(x', 0)$, where $x' = (x_1, \ldots, x_{n-1})$. An algorithm of finding generators of J from those of I is given in [11]. By an elimination, we can find a system of equations for $u(x', 0)$ from J [18, §5.2].

Take an integer m such that $0 \leq m < n$. Let Z be the algebraic set $\{(x_1, \ldots, x_n) \,|\, x_{m+1} = \cdots = x_n = 0\}$ and M a left D_n-module D_n^r/I where I is a left submodule of D_n^r. The *restriction* of M to Z is defined by $M/(x_{m+1}M + \cdots + x_n M)$ and is denoted by M_Z as in the case of the restriction to a hypersurface. It follows from the definition we have $M/(x_{n-1}M + x_n M) \simeq ((M/x_n M)/x_{n-1}(M/x_n M))$, $M/(x_{n-2}M + x_{n-1}M + x_n M) \simeq (((M/x_n M)/x_{n-1}(M/x_n M))\, x_{n-2}\, ((M/x_n M)/x_{n-1}(M/x_n M)))$ and so on. Therefore, the iterative application of the restriction algorithm for the hypersurface case provides an algorithm to get the restriction M_Z. Yet another algorithm which computes the restriction M_Z without the iteration is given in [18, §5.5], which uses weight vectors. It is an interesting question to compare the two methods from the efficiency point of view. We finally note that the book [18] and our discussion consider restrictions of a singly generated left D-module D/I, but it is straightforward to generalize it in the case of D^m/I.

3.2. *Integrals of holonomic functions with parameters*

Let $f(x)$ be a holonomic function and let I be a left ideal of D_n such that $M := D_n/I$ is a holonomic system and $I \subset \mathrm{Ann}(f)$. For the sake of simplicity, let us assume that $f(x)$ is infinitely differentiable on $\mathbf{R}^n$ and rapidly decreasing with respect to x_n, i.e., $\lim_{x_n \to \infty} x_n^j \partial_n^k f(x) = 0$ holds for any $x' := (x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $j, k \in \mathbf{N}$. Put $g_k(x') :=$

$\int_{-\infty}^{\infty} x_n^k f(x',t)\,dt \qquad (k \in \mathbf{N})$. Then $g_0(x'), g_1(x'), \ldots, g_{k_0}(x')$ are solutions of the holonomic system $M/\partial_n M$ where k_0 is the maximal non-negative integral root of the associated b-function (see also [1],[19] although only g_0 is considered there). Computation of $M/\partial_n M$ can be reduced to that of $M/x_n M$ by an isomorphism of D_n induced by the Fourier transform. See, e.g., [18, §5.5] for details.

3.3. *Sum and product of holonomic functions*

Let u be a holonomic function and suppose that a left ideal I of D_n is given so that $I \subset \mathrm{Ann}(u)$ and $M := D_n/I$ is holonomic. First, for a given $Q \in D_n$, we show that we can compute a holonomic system for Qu. The fact that Qu is holonomic follows from $D_n Qu \subset D_n u$. Let $P_1, \ldots, P_r$ be generators of I. Then for $P \in D_n$, $PQ \in I$ holds if and only if there exist $Q_1, \ldots, Q_r \in D_n$ such that $PQ + Q_1 P_1 + \ldots + Q_r P_r = 0$. By computing a Gröbner basis of the ideal generated by $Q, P_1, \ldots, P_r$, we can obtain generators of their syzygy module

$$S := \{(P, Q_1, \ldots, Q_r) \in D_n^{r+1} \mid PQ + Q_1 P_1 + \ldots + Q_r P_r = 0\}.$$

Then the projections of generators of S to the first component generate the left ideal $I : Q = \{P \in D_n \mid PQ \in I\}$. Thus we have $I : Q \subset \mathrm{Ann}(Qu)$. The left D_n-homomorphism $D_n \ni P \mapsto PQ \in D_n$ induces a homomorhism $D_n/(I : Q) \to D_n/I$, which is injective by the definition of $I : Q$. Hence $D_n/(I : Q)$ is holonomic.

Now let v be another holonomic function with an explicitly given left ideal $J \subset \mathrm{Ann}(v)$ so that D_n/J is a holonomic system. Our first aim is to compute a holonomic system for $Pu + Qv$ for given $P, Q \in D_n$. Since the holonomic systems for $I : P$ and $J : Q$ are computed in the way described above, we may assume that $P = Q = 1$. Then we have $I \cap J \subset \mathrm{Ann}(u + v)$. This ideal intersection can be computed by the Buchberger algorithm in the same way as in the polynomial ring (see, e.g., [4]). $D_n/(I \cap J)$ is a holonomic system since the homomorphism $D_n \ni P \mapsto (P, P) \in D_n^2$ induces an injective homomorphism

$$D_n/(I \cap J) \longrightarrow (D_n/I) \bigoplus (D_n/J).$$

Next let us consider an algorithm to find a holonomic system for the product uv. Let G_u and G_v be finite sets of generators of I and J respectively. Put $D_{2n} = \mathbf{C}[x, y]\langle \partial_x, \partial_y \rangle$ with $y = (y_1, \ldots, y_n)$ and $\partial_x := (\partial_{x_1}, \ldots, \partial_{x_n})$, $\partial_y := (\partial_{y_1}, \ldots, \partial_{y_n})$. Let $I_{u \otimes v}$ be a left ideal of D_{2n}

generated by both $G_u(x) := \{P(x, \partial_x) \mid P \in G_u\}$ and $G_u(y) := \{P(y, \partial_y) \mid P \in G_v\}$. Then it is easy to see that $I_{u\otimes v} \subset \mathrm{Ann}(u(x)v(y))$ and that $M_{u\otimes v} := D_{2n}/I_{u\otimes v}$ is holonomic. Put $\Delta := \{(x, y) \in \mathbf{C}^{2n} \mid x = y\}$. Then the restriction of M to Δ:

$$M_\Delta := D_{2n}/((x_1 - y_1)D_{2n} + \cdots + (x_n - y_n)D_{2n} + I_{u\otimes v})$$

can be computed by performing coordinate transformation $x_i - y_i \to y_i$ and $x_i \to x_i$ and then applying the restriction algorithm with respect to the variables $y_1, \ldots, y_n$. Note that M_Δ is holonomic since holonomicity is preserved under restriction. In fact, M_Δ is nothing but the tensor product of D_n/I and D_n/J over $\mathbf{C}[x]$, and the above algorithm was introduced in [12]. From M_Δ, we can compute a left ideal I_{uv} of D_n so that D_n/I_{uv} is a holonomic system for $u(x)v(x)$ by elimination.

The above algorithm for I_{uv} is for general purpose but is not efficient since it involves restriction to the n-dimensional linear space in the $2n$-dimensional space. Hence possible short cuts for some particular cases would be worth mentioning. As one of such cases, consider $v := e^f u$ with a holonomic function u and a polynomial f. Suppose given a left ideal $I \subset \mathrm{Ann}(u)$ such that D_n/I is holonomic. Put $f_i := \partial f/\partial x_i$. Then the left ideal J of D_n generated by

$$\{P(x_1, \ldots, x_n; \partial_1 - f_1, \ldots, \partial_n - f_n) \mid P(x_1, \ldots, x_n; \partial_1, \ldots, \partial_n) \in I\}$$

is contained in $\mathrm{Ann}(v)$ since $(\partial_i - f_i) \bullet (e^f u) = e^f(\partial_i \bullet u)$. The characteristic variety of D_n/J is

$$\{(x, \xi_1 - f_1(x), \ldots, \xi_n - f_n(x)) \in \mathbf{C}^{2n} \mid (x, \xi) \in \mathrm{Char}(D_n/I)\}.$$

Hence D_n/J is holonomic. As another case, the product of a holonomic function and the Heaviside function will be discussed later.

4. Holonomic distributions and their integrals

Since some important analytic holonomic functions are expressed as definite integrals of distributions, the notion of holonomic function should be generalized; we will introduce holonomic distributions. They are closed under four operations if the result of an operation is well-defined. Computation of these operations can be done by the same algorithms as in the case of holonomic functions.

Definition 4.1. Let u be a distribution (in the sense of Schwartz) defined on $\mathbf{R}^n$. Then u is said to be a *holonomic distribution* if there is a left ideal

I of D_n so that D_n/I is holonomic and $Pu = 0$ holds as distribution for any $P \in I$.

For example Dirac's delta function $\delta(x) = \delta(x_1)\cdots\delta(x_n)$ is a holonomic distribution since $x_1\delta(x) = \cdots = x_n\delta(x) = 0$. Let us introduce the Heaviside function $Y(x_1)$ defined by $Y(x_1) = 0$ for $x_1 < 0$ and $Y(x_1) = 1$ for $x_1 \geq 0$. Then we have $\partial_1 Y(x_1) = \delta(x_1)$ as distribution derivative. The Heaviside function is a holonomic distribution since it satisfies the holonomic system $x_1\partial_1 Y(x_1) = \partial_2 Y(x_1) = \cdots = \partial_n Y(x_1) = 0$. As another example of holonomic distribution, let $f(x)$ be a polynomial with real coefficients and λ be a complex numer. Then we introduce the symbol

$$f(x)_+^\lambda := \begin{cases} f(x)^\lambda & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0. \end{cases}$$

It is easy to see that $f(x)_+^\lambda$ is well-defined as a tempered distribution if the real part of λ is positive by the pairing

$$\langle f(x)_+^\lambda, \psi(x)\rangle = \int_{\mathbf{R}^n} f(x)_+^\lambda \psi(x)\, dx$$

for rapidly decreasing smooth functions $\psi(x)$. By virtue of the identity of the Bernstein-Sato polynomial

$$P(\lambda)f(x)_+^{\lambda+1} = b_f(\lambda)f(x)_+^\lambda$$

with the Bernstein-Sato polynomial $b_f(s) \in \mathbf{C}[s]$ of $f(x)$ and some $P(s) \in D_n[s]$, the tempered distribution $f(x)_+^\lambda$ can be analytically continued to the whole complex plane as a meromorphic function with respect to the parameter λ. The possible poles are contained in the set

$$\{r - \nu \mid r \in \mathbf{C},\, b_f(r) = 0,\, \nu = 0, 1, 2, \ldots\}, \tag{4.2}$$

which is in fact a subset of the negative rational numbers according to the celebrated theorem of Kashiwara [6]. Put $\mathrm{Ann}(f^s) := \{P(s) \in D_n[s] \mid P(s)f^s = 0\}$. Then the algorithm in [10] produces a set G of generators of $\mathrm{Ann}(f^s)$. If λ does not belong to the exceptional set (4.2), then we have $P(\lambda)f(x)_+^\lambda = 0$ for any $P(s) \in G$. This follows easily from the definition of the action of $P(s)$ on f^s viewed as a multi-valued analytic function together with analytic continuation. However, even if $P \in D_n$ annihilates f^λ as an analytic function, it does not necessarily annihilates $f(x)_+^\lambda$ as a distribution. For example, we have $\partial_x(1) = 0$ but $\partial_x 1_+ = \partial_x Y(x) = \delta(x) \neq 0$ with $n = 1$. Anyway, it is known that the ideal generated by $\{P(\lambda) \mid P(s) \in \mathrm{Ann}(f^s)\}$

is holonomic [6, Prop 6.1]. Hence the distribution $f(x)_+^\lambda$ is holonomic if λ does not belong to (4.2).

The integral of a holonomic distribution with respect to some variables is again holonomic and can be computed by the integration algorithm. In general, let $u = u(x_1, \dots, x_m)$ be a holonomic distribution on $\mathbf{R}^n$ such that the projection $\pi_m : \mathbf{R}^n \ni x \mapsto (x_1, \dots, x_m) \in \mathbf{R}^m$ restricted to the support of u is proper. Then the integral

$$v(x_1, \dots, x_m) := \int_{\mathbf{R}^{n-m}} u(x_1, \dots, x_m, x_{m+1}, \dots, x_n)\, dx_{m+1} \cdots dx_n$$

is well-defined as a distribution on $\mathbf{R}^m$. In fact, it is defined by the pairing

$$\langle v, \psi \rangle := \langle u, 1 \otimes \psi \rangle$$

for a smooth function $\psi(x_1, \dots, x_m)$ with compact support, where $1 \otimes \psi$ means regarding $\psi(x_1, \dots, x_m)$ as a function on $\mathbf{R}^n$. We have

$$\langle \partial_i P u, 1 \otimes \psi \rangle = \langle u, -{}^t P \partial_i (1 \otimes \psi) \rangle = 0$$

for any $P \in D_n$ and $i = m_1, \dots, n$, where ${}^t P$ denotes the formal adjoint of P. It follows that v satisfies the integral of the D-module for u. In particular, if u is a holonomic distribution, then so is its integral v.

Example 4.3. Put $u = \delta(t - x_1^4 - x_2^4)$ and

$$v(t) := \int_{\mathbf{R}^2} \delta(t - x_1^4 - x_2^4)\, dx_1 dx_2.$$

Then by the integration algorithm, we know that the distribution $v(t)$ satisfies $(2t\partial_t + 1)v(t) = 0$ on $\mathbf{R}$. From the definition, it follows that $u(t) = 0$ on $t < 0$. Hence $v(t)$ is written in the form $v(t) = Ct_+^{-1/2}$ with some constant C.

5. Definite integral by using the Heaviside function

We can compute the definite integral of the form

$$\int_a^b u(x)\, dx_1 = \int_{-\infty}^{\infty} Y(x_1 - a) Y(b - x_1) u(x)\, dx_1,$$

where $u(x)$ is a smooth function defined on an open neighborhood of $[a, b] \times U$ with an open set U of $\mathbf{R}^{n-1}$. The integrand $Y(x_1 - a)Y(b - x_1)u(x)$ is well-defined as a distribution on $\mathbf{R} \times U$ with a proper support with respect to the projection to U. In the extreme case $b = \infty$, we can define

$$v(x_2, \dots, x_n) := \int_a^\infty u(x)\, dx_1 = \int_{-\infty}^\infty Y(x_1 - a) u(x)\, dx_1,$$

which is a smooth function on U if $u(x)$ is a smooth function on a neighborhood of $[a,\infty) \times U$ which is rapidly decreasing as x_1 tends to infinity. More precisely, we assume that $\lim_{x_1 \to \infty} Pu(x) = 0$ for any $P \in D_n$ and $(x_2, \dots, x_n) \in U$. The distribution $Y(x_1 - a)u(x)$ satisfies a holonomic system $M = D_n/\mathrm{Ann}(Y(x_1 - a)u(x))$. Then we can see that $v(x_2, \dots, x_n)$ satisfies the integral $M/\partial_1 M$ in the same way as for a distribution with proper support discussed in the previous section. A possible bottle neck in this computation is that of the product of $Y(x_1 - a)u(x)$. So let us present a short cut for this computation. Let I be a left ideal of D_n which annihilates $u(x)$ such that D_n/I is holonomic. We assume $a = 0$ for the sake of simplicity. First recall the formulae

$$x_1^j \delta^{(k)}(x_1) = \begin{cases} (-1)^j k(k-1)\cdots(k-j+1)\delta^{(k-j)}(x_1) & (j \le k) \\ 0 & (j > k), \end{cases}$$

$$\partial_1^m (Y(x_1)u(x)) = Y(x_1)\partial_1^m u(x) + \sum_{k=1}^{m} \binom{m}{k} \delta^{(k-1)}(x_1)\partial_1^{m-k} u(x).$$

Let P be an element of I whose order with respect to the weight vector $(-1, 0, \dots, 0; 1, 0, \dots, 0)$ is m. Using the above formulae, we get

$$P(Y(x_1)u(x)) = Y(x_1)Pu(x) + \sum_{k=1}^{\max\{m,0\}} \delta^{(k-1)}(x_1) Q_k u(x) = \sum_{k=1}^{\max\{m,0\}} \delta^{(k-1)}(x_1) Q_k u(x)$$

with some $Q_0, \dots, Q_m \in D_n$. It follows that

$$\tilde{I} := \{\mathrm{sat}(P) := x_1^{\max\{m,0\}} P \mid P \in I,\, m = \mathrm{ord}_{(-1,0,\dots,0;1,0\dots,0)} P\} \subset \mathrm{Ann}(Y(x_1)u(x)).$$

We conjecture that $D_n/\tilde{I}$ is holonomic. In practice, we can take a generating set G of I and compute $\tilde{G} := \{\mathrm{sat}(P) \mid P \in G\}$, the ideal which generates is contained in $\mathrm{Ann}(Y(x_1)u(x))$. We can easily extend the arguments so far to integrals of the form

$$\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} u(x)\, dx_1 \cdots dx_m.$$

Example 5.1. Let t, x be real variables and put

$$v(x) := \int_0^\infty e^{(-t^3+t)x}\, dt,$$

which is a smooth function on $x > 0$. Then $u := e^{(-t^3+t)x}$ satisfies a holonomic system

$$(\partial_t + (3t^2 - 1)x)u = (\partial_x + t^3 - t)u = 0.$$

By using the argument above, we know that $Y(t)u$ satisfies

$$(t\partial_t + (3t^3 - t)x)u = (\partial_x + t^3 - t)u = 0.$$

By the integration algorithm, we can conclude that $v(x)$ satisfies

$$(27x^3\partial_x^3 - 4x^3\partial_x + 54x^2\partial_x^2 - 4x^2 - 3x\partial_x + 3)v(x) = 0.$$

6. Mellin transform and *z*-transform

Let C be a path in the complex plane. The C-Mellin transform of a function $f(x)$ is defined as

$$f(x) \longmapsto g[k] = \int_C f(x)x^{k-1}dx. \tag{6.1}$$

When the path C can be regarded as a twisted cycle with respect to $f(x)x^{k-1}$, we have the following identities:

$$(k-1)E_k^{-1} \bullet g[k] = -\int_C (\partial_x f(x))x^{k-1}dx, \quad E_k \bullet g[k] = \int_C xf(x)x^{k-1}dx$$

where $E_k \bullet g[k] = g[k+1]$. The identities induce the correspondence

$$(k-1)E_k^{-1} \longleftrightarrow -\partial_x, \qquad E_k \longleftrightarrow x$$

In other words, if the function $f(x)$ is a solution of a differential equation $\sum_{i=0}^m a_i(x)\partial_x^i f = 0$, then the function $g(k)$ satisfies the difference equation $\sum_{i=0}^m a_i(E_k)(-(k-1)E_k^{-1})^i f = 0$. Conversely, if the function $g(k)$ satisfies a difference equation $\sum_{i=0}^m b_i[k-1]E_k^i g = 0$, then the function $f(x)$ satisfies the differential equation $\sum_{i=0}^m b_i(-\partial_x x)x^i g = 0$.

Following these observations, we can prove, by a purely algebraic discussion, that $\mathbf{C}\langle k, E_k\rangle \simeq \mathbf{C}\langle -\theta_x, x\rangle$ and

$$\mathbf{C}\langle k, E_k, E_k^{-1}\rangle \simeq \mathbf{C}\langle x, \partial_x\rangle. \tag{6.3}$$

Let us consider a function $f[k,n]$ which satisfies a system of difference operators J. We apply the Mellin transform

$$k \leftrightarrow -\theta_x,\ E_k \leftrightarrow x,\ -E_k^{-1}k \leftrightarrow \partial_x,\ n \leftrightarrow -\theta_y,\ E_n \leftrightarrow n,\ -E_n^{-1}n \leftrightarrow \partial_y$$

to J and obtain the ideal $\hat{J}$ in the ring of differential operators.

Theorem 6.4. *We assume $f[k,n] = 0$ for a sufficiently large $|k|$. Put*

$$I = \left(\hat{J} + (x-1)D_2\right) \cap \mathbf{C}\langle y, \partial_y\rangle$$

By applying the inverse Mellin transform to I, we obtain a difference equation for $F[n] = \sum_k f[k,n]$.

Example 6.5. Put $f[k,n] = \binom{n}{k}$. Then, we have

$$(E_n - 2)\sum_k f[k,n] = 0.$$

The function $f[k,n]$ satisfies the system of difference equations $\{(n-k+1)E_n - (n+1)\}f = 0$ and $\{(k+1)E_k - (n-k)\}f = 0$. Let J the ideal generated by the two difference operators above. Consider the inverse Mellin transform of J. Apply the algorithm of restriction to obtain the restriction $\hat{J} + (x-1)D_2$. From the output of the algorithm, we can see that the ideal I is generated by $-y^2\partial_y + 2y\partial_y - 2 = -y\theta_y + 2\theta_y - 2$. Hence, the sum is annihilated by $E_n n - 2n - 2 = (n+1)(E_n - 2)$.

The inverse Mellin transform is called the z-transform in the theory of signal processing. Let $\{s[k]\}$ be a sequence of complex numbers indexed by $k = (k_1, k_2, \ldots, k_n) \in \mathbf{Z}^n$, which we call a (multidimensional) *discrete signal.* The z-transform of $\{s[k]\}$ is the formal series

$$Z(s)(z) = \sum_{k\in\mathbf{Z}^n} = \sum_{k\in\mathbf{Z}^n} s[(k_1,\ldots,k_n)]z_1^{k_1}\cdots z_n^{k_n}.$$

If the z-transform $S(z) = Z(s)(z)$ is convergent around $z = 0$, then we have

$$s[k] = \frac{1}{(2\pi\sqrt{-1})^n}\int_C z_1^{-k_1-1}\cdots z_n^{-k_n-1}S(z)dz_1\cdots dz_n$$

by the residue theorem where C is the product of n circles with the center at 0. The inverse z-transform is nothing but a multi-variable generalization of C-Mellin transform. A signal $s[k]$ is called bounded when $s[k] = 0$ for $k_1, \ldots, k_n \ll 0$.

A bounded discrete signal is called holonomic if the annihilating set of the difference operators of the signal is holonomic under the n-variable generalization of the isomorphism (6.3).

Let $x[k]$ and $y[k]$ be one dimensional holonomic signals and $X(z)$ and $Y(z)$ be the z-transforms of $x[k]$ and $y[k]$ respectively. Since we have

$$Z(x[k] * y[k]) = X(z)Y(z)$$

and

$$Z(x[k]y[k]) = \frac{1}{2\pi\sqrt{-1}}\int_C X(w)Y(z/w)w^{-1}dw,$$

the product and the convolution of holonomic signals are again holonomic signal. It follows from discussions of the previous and this sections that we have the following theorem.

Theorem 6.6.

$S(z)$	$Z^{-1}(S)[k]$
sum and subtraction	*sum and subtraction*
product	*convolution*
multiplicative convolution	*(element-wise) product*
restriction to $z_n = 1$	*Sum with respect to* k_n
integration w.r.t. z_n *(the inverse of* z*-transform)*	

Holonomic signals are closed under the operations listed above if they are well-defined and holonomic systems for new signals under these operations are computable.

We consider a one dimensional discrete signal system with the impulse response $h[k]$. Then, for an input signal $x[k]$, the system outputs the signal $y[k] = h[k] * x[k]$. Let $H(z)$, $X(z)$, and $Y(z)$ be the z-transforms of $h[k]$, $x[k]$, and $y[k]$ respectively. Then, we have $Y(z) = H(z)X(z)$. The function $H(z)$ is called the *transfer function* of the system. In the theory of discrete signals, rational functions are usually appear as transfer functions and a beautiful theory is established for this class of transfer functions. We may try to replace rational functions by holonomic functions. This idea is not only mathematically natural, but has also been used in signal processing; an example is the Kaiser window, which is expressed in terms of the zeroth-order modified Bessel function of the first kind (see, e.g., [15]). The Bessel function is no longer rational, but it is a holonomic function. Our holonomic approach will give a systematic framework to design filters out of rational functions. As the first step, numerical evaluation of holonomic functions is necessary to design and evaluate a new filter. In the next section, we will see that our holonomic approach gives an effective method of numerical evaluations of holonomic functions.

7. Numerical evaluation of holonomic functions

Let us compare several computational techniques to evaluate a definite integral. We consider the problem of checking numerically the identity $F(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{1323}{1331}) = \frac{3}{4}\sqrt[4]{11}$ (F.Beukers, 1990) where

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha} dt$$

The function F is a holonomic function with respect to z; F satisfies the Gauss differential equation

$$z(1-z)f'' + (\gamma - (\alpha+\beta+1)z)f' - \alpha\beta f = 0, \quad f(0) = 1. \tag{7.1}$$

Let us try a numerical integration over $[0,1]$ by the adaptive Gauss method; we do not utilize the differential equation. Since the integrand is singular at the boundary, we use the following contiguity relation and evaluate the two hypergeometric functions below [21]

$$\begin{aligned} &F(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{1323}{1331}) \\ &= -\frac{555146934690291893170809321}{77265229938688} F(-\frac{31}{12}, \frac{37}{12}, \frac{13}{2}; \frac{1323}{1331}) \\ &\quad \frac{230084970555301908546825319 19}{4017791956811776} F(-\frac{31}{12}, \frac{37}{12}, \frac{15}{2}; \frac{1323}{1331}) \end{aligned}$$

It takes about 9 seconds to get the value in the accuracy 10^{-4}.

Let us evaluate the value by solving (7.1). The fourth order adaptive Runge Kutta method [17] takes about 2 seconds to get the value in the accuracy 10^{-4}.

We can find the series solution of (7.1) in an algorithmic way. The evaluation of the series expansion at $z = \frac{1323}{1331}$ gives the value in the accuracy 10^{-4} in less than 1 second [21].

This example shows that differential equations give substantial information for effective numerical evaluations and leads us to the following method to evaluate a holonomic function f at $x = b$ numerically.

(1) Find a system of differential equations for holonomic function f. Let r be the rank of the system of differential equations.
(2) Choose a point $x = a$. Evaluate $f(a)$, $f^{(1)}(a)$, $\ldots$, $f^{(r-1)}(a)$. *This step is not algorithmic.*
(3) Find the value $f(b)$ by an adaptive Runge-Kutta method by the system of differential equations and the initial values at $x = a$.

If we can find series solution at $x = a$ and it converges at $x = b$ rapidly, we may replace the last step by a computation of a series solution and its evaluation. As to methods to find series solutions, see the Chapter 2 of [18] and references of it, however there remains some fundamental unsolved problems. As a demonstration of our method, we close this paper with showing a graph of a solution of a Bessel differential equation in two variables [14], which is drawn by using our method (Figure 1).

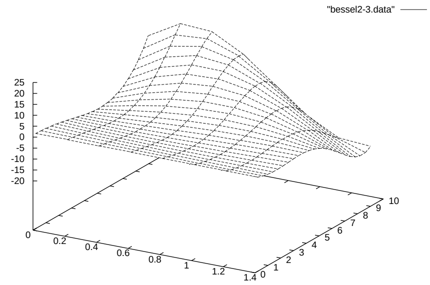

Figure 1. Bessel function in two variables: We consider the integral $f(a; x, y) = \int_C \exp(-\frac{1}{4}t^2 - xt - y/t)t^{-a-1}dt$, where $C = \vec{01} + \{e^{2\pi\sqrt{-1}\theta} \,|\, \theta \in [0, 2\pi]\} + \vec{10}$. The function $f(a; x, y)$ satisfies the holonomic system $\partial_x\partial_y + 1, \partial_x^2 - 2x\partial_x + 2y\partial_y + 2a, 2y\partial_y^2 + 2(a+1)\partial_y - \partial_x + 2x$. The rank of the system is 3. Take $a = 1/2$. It admits a unique solution of the form $y^{-a}g(x, y)$ such that g is holomorphic at the origin and $g(0, 0) = 1$. This is the graph of g for $(x, y) \in [0, 1.4] \times [0, 9]$. The function $f(1/2; x, y)$ is a constant multiple of $y^{-a}g(x, y)$. The normal form computation in D is used to derive ODE's to apply for the adaptive Runge-Kutta method.

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