A global representation of the solutions of the system of hypergeometric equations $E_{2,5}$ and the Appell function F_1

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Introduction

Let $M_{2,5} = \{(z_{ij}) \,|\, z_{ij} \in \mathbf{C}, 1 \le i \le 2, 1 \le j \le 5\}$ be the variety of 2×5 matrices of complex numbers which is biholomorphic to the 10 dimensional complex space $\mathbf{C}^{10}$. The group $\mathbf{S_5}$ of all permutations of five letters acts naturally on $M_{2,5}$:

$$M_{2,5} \ni z = (z_{ij}) \mapsto z^{\sigma} = (z_{i\sigma(j)}) \in M_{2,5}, \quad \sigma \in \mathbf{S_5}.$$

The system of hypergeometric equations $E_{2,5}(\alpha)$ (see [G1], [GG1], [GG2] or (3.a1) $\sim$ (3.a3)) is a system of differential equations defined on $M_{2,5}$ and the group $\mathbf{S_5}$ acts on the space of solutions of the system $E_{2,5}(\alpha)$. It is known that the system $E_{2,5}(\alpha)$ has a fundamental set of solutions corresponding to each regular triangulation of the prism (see [GZK], [BFS] or Proposition 3.4). Let Ψ be a fundamental set of solutions obtained from a regular triangulation of the prism. Then the function Ψ^{σ}, $\sigma \in \mathbf{S_5}$, is also a solution of the system $E_{2,5}(\alpha)$ (Proposition 3.1). The series expansion of the function Ψ is given in (4.a90) and (4.a91) explicitly.

We define branch cuts on the variety $M_{2,5}$ (see Definition 2.2). We can uniquely specify a branch of the analytic continuations of the function Ψ outside of the cuts. Let q be a point of $M_{2,5}$ that is not on the branch cuts and U a sufficiently small simply connected neighborhood of the point q. Since the solutions Ψ and Ψ^{σ} are fundamental sets of solutions, there exists a matrix $C(\sigma, q, \alpha)$ that satisfies the relation

$$\Psi = C(\sigma, q, \alpha)\Psi^{\sigma} \quad \text{on } U.$$

The purpose of this paper is to give an explicit expression of the connection matrix $C(\sigma, q, \alpha)$.

In order to find the matrix, we consider a regular graph of 15 vertices which can be obtained from a blowing-up space considered in Section 1 and show that the matrix $C(\sigma, q, \alpha)$ can be decomposed into the connection matrices between the solutions Ψ and $\Psi^{(45)}$, $(45) \in \mathbf{S_5}$, and between the solutions Ψ and Ψ^{τ}, $\tau \in I$, where I is the subgroup

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of $\mathbf{S_5}$ generated by $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$ and (13) . We derive a connection matrix between the solutions Ψ and $\Psi^{(45)}$, $(45) \in \mathbf{S_5}$, in Section 6 (Theorem 6.1) by using a uniqueness of a solution of a partial differential equation with regular singularity. In Section 7, we derive connection matrices between the functions Ψ and Ψ^{τ}, $\tau \in I$ (Theorem 7.3). The relation between Ψ and Ψ^{τ} can be considered as a generalization of Kummer's relations for the Gauss hypergeometric function.

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1. Geometry of a blowing-up space

Let $(\xi_1 : \xi_2 : \xi_3)$ be a system of homogeneous coordinates of the two-dimensional projective space $\mathbf{P}^2$. Put

$$S = \{(\xi_1 : \xi_2 : \xi_3) \in \mathbf{P}^2 \quad | \quad \xi_1\xi_2\xi_3(\xi_2 - \xi_3)(\xi_3 - \xi_1)(\xi_1 - \xi_2) = 0\}$$

and $X' = \mathbf{P}^2 \setminus S$. Consider the algebraic variety

$$Z = \{((\xi_1 : \xi_2 : \xi_3),(\eta_1 : \eta_2 : \eta_3),(\zeta_1 : \zeta_2 : \zeta_3)) \in \mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2 \quad | \quad \xi_1\eta_1 = \xi_2\eta_2 = \xi_3\eta_3,\ \xi_1\zeta_1 + \xi_2\zeta_2 + \xi_3\zeta_3 = 0,\ \zeta_1 + \zeta_2 + \zeta_3 = 0\}$$

and the projection

$$\pi : Z \ni (\xi, \eta, \zeta) \mapsto \xi \in \mathbf{P}^2.$$

PROPOSITION 1.1.
(1) *The space* $\pi^{-1}(X')$ *is biholomorphic to* X'.
(2) *The complement* $Z \setminus \pi^{-1}(X')$ *is a union of* 10 *irreducible curves that are normally crossing (see figure 1.1.)*

Put

$$g_1 : \mathbf{P}^2 \ni (\xi_1 : \xi_2 : \xi_3) \mapsto (1/\xi_1 : 1/\xi_2 : 1/\xi_3) \in \mathbf{P}^2$$
$$g_2 : \mathbf{P}^2 \ni (\xi_1 : \xi_2 : \xi_3) \mapsto (\xi_1 : \xi_1 - \xi_2 : \xi_1 - \xi_3) \in \mathbf{P}^2$$
$$g_3 : \mathbf{P}^2 \ni (\xi_1 : \xi_2 : \xi_3) \mapsto (\xi_2 : \xi_1 : \xi_3) \in \mathbf{P}^2$$
$$g_4 : \mathbf{P}^2 \ni (\xi_1 : \xi_2 : \xi_3) \mapsto (\xi_1 : \xi_3 : \xi_2) \in \mathbf{P}^2.$$

The morphisms g_i are birational on $\mathbf{P}^2$ and the restrictions of g_i to X' are biholomorphic maps. The maps $g_{i|_{X'}}$ are also denoted by g_i. It is well known (see [Ter2; Theorem 1]) that $\mathrm{Aut}(X')$ is generated by g_i and isomorphic to the permutation group $\mathbf{S_5}$. The isomorphism is given by

$$g_i \mapsto s_i = (i, i+1) \in \mathbf{S_5}.$$

LEMMA 1.2. *There exist holomorphic transformations* t_i $(i = 1, \ldots, 4)$ *of the blowing up space* Z *such that*

$$\rho \circ t_i = g_i \circ \rho \quad (i = 1, \ldots, 4)$$

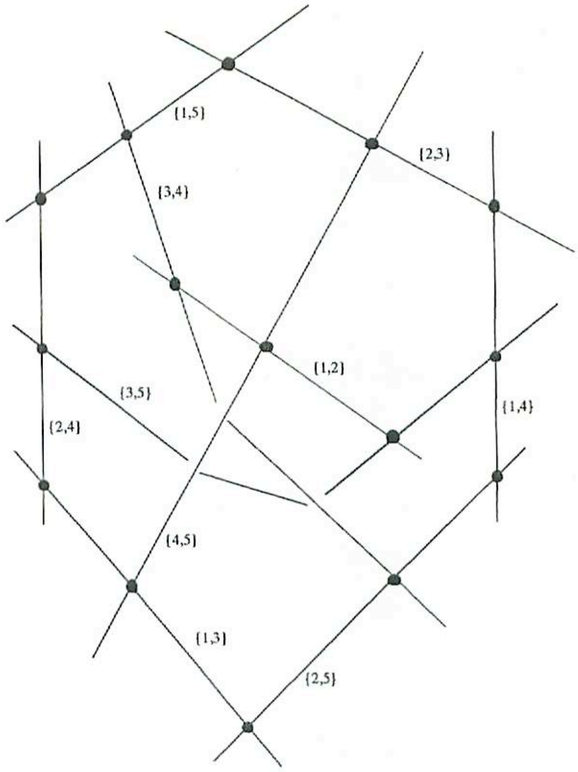


Figure 1.1

where ρ is the restriction of the projection π to $\pi^{-1}(X')$.

The holomorphic transformations t_i of Z induce permutations of the ten curves in Z. In order to see the action of $\mathbf{S_5}$, we name each of the ten curves a pair of integers in the following way.

$$\xi_1 = \xi_2 = \xi_3,\ \eta_1 = \eta_2 = \eta_3 \qquad \{1,2\}$$
$$\xi_2 = \xi_3 = \eta_1 = 0 \qquad \{1,3\}$$
$$\xi_1 = \xi_3 = \eta_2 = 0 \qquad \{1,4\}$$
$$\xi_1 = \xi_2 = \eta_3 = 0 \qquad \{1,5\}$$
$$\xi_1 = \eta_2 = \eta_3 = 0 \qquad \{2,3\}$$
$$\xi_2 = \eta_1 = \eta_3 = 0 \qquad \{2,4\}$$
$$\xi_3 = \eta_1 = \eta_2 = 0 \qquad \{2,5\}$$
$$\xi_1 = \xi_2,\ \eta_1 = \eta_2 \qquad \{3,4\}$$
$$\xi_1 = \xi_3,\ \eta_1 = \eta_3 \qquad \{3,5\}$$
$$\xi_2 = \xi_3,\ \eta_2 = \eta_3 \qquad \{4,5\}.$$

It is convenient to put $\{i,j\} = \{j,i\}$, if $i > j$.

PROPOSITION 1.3.

(1)
$$Z \setminus \pi^{-1}(X') \simeq \bigcup_{1\le i<j\le 5} \{i,j\}.$$

(2) $t_k(\{i,j\}) = \{i^\sigma, j^\sigma\} = \{\sigma(i), \sigma(j)\}$ *where* $\sigma = (k, k+1) \in \mathbf{S_5}$.

REMARK. There are two conventions to write the product of two elements of the permutation group. In this paper, the product is defined by

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \tau(s_1) & \tau(s_2) & \tau(s_3) & \tau(s_4) & \tau(s_5) \end{pmatrix}$$

where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ s_1 & s_2 & s_3 & s_4 & s_5 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ t_1 & t_2 & t_3 & t_4 & t_5 \end{pmatrix}.$$

REMARK. Let p and p' be points of 15 normally crossing points. We assume that the points p and p' are on a curve $\{i,j\}$ and $p \neq p'$. In the case, there exist numbers r, s, t such that

$$p = \{i,j\} \cap \{r,s\} \quad p' = \{i,j\} \cap \{r,t\}.$$

Therefore both of the substitutions $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ i & r & j & t & s \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ j & r & i & t & s \end{pmatrix}$ transform the points $\{1,3\} \cap \{2,5\}$ and $\{1,3\} \cap \{2,4\}$ into the points p and p'.

Let us introduce $120 (= |\mathbf{S_5}|)$ local coordinate systems on the blowing up space Z to use in later sections. The system of functions $w = \xi_2/\xi_1, v = \eta_2/\eta_3$ is a system of local coordinates of Z defined in a neighborhood of the point $\{1,3\} \cap \{2,5\}$. We define automorphisms $\{T_\sigma \,|\, \sigma \in \mathbf{S_5}\}$ on Z inductively by the relations

$$T_{\sigma\tau} = T_\sigma \circ T_\tau$$

and

$$T_{s_i} = t_i, \quad s_i = (i, i+1) \in \mathbf{S_5}.$$

The automorphisms T_σ are well-defined, i.e. if $\sigma = \prod_{k=1}^{m_1} s_{i_k} = \prod_{k=1}^{m_2} s_{j_k}$, then $T_{s_{i_1}} \circ \cdots \circ T_{s_{i_{m_1}}} = T_{s_{j_1}} \circ \cdots \circ T_{s_{j_{m_2}}}$. Notice that we have

$$(T_{s_i})^{-1} = T_{s_i}$$

and

$$(T_\sigma)^{-1}(\{i,j\}) = \{i^\sigma, j^\sigma\}.$$

Put

$$\begin{cases} w_\sigma = w \circ T_\sigma \\ v_\sigma = v \circ T_\sigma \end{cases} \quad \begin{cases} w'_\sigma = w_\sigma v_\sigma \\ v'_\sigma = 1/v_\sigma. \end{cases}$$

Notice that

$$\begin{cases} w'_\sigma = w_{s_4\sigma} \\ v'_\sigma = v_{s_4\sigma}. \end{cases}$$

Put $x = w = \xi_2/\xi_1$ and $y = wv = \xi_3/\xi_1$. The functions x and y defined in a neighborhood of the point $\{2,3\} \cap \{2,5\}$ have unique extensions on $\pi^{-1}(X')$ which are also denoted by x and y.

Put

$$\begin{aligned}
\varphi_1 &= |\xi_1\xi_2\xi_3|^2 \mathrm{Im}\left(\frac{1}{\bar{\xi}_2} - \frac{1}{\bar{\xi}_1}\right)\left(\frac{1}{\xi_3} - \frac{1}{\xi_1}\right) \\
&= \mathrm{Im}(|\xi_1|^2\xi_2\bar{\xi}_3 - \xi_1\bar{\xi}_3|\xi_2|^2 + |\xi_3|^2\xi_1\bar{\xi}_2) \\
&= \mathrm{Im}\,(wv - v - |v|^2 w) \\
\varphi_2 &= \mathrm{Im}(\xi_1 - \xi_3)(\bar{\xi}_1 - \bar{\xi}_2) = \mathrm{Im}\,(1 - wv)(1 - \bar{w}) \\
\varphi_3 &= \mathrm{Im}\bar{\xi}_2\xi_3 = \mathrm{Im}\, v \\
\varphi_4 &= \mathrm{Im}\bar{\xi}_1\xi_3 = \mathrm{Im}\, wv \\
\varphi_5 &= \mathrm{Im}\bar{\xi}_1\xi_2 = \mathrm{Im}\, w,
\end{aligned}$$

$$U_i^\varepsilon = \{\xi \in \mathbf{P}^2 \mid \varepsilon\varphi_i(\xi) > 0\}, \quad \varepsilon = \pm.$$

THEOREM 1.1. (1) *The permutation group* $\mathbf{S}_5$ *acts on the set of* U_i^ε *as follows.*

$$T_{s_i}(U_i^\pm) = U_{i+1}^\pm, \quad i = 1,2,3,4,$$

$$T_{s_i}(U_1^+) = U_1^-, \quad i = 2,3,4,$$

$$T_{s_i}(U_2^+) = U_2^-, \quad i = 3,4,$$

$$T_{s_i}(U_3^+) = U_3^-, \quad i = 1,4,$$

$$T_{s_i}(U_4^+) = U_4^-, \quad i = 1,2,$$

$$T_{s_i}(U_5^+) = U_5^-, \quad i = 1,2,3.$$

(2) *The intersection* $U_1^{\varepsilon_1} \cap \cdots \cap U_5^{\varepsilon_5}$ $(\varepsilon_i = \pm)$ *is simply connected or empty.*

(3) *The twenty open sets given below are disjoint and the union of them are open dense in* X'.

$$(-5,-4,-3,-2,-1),(-5,-4,-3,-2,1),(-5,-4,-3,1,2),(-5,-4,-2,-1,3),$$
$$(-5,-4,-1,2,3),(-5,-4,1,2,3),(-5,-3,-2,1,4),(-5,-2,-1,3,4),$$
$$(-5,-2,1,3,4),(-5,1,2,3,4),(-4,-3,-2,-1,5),(-4,-3,-1,2,5),$$
$$(-4,-3,1,2,5),(-4,-1,2,3,5),(-3,-2,-1,4,5),(-3,-2,1,4,5),$$
$$(-3,1,2,4,5),(-2,-1,3,4,5),(-1,2,3,4,5),(1,2,3,4,5)$$

where

$$(s_1,\ldots,s_5)=\{(w,v)\,|\,\varphi_{s_i}>0,\ 1\le i\le 5\}$$

and $\varphi_{-k}:=-\varphi_k,\quad k>0.$

As for a proof and a detailed study of the twenty simply connected domains, see [Sek2; Theorem 2.3].

The set of the twenty simply connected domains is denoted by $\mathcal{D}_{20}$.

REMARK. Another study of a blowing up space is given in [Ter1].

2. A map from $M'_{2,5}$ to X'

Put

$$M_{2,5} := \{z \mid z = \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \end{pmatrix},\ z_{ij} \in \mathbf{C}\},$$

$$[ij] := \begin{vmatrix} z_{1i} & z_{1j} \\ z_{2i} & z_{2j} \end{vmatrix},$$

$$M'_{2,5} := \{z \in M_{2,5} |\ [ij] \neq 0, \forall i \neq j\}.$$

We define a map from $M'_{2,5}$ to $\pi^{-1}(X')$ as follows.

$$\varphi : z \mapsto (\xi', \eta', \zeta') \in Z \tag{2.a1}$$

where

$$\begin{aligned} &\xi'_1 = [41][32][51], \quad \xi'_2 = [42][31][51], \quad \xi'_3 = [52][31][41], \\ &\eta'_1 = [24][25][13], \quad \eta'_2 = [14][23][25], \quad \eta'_3 = [15][23][24], \\ &\zeta'_1 = [13][45], \quad \zeta'_2 = [14][53], \quad \zeta'_3 = [15][34]. \end{aligned}$$

Note that we have Plücker's relation:

$$|\lambda_2\lambda_3||\lambda_1\mu_1| - |\lambda_1\lambda_3||\lambda_2\mu_1| + |\lambda_1\lambda_2||\lambda_3\mu_1| \equiv 0,\ \lambda_1, \lambda_2, \lambda_3, \mu_1 \in \mathbf{C}^2.$$

We can show that the map (2.a1) is well-defined by utilizing Plücker's relation.

The general linear group $GL(2, \mathbf{C})$ and $(\mathbf{C}^*)^5$ act on $M'_{2,5}$ from the left-hand side and the right-hand side respectively. The equivalence relation of the action above is denoted by $\sim$. Notice that we have

$$\varphi(gzh) = \varphi(z)$$

where

$$g \in GL(2, \mathbf{C}),\ h \in \mathrm{diag}(h_1, \ldots, h_5), h_i \in \mathbf{C}^*.$$

The permutation group $\mathbf{S_5}$ induces a set of automorphisms $\{S_\sigma \,|\, \sigma \in \mathbf{S_5}\}$ of $M'_{2,5}$ as follows.

$$S_\sigma : M'_{2,5} \ni z = (z_{ij}) \mapsto (z_{i\sigma(j)}) \in M'_{2,5},\ \sigma \in \mathbf{S_5}.$$

If we put

$$z = S_\sigma(Z) \ \text{ and } \ Z = S_\tau(Z')$$

then we have

$$z_{ij} = Z_{ij^\sigma} \ \text{ and } \ Z_{ij} = Z'_{ij^\tau}.$$

Changing j into j^σ, we have $Z_{ij^\sigma} = Z'_{ij^{\sigma\tau}}$, which yields $z_{ij} = Z'_{ij^{\sigma\tau}}$. We have shown that the set of the automorphisms satisfies

$$S_{\sigma\tau} = S_\sigma \circ S_\tau.$$

PROPOSITION 2.1. ([G1; §6], [MSY; Part II, 1-3]) (1) *The space* $M'_{2,5}/\sim$ *is biholomorphic to* $\pi^{-1}(X') \simeq X'$. *The isomorphism is given by* φ.
(2)

$$\begin{array}{ccc} M'_{2,5}/\sim & \xrightarrow{\varphi} & \pi^{-1}(X') \\ \Big\downarrow S_\sigma & & \Big\downarrow T_\sigma \\ M'_{2,5}/\sim & \xrightarrow{\varphi} & \pi^{-1}(X') \end{array}$$

where $\sigma \in \mathbf{S_5}$.

REMARK. We give a table of the functions $\varphi_i \circ \varphi, w \circ \varphi, v \circ \varphi, (wv) \circ \varphi, (1-w) \circ \varphi, (1-wv) \circ \varphi, ((1-w)/(1-wv)) \circ \varphi$.

$$\varphi_1 \circ \varphi = \mathrm{Im}\frac{[53][42]}{[43][52]},$$
$$\varphi_2 \circ \varphi = \mathrm{Im}\frac{[41][53]}{[51][43]},$$
$$\varphi_3 \circ \varphi = \mathrm{Im}\frac{[52][41]}{[42][51]},$$
$$\varphi_4 \circ \varphi = \mathrm{Im}\frac{[52][31]}{[32][51]},$$
$$\varphi_5 \circ \varphi = \mathrm{Im}\frac{[42][31]}{[32][41]}.$$

$$w \circ \varphi = \frac{[24][13]}{[14][23]}, \qquad (1-w) \circ \varphi = \frac{[12][34]}{[32][14]},$$
$$v \circ \varphi = \frac{[25][14]}{[15][24]}, \qquad (1-wv) \circ \varphi = \frac{[12][35]}{[32][15]},$$
$$(wv) \circ \varphi = \frac{[13][25]}{[23][15]}, \qquad \left(\frac{1-w}{1-wv}\right) \circ \varphi = \frac{[34][15]}{[14][35]},$$
$$w' = wv,\ v' = 1/v, \qquad \left(\frac{(1-w)v}{1-wv}\right) \circ \varphi = \frac{[25][34]}{[24][35]}.$$

DEFINITION 2.1. Given a point $p \in M'_{2,5}$ and an open set $\Omega \subset \pi^{-1}(X')$, we put

$$\mathcal{S}(\Omega, p) = \{s : \Omega \longrightarrow M'_{2,5} \mid s \text{ is a holomorphic function}, s(\varphi(p)) = p, \varphi \circ s = id_\Omega\}.$$

REMARK. There exists a function $\delta(\varepsilon)$ that satisfies the following condition. If

$$||A - B|| < \varepsilon, A \sim B \text{ and } A, B \in M'_{2,5},$$

then there exist $g \in GL(2, \mathbf{C})$ and $h \in \operatorname{diag}((\mathbf{C}^*)^5)$ such that

$$gBh = A, ||g - E|| < \delta(\varepsilon) \text{ and } ||h - E|| < \delta(\varepsilon).$$

We define branch cuts on $M'_{2,5}$ and denote the points which are outside of the branch cuts by $M''_{2,5}$.

DEFINITION 2.2.

$$\begin{aligned} M''_{2,5} = \{z \in M'_{2,5} \mid & \operatorname{Im}\frac{[ij][k\ell]}{[kj][i\ell]} \neq 0, \operatorname{Im}\frac{[ij]}{[kj]} \neq 0, \operatorname{Im}\frac{[k\ell]}{[i\ell]} \neq 0, \\ & \operatorname{Im}[ij] \neq 0, \operatorname{Im}[kj] \neq 0, \operatorname{Im}[k\ell] \neq 0, \operatorname{Im}[i\ell] \neq 0, \\ & \text{for all } \{i, j, k, \ell\} \subseteq \{1, 2, 3, 4, 5\} \text{ where } i, j, k, \ell \text{ are different numbers.}\} \end{aligned}$$

3. The system of hypergeometric equations $E_{2,5}$

Let α_i $(i = 1, \ldots, n)$ be complex numbers that satisfy

$$\sum_{i=1}^{n} \alpha_i = n - k$$

and α be $(\alpha_1, \ldots, \alpha_n)$. Let $z = (z_{ij})$ be a $k \times n$ matrix. The system of differential equations

$$\sum_{i=1}^{k} z_{ip} \frac{\partial}{\partial z_{ip}} F = (\alpha_p - 1)F, \quad p = 1, \ldots, n \tag{3.a1}$$

$$\sum_{p=1}^{n} z_{ip} \frac{\partial}{\partial z_{jp}} F = -\delta_{ij} F, \quad i, j = 1, \ldots, k \tag{3.a2}$$

$$\frac{\partial^2}{\partial z_{ip} \partial z_{jq}} F = \frac{\partial^2}{\partial z_{iq} \partial z_{jp}} F, \quad i, j = 1, \ldots, k, \ p, q = 1, \ldots, n \tag{3.a3}$$

is called the system of hypergeometric equations $E_{k,n}(\alpha)$ ([GG1; §1.1]).

Proposition 3.1. ([G1; §4]) *If a function $F(\alpha; z)$ is a solution of the system $E_{k,n}(\alpha)$, then the function $F(\alpha^\sigma; z^\sigma)$ is a solution of $E_{k,n}(\alpha)$ where $z^\sigma = (z_{i\sigma(j)})$, $\alpha^\sigma = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)})$ and $\sigma \in \mathbf{S_n}$.*

Put

$$\chi = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The system of differential equations

$$\chi \begin{pmatrix} z_{13}\partial_{13} \\ z_{14}\partial_{14} \\ z_{15}\partial_{15} \\ z_{23}\partial_{23} \\ z_{24}\partial_{24} \\ z_{25}\partial_{25} \end{pmatrix} G = \begin{pmatrix} -\alpha_1 \\ -\alpha_2 \\ \alpha_3 - 1 \\ \alpha_4 - 1 \\ \alpha_5 - 1 \end{pmatrix} G, \quad \partial_{ij} = \frac{\partial}{\partial z_{ij}} \tag{3.a4}$$

$$\frac{\partial^2}{\partial z_{ip} \partial z_{jq}} G = \frac{\partial^2}{\partial z_{iq} \partial z_{jp}} G, \quad i, j \in \{1, 2\}, \ p, q \in \{3, 4, 5\} \tag{3.a5}$$

is denoted by $E_{2,5}(\alpha; J)$, $J = \{1,2\}$ ([GG1; §1.1]). The system $E_{2,5}(\alpha; J)$ is the restriction of the system $E_{2,5}(\alpha)$ on the subvariety $\begin{pmatrix} 1 & 0 & * & * & * \\ 0 & 1 & * & * & * \end{pmatrix}$ ([GG1; §1.1]).

Proposition 3.2. [GG2; §3] (1) *The dimension of the solution space of $E_{2,5}(\alpha; J)$ is 3. The locus of the singularity is*

$$\prod_{i \in \{1,2\}, j \in \{3,4,5\}} z_{ij}[34][35][45] = 0.$$

(2) *The dimension of the solution space of $E_{2,5}(\alpha)$ is 3. The locus of the singularity is*

$$\prod_{1 \le i < j \le 5} [ij] = 0.$$

Proposition 3.3. [GG1; Proposition 1.1] *Let F be a solution of the system $E_{2,5}(\alpha)$. We have*

$$F(\alpha; z) = [12]^{-1} F\left(\alpha; \begin{pmatrix} 1 & 0 & \frac{[32]}{[12]} & \frac{[42]}{[12]} & \frac{[52]}{[12]} \\ 0 & 1 & \frac{[13]}{[12]} & \frac{[14]}{[12]} & \frac{[15]}{[12]} \end{pmatrix}\right).$$

Corollary 3.3. *Let G be a solution of the system $E_{2,5}(\alpha; J)$. The function*

$$[12]^{-1} G\left(\alpha; \begin{pmatrix} \frac{[32]}{[12]} & \frac{[42]}{[12]} & \frac{[52]}{[12]} \\ \frac{[13]}{[12]} & \frac{[14]}{[12]} & \frac{[15]}{[12]} \end{pmatrix}\right)$$

is a solution of the system $E_{2,5}(\alpha)$.

Let us proceed on a construction of series solutions of the system of hypergeometric equations $E_{2,5}(\alpha)$. Before accomplishing this purpose, we specify a branch of the power function v^μ. Let $p(\mu; v)$ be the single-valued holomorphic function on the domain $\mathbf{C} \setminus \mathbf{R}_{\le 0}$ such that $\lim_{y \to 0} p(\mu; x + iy) = e^{\mu \log x}$ on $x > 0$. The following Lemma is given in [Sek1].

Lemma 3.1.

(1) $$p(\mu; -v) = \begin{cases} e^{-\pi i \mu} p(\mu; v) & \text{if } \operatorname{Im} v > 0 \\ e^{\pi i \mu} p(\mu; v) & \text{if } \operatorname{Im} v < 0. \end{cases}$$

(2) *We have*

$$p(\mu; wv) = \begin{cases} e^{-2\pi i\mu} p(\mu; w)p(\mu; v) & \text{if } \mathrm{Im}\, w > 0, \mathrm{Im}\, w > 0, \mathrm{Im}\, wv < 0, \\ e^{2\pi i\mu} p(\mu; w)p(\mu; v) & \text{if } \mathrm{Im}\, w < 0, \mathrm{Im}\, w < 0, \mathrm{Im}\, wv > 0. \end{cases}$$

In other cases, we have

$$p(\mu; wv) = p(\mu; w)p(\mu; v).$$

We note that the system $E_{2,5}(\alpha; J)$ can be written as follows.

$$\text{(3.a6)} \qquad \chi \begin{pmatrix} v_1 \frac{\partial}{\partial v_1} \\ \cdot \\ \cdot \\ \cdot \\ v_6 \frac{\partial}{\partial v_6} \end{pmatrix} G = \beta G, \quad \chi = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} -\alpha_1 \\ -\alpha_2 \\ \alpha_3 - 1 \\ \alpha_4 - 1 \\ \alpha_5 - 1 \end{pmatrix}$$

$$\text{(3.a7)} \qquad \Diamond_a G = 0, \quad a \in \mathrm{Ker}\chi \cap \mathbf{Z}^6, \ \Diamond_a = \prod_{a_i > 0} \left(\frac{\partial}{\partial v_i}\right)^{a_i} - \prod_{a_i < 0} \left(\frac{\partial}{\partial v_i}\right)^{-a_i}$$

where

$$v_1 = z_{13}, v_2 = z_{14}, v_3 = z_{15}, v_4 = z_{23}, v_5 = z_{24}, v_6 = z_{25}.$$

[GZK] showed that the regular triangulations of a set of points determined by the matrix χ yield solutions of (3.a6) and (3.a7).

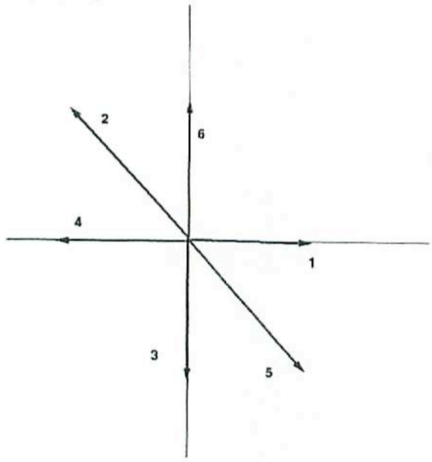


Figure 3.f1

We compute regular triangulations of the prism by the method of [BFS; §4] (see Figure 3.f1) and derive solutions of the system (3.a6) and (3.a7). We can obtain solutions of the system $E_{2,5}(\alpha)$ by virtue of Corollary 3.3. Carrying out these computations, we obtain the following result.

Put

$$F(\gamma; z) = p(-\sum \gamma_i - 1; [12])$$
$$\times \sum_{\ell \in \mathrm{Ker}\chi \cap \mathbf{Z}^6} \prod_{i=1}^{6} p(\gamma_i + \ell_i; u_i) / \prod_{i=1}^{6} \Gamma(\gamma_i + \ell_i + 1)$$

where

$$\ell = (\ell_1, \ldots, \ell_6),$$
$$\gamma = (\gamma_1, \ldots, \gamma_6),$$
$$\begin{array}{lll} u_1 = [13] & u_2 = [14] & u_3 = [15] \\ u_4 = [23] & u_5 = [24] & u_6 = [25]. \end{array}$$

Put

$$\Psi = \begin{pmatrix} F(\gamma_{23}; z) \\ F(\gamma_{45}; z) \\ F(\gamma_{34}; z) \end{pmatrix}, \quad \Psi' = \begin{pmatrix} F(\gamma_{23}; z) \\ F(\gamma_{46}; z) \\ F(\gamma_{24}; z) \end{pmatrix} = \Psi^{(45)} \quad \text{where } (45) \in \mathbf{S_5}.$$

Proposition 3.4. ([GZK]) *The functions Ψ and Ψ' are solutions of the system of hypergeometric equations $E_{2,5}(\alpha)$ where the vector γ_{ij} is a unique solution of the linear equation*

$$\chi\gamma = \beta = \begin{pmatrix} -\alpha_1 \\ -\alpha_2 \\ \alpha_3 - 1 \\ \alpha_4 - 1 \\ \alpha_5 - 1 \end{pmatrix}$$

such that

$$\gamma_i = \gamma_j = 0.$$

We explicitly give the vectors γ_{ij}.

$$\gamma_{23} = \begin{pmatrix} -\alpha_1 \\ 0 \\ 0 \\ \alpha_1 + \alpha_3 - 1 \\ \alpha_4 - 1 \\ \alpha_5 - 1 \end{pmatrix}, \ \gamma_{45} = \begin{pmatrix} \alpha_3 - 1 \\ \alpha_4 - 1 \\ \alpha_2 + \alpha_5 - 1 \\ 0 \\ 0 \\ -\alpha_2 \end{pmatrix}, \ \gamma_{34} = \begin{pmatrix} \alpha_3 - 1 \\ -\alpha_1 - \alpha_3 + 1 \\ 0 \\ 0 \\ -\alpha_2 - \alpha_5 + 1 \\ \alpha_5 - 1 \end{pmatrix},$$

$$\gamma_{46} = \begin{pmatrix} \alpha_3 - 1 \\ \alpha_2 + \alpha_4 - 1 \\ \alpha_5 - 1 \\ 0 \\ -\alpha_2 \\ 0 \end{pmatrix}, \ \gamma_{24} = \begin{pmatrix} \alpha_3 - 1 \\ 0 \\ -\alpha_1 - \alpha_3 + 1 \\ 0 \\ \alpha_4 - 1 \\ -\alpha_2 - \alpha_4 + 1 \end{pmatrix}.$$

4. The system of differential equations for the Appell function F_1

The function

$$F_1\left(\begin{matrix}\alpha_1 & -\alpha_4+1 & -\alpha_5+1 \\ \alpha_1+\alpha_3 & & \end{matrix}; x, y\right) = \sum_{k,n=0}^{\infty} \frac{(\alpha_1)_{k+n}(-\alpha_4+1)_k(-\alpha_5+1)_n}{(\alpha_1+\alpha_3)_{k+n}(1)_k(1)_n} x^k y^n$$

is called the Appell function F_1 which is denoted by $\hat{f}_0(\alpha; x, y)$ in the sequel ([AK]).

The system of differential equations for the Appell function F_1 can be written as follows.

(4.1) $$[\theta_x(\theta_x+\theta_y+\alpha_1+\alpha_3-1)-x(\theta_x+\theta_y+\alpha_1)(\theta_x-\alpha_4+1)]f=0$$

(4.2) $$[\theta_y(\theta_x+\theta_y+\alpha_1+\alpha_3-1)-y(\theta_x+\theta_y+\alpha_1)(\theta_y-\alpha_5+1)]f=0$$

(4.3) $$[(x-y)\partial_x\partial_y-(-\alpha_5+1)\partial_x+(-\alpha_4+1)\partial_y]f=0$$

where $\theta_x = x\partial_x$, $\theta_y = y\partial_y$ and $\sum_{i=1}^{5}\alpha_i = 3$.

The system of equations above is denoted by $A(\alpha)$.

Put

(4.a90)

$$f_0(\alpha; w, wv) = \sum_{k,n=0}^{\infty} w^k (wv)^n/\Gamma(1+k)\Gamma(1+n)c_{23}^{kn}$$

$$c_{23}^{kn} = \Gamma(\alpha_1+\alpha_3+k+n)\Gamma(1-\alpha_1-k-n)\Gamma(\alpha_4-k)\Gamma(\alpha_5-n)$$

$$f_1(\alpha; wv, v) = \sum_{k,n=0}^{\infty} (wv)^k v^n/\Gamma(1+k)\Gamma(1+n)c_{45}^{kn}$$

$$c_{45}^{kn} = \Gamma(\alpha_2+\alpha_5+k+n)\Gamma(1-\alpha_2-k-n)\Gamma(\alpha_3-k)\Gamma(\alpha_4-n)$$

$$g_1(\alpha; w, v) = \sum_{k,n=0}^{\infty} w^k v^n/\Gamma(1+k)\Gamma(1+n)c_{34}^{kn}$$

$$c_{34}^{kn} = \Gamma(2-\alpha_1-\alpha_3+k-n)\Gamma(\alpha_3-k)\Gamma(\alpha_5-n)\Gamma(2-\alpha_2-\alpha_5+n-k)$$

and

$$f_2(\alpha; w'v', v') = \sum_{k,n=0}^{\infty} (w'v')^k v'^n/\Gamma(1+k)\Gamma(1+n)c_{46}^{kn}$$

$$c_{46}^{kn} = \Gamma(\alpha_2+\alpha_4+k+n)\Gamma(1-\alpha_2-k-n)\Gamma(\alpha_3-k)\Gamma(\alpha_5-n)$$

$$g_2(\alpha; w', v') = \sum_{k,n=0}^{\infty} w'^k v'^n/\Gamma(1+k)\Gamma(1+n)c_{24}^{kn}$$

$$c_{24}^{kn} = \Gamma(2-\alpha_1-\alpha_3+k-n)\Gamma(\alpha_3-k)\Gamma(\alpha_4-n)\Gamma(2-\alpha_2-\alpha_4+n-k).$$

Note that we have

$$\hat{f}_0(\alpha; w, wv) = \Gamma(\alpha_1 + \alpha_3)\Gamma(1 - \alpha_1)\Gamma(\alpha_4)\Gamma(\alpha_5) f_0(\alpha; w, wv).$$

We have the following fundamental system of solutions of the equations $A(\alpha)$.

PROPOSITION 4.1.
(1) *Assume* $\alpha_1 + \alpha_3, \alpha_2 + \alpha_5 \notin \mathbf{Z}$. *The system of functions*

$$\Phi = \begin{pmatrix} f_0(\alpha; w, wv) \\ p(1 - \alpha_1 - \alpha_3; w) p(-1 + \alpha_2 + \alpha_5; v) f_1(\alpha; wv, v) \\ p(1 - \alpha_1 - \alpha_3; w) g_1(\alpha; w, v) \end{pmatrix}$$

is a fundamental system of solutions of $A(\alpha)$ *at the point* $\{1,3\} \cap \{2,5\}$ *where* $w = x$ *and* $v = y/x$.
(2) *Assume* $\alpha_1 + \alpha_3, \alpha_2 + \alpha_4 \notin \mathbf{Z}$. *The system of functions*

$$\Phi' = \begin{pmatrix} f_0(\alpha; w'v', w') \\ p(1 - \alpha_1 - \alpha_3; w'v') p(-1 + \alpha_5; 1/v') f_2(\alpha; w'v', v') \\ p(1 - \alpha_1 - \alpha_3; w'v') p(1 - \alpha_1 - \alpha_3; 1/v') g_2(\alpha; w', v') \end{pmatrix}$$

is a fundamental system of solutions of $A(\alpha)$ *at the point* $\{1,3\} \cap \{2,4\}$ *where* $w' = wv = y$ *and* $v' = 1/v = x/y$.

We can easily prove Proposition 4.1 by showing that each element of Φ and Φ' satisfies the system $A(\alpha)$ and we can find these expressions by Theorem 4.1 and the expressions of the functions Ψ and Ψ' given in Proposition 3.4. Note that we can also use the method of [Tak1; section 2] to find them.

The function $f_0(\alpha; w, wv)$ defines a holomorphic function on the domain:

$$D_0 = \{(w, v) \in \mathbf{C}^2 \mid |w|, |v| << 1\}.$$

LEMMA 4.1. *The domain* $U_3^{\sigma} \cap U_4^{\sigma'} \cap U_5^{\sigma''} \cap D_0$ $(\sigma, \sigma', \sigma'' = \pm)$ *is simply connected.*

Since the domain $U_3^{\sigma} \cap U_4^{\sigma'} \cap U_5^{\sigma''}$ $(\sigma, \sigma', \sigma'' = \pm)$ is simply connected and has a simply connected intersection with the domain D_0, there is a unique holomorphic function on the domain $U_3^{\sigma} \cap U_4^{\sigma'} \cap U_5^{\sigma''}$ of which restriction to $U_3^{\sigma} \cap U_4^{\sigma'} \cap U_5^{\sigma''} \cap D_0$ coincides with $f_0(\alpha; w, wv)$. In this way, we have a holomorphic function defined on the domain

$$\bigsqcup_{D \in \mathcal{D}_8} D, \quad \mathcal{D}_8 = \{U_3^{\sigma} \cap U_4^{\sigma'} \cap U_5^{\sigma''} \mid \sigma, \sigma', \sigma'' = \pm\}$$

of which powerseries expansion around the point $w = v = 0$ is $f_0(\alpha; w, wv)$. We also denote the holomorphic function thus obtained $f_0(\alpha; w, wv)$. Restricting the function f_0 to each of the twenty simply connected domains $\delta \in \mathcal{D}_{20}$, we have a holomorphic function on δ that is also denoted by f_0. Similarly, we have unique extensions of the functions f_i and g_i to each of the eight simply connected domains $\mathcal{D}_8$ and to each of the twenty simply connected domains $\mathcal{D}_{20}$ and we have unique extensions of the fundamental systems of solutions Φ and Φ'. If we need to specify the domain of the definitions, we often denote the extensions of the functions Φ and Φ' to the domain δ by Φ_δ and Φ'_δ where $\delta \in \mathcal{D}_{20}$.

Now, we become to be able to specify a branch of the function Ψ at a point of $M''_{2,5}$. We have specified the branch of the functions f_i, g_i on the simply connected domain $\delta \in \mathcal{D}_{20}$. Hence, the values of the functions

$$f_i \circ \varphi, \quad g_i \circ \varphi$$

are uniquely specified at any point of $M''_{2,5}$. Writing the function Ψ (resp. Ψ') in terms of f_i and g_i, we can specify a branch of the function Ψ (resp. Ψ') at any point of $M''_{2,5}$. Indeed, the functions Ψ and Ψ' can be written as follows.

$$\Psi = \begin{pmatrix} [12]^{\alpha_1+\alpha_2-1}[13]^{\alpha_1+\alpha_3-1}[14]^{\alpha_4-1}[15]^{\alpha_5-1}[23]^{-\alpha_1} f_0(\alpha; \bar{w}, \bar{w}\bar{v}) \\ [12]^{\alpha_1+\alpha_2-1}[15]^{-\alpha_2}[23]^{\alpha_3-1}[24]^{\alpha_4-1}[25]^{\alpha_2+\alpha_5-1} f_1(\alpha; \bar{w}\bar{v}, \bar{v}) \\ [12]^{\alpha_1+\alpha_2-1}[14]^{-\alpha_2-\alpha_5+1}[15]^{\alpha_5-1}[23]^{\alpha_3-1}[24]^{-\alpha_1-\alpha_3+1} g_1(\alpha; \bar{w}, \bar{v}) \end{pmatrix}$$

$$(4.a91) \qquad \Psi' = \begin{pmatrix} [12]^{\alpha_1+\alpha_2-1}[13]^{\alpha_1+\alpha_3-1}[14]^{\alpha_4-1}[15]^{\alpha_5-1}[23]^{-\alpha_1} f_0(\alpha; \bar{w}'\bar{v}', \bar{w}') \\ [12]^{\alpha_1+\alpha_2-1}[14]^{-\alpha_2}[23]^{\alpha_3-1}[25]^{\alpha_5-1}[24]^{\alpha_2+\alpha_4-1} f_2(\alpha; \bar{w}'\bar{v}', \bar{v}') \\ [12]^{\alpha_1+\alpha_2-1}[15]^{-\alpha_2-\alpha_4+1}[14]^{\alpha_4-1}[23]^{\alpha_3-1}[25]^{-\alpha_1-\alpha_3+1} g_2(\alpha; \bar{w}', \bar{v}') \end{pmatrix}$$

where

$$\bar{w} = w \circ \varphi = \frac{[24][13]}{[14][23]}$$
$$\bar{v} = v \circ \varphi = \frac{[25][14]}{[15][24]}$$
$$\bar{w}\bar{v} = \frac{[13][25]}{[23][15]}$$
$$\bar{w}' = \bar{w}\bar{v}, \; \bar{v}' = 1/\bar{v}.$$

Here, we do not use the notation $p(*;*)$ to avoid long expressions, but note that we had specified a branch of powerfunction w^a. For example, the function $[12]^{\alpha_1+\alpha_2-1}$ above means $p(\alpha_1+\alpha_2-1;[12])$.

Remark. We have

$$\Psi^{(45)} = \Psi', \; (45) \in \mathbf{S_5}$$

and

$$\Psi^{(13)} = \begin{pmatrix} [32]^{\alpha_3+\alpha_2-1}[31]^{\alpha_3+\alpha_1-1}[34]^{\alpha_4-1}[35]^{\alpha_5-1}[21]^{-\alpha_3} f_0^{(13)} \\ [32]^{\alpha_3+\alpha_2-1}[35]^{-\alpha_2}[21]^{\alpha_1-1}[24]^{\alpha_4-1}[25]^{\alpha_2+\alpha_5-1} f_1^{(13)} \\ [32]^{\alpha_3+\alpha_2-1}[34]^{-\alpha_2-\alpha_5+1}[35]^{\alpha_5-1}[21]^{\alpha_1-1}[24]^{-\alpha_3-\alpha_1+1} g_1^{(13)} \end{pmatrix}.$$

Let us study a correspondence between solutions of the system $E_{2,5}(\alpha)$ and the system $A(\alpha)$. Put

$$\eta = [12]^{\alpha_1+\alpha_2-1}[13]^{\alpha_1+\alpha_3-1}[14]^{\alpha_4-1}[15]^{\alpha_5-1}[23]^{-\alpha_1}.$$

THEOREM 4.1. (c.f. [Sas1; §3])
(1) *Suppose that a point* $z \in M''_{2,5}$ *satisfies the condition:*

(4.c1) *The point* $\varphi(z)$ *is sufficiently close to the point* $\{1,3\} \cap \{2,5\}$.

Let Ω *be a sufficiently small neighborhood of the point* $\varphi(z)$. *Then there exists a unique diagonal matrix* Λ *that satisfies*

$$(\Psi/\eta) \circ s = \Lambda\Phi \quad \text{on } \Omega$$

where

$$s \in \mathcal{S}(\Omega, z)$$

and the matrix Λ *does not depend on the choice of the section* s.
(2) *Suppose* z *is a point of* $M''_{2,5}$ *and let* Ω *be a sufficiently small neighborhood of the point* $\varphi(z)$. *If a function* F *on* $M'_{2,5}$ *is a solution of the system* $E_{2,5}(\alpha)$, *then the function* $(F/\eta) \circ s$ *is a solution of the system* $A(\alpha)$ *where*

$$s \in \mathcal{S}(\Omega, z)$$

and we have

$$(F/\eta) \circ s = (F/\eta) \circ s' \quad \text{for } s, s' \in \mathcal{S}(\Omega, z).$$

Proof of (1). The i-th rows of Ψ and Φ are denoted by Ψ_i and Φ_i respectively. Since we have

$$\Psi_1/\eta = f_0(\alpha; \frac{[24][13]}{[14][23]}, \frac{[13][25]}{[23][15]}),$$

$$\frac{[24][13]}{[14][23]} \circ s = \xi_2/\xi_1 = w \text{ and } \frac{[13][25]}{[23][15]} \circ s = \xi_3/\xi_1 = wv,$$

we obtain

$$(\Psi_1/\eta) \circ s = \Phi_1.$$

Next, let us show that $((\Psi_2/\eta) \circ s)/\Phi_2$ is a constant on Ω. Since we have

$$\begin{aligned} \frac{\Psi_2}{\eta} &= \frac{[12]^{\alpha_1+\alpha_2-1}[15]^{-\alpha_2}[23]^{\alpha_3-1}[24]^{\alpha_4-1}[25]^{\alpha_2+\alpha_5-1}}{[12]^{\alpha_1+\alpha_2-1}[13]^{\alpha_1+\alpha_3-1}[14]^{\alpha_4-1}[15]^{\alpha_5-1}[23]^{-\alpha_1}} f_1(\alpha; (wv) \circ \varphi, v \circ \varphi) \\ &= \frac{[23]^{\alpha_1+\alpha_3-1}[14]^{\alpha_1+\alpha_3-1}[14]^{\alpha_2+\alpha_5-1}[25]^{\alpha_2+\alpha_5-1}}{[13]^{\alpha_1+\alpha_3-1}[24]^{\alpha_1+\alpha_3-1}[24]^{\alpha_2+\alpha_5-1}[15]^{\alpha_2+\alpha_5-1}} f_1(\alpha; (wv) \circ \varphi, v \circ \varphi), \end{aligned}$$

there exists a constant c such that

$$
\begin{aligned}
&c\left(\frac{[23][14]}{[13][24]}\right)^{\alpha_1+\alpha_3-1}\left(\frac{[14][25]}{[24][15]}\right)^{\alpha_2+\alpha_5-1}\\
=&\frac{[23]^{\alpha_1+\alpha_3-1}[14]^{\alpha_1+\alpha_3-1}[14]^{\alpha_2+\alpha_5-1}[25]^{\alpha_2+\alpha_5-1}}{[13]^{\alpha_1+\alpha_3-1}[24]^{\alpha_1+\alpha_3-1}[24]^{\alpha_2+\alpha_5-1}[15]^{\alpha_2+\alpha_5-1}}
\end{aligned}
$$

around the point z by virtue of Lemma 3.1. Therefore we have

$$(\Psi_2/\eta)\circ s = c\cdot p(\alpha_1+\alpha_3-1;w^{-1})p(\alpha_2+\alpha_5-1;v)f_1(\alpha;wv,v).$$

Similarly, we can show that the function $(\Psi_3/\eta)\circ/\Phi_3$ is a constant. ▯

Proof of (2). Since the function Ψ is a fundamental set of solutions of the system $E_{2,5}(\alpha)$, the function F can be written as a linear combination of Ψ_i $(i=1,2,3)$. Then (2) follows from (1). ▯

We will study a symmetry of the system $A(\alpha)$ through the correspondence between $E_{2,5}(\alpha)$ and $A(\alpha)$.

Let W be the set of free words generated by s_i $(i=1,\dots,4)$.

Definition 4.1. Given a word $s\in W$, we define a function m_s inductively as follows.

$$
\begin{aligned}
m_{s_1}&=p(\alpha_4-1;x)p(\alpha_5-1;y),\\
m_{s_2}&=1,\\
m_{s_3}&=p(-\alpha_1;x),\\
m_{s_4}&=1,\\
m_{ss_i}&=m_{s_i}(m_s)^{s_i}
\end{aligned}
$$

where

$$f(\alpha;x,y)^s=f(\alpha^\sigma;w_\sigma,w_\sigma v_\sigma),\quad s=\sigma \text{ in } \mathbf{S_5}.$$

Note that the function $p(a;w)$ is undefined on $w<0$. We can see that the functions $m_{s_1}(w_\sigma,w_\sigma v_\sigma)$ and $m_{s_3}(w_\sigma,w_\sigma v_\sigma)$ have no undefined point on $\delta\in\mathcal{D}_{20}$ for all $\sigma\in\mathbf{S_5}$ by using a list of w_σ and v_σ. Then we have the following Lemma.

Lemma 4.2. *Given a word $s\in W$ and a domain $\delta\in\mathcal{D}_{20}$, the function m_s has no undefined point on δ.*

Remark. Let s and s' be words in W. The identity $s=s'$ in $\mathbf{S_5}$ does not always imply $m_s=m_{s'}$. For example, we have $s_1s_3=s_3s_1$ in $\mathbf{S_5}$. However we have

$$
\begin{aligned}
m_{s_1s_3}&=p(-\alpha_1;x)p(-\alpha_3+1;x)p(\alpha_5-1;y/x)\\
&=e^{-2\pi i(\alpha_5-1)}p(\alpha_2+\alpha_4-1;x)p(\alpha_5-1;y)
\end{aligned}
$$

and

$$\begin{aligned} m_{s_3 s_1} &= p(\alpha_4 - 1; x)p(\alpha_5 - 1; y)p(\alpha_2; x) \\ &= p(\alpha_2 + \alpha_4 - 1; x)p(\alpha_5 - 1; y) \end{aligned}$$

on

$$\operatorname{Im} 1/x > 0, \operatorname{Im} y > 0 \text{ and } \operatorname{Im} y/x < 0.$$

Conjecture

PROPOSITION 4.2. *Suppose z is a point of $M''_{2,5}$ and let Ω be a sufficiently small simply connected neighborhood of the point $\varphi(z)$. Given a word $t \in W$, we have*

$$((\eta^\tau/\eta) \circ s)/m_t = \text{constant on } \Omega$$

where $t = \tau$ in $\mathbf{S_5}$ and $s \in \mathcal{S}(\Omega, z)$.

PROPOSITION 4.3. (c.f. [AK; 55p, the method of M.Goursat]) *If a function g is a solution $A(\alpha)$, then so is the function $m_t g^\tau$ where $t \in W$ and $t = \tau$ in $\mathbf{S_5}$.*

We mention integral representations of the function $\hat{f}_0$ and induced solutions of $A(\alpha)$ from $\hat{f}_0$ by the $\mathbf{S_5}$ action.

PROPOSITION 4.4. *The function $\hat{f}_0(\alpha; x, y)$ is identically equal to 1 on the curve $\{1, 3\}$ where $x = w$ and $y = wv$.*

Put

$$\begin{array}{llll} \sigma_1 = \{1,2,3,4,5\} & \sigma_2 = \{1,3,2,4,5\} & \sigma_3 = \{2,1,3,4,5\} & \sigma_4 = \{4,1,2,3,5\} \\ \sigma_5 = \{5,1,2,4,3\} & \sigma_6 = \{4,1,3,2,5\} & \sigma_7 = \{5,1,3,4,2\} & \sigma_8 = \{1,2,4,3,5\} \\ \sigma_9 = \{1,2,5,4,3\} & \sigma_{10} = \{5,1,4,2,3\}. & & \end{array}$$

PROPOSITION 4.5. *The function $m_{t_i} \hat{f}_0^{\sigma_i}$ is a constant multiple of the function z_i in* [AK; 62p] *where $t_i \in W$ and $t_i = \sigma_i$ in $\mathbf{S_5}$.*

We remark that the ten functions above have integral representations which are given in [AK; 58p].

5. An elemental connection formula of the system of F_1

Put

$$M_{\pm}(\alpha) = \begin{pmatrix} e^{i\pi(\alpha_4 \pm \alpha_2)} - e^{-i\pi(\alpha_4 \mp \alpha_2)} & e^{i\pi(\alpha_2 \mp \alpha_4)} - e^{-i\pi(\alpha_2 \pm \alpha_4)} \\ e^{i\pi(\alpha_1+\alpha_3 \mp \alpha_5)} - e^{-i\pi(\alpha_1+\alpha_3 \pm \alpha_5)} & e^{i\pi(\alpha_5 \pm \alpha_1 \pm \alpha_3)} - e^{-i\pi(\alpha_5 \mp \alpha_1 \mp \alpha_3)} \end{pmatrix} / \xi,$$

where

$$\xi = e^{i\pi(\alpha_4+\alpha_2)} - e^{-i\pi(\alpha_4+\alpha_2)}.$$

Let $\delta \in \mathcal{D}_{20}$ be a simply connected domain of the twenty simply connected domains. When $\delta \subset U_3^+$, we put $M_\delta(\alpha) = M_+(\alpha)$ and when $\delta \subset U_3^-$, we put $M_\delta(\alpha) = M_-(\alpha)$. Notice that

$$|M_{\pm}(\alpha)| = -e^{\mp i\pi(\alpha_4+\alpha_5)}(e^{i\pi(\alpha_2+\alpha_5)} - e^{-i\pi(\alpha_2+\alpha_5)})/\xi,$$

$$(M_\delta(\alpha))^{-1} = M_{\delta^{(45)}}(\alpha^{(45)}),\ (45) \in \mathbf{S_5},$$

$$(M_{\mp}(\alpha))^{-1} = \begin{pmatrix} e^{i\pi(\alpha_5 \pm \alpha_2)} - e^{-i\pi(\alpha_5 \mp \alpha_2)} & e^{i\pi(\alpha_2 \mp \alpha_5)} - e^{-i\pi(\alpha_2 \pm \alpha_5)} \\ e^{i\pi(\alpha_1+\alpha_3 \mp \alpha_4)} - e^{-i\pi(\alpha_1+\alpha_3 \pm \alpha_4)} & e^{i\pi(\alpha_4 \pm \alpha_1 \pm \alpha_3)} - e^{-i\pi(\alpha_4 \mp \alpha_1 \mp \alpha_3)} \end{pmatrix} / \xi',$$

$$\xi' = e^{i\pi(\alpha_5+\alpha_2)} - e^{-i\pi(\alpha_5+\alpha_2)},$$

and

$$M_\delta(\alpha) = \begin{pmatrix} e^{\varepsilon\alpha_2\pi i} \frac{\sin \pi\alpha_4}{\sin \pi(-\alpha_2-\alpha_4+1)} & e^{\varepsilon(-\alpha_4+1)\pi i} \frac{\sin \pi(-\alpha_2+1)}{\sin \pi(\alpha_2+\alpha_4-1)} \\ e^{\varepsilon(-\alpha_5+1)\pi i} \frac{\sin \pi(\alpha_1+\alpha_3-1)}{\sin \pi(-\alpha_2-\alpha_4+1)} & e^{\varepsilon(\alpha_1+\alpha_3-1)\pi i} \frac{\sin \pi(-\alpha_5+1)}{\sin \pi(\alpha_2+\alpha_4-1)} \end{pmatrix}$$

where $\varepsilon = +$ when $\delta \subset U_3^+$ and $\varepsilon = -$ when $\delta \subset U_3^-$.

THEOREM 5.1. *Suppose that*

$$\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_2 + \alpha_5 \notin \mathbf{Z}.$$

We have

$$\Phi_\delta = \begin{pmatrix} 1 & 0 \\ 0 & M_\delta(\alpha) \end{pmatrix} \Phi'_\delta.$$

In order to prove Theorem 5.1, we need a connection formula of the Gauss hypergeometric function. Put

$$\hat{F}\begin{pmatrix} a & b \\ c & \end{pmatrix}; v\Big) = \sum_{n=0}^{\infty} \frac{v^n}{\Gamma(1+n)\Gamma(c+n)\Gamma(1-a-n)\Gamma(1-b-n)}.$$

Lemma 5.1. (Connection formula of the Gauss hypergeometric function, see, e.g. [IKSY; Chapter 2])

$$\hat{F}\begin{pmatrix} a & b \\ c & ;v \end{pmatrix} = e^{\pm a\pi i}\frac{\sin \pi b}{\sin \pi(b-a)}p(-a;v)\hat{F}\begin{pmatrix} a & 1+a-c \\ 1+a-b & ;1/v \end{pmatrix}$$
$$+ e^{\pm b\pi i}\frac{\sin \pi a}{\sin \pi(a-b)}p(-b;v)\hat{F}\begin{pmatrix} b & 1+b-c \\ 1+b-a & ;1/v \end{pmatrix},$$

$$p(1-c;v)\hat{F}\begin{pmatrix} a+1-c & b+1-c \\ 2-c & ;v \end{pmatrix}$$
$$= e^{\pm(a-c+1)\pi i}\frac{\sin \pi(c-b)}{\sin \pi(b-a)}p(-a;v)\hat{F}\begin{pmatrix} a & 1+a-c \\ 1+a-b & ;1/v \end{pmatrix}$$
$$+ e^{\pm(b-c+1)\pi i}\frac{\sin \pi(c-a)}{\sin \pi(a-b)}p(-b;v)\hat{F}\begin{pmatrix} b & 1+b-c \\ 1+b-a & ;1/v \end{pmatrix}.$$

where $\pm\mathrm{Im}\, v > 0$ *and* $a-b, c \notin \mathbf{Z}$.

Proof of Theorem 5.1. Putting $a = \alpha_1 + \alpha_3 - 1$, $b = 1 - \alpha_5$ and $c = 2 - \alpha_2 - \alpha_5$, we apply Lemma 5.1 to the functions $g_1(\alpha; 0, v)$ and $p(\alpha_2 + \alpha_5 - 1; v)f_1(\alpha; 0, v)$. Then we have

$$\begin{pmatrix} p(\alpha_2+\alpha_5-1;v)f_1(\alpha;0,v) \\ g_1(\alpha;0,v) \end{pmatrix} = M_\delta(\alpha)\begin{pmatrix} p(1-\alpha_5;1/v)f_2(\alpha;0,v) \\ p(\alpha_1+\alpha_3-1;1/v)g_2(\alpha;0,v) \end{pmatrix} \tag{5.a1}$$

where $\mathrm{Im}\, v > 0 \Leftrightarrow \delta \subset U_3^+$ and $\mathrm{Im}\, v < 0 \Leftrightarrow \delta \subset U_3^-$. Let f be a solution of the system $A(\alpha)$. Changing the variables x to w and y to wv in (4.1) and (4.2), we have

$$[(\theta_w - \theta_v)(\theta_w + \alpha_1 + \alpha_3 - 1) - w(\theta_w + \alpha_1)(\theta_w - \theta_v - \alpha_4 + 1)]f = 0 \tag{5.a2}$$
$$[\theta_v(\theta_w + \alpha_1 + \alpha_3 - 1) - wv(\theta_w + \alpha_1)(\theta_v - \alpha_5 + 1)]f = 0. \tag{5.a3}$$

Adding the two equations above, we obtain $\ell f = 0$ where

$$\begin{aligned} \ell = {} & \theta_w(\theta_w + \alpha_1 + \alpha_3 - 1) \\ & - w\left((\theta_w + \alpha_1)(\theta_w - \theta_v - \alpha_4 + 1) + v(\theta_w + \alpha_1)(\theta_v - \alpha_5 + 1)\right). \end{aligned} \tag{5.a5}$$

Put

$$\begin{aligned} h = {} & p(-1 + \alpha_2 + \alpha_5; v)f_1(\alpha; wv, v) \\ & - M_\delta(\alpha)_{11}p(-1 + \alpha_5; v)f_2(\alpha; w, 1/v) \\ & - M_\delta(\alpha)_{12}p(1 - \alpha_1 - \alpha_3; v)g_2(\alpha; wv, 1/v) \end{aligned}$$

where $M_\delta(\alpha)_{ij}$ is the (i,j)-th element of the matrix $M_\delta(\alpha)$.

The function $h(w, v)$ is holomorphic function at $(w, v) = (0, a)$, $a \notin \mathbf{R}$. We have

$$\ell w^{-\alpha_1-\alpha_3+1}h = 0$$

and $h(0,v)\equiv 0$ from (5.a1). Put

$$h(w,v)=\sum_{k=0}^{\infty} h_k(v)w^k.$$

Then the function $h_k(v)$ satisfies

$$\begin{aligned}&(k+1-\alpha_1-\alpha_3+1)(k+1)h_{k+1}(v)\\&\quad-\left((k-\alpha_3+1)(k-\theta_v-\alpha_1-\alpha_3-\alpha_4)+v(k-\alpha_3+1)(\theta_v-\alpha_5+1)\right)h_k(v)=0.\end{aligned}$$

Since $h_0(v)\equiv 0$, we have $h_k(v)\equiv 0$, which shows $h\equiv 0$ and

$$p(1-\alpha_1-\alpha_3;w)h=\Phi_2-M_\delta(\alpha)_{11}\Phi'_2-M_\delta(\alpha)_{12}\Phi'_3=0.$$

Similarly, we can show

$$\Phi_3=M_\delta(\alpha)_{21}\Phi'_2+M_\delta(\alpha)_{22}\Phi'_3. \quad \square$$

In Section one, we defined the action of $\mathbf{S_5}$ on the open variety $\pi^{-1}(X')$. The permutation group $\mathbf{S_5}$ induces 120 biholomorphic transformations on $\pi^{-1}(X')$. The group $\mathbf{S_5}$ acts on simply connected domain $\delta\in\mathcal{D}_{20}$. Given a permutation $\sigma\in\mathbf{S_5}$ and a domain $\delta\in\mathcal{D}_{20}$, it follows from Theorem 1.1 (1) (3) that there exists a domain β that satisfies $\delta=T_\sigma(\beta)$. The domain β is denoted by δ^σ. Put

$$\begin{aligned}\Phi_{s,\delta^\sigma}&=m_s\Phi_\delta(\alpha^\sigma;w_\sigma,v_\sigma)\\ \Phi'_{s,\delta^\sigma}&=m_s\Phi'_\delta(\alpha^\sigma;w'_\sigma,v'_\sigma)\end{aligned}$$

where $s\in W$ and $s=\sigma$ in $\mathbf{S_5}$.

We have the following connection formula.

THEOREM 5.2. *We have*

$$\Phi_{s,\delta^\sigma}=\begin{pmatrix}1 & 0\\ 0 & (M_\delta(\alpha))^\sigma\end{pmatrix}\Phi'_{s,\delta^\sigma}$$

on the domain δ^σ where $s\in W$ and $s=\sigma$ in $\mathbf{S_5}$.

6. A connection formula for the system of hypergeometric equations $E_{2,5}$

We can derive a linear relation between the functions Ψ and Ψ' by utilizing Theorem 5.1. To show the linear relation, we need to define the following functions c_1, c_2 and c_3.

DEFINITION 6.1.

$$c_1(\mu; a) = \begin{cases} e^{\pi i \mu} & \text{when} \quad \operatorname{Im} a > 0 \\ e^{-\pi i \mu} & \text{when} \quad \operatorname{Im} a < 0, \end{cases}$$

$$c_2(\mu; a, b, c) = \begin{cases} e^{-2\pi i \mu} & \text{when } \operatorname{Im} a > 0, \operatorname{Im} b > 0, \operatorname{Im} c < 0 \\ e^{2\pi i \mu} & \text{when } \operatorname{Im} a < 0, \operatorname{Im} b < 0, \operatorname{Im} c > 0 \\ 1 & \text{in other cases,} \end{cases}$$

$$c_3\begin{pmatrix} \mu; & [ij] & [k\ell] \\ & [kj] & [i\ell] \end{pmatrix} = \left(\frac{[ij][k\ell]}{[kj][i\ell]}\right)^{\alpha} [ij]^{-\alpha}[k\ell]^{-\alpha}[kj]^{\alpha}[i\ell]^{\alpha}.$$

Note that

$$\begin{aligned} x^{\mu} &= c_1(\mu; x)(-x)^{\mu}, \\ (xy)^{\mu} &= c_2(\mu; x, y, xy)x^{\mu}y^{\mu}, \end{aligned}$$

$$c_3\left(\mu; \begin{matrix} [ij] & [k\ell] \\ [kj] & [i\ell] \end{matrix}\right) = c_2\left(\mu; \frac{[ij]}{[kj]}, \frac{[k\ell]}{[i\ell]}, \frac{[ij][k\ell]}{[kj][i\ell]}\right) c_2\left(\mu; [ij], \frac{1}{[kj]}, \frac{[ij]}{[kj]}\right) \cdot c_2\left(\mu; [k\ell], \frac{1}{[i\ell]}, \frac{[k\ell]}{[i\ell]}\right).$$

THEOREM 6.1. *Suppose $p \in M''_{2,5}$ and let U be a sufficiently small simply connected neighborhood of the point p. We have*

$$\Psi(\alpha; z) = M(p, \alpha)\Psi'(\alpha; z) \text{ on } U$$

where

$$M(p, \alpha) = D^{-1}\begin{pmatrix} 1 & 0 \\ 0 & M_{\delta}(\alpha) \end{pmatrix} D', \quad \varphi(p) \in \delta \in \mathcal{D}_{20}.$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_2 \end{pmatrix}, \quad D' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d'_1 & 0 \\ 0 & 0 & d'_2 \end{pmatrix}$$

$$d_1 = c_3\left(\alpha_2 + \alpha_5 - 1; \begin{matrix} [25] & [14] \\ [15] & [24] \end{matrix}\right)$$

$$d_2 = 1$$

$$d'_1 = c_3\left(\alpha_5 - 1; \begin{matrix} [25] & [14] \\ [15] & [24] \end{matrix}\right)$$

$$d'_2 = c_3\left(1 - \alpha_1 - \alpha_3; \begin{matrix} [25] & [14] \\ [15] & [24] \end{matrix}\right)$$

and

$$[ij] = [ij](p), \text{ i.e. } [ij] = \begin{vmatrix} p_{11} & p_{13} \\ p_{21} & p_{23} \end{vmatrix}, \ldots, \ p = (p_{ij}) \in M''_{2,5}.$$

Proof. The i-th rows of the functions Φ, Φ', Ψ and Ψ' are denoted by Φ_i, Φ'_i, Ψ_i and Ψ'_i respectively. It follows from the definition of the function Ψ, we have

$$\eta\Phi_1 = \Psi_1 \text{ and } \eta\Phi'_1 = \Psi'_1.$$

We will show

$$\eta\Phi_2 = c_4 d_1 \Psi_2 \text{ and } \eta\Phi'_2 = c_4 d'_1 \Psi'_2 \tag{6.a1}$$

where

$$c_4 = c_3\left(1 - \alpha_1 - \alpha_3; \begin{matrix} [24] & [13] \\ [14] & [23] \end{matrix}\right).$$

It follows from Lemma 3.1 that we have

$$\begin{aligned} &\eta\Phi_2 \\ =&\eta p(1 - \alpha_1 - \alpha_3; w)p(\alpha_2 + \alpha_5 - 1; v)f_1 \\ =&\eta\left(\frac{[24][13]}{[14][23]}\right)^{1-\alpha_1-\alpha_3}\left(\frac{[25][14]}{[15][24]}\right)^{\alpha_2+\alpha_5-1} f_1 \\ =&\eta c_3\left(1 - \alpha_1 - \alpha_3; \begin{matrix} [24] & [13] \\ [14] & [23] \end{matrix}\right) c_3\left(\alpha_2 + \alpha_5 - 1; \begin{matrix} [25] & [14] \\ [15] & [24] \end{matrix}\right) \\ &\times [24]^{1-\alpha_1-\alpha_3}[14]^{\alpha_1+\alpha_3-1}[13]^{1-\alpha_1-\alpha_3}[23]^{\alpha_1+\alpha_3-1} \\ &\times [25]^{\alpha_2+\alpha_5-1}[15]^{-\alpha_2-\alpha_5+1}[14]^{\alpha_2+\alpha_5-1}[24]^{-\alpha_2-\alpha_5+1} f_1. \end{aligned}$$

and

$$\begin{aligned} &\eta\Phi'_2 \\ =&\eta p(1 - \alpha_1 - \alpha_3; w)p(\alpha_5 - 1; v)f_2 \\ =&c_3\left(1 - \alpha_1 - \alpha_3; \begin{matrix} [24] & [13] \\ [14] & [23] \end{matrix}\right) c_3\left(\alpha_5 - 1; \begin{matrix} [25] & [14] \\ [15] & [24] \end{matrix}\right) \Psi'_2, \end{aligned}$$

which yields (6.a1).

Similary, we can show that

$$\eta\Phi_3 = c_5 d_2 \Psi_3 \text{ and } \eta\Phi'_3 = c_5 d'_2 \Psi'_3$$

where

$$c_5 = c_3\left(1 - \alpha_1 - \alpha_3; \begin{matrix} [24] & [13] \\ [14] & [23] \end{matrix}\right). \quad \square$$

7. Local connection matrices for the system of hypergeometric equations $E_{2,5}$

Kummer's relation for the Gauss hypergeometric function

$$F\begin{pmatrix} a & b \\ c & \end{pmatrix}; x\Big) = (1-x)^{-b} F\begin{pmatrix} c-a & b \\ c & \end{pmatrix}; \frac{x}{x-1}\Big) \tag{7.a1}$$

is well known. In this section, we give similar formulas for solutions of the system $E_{2,5}(\alpha)$.

THEOREM 7.1. *The isotropy group I of the point $\{1,3\} \cap \{2,5\}$ by the action of $\mathbf{S_5}$ on Z is generated by*

$$\tau_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix} \text{ and } \tau_2 = (13).$$

We have $(\tau_1)^2 = (\tau_2)^2 = 1$ and

$$I = \{id, \tau_1, \tau_2, \tau_1\tau_2, \tau_2\tau_1, \tau_1\tau_2\tau_1, \tau_2\tau_1\tau_2, (\tau_1\tau_2)^2 = (\tau_2\tau_1)^2\}.$$

REMARK. The group I is the Weyl group of the root system B_2.

Suppose that $\alpha_1+\alpha_3, \alpha_2+\alpha_5 \notin \mathbf{Z}$ and $p \in M''_{2,5}$. Let U be a sufficiently small simply connected neighborhood of the point p. Given an element $\tau \in I$, there exists an unique matrix $N(\tau, p, \alpha) \in GL(3, \mathbf{C})$ such that

$$\Psi^\tau = N(\tau, p, \alpha)\Psi \quad \text{on } U,$$

because Ψ is a fundamental set of solutions of the system $E_{2,5}(\alpha)$.

THEOREM 7.2. *Given elements $\sigma, \tau \in I$, we have*

$$N(\sigma\tau, p, \alpha) = (N(\sigma, q, \alpha))^\tau N(\tau, p, \alpha) \tag{7.a2}$$

where $q = p^\tau$.

Proof. We have

$$\Psi^\sigma = N(\sigma, q, \alpha)\Psi$$

around the point q from the definition. Acting τ on the both sides, we have

$$\Psi^{\sigma\tau} = (N(\sigma, q, \alpha))^\tau \, \Psi^\tau$$

on U. Since the domain U is simply connected and we have

$$\Psi^\tau = N(\tau, p, \alpha)\Psi \quad \text{on } U,$$

we obtain

$$\Psi^{\sigma\tau} = (N(\sigma, q, \alpha))^{\tau} N(\tau, p, \alpha)\Psi \quad \text{on } U. \ \Box$$

REMARK. We call the condition (7.a2) *pseudo-cocycle condition.* The system of the matrices (7.a30) and (7.a40) given below is a solution of the pseudo-cocycle condition (7.a2).

We will derive explicit formulas of $N(\sigma, p, \alpha)$, $\sigma \in I$. It follows from Theorems 7.1 and 7.2 that $N(\sigma, p, \alpha)$ can be expressed in terms of $N(\tau_1, q, \alpha)$ and $N(\tau_2, r, \alpha)$. Hence, it is sufficient for our purpose to derive explicit formulas of $N(\tau_1, q, \alpha)$ and $N(\tau_2, r, \alpha)$.

THEOREM 7.3. *Suppose $p \in M''_{2,5}$. Let U be a sufficiently small simply connected neighborhood of the point p. We have*

$$\Psi^{\tau_1} = N(\tau_1, p, \alpha)\Psi,$$

$$\Psi^{\tau_2} = N(\tau_2, p, \alpha)\Psi \quad \text{on } U$$

where

(7.a30)
$$N(\tau_1, p, \alpha) = \begin{pmatrix} 0 & a_1 & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & a_3 \end{pmatrix},$$

$$a_1 = c_1(\alpha_2 + \alpha_1 - 1; -[12]),$$
$$a_2 = c_1(\alpha_2 + \alpha_1 - 1; -[12]),$$
$$a_3 = c_1(\alpha_2 + \alpha_1 - 1; -[12]),$$

(7.a40)
$$N(\tau_2, p, \alpha) = \frac{\Gamma(\alpha_3)}{\Gamma(a_1)} \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix},$$

$$b_1' = \frac{c_1(\alpha_1 + \alpha_3 - 1; [31])c_1(-\alpha_3; [21])}{c_1(-\alpha_1; [23])c_3\left(\alpha_4 - 1; \begin{matrix} [34] & [12] \\ [32] & [14] \end{matrix}\right) c_3\left(\alpha_5 - 1; \begin{matrix} [35] & [12] \\ [32] & [15] \end{matrix}\right)},$$

$$b_1 = b_1' \frac{e^{\pi i \alpha_3} - e^{-\pi i \alpha_3}}{e^{\pi i \alpha_1} - e^{-\pi i \alpha_1}},$$

$$b_2 = \frac{c_1(\alpha_1 - 1; [21])}{c_1(\alpha_3 - 1; [23])c_3\left(\alpha_2; \begin{matrix} [15] & [32] \\ [12] & [35] \end{matrix}\right)},$$

$$b_3 = \frac{c_1(\alpha_3 - 1; [32])}{c_1(\alpha_1 - 1; [12])c_3\left(-\alpha_2 - \alpha_5 + 1; \begin{matrix} [34] & [12] \\ [32] & [14] \end{matrix}\right) c_3\left(\alpha_5 - 1; \begin{matrix} [35] & [12] \\ [32] & [15] \end{matrix}\right)}.$$

In order to prove the theorem, we need lemmas and propositions. Put

$$\eta_1 = [12]^{\alpha_1+\alpha_2-1}[13]^{\alpha_1+\alpha_3-1}[14]^{\alpha_4-1}[15]^{\alpha_5-1}[23]^{-\alpha_1},$$
$$\eta_2 = [12]^{\alpha_1+\alpha_2-1}[15]^{-\alpha_2}[23]^{\alpha_3-1}[24]^{\alpha_4-1}[25]^{\alpha_2+\alpha_5-1},$$
$$\eta_3 = [12]^{\alpha_1+\alpha_2-1}[14]^{-\alpha_2-\alpha_5+1}[15]^{\alpha_5-1}[23]^{\alpha_3-1}[24]^{-\alpha_1-\alpha_3+1}.$$

Then we have the following lemma.

LEMMA 7.1. *Suppose that* $z \in M''_{2,5}$. *Then*

$$\frac{\eta_1^{(13)}}{\eta_1} = b_1' \left[(1-w)^{\alpha_4-1}(1-wv)^{\alpha_5-1}\right] \circ \varphi,$$
$$\frac{\eta_2^{(13)}}{\eta_2} = b_2 \left[(1-wv)^{-\alpha_2}\right] \circ \varphi,$$
$$\frac{\eta_3^{(13)}}{\eta_3} = b_3 \left[(1-w)^{-\alpha_2-\alpha_5+1}(1-wv)^{\alpha_5-1}\right] \circ \varphi.$$

Proof. The lemma can be proved by the following computation.

$$\begin{aligned}
\frac{\eta_1^{(13)}}{\eta_1} &= \frac{c_1(\alpha_1+\alpha_3-1;[31])c_1(-\alpha_3;[21])}{c_1(-\alpha_1;[23])} \\
&\quad \times [32]^{\alpha_1+\alpha_2+\alpha_3-1}[34]^{\alpha_4-1}[35]^{\alpha_5-1} \\
&\quad \times [12]^{-\alpha_3-\alpha_1-\alpha_2+1}[14]^{-\alpha_4+1}[15]^{-\alpha_5+1} \\
&= \frac{c_1(\alpha_1+\alpha_3-1;[31])c_1(-\alpha_3;[21])}{c_1(-\alpha_1;[23])} \\
&\quad \times [32]^{-\alpha_4+1}[34]^{\alpha_4-1}[12]^{\alpha_4-1}[14]^{-\alpha_4+1} \\
&\quad \times [32]^{-\alpha_5+1}[35]^{\alpha_5-1}[12]^{\alpha_5-1}[15]^{-\alpha_5+1} \\
&= b_1' \left(\frac{[34][12]}{[32][14]}\right)^{\alpha_4-1} \left(\frac{[35][12]}{[32][15]}\right)^{\alpha_5-1}.
\end{aligned}$$

Thus, we obtain the first formula. The other formulas can be proved in a similar way. ▯

In Section 4, we show that the functions f_i and g_i defined around the point $(w,v) = (0,0) \in Z$ have the unique analytic continuations to the domain $U_3^{\sigma} \cap U_4^{\sigma'} \cap U_5^{\sigma''}$ $(\sigma, \sigma', \sigma'' = \pm)$. We also denote the analytic continuations by f_i and g_i, i.e., the functions f_i and g_i are holomorphic functions defined on the domain $\bigsqcup_{D\in\mathcal{D}_8} D$, which is a disjoint sum of 8 simply connected domains, and have powerseries expansions given in (4.a90) around the point $w = v = 0$. Here,

$$\mathcal{D}_8 = \{U_3^{\sigma} \cap U_4^{\sigma'} \cap U_5^{\sigma''} \mid \sigma, \sigma', \sigma'' = \pm\}.$$

PROPOSITION 7.1. *Let D, D' be elements of $\mathcal{D}_8$. Then the domain*

$$D \cap T_{\tau_i}(D')$$

is empty or connected. If the domain is not empty, then the domain

$$D_0(\varepsilon) \cap D \cap T_{\tau_i}(D')$$

is not empty for any positive number ε where

$$D_0(\varepsilon) = \{(w,v) \,|\, |w|, |v| < \varepsilon\} \subset Z.$$

LEMMA 7.2. *Let f and m be holomorphic functions on $\bigsqcup_{D \in \mathcal{D}_8} D$. If*

$$m \cdot (f \circ T_{\tau_i}) = f \quad \text{on } T_{\tau_i}^{-1}(D') \cap D \cap D_0(\varepsilon) \neq \emptyset, \ 0 < \varepsilon << 1,$$

then

$$m \cdot (f \circ T_{\tau_i}) = f \quad \text{on } T_{\tau_i}^{-1}(D') \cap D.$$

Proof. The domain $T_{\tau_i}^{-1}(D') \cap D$ is connected from Proposition 7.1. Since the functions f and $m \cdot (f \circ T_{\tau_i})$ are holomorphic in the domain $T_{\tau_i}^{-1}(D') \cap D$, then we have the conclusion. ▯

In order to prove Theorem 7.3, we need to find operators, which are elements of the left ideal generated by (4.1), (4.2) and (4.3), of the forms

$$\theta_w(\theta_w + e_1) - wp_1(w, v, \theta_w, \theta_v) \tag{7.a2}$$

$$\theta_v(\theta_v + e_2) - vp_2(w, v, \theta_w, \theta_v), \tag{7.a3}$$

where $e_1, e_2 \in \mathbf{C}$ and p_i $(i = 1, 2)$ are polynomials in w, v, θ_w and θ_v.

The operator of the form (7.a2) is given in (5.a5). Let us find the operator of the form (7.a3). Multiplying (4.3) by wv and changing the variables x to w and y to wv, we have

$$\theta_v(\theta_w - \theta_v - \alpha_4 + 1) + v(-\theta_w\theta_v + \theta_v^2 - (-\alpha_5 + 1)\theta_w + (-\alpha_5 + 1)\theta_v). \tag{7.a4}$$

Substituting (7.a4) from (5.a3), we obtain

$$\theta_v(\theta_v - \alpha_2 - \alpha_5 + 1) + v[-w(\theta_w + \alpha_1)(\theta_v - \alpha_5 + 1) + \theta_w\theta_v - \theta_v + (-\alpha_5 + 1)\theta_w - (-\alpha_5 + 1)\theta_v] =: \ell'. \tag{7.a5}$$

LEMMA 7.3. *Let h be a holomorphic function at $(w,v)=(0,0)$. We suppose that $h(0,0)=0$ and $\alpha_1+\alpha_3, \alpha_2+\alpha_5 \notin \mathbf{Z}$. Then we have*
(1) *If $\ell h = \ell' h = 0$, then $h=0$.*
(2) *If $\ell w^{1-\alpha_1-\alpha_3} v^{\alpha_2+\alpha_5-1} h = \ell' w^{1-\alpha_1-\alpha_3} v^{\alpha_2+\alpha_5-1} h = 0$, then $h=0$.*
(3) *If $\ell w^{1-\alpha_1-\alpha_3} h = \ell' w^{1-\alpha_1-\alpha_3} h = 0$, then $h=0$.*

Proof. Put

$$h(w,v) = \sum_{k=0}^{\infty} h_k(v) w^k.$$

Since $\ell h = 0$, we have

$$\begin{aligned}(k+1)(k+1+\alpha_1+\alpha_3-1)h_{k+1}(v)\\ -\left((k+\alpha_1)(k-\theta_v-\alpha_4+1)+v(k+\alpha_1)(\theta_v-\alpha_5+1)\right)h_k(v)=0.\end{aligned}$$

Therefore if $h_0(v)=0$, then $h_k(v)=0$. It follows from (7.a5) that the function $h_0(v)$ satisfies

$$\left[\theta_v(\theta_v-\alpha_2-\alpha_5+1)+v\{-\theta_v-(-\alpha_5+1)\theta_v\}\right]h_0=0.$$

Since $h_0(0)=h(0,0)=0$, we have $h_0(v)=0$. We have completed the proof of (1).
Similarly, we can show (2) and (3). □

LEMMA 7.4. *Put*

$$\begin{aligned}
h_0 &= (1-w)^{\alpha_4-1}(1-wv)^{\alpha_5-1} f_0(\alpha^{(13)}; \frac{w}{w-1}, \frac{wv}{wv-1}),\\
h_1 &= (1-wv)^{-\alpha_2} f_1(\alpha^{(13)}; \frac{wv}{wv-1}, \frac{(1-w)v}{1-wv}),\\
h_2 &= (1-w)^{-\alpha_2-\alpha_5+1}(1-wv)^{\alpha_5-1} g_1(\alpha^{(13)}; \frac{w}{w-1}, \frac{(1-w)v}{1-wv}).
\end{aligned}$$

Then the functions h_i are holomorphic at $(w,v)=(0,0)$ and

$$\begin{aligned}
h_0(0,0) &= 1/(\Gamma(\alpha_1+\alpha_3)\Gamma(1-\alpha_3)\Gamma(\alpha_4)\Gamma(\alpha_5)),\\
h_1(0,0) &= 1/(\Gamma(\alpha_2+\alpha_5)\Gamma(1-\alpha_2)\Gamma(\alpha_1)\Gamma(\alpha_4)),\\
h_2(0,0) &= 1/(\Gamma(2-\alpha_1-\alpha_3)\Gamma(1-\alpha_2)\Gamma(\alpha_1)\Gamma(\alpha_5)).
\end{aligned}$$

Moreover, the functions h_i satisfies

$$\begin{aligned}
&\ell h_0 = \ell' h_0 = 0,\\
&\ell w^{1-\alpha_1-\alpha_3} v^{\alpha_2+\alpha_5-1} h_1 = \ell' w^{1-\alpha_1-\alpha_3} v^{\alpha_2+\alpha_5-1} h_1 = 0,\\
&\ell w^{1-\alpha_1-\alpha_3} h_2 = \ell w^{1-\alpha_1-\alpha_3} h_2 = 0.
\end{aligned}$$

Proposition 7.2. *Let D and D' be elements of $\mathcal{D}_8$. If the domain $T_{\tau_2}^{-1}(D') \cap D$ is not empty, then the identies*

$$h_0 = f_0(\alpha; w, wv)\frac{\Gamma(\alpha_3)(e^{i\pi\alpha_3} - e^{-i\pi\alpha_3})}{\Gamma(\alpha_1)(e^{i\pi\alpha_1} - e^{-i\pi\alpha_1})},$$
$$h_1 = f_1(\alpha; wv, v)\frac{\Gamma(\alpha_3)}{\Gamma(\alpha_1)},$$
$$h_2 = g_1(\alpha; w, v)\frac{\Gamma(\alpha_3)}{\Gamma(\alpha_1)}$$

hold on the domain.

Proof. Let us show the first identity. Put

$$c_0 = \frac{\Gamma(\alpha_3)(e^{i\pi\alpha_3} - e^{-i\pi\alpha_3})}{\Gamma(\alpha_1)(e^{i\pi\alpha_1} - e^{-i\pi\alpha_1})}.$$

Since $\ell(h_0 - c_0 f_0) = \ell'(h_0 - c_0 f_0) = 0$ and $h_0 - c_0 f_0$ is holomorphic and equal to 0 at the point $(w, v) = (0, 0)$, we have

$$h_0 - c_0 f_0 = 0$$

on the domain $D_0(\varepsilon)$ $(0 < \varepsilon << 1)$ from Lemma 7.3. Since

$$h_0 = (1-w)^{\alpha_4 - 1}(1-wv)^{\alpha_5 - 1}\left(f_0(\alpha^{(13)}; \cdot, \cdot) \circ T_{\tau_2}\right)(w, v),$$

we have

$$h_0 - c_0 f_0 = 0$$

on the domain $T_{\tau_2}^{-1}(D') \cap D$ from Lemma 7.2. Other identities can be proved in a similar way. ▯

Remark. We have

$$\begin{aligned} &p(\alpha_4 - 1; 1-w)p(\alpha_5 - 1; 1-wv)\hat{f}_0(\alpha_3, \alpha_2, \alpha_1, \alpha_4, \alpha_5; \frac{w}{w-1}, \frac{wv}{wv-1}) \\ =&\hat{f}_0(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5; w, wv) \end{aligned}$$

from Proposition 7.2. The formula above is known (see, for example, [Mill1] or [Ue1; §3]). Putting $v = 0$, we have

$$p(\alpha_4 - 1; 1-w)\hat{f}_0(\alpha_3, \alpha_2, \alpha_1, \alpha_4, \alpha_5; \frac{w}{w-1}, 0) = \hat{f}_0(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5; w, 0),$$

which is equivalent to (7.a1).

Proof of Theorem 7.3. We can easily show (7.a30) from the expression of Ψ^{τ_1}. Let us show (7.a40). We have

$$b_1'(h_0 \circ \varphi) = \frac{\eta_1^{(13)}}{\eta_1} \Psi_1^{(13)}$$

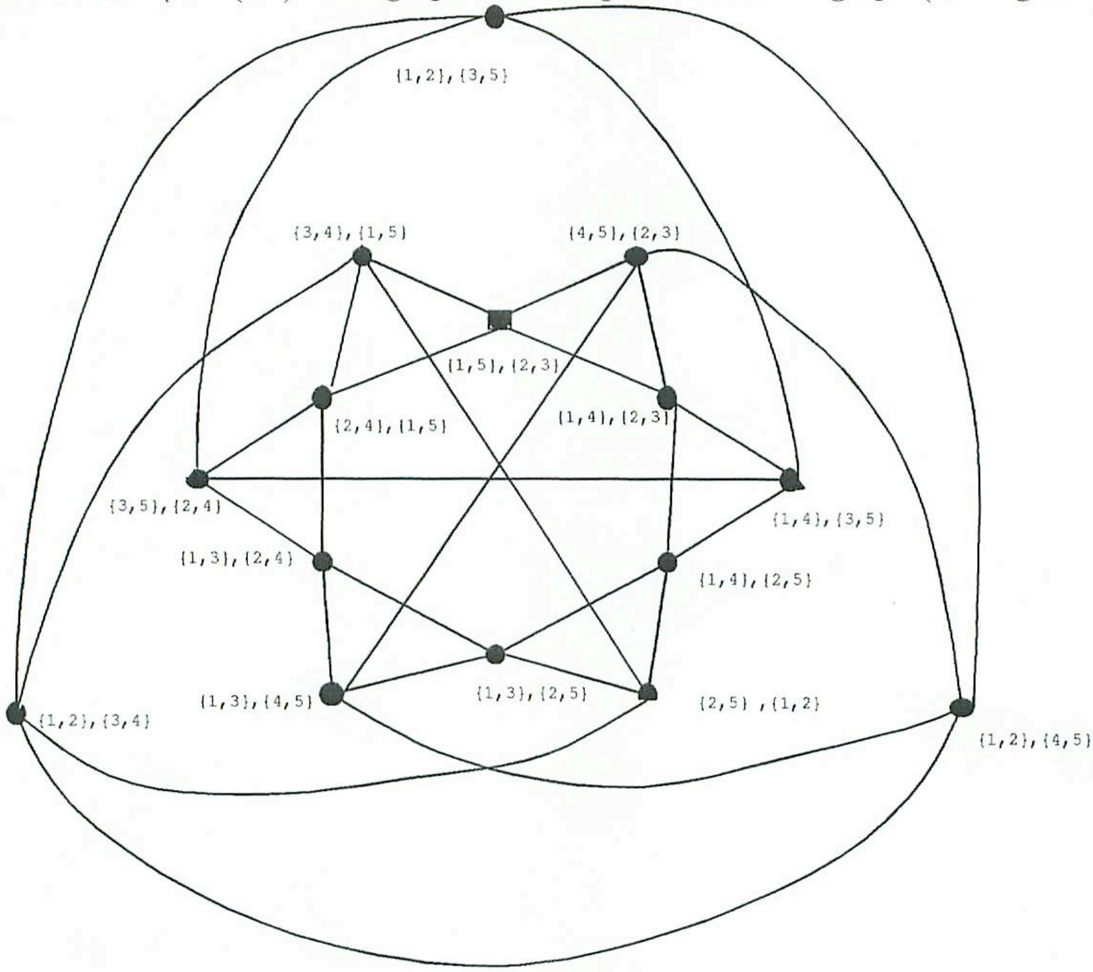
from Lemma 7.1. Hence, we have

$$\frac{\eta_1^{(13)}}{\eta_1} \Psi_1^{(13)} = b_1' \Psi_1 \frac{\Gamma(\alpha_3)(e^{i\pi\alpha_3} - e^{-i\pi\alpha_3})}{\Gamma(\alpha_1)(e^{i\pi\alpha_1} - e^{-i\pi\alpha_1})}$$

from Proposition 7.2. Similarly, we can derive the other elements of the matrix $N(\tau_2, p, \alpha)$. □

8. The connection matrix between Ψ and Ψ^σ

In section 1, we constructed a blowing up space Z of the projective space $\mathbf{P}^2$. We construct a connected graph G from the blowing up space as follows. The space $Z\setminus\pi^{-1}(X')$ consists of ten irreducible curves which have 15 normally crossing points. The vertices of the graph G correspond to the 15 normally crossing points. We use the naming of the normally crossing points to name the vertices, i.e. we name each of the vertices $\{i,j\}\cap\{k,\ell\}$ (or $\{i,j\},\{k,\ell\}$). Two vertices are connected if and only if the corresponding two normally crossing points are on a curve of 10 curves, i.e. the two points are on an irreducible component of $Z\setminus\pi^{-1}(X')$. The graph G is a regular connected graph (see Figure f.8.1).



The graph G

Figure f.8.1

Given an element σ of $\mathbf{S_5}$, put

$$p = \{1^\sigma, 3^\sigma\}\cap\{2^\sigma, 4^\sigma\}$$

and

$$p_1 = \{1,3\}\cap\{2,5\}.$$

Since the graph G is connected, there exists a walk from the vertex p_1 to the vertex p. The vertices on the walk are denoted by $p_1, \ldots, p_m, p_{m+1} = p$. It follows from the naming of the vertices that there exist permutations $\sigma_k \in \mathbf{S_5}$ that satisfy the condition

$$\begin{aligned} p_k &= \{1^{\sigma_k}, 3^{\sigma_k}\} \cap \{2^{\sigma_k}, 5^{\sigma_k}\} \\ &= \{1^{\sigma_{k-1}}, 3^{\sigma_{k-1}}\} \cap \{2^{\sigma_{k-1}}, 4^{\sigma_{k-1}}\} \end{aligned}$$

for $k = 1, \ldots, m+1$.

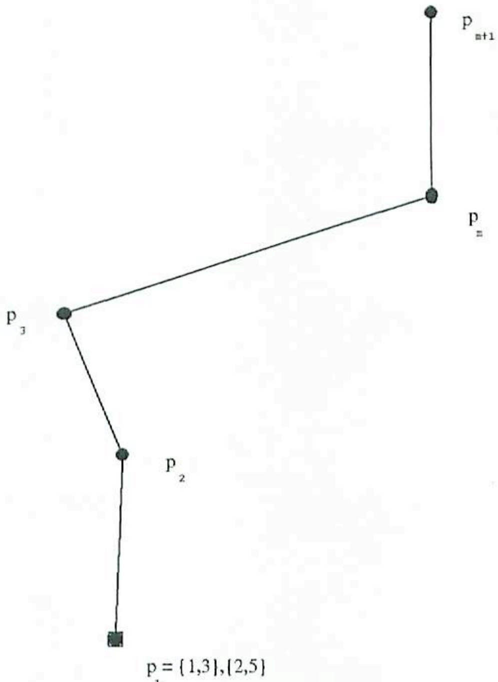


Figure f.8.2

LEMMA 8.1.

$$(45)\sigma_k(\sigma_{k+1})^{-1} \in I$$

where I is the isotropy group given in Theorem 7.2.

Put $\tau_k = (45)\sigma_k(\sigma_{k+1})^{-1}$. Let $q \in M'_{2,5}$ be a non-split point and U be a sufficiently small simply connected neighborhood of the point q. We have

$$\Psi = M(q^{\sigma_k}, \alpha)\Psi^{(45)} \tag{8.c1}$$

on $S_{\sigma_k}(U)$ where the matrix $M(q^{\sigma_k}, \alpha)$ is explicitly given in Theorem 6.1. Acting σ_k on the both sides of (8.c1), we obtain

$$\Psi^{\sigma_k} = (M(q^{\sigma_k}, \alpha))^{\sigma_k} \Psi^{(45)\sigma_k} \tag{8.c2}$$

on U. We have

$$\Psi^{\tau_k} = N(\tau_k, q^{\sigma_{k+1}}, \alpha)\Psi \tag{8.c3}$$

on $S_{\sigma_{k+1}}(U)$. The explicit expression of $N(\tau_k, q^{\sigma_{k+1}}, \alpha)$ is given in Theorem 7.3 and 7.4. Acting σ_{k+1} on the both sides of (8.c3), we obtain

$$\Psi^{\tau_k \sigma_{k+1}} = \Psi^{(45)\sigma_k} = (N(\tau_k, q^{\sigma_{k+1}}, \alpha))^{\sigma_{k+1}} \Psi^{\sigma_{k+1}} \quad \text{on } U. \tag{8.c4}$$

Therefore we have

$$\begin{aligned}
\Psi^{\sigma_1} =& (M(q^{\sigma_1}, \alpha))^{\sigma_1} (N(\tau_1, q^{\sigma_2}, \alpha))^{\sigma_2} \\
& \times (M(q^{\sigma_2}, \alpha))^{\sigma_2} (N(\tau_2, q^{\sigma_3}, \alpha))^{\sigma_3} \\
& \times \cdots \\
& \times (M(q^{\sigma_m}, \alpha))^{\sigma_m} (N(\tau_m, q^{\sigma_{m+1}}, \alpha))^{\sigma_{m+1}} \Psi^{\sigma_{m+1}} \qquad \text{on } U.
\end{aligned}$$

Since $\sigma_1 \in I$ and $\sigma\sigma_{m+1} = \tau_{m+1} \in I$, we have

$$\begin{aligned}
\Psi^{\sigma_1} &= N(\sigma_1, q, \alpha)\Psi \quad \text{on } U \\
\Psi^{\sigma} &= (N(\tau_{m+1}, q^{\sigma_{m+1}}, \alpha))^{\sigma_{m+1}} \Psi^{\sigma_{m+1}} \quad \text{on } U.
\end{aligned}$$

Now, we have proved the following fact.

Theorem 8.1.

$$\Psi = C(\sigma, q, \alpha)\Psi^{\sigma} \quad \text{on } U$$

where

$$C(\sigma, q, \alpha) = (N(\sigma_1, q, \alpha))^{-1} \left(\prod_{k=1}^{m} (M(q^{\sigma_k}, \alpha))^{\sigma_k} (N(\tau_k, q^{\sigma_{k+1}}, \alpha))^{\sigma_{k+1}} \right) \{(N(\tau_{m+1}, q^{\sigma_{m+1}}, \alpha))^{\sigma_{m+1}}\}^{-1}.$$

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