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Minimal Free Resolutions of Homogenized D-Modules

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Abstract

Homogenizing a module over the ring of differential operators, we define the notion of a minimal free resolution that is adapted to a filtration. We show that one can apply a modification of the algorithm of La Scala and Stillman to compute such a free resolution. By dehomogenization, one gets a free resolution of the original module that is small enough to compute, e.g., its restriction and integration. We have implemented our algorithm in a computer algebra system Kan and give examples by using this implementation.

1. Introduction

We denote by $D = D_n$ the Weyl algebra on n indeterminates $x_1, \ldots, x_n$ over a field K of characteristic 0; i.e., a K-algebra generated by $x_1, \ldots, x_n$ and $\partial_1, \ldots, \partial_n$ with fundamental relations

$$x_i x_j = x_j x_i, \quad \partial_i \partial_j = \partial_j \partial_i, \quad \partial_i x_j = x_j \partial_i + \delta_{ij}$$

for $i, j = 1, \ldots, n$. Let M be a left D-module of finite type. A presentation of M is an exact sequence

$$D^{r_1} \xrightarrow{\varphi_1} D^{r_0} \xrightarrow{\varphi_0} M \to 0 \tag{1}$$

of left D-modules. The homomorphism φ_1 is defined by

$$\varphi_1 : D^{r_1} \ni U = (U_1, \ldots, U_{r_1}) \longmapsto UP \in D^{r_0}$$

with an $r_1 \times r_0$ matrix $P = (P_{ij})$ whose elements are in D. Hence we often identify the homomorphism φ_1 with the matrix P. Hence giving a presentation of M is equivalent to giving such a matrix P. It is the starting point of the

D-module theory to regard M as a system of linear differential equations

$$\sum_{j=1}^{r_0} P_{ij} u_j = 0 \quad (i = 1, \ldots, r_1)$$

for unknown functions u_1, ..., u_{r_0}. In algorithms for D-modules, one of the basic tools is the computation of free resolutions. For example, in order to compute the dual of a holonomic system M, we can use an arbitrary free resolution of M. On the other hand, for the computation of the cohomology groups associated with the restriction (inverse image) and the integration (direct image) of a D-module M, we need a free resolution that is adapted to (or strict with respect to) the filtration of D defined by a certain weight vector (cf. Oaku and Takayama (in press), Oaku and Takayama (1999)). In Oaku and Takayama (in press), we proposed to use the so called Schreyer resolution to obtain an adapted free resolution. However, it often produces a resolution that is too big to complete the resolution computation, or to pass to the next step.

To overcome this bottleneck, we need an adapted free resolution that is as small as possible. However, for a D-module the notion of minimal free resolution is not defined directly. For this purpose, we use the homogenized Weyl algebra (or the homogenized ring of differential operators); it was implemented in a computer algebra system Kan/sm1 and its fundamental properties were studied by Castro-Jimenez and Macarro (1997).

Definition (homogenized Weyl algebra): We denote by $D_n^{(h)} = D^{(h)}$ the K-algebra generated by h, $x = (x_1, \ldots, x_n)$, and $\partial = (\partial_1, \ldots, \partial_n)$ with the fundamental relations

$$\begin{array}{lll} x_i x_j - x_j x_i = 0, & \partial_i \partial_j - \partial_j \partial_i = 0, & x_i \partial_j - \partial_j x_i = -\delta_{ij} h^2, \\ h x_i - x_i h = 0, & h \partial_i - \partial_i h = 0 & (i, j = 1, \ldots, n). \end{array}$$

We call $D^{(h)}$ *the homogenized Weyl algebra.*

Note that $D^{(h)}$ is isomorphic to the Rees ring associated with D. An element P of $D^{(h)}$ is written uniquely in the form

$$P = \sum_{\lambda \in \mathbb{N}, \alpha, \beta \in \mathbb{N}^n} a_{\lambda\alpha\beta} h^\lambda x^\alpha \partial^\beta$$

with $a_{\lambda\alpha\beta} \in K$. The *total degree* of P is defined by

$$\deg(P) := \max\{\lambda + |\alpha| + |\beta| \mid a_{\lambda\alpha\beta} \neq 0\}.$$

Let $(D^{(h)})_i$ be the set of elements of $D^{(h)}$ homogeneous of degree i with respect to the total degree. Then we get a decomposition into direct sum

$$D^{(h)} = \bigoplus_{i \geq 0} (D^{(h)})_i,$$

which makes $D^{(h)}$ a non-commutative graded ring. We have $(D^{(h)})_0 = K$, and $(D^{(h)})_i$ is a finite dimensional K-vector space. Hence we can define the notion of minimal free resolution for $D^{(h)}$-modules. Moreover, we shall show that we can define a minimal one among the adapted free resolutions of a $D^{(h)}$-module. We can compute such a free resolution by modifying the algorithm of La Scala and Stillman (1998) (see also La Scala (1994)).

The substitution $h = 1$ induces a ring homomorphism

$$\rho : D^{(h)} \ni P \longmapsto P|_{h=1} \in D.$$

We call this homomorphism the *dehomogenization*.

Dehomogenizing a presentation

$$(D^{(h)})^{r_1} \xrightarrow{\psi_1} (D^{(h)})^{r_0} \xrightarrow{\psi_0} M' \to 0 \tag{2}$$

of a left graded $D^{(h)}$-module M', we get an exact sequence

$$D^{r_1} \xrightarrow{\psi_1|_{h=1}} D^{r_0} \xrightarrow{\psi_0|_{h=1}} M'|_{h=1} \to 0$$

of a left D-module $M'|_{h=1}$, which is unique up to isomorphism, independent of a presentation of M'. Conversely, given a presentation (1) of a D-module M, there exists an exact sequence (2) of $D^{(h)}$-modules whose dehomogenization coincides with (1). However, such an M' depends on the presentation of M.

Further, dehomogenizing a free resolution

$$\cdots \xrightarrow{\psi_3} (D^{(h)})^{r_2} \xrightarrow{\psi_2} (D^{(h)})^{r_1} \xrightarrow{\psi_1} (D^{(h)})^{r_0} \xrightarrow{\psi_0} M' \to 0 \tag{3}$$

of a $D^{(h)}$-module M', we get a free resolution of the D-module $M'|_{h=1}$.

Hence, given a presentation (1) of a left D-module M, we propose to compute a free resolution of M as follows:

1. Take a presentation (2) of a left $D^{(h)}$-module M' whose dehomogenization coincides with (1);
2. Compute a minimal free resolution (3) of M' that is adapted to a filtration if necessary;
3. Dehomogenizing (3), we get a free resolution of M (adapted to a filtration).

The reason why we compute free resolutions via homogenization consists in the fact that a minimal free resolution (adapted to a filtration) is defined and computable for a graded $D^{(h)}$-module of finite type by using (a modification of) the algorithm of La Scala and Stillman (1998). We can also define the notion of minimal free resolution of a D-module M adapted to the filtration defined by the total degree. We prove that this coincides with the dehomogenization of a minimal free resolution in $D^{(h)}$ of a homogenization of M. Hence, at least for the ordinary minimal resolutions, the homogenization does not increase the size of the free resolution although it increments the number of indeterminates by one.

We have implemented these algorithms in a computer algebra system `Kan/k0`, which is obtainable from the web page of OpenXM project (2000). Examples computed by this implementation suggest that this method for adapted free resolutions is almost optimal as far as the size of the resolution is concerned.

2. Minimal free resolutions of $D^{(h)}$-modules

By assigning a vector $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r$, we define the i-th homogeneous part of the free module $(D^{(h)})^r$ to be

$$((D^{(h)})^r)_i = (D^{(h)})_{i-n_1} \oplus \cdots \oplus (D^{(h)})_{i-n_r}.$$

This defines a structure of graded left $D^{(h)}$-module, which we denote by $(D^{(h)})^r[\mathbf{n}]$. We call such a graded module a *graded free $D^{(h)}$-module of finite type*. In general, for graded $D^{(h)}$-modules $M = \oplus_{i\in\mathbb{Z}} M_i$ and $N = \oplus_{i\in\mathbb{Z}} N_i$, a $D^{(h)}$-homomorphism $\varphi : M \to N$ is *homogeneous* by definition if $\varphi(M_i) \subset N_i$ holds for any $i \in \mathbb{Z}$.

Definition (minimal free resolution): Let M' be a graded left $D^{(h)}$-module of finite type. A free resolution

$$\cdots \xrightarrow{\varphi_3} (D^{(h)})^{r_2} \xrightarrow{\varphi_2} (D^{(h)})^{r_1} \xrightarrow{\varphi_1} (D^{(h)})^{r_0} \xrightarrow{\varphi_0} M' \to 0 \qquad (4)$$

of M' is called a *minimal free resolution* if and only if there exist $\mathbf{n}_i \in \mathbb{Z}^{r_i}$ $(i \geq 0)$ such that each $\varphi_i : (D^{(h)})^{r_i}[\mathbf{n}_i] \to (D^{(h)})^{r_{i-1}}[\mathbf{n}_{i-1}]$ is homogeneous and does not contain nonzero constants when regarded as a matrix.

The following fact follows easily; See e.g., Eisenbud (1994, Lemma 19.4) for the commutative case:

LEMMA 2.1: *A homogeneous free resolution (4) of a graded $D^{(h)}$-module M' is minimal if and only if $\varphi_i(1, 0, \ldots, 0)$, …, $\varphi_i(0, \ldots, 0, 1)$ form a minimal set of generators of* $\mathrm{Ker}\,\varphi_{i-1}$ *for each $i \geq 1$, and $\varphi_0(1, 0, \ldots, 0)$, …, $\varphi_0(0, \ldots, 0, 1)$ are a minimal set of generators of M.*

PROPOSITION 2.1: *A minimal free resolution (4) of a left graded $D^{(h)}$-module M' is unique up to isomorphism. In particular, the Betti numbers $r_0, r_1, \ldots$ are uniquely determined by M'.*

This proposition can be proved in the same way as in the commutative case (Eisenbud (1994, Theorem 20.2)). To this proof, the following lemma is essential, which can be also proved in the same way as in the commutative case:

LEMMA 2.2: *Let N be a graded left $D^{(h)}$-module of finite type. If N is projective, then it is a graded free $D^{(h)}$-module.*

PROPOSITION 2.2: *A minimal free resolution of a finitely generated $D^{(h)}$-module M' is computable.*

Proof: We can apply Algorithms 4.1 and 4.6 of La Scala and Stillman (1998). In applying Algorithm 4.1, we can use an arbitrary term order for $(D^{(h)})^{r_0}$; As deg in Algorithm 4.1 to determine the strategy (the order to perform the reduction), we adopt the total degree (with appropriate shift vectors determined by the Schreyer resolution).

However, for the output of Algorithm 4.6 executed after Algorithm 4.1 to be a minimal resolution, it is necessary that $\psi_0(1,0,\ldots,0)$, $\cdots$, $\psi_0(0,\ldots,0,1)$ be a minimal set of generators of M' in the initial presentation (2) ; i.e., that ψ_1 does not contain nonzero constants as its components. If (2) does not meet this condition, we apply the following pre-process to (2): Suppose that e.g. the $(1,r_0)$-component of ψ_1 is a nonzero constant. Then compute a set of generators of the $D^{(h)}$-module

$$\{(U_1,\ldots,U_{r_0-1}) \mid (U_1,\ldots,U_{r_0-1},0) \in \mathrm{Ker}\,\psi_0\}$$

and let ψ_1' be the matrix with these generators as row vectors. Replace the presentation (2) by

$$(D^{(h)})^{r_1'} \xrightarrow{\psi_1'} (D^{(h)})^{r_0-1} \to M' \to 0.$$

Continue this procedure until ψ_1 does not contain nonzero constants as its components, or $r_0 = 0$. If $r_0 = 0$, then the minimal free resolution is $0 \to M' \to 0$. □

3. Minimal resolutions adapted to a weight vector

We call (u,v) an *(admissible) weight vector* if $u,v \in \mathbb{Z}^n$ satisfy $u+v \geq 0$ (i.e., each component of $u+v$ is non-negative) and $(u,v) \neq (0,0)$. The weight of a monomial $x^\alpha\partial^\beta$ of D as well as of a monomial $h^\nu x^\alpha \partial^\beta$ of $D^{(h)}$ is defined by $\langle u,\alpha\rangle + \langle v,\beta\rangle$. In general, for an element P of D or of $D^{(h)}$, we define the (u,v)-*order* $\mathrm{ord}_{(u,v)}(P)$ of P to be the maximum weight of its monomials. This defines filtrations on D and on $D^{(h)}$ by

$$\begin{aligned} F^k_{(u,v)}(D) &:= \{P \in D \mid \mathrm{ord}_{(u,v)}(P) \leq k\}, \\ F^k_{(u,v)}(D^{(h)}) &:= \{P \in D^{(h)} \mid \mathrm{ord}_{(u,v)}(P) \leq k\} \end{aligned}$$

for $k \in \mathbb{Z}$. The graded rings with respect to these filtrations are defined by

$$\begin{aligned} \mathrm{gr}_{(u,v)}(D) &:= \bigoplus_{k\in\mathbb{Z}} F^k_{(u,v)}(D)/F^{k-1}_{(u,v)}(D), \\ \mathrm{gr}_{(u,v)}(D^{(h)}) &:= \bigoplus_{k\in\mathbb{Z}} F^k_{(u,v)}(D^{(h)})/F^{k-1}_{(u,v)}(D) \end{aligned}$$

respectively. For example, we have

$$\mathrm{gr}_{(u,v)}(D) \simeq D, \qquad \mathrm{gr}_{(u,v)}(D^{(h)}) \simeq D^{(h)}$$

if $u+v=0$, and

$$\mathrm{gr}_{(u,v)}(D) \simeq K[x,\xi], \qquad \mathrm{gr}_{(u,v)}(D^{(h)}) \simeq K[x,\xi,h]$$

if $u+v>0$ (componentwise positive), where $\xi = (\xi_1,\dots,\xi_n)$ denotes commutative indeterminates.

Moreover, by assigning $\mathbf{m} = (m_1,\dots,m_r) \in \mathbb{Z}^r$, which we call a *shift vector*, we define filtrations on D^r and on $(D^{(h)})^r$ by

$$\begin{aligned} F^k_{(u,v)}[\mathbf{m}](D^r) &:= \{(P_1,\dots,P_r) \in D^r \mid \mathrm{ord}_{(u,v)}(P_i) + m_i \le k \ (i=1,\dots,r)\}, \\ F^k_{(u,v)}[\mathbf{m}]((D^{(h)})^r) &:= \{(P_1,\dots,P_r) \in (D^{(h)})^r \mid \mathrm{ord}_{(u,v)}(P_i) + m_i \le k \ (i=1,\dots,r)\} \end{aligned}$$

respectively. We denote by $\mathrm{gr}_{(u,v)}[\mathbf{m}](D^r)$, $\mathrm{gr}_{(u,v)}[\mathbf{m}]((D^{(h)})^r)$ the graded modules associated with these filtrations. For

$$P = (P_1,\dots,P_r) \in (D^{(h)})^r, \quad P_i = \sum_{\lambda\in\mathbb{N},\alpha,\beta\in\mathbb{N}^n} a_{i\lambda\alpha\beta} h^\lambda x^\alpha \partial^\beta,$$

putting

$$k = \mathrm{ord}_{(u,v)}[\mathbf{m}](P) := \max\{\mathrm{ord}_{(u,v)}(P_i) + m_i \mid 1 \le i \le r\},$$

we define the *initial part* $\mathrm{in}_{(u,v)}[\mathbf{m}](P)$ to be the residue class of $P \in F^k_{(u,v)}[\mathbf{m}]((D^{(h)})^r)$ in $F^k_{(u,v)}[\mathbf{m}]((D^{(h)})^r)/F^{k-1}_{(u,v)}[\mathbf{m}]((D^{(h)})^r) \subset \mathrm{gr}_{(u,v)}[\mathbf{m}]((D^{(h)})^r)$. More concretely the initial part is written in the form

$$\mathrm{in}_{(u,v)}[\mathbf{m}](P) = \left(\sum_{\langle u,\alpha\rangle+\langle v,\beta\rangle=k-m_1} a_{1\lambda\alpha\beta} h^\lambda x^\alpha \partial^\beta, \cdots, \sum_{\langle u,\alpha\rangle+\langle v,\beta\rangle=k-m_r} a_{r\lambda\alpha\beta} h^\lambda x^\alpha \partial^\beta \right)$$

if $u+v=0$, and

$$\mathrm{in}_{(u,v)}[\mathbf{m}](P) = \left(\sum_{\langle u,\alpha\rangle+\langle v,\beta\rangle=k-m_1} a_{1\lambda\alpha\beta} h^\lambda x^\alpha \xi^\beta, \cdots, \sum_{\langle u,\alpha\rangle+\langle v,\beta\rangle=k-m_r} a_{r\lambda\alpha\beta} h^\lambda x^\alpha \xi^\beta \right)$$

if $u+v>0$.

Definition (free resolution adapted to a weight vector): Let M be a left D-module of finite type. A free resolution

$$\cdots \xrightarrow{\varphi_2} D^{r_1} \xrightarrow{\varphi_1} D^{r_0} \xrightarrow{\varphi_0} M \to 0 \tag{5}$$

of M is said to be *adapted to* the weight vector (u,v), or *strict with respect to* (u,v) (also (u,v)-adapted or (u,v)-strict for short) if there exist $\mathbf{m}_i = (m_{i1},\dots,m_{ir_i}) \in \mathbb{Z}^{r_i}$ such that (5) induces an exact sequence

$$\cdots \xrightarrow{\varphi_2} F^k_{(u,v)}[\mathbf{m}_1](D^{r_1}) \xrightarrow{\varphi_1} F^k_{(u,v)}[\mathbf{m}_0](D^{r_0}) \xrightarrow{\varphi_0} F^k_{(u,v)}[\mathbf{m}_0](M) \to 0 \tag{6}$$

of abelian groups for any $k \in \mathbb{Z}$, where we put

$$F^k_{(u,v)}[\mathbf{m}_0](M) := \varphi_0(F^k_{(u,v)}[\mathbf{m}_0](D^{r_0})).$$

Under the condition that (5) is exact and

$$\varphi_i(F^k_{(u,v)}[\mathbf{m}_i]((D^{(h)})^{r_i})) \subset F^k_{(u,v)}[\mathbf{m}_{i-1}]((D^{(h)})^{r_{i-1}})$$

holds for any $k \in \mathbb{Z}$ and $i \geq 1$, the free resolution (5) is adapted to (u, v) if and only if the complex

$$\cdots \xrightarrow{\overline{\varphi}_2} \mathrm{gr}_{(u,v)}[\mathbf{m}_1](D^{r_1}) \xrightarrow{\overline{\varphi}_1} \mathrm{gr}_{(u,v)}[\mathbf{m}_0](D^{r_0}) \xrightarrow{\overline{\varphi}_0} \mathrm{gr}_{(u,v)}[\mathbf{m}_0](M) \to 0$$

of graded $\mathrm{gr}_{(u,v)}(D)$-modules induced by (5) is exact (Oaku and Takayama (in press, Theorem 10.7)). We denote by $(\varphi_i)_j$ the j-th row vector of φ_i regarded as a matrix. If (5) is adapted to (u, v), we have

$$\mathrm{ord}_{(u,v)}[\mathbf{m}_{i-1}]((\varphi_i)_j) \leq m_{ij} \quad (i \geq 1,\ 1 \leq j \leq r_i).$$

More strictly, if

$$\mathrm{ord}_{(u,v)}[\mathbf{m}_{i-1}]((\varphi_i)_j) = m_{ij} \quad (i \geq 1,\ 1 \leq j \leq r_i)$$

holds, then (5) is said to be *properly adapted to* (u, v) with respect to the shift vectors $\mathbf{m}_i$. This condition is equivalent to no row vectors of each $\overline{\varphi}_i$ being zero vectors. If this condition holds, we have

$$\overline{\varphi}_i = \mathrm{in}_{(u,v)}[\mathbf{m}_{i-1}](\varphi_i) := \begin{pmatrix} \mathrm{in}_{(u,v)}[\mathbf{m}_{i-1}]((\varphi_i)_1) \\ \vdots \\ \mathrm{in}_{(u,v)}[\mathbf{m}_{i-1}]((\varphi_i)_{r_i}) \end{pmatrix}.$$

We define a free resolution of a left $D^{(h)}$-module to be (properly) adapted to (u, v) in the same way.

For computing the cohomology groups associated with the restriction or the integration of a D-module, we need a free resolution adapted to a weight vector (u, v) with $u + v = 0$ (Oaku and Takayama (in press), Oaku and Takayama (1999)).

Lemma 3.1: *Substituting* 1 *for* h *in a free resolution*

$$\cdots \xrightarrow{\psi_2} (D^{(h)})^{r_1} \xrightarrow{\psi_1} (D^{(h)})^{r_0} \xrightarrow{\psi_0} M' \to 0 \qquad (7)$$

of a $D^{(h)}$*-module* M'*, we get an exact sequence*

$$\cdots \xrightarrow{\varphi_2} D^{r_1} \xrightarrow{\varphi_1} D^{r_0} \xrightarrow{\varphi_0} M \to 0 \qquad (8)$$

of D*-modules, which we call the* dehomogenization *of (7). We denote this* M *by* $M'|_{h=1}$*. Then the* D*-module* $M'|_{h=1}$ *is uniquely determined by* M' *up to isomorphism independently of the free resolution (7). Moreover, if (7) is (properly) adapted to* (u, v)*, then (8) is also (properly) adapted to* (u, v)*.*

Proof: The exactness of (7) easily implies the exactness of (8). Since two free resolutions of M' are homotopic, so are their dehomogenizations. Hence $M'|_{h=1}$ is unique up to isomorphism. Moreover, if (7) is adapted to (u,v), there exist $\mathbf{m}_i \in \mathbb{Z}^{r_i}$ such that (7) induces an exact sequence

$$\cdots \xrightarrow{\overline{\psi}_2} \mathrm{gr}_{(u,v)}[\mathbf{m}_1]((D^{(h)})^{r_1}) \xrightarrow{\overline{\psi}_1} \mathrm{gr}_{(u,v)}[\mathbf{m}_0]((D^{(h)})^{r_0}) \xrightarrow{\overline{\psi}_0} \mathrm{gr}_{(u,v)}[\mathbf{m}_0](M') \to 0.$$

Its dehomogenization coincides with the complex

$$\cdots \xrightarrow{\overline{\varphi}_2} \mathrm{gr}_{(u,v)}[\mathbf{m}_1](D^{r_1}) \xrightarrow{\overline{\varphi}_1} \mathrm{gr}_{(u,v)}[\mathbf{m}_0](D^{r_0}) \xrightarrow{\overline{\varphi}_0} \mathrm{gr}_{(u,v)}[\mathbf{m}_0](M) \to 0$$

induced by (8). Hence this last complex is exact. This proves that (8) is adapted to (u,v). □

By definition $\mathrm{gr}_{(u,v)}(D^{(h)})$ has a structure of graded ring defined by

$$\mathrm{gr}_{(u,v)}(D^{(h)}) = \bigoplus_{k\in\mathbb{Z}} F^k_{(u,v)}(D^{(h)})/F^{k-1}_{(u,v)}(D^{(h)})$$

while it has another structure of graded ring by the decomposition

$$\mathrm{gr}_{(u,v)}(D^{(h)}) = \bigoplus_{i\geq 0} \mathrm{gr}_{(u,v)}(D^{(h)})_i$$

defined by the total degree. For finitely generated modules over $\mathrm{gr}_{(u,v)}(D^{(h)})$ regarded as a graded ring with respect to the total degree, we can define minimal free resolutions and they are unique up to isomorphism.

Definition ((u,v)-minimal free resolution): A free resolution

$$\cdots \xrightarrow{\psi_2} (D^{(h)})^{r_1}[\mathbf{n}_1] \xrightarrow{\psi_1} (D^{(h)})^{r_0}[\mathbf{n}_0] \xrightarrow{\psi_0} M' \to 0 \qquad (9)$$

of a graded left $D^{(h)}$-module M' with $\mathbf{n}_i \in \mathbb{Z}^{r_i}$ is said to be a (u,v)-*minimal free resolution* if (9) is adapted to (u,v) with shift vectors $\mathbf{m}_i \in \mathbb{Z}^{r_i}$, and the induced exact sequence

$$\cdots \xrightarrow{\overline{\psi}_2} \mathrm{gr}_{(u,v)}[\mathbf{m}_1](D^{r_1})[\mathbf{n}_1] \xrightarrow{\overline{\psi}_1} \mathrm{gr}_{(u,v)}[\mathbf{m}_0](D^{r_0})[\mathbf{n}_0] \xrightarrow{\overline{\psi}_0} \mathrm{gr}_{(u,v)}[\mathbf{m}_0](M') \to 0$$

is a minimal free resolution of $\mathrm{gr}_{(u,v)}[\mathbf{m}_0](M')$ regarded as a graded $\mathrm{gr}_{(u,v)}(D^{(h)})$-module with respect to the total degree. When a free resolution (9) of M' is properly adapted to (u,v), then (9) is (u,v)-minimal if and only if each $\mathrm{in}_{(u,v)}[\mathbf{m}_{i-1}](\psi_i)$ does not have nonzero constants as its components.

A (u,v)-minimal free resolution is not necessarily a minimal free resolution.

EXAMPLE 3.1: Putting $n = 2$, $x = x_1$, $y = x_2$, $\partial_x = \partial_1$, $\partial_y = \partial_2$, define a $D^{(h)}$-module M' by

$$M' = D^{(h)}/(D^{(h)}(h\partial_x - x\partial_x - y\partial_y) + D^{(h)}(h\partial_y - x\partial_x - y\partial_y)).$$

As a $(-\mathbf{1}, \mathbf{1}) = (-1, -1, 1, 1)$-minimal free resolution of M', we obtain

$$0 \to (D^{(h)})^2 \xrightarrow{\psi_2} (D^{(h)})^3 \xrightarrow{\psi_1} D^{(h)} \to M' \to 0,$$

$$\begin{aligned}
\psi_1 &= \begin{pmatrix} h\partial_x - x\partial_x - y\partial_y \\ h\partial_y - x\partial_x - y\partial_y \\ x\partial_x^2 - x\partial_x\partial_y + y\partial_x\partial_y - y\partial_y^2 \end{pmatrix}, \\
\psi_2 &= \begin{pmatrix} x\partial_x - x\partial_y + y\partial_y + hx & -y\partial_y - hx & -h + x \\ -\partial_y + h & \partial_x - h & 1 \end{pmatrix}.
\end{aligned}$$

This is not a minimal resolution of M' since ψ_2 has 1 as its component. The associated shift vectors are $\mathbf{m}_0 = (0)$, $\mathbf{m}_1 = (1, 1, 1)$, $\mathbf{m}_2 = (1, 2)$. Hence the associated minimal resolution of $\mathrm{gr}_{(-\mathbf{1},\mathbf{1})}(M')$ is given by

$$0 \to (D^{(h)})^2 \xrightarrow{\overline{\psi}_2} (D^{(h)})^3 \xrightarrow{\overline{\psi}_1} D^{(h)} \to \mathrm{gr}_{(-1,1)}[\mathbf{m}_0](M') \to 0,$$

$$\begin{aligned}
\overline{\psi}_1 &= \begin{pmatrix} h\partial_x \\ h\partial_y \\ x\partial_x^2 - x\partial_x\partial_y + y\partial_x\partial_y - y\partial_y^2 \end{pmatrix}, \\
\overline{\psi}_2 &= \begin{pmatrix} x\partial_x - x\partial_y + y\partial_y & -y\partial_y & -h \\ -\partial_y & \partial_x & 0 \end{pmatrix}.
\end{aligned}$$

On the other hand, a minimal free resolution of M' is

$$0 \to D^{(h)} \xrightarrow{\psi_2'} (D^{(h)})^2 \xrightarrow{\psi_1'} D^{(h)} \to M' \to 0,$$

$$\psi_1' = \begin{pmatrix} -x\partial_x - y\partial_y + h\partial_y \\ -h\partial_x + h\partial_y \end{pmatrix}, \quad \psi_2' = \begin{pmatrix} -h\partial_x + h\partial_y, & x\partial_x + y\partial_y - h\partial_y + h^2 \end{pmatrix}.$$

LEMMA 3.2: *Let N be a graded left submodule of $(D^{(h)})^r[\mathbf{n}]$ with $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^r$. Let $P_1, \ldots, P_s$ be homogeneous elements of N. Assume that for any $P \in N$, $\mathrm{in}_{(u,v)}[\mathbf{m}](P)$ is contained in the left $\mathrm{gr}_{(u,v)}[\mathbf{m}]((D^{(h)})^r)$-module generated by $\mathrm{in}_{(u,v)}[\mathbf{m}](P_1)$, $\ldots$, $\mathrm{in}_{(u,v)}[\mathbf{m}](P_s)$. Then for any $P \in N$, there exist $Q_1, \ldots, Q_s \in D^{(h)}$ satisfying*

$$P = \sum_{i=1}^{s} Q_i P_i, \qquad \mathrm{ord}_{(u,v)}(Q_i) + \mathrm{ord}_{(u,v)}[\mathbf{m}](P_i) \leq \mathrm{ord}_{(u,v)}[\mathbf{m}](P) \quad (i = 1, \ldots, s).$$

Proof: Suppose P is a homogeneous element of N and put $k = \mathrm{ord}_{(u,v)}[\mathbf{m}](P)$. By the assumption, there exist homogeneous elements $Q_1^{(0)}$, ..., $Q_s^{(0)}$ of $D^{(h)}$ satisfying

$$P^{(1)} := P - \sum_{i=1}^{s} Q_i^{(0)} P_i \in F_{(u,v)}^{k-1}[\mathbf{m}]((D^{(h)})^r),$$
$$\mathrm{ord}_{(u,v)}(Q_i^{(0)}) + \mathrm{ord}_{(u,v)}[\mathbf{m}](P_i) \le k \quad (i = 1, \ldots, s).$$

Then $P^{(1)}$ belongs to $\in N$ and $\mathrm{ord}_{(u,v)}[\mathbf{m}](P^{(1)}) \le k-1$ holds. Hence there exist homogeneous elements $Q_1^{(1)}, \ldots, Q_s^{(1)}$ of $D^{(h)}$ satisfying

$$P^{(1)} - \sum_{i=1}^{s} Q_i^{(1)} P_i \in F_{(u,v)}^{k-2}[\mathbf{m}]((D^{(h)})^r),$$
$$\mathrm{ord}_{(u,v)}(Q_i^{(1)}) + \mathrm{ord}_{(u,v)}[\mathbf{m}](P_i) \le k-1 \quad (i = 1, \ldots, s).$$

In the same way, we can take $Q_i^{(2)}$, $Q_i^{(3)}$, ... successively. For any $j \in \mathbb{Z}$, we have

$$((D^{(h)})^r[\mathbf{n}])_j \cap F_{(u,v)}^{k}[\mathbf{m}]((D^{(h)})^r) = 0$$

for a sufficiently small $k \in \mathbb{Z}$. Hence the above procedure terminates and the conclusion of the proposition holds for the finite sum $Q_i = \sum_{j \ge 0} Q_i^{(j)}$. □

Definition (involutive base): We call $\{P_1, \ldots, P_s\}$ satisfying the assumption of Lemma 3.2 a (u,v)-*involutive* base of N with respect to $\mathbf{m}$. In case of D, we impose the additional condition that $\{P_1, \ldots, P_s\}$ generate N (cf. Oaku and Takayama (in press, Definition 10.1)).

We have the following criteria for (u,v)-adapted and (u,v)-minimal resolutions:

PROPOSITION 3.1: *Let M' be a graded left $D^{(h)}$-module of finite type and (u,v) be an admissible weight vector. Assume that*

$$\cdots \xrightarrow{\psi_2} (D^{(h)})^{r_1} \xrightarrow{\psi_1} (D^{(h)})^{r_0} \xrightarrow{\psi_0} M' \to 0 \tag{10}$$

is a complex of graded left $D^{(h)}$-modules, i.e., that $\psi_{i-1} \circ \psi_i = 0$ holds for $i \ge 1$. Assume moreover that $\mathrm{Im}\,\psi_1 = \mathrm{Ker}\,\psi_0$ and

$$\psi_i(F_{(u,v)}^{k}[\mathbf{m}_i]((D^{(h)})^{r_i}) \subset F_{(u,v)}^{k}[\mathbf{m}_{i-1}]((D^{(h)})^{r_{i-1}}) \qquad (i \ge 1,\ k \in \mathbb{Z})$$

hold with $\mathbf{m}_i \in \mathbb{Z}^{r_i}$. Then (10) is a (u,v)-adapted free resolution with respect to the shift vectors $\mathbf{m}_i$ if and only if ψ_1 is (u,v)-involutive with respect to $\mathbf{m}_0$, i.e., the row vectors of ψ_1 constitute a (u,v)-involutive base of $\mathrm{Im}\,\psi_1$ with respect to $\mathbf{m}_0$, and (10) induces an exact sequence

$$\cdots \xrightarrow{\overline{\psi}_2} \mathrm{gr}_{(u,v)}[\mathbf{m}_1]((D^{(h)})^{r_1}) \xrightarrow{\overline{\psi}_1} \mathrm{gr}_{(u,v)}[\mathbf{m}_0]((D^{(h)})^{r_0}). \tag{11}$$

In addition, under the above conditions, (10) is a (u,v)-minimal free resolution of M' if and only if (11) is a minimal free resolution of $\mathrm{Coker}\,\overline{\psi}_1$ *and ψ_1 is (u,v)-involutive with respect to $\mathbf{m}_0$.*

Proof: Suppose that ψ_1 is (u,v)-involutive and (11) is exact. For $i \geq 2$, $N = \mathrm{Ker}\,\psi_{i-1}$ and $\{(\psi_i)_1, \dots, (\psi_i)_{r_i}\}$ satisfy the assumptions of Lemma 3.2. Hence in view of this lemma, $\mathrm{Ker}\,\psi_{i-1} = \mathrm{Im}\,\psi_i$ holds and (10) is (u,v)-adapted as well.

Next, since $\{(\psi_1)_1, \dots, (\psi_1)_{r_i}\}$ is a (u,v)-involutive base of $N = \mathrm{Im}\,\psi_1$ with respect to $\mathbf{m}_0$, we get

$$\mathrm{Coker}\,\overline{\psi}_1 \simeq \mathrm{gr}_{(u,v)}[\mathbf{m}_0]((D^{(h)})^{r_0})/\mathrm{gr}_{(u,v)}[\mathbf{m}_0](N) = \mathrm{gr}_{(u,v)}[\mathbf{m}_0](M).$$

The converse implication is obvious. □

The above proposition also holds for D-modules if $u, v \geq 0$.

PROPOSITION 3.2 (LIFTING): *(1) Let M' be a graded left $D^{(h)}$-module of finite type and $(u,v) \in \mathbb{Z}^{2n}$ be an arbitrary admissible weight vector. In a presentation*

$$(D^{(h)})^{r_1} \xrightarrow{\psi_1} (D^{(h)})^{r_0} \xrightarrow{\psi_0} M' \to 0 \tag{12}$$

which is homogeneous with respect to the total degree, suppose that ψ_1 is (u,v)-involutive with respect to $\mathbf{m}_0 \in \mathbb{Z}^{r_0}$. Moreover, assume that there exist $\mathbf{m}_i \in \mathbb{Z}^{r_i}$ $(i \geq 1)$ and an exact sequence

$$\cdots \xrightarrow{\varphi_2} \mathrm{gr}_{(u,v)}[\mathbf{m}_1]((D^{(h)})^{r_1}) \xrightarrow{\varphi_1} \mathrm{gr}_{(u,v)}[\mathbf{m}_0]((D^{(h)})^{r_0}) \tag{13}$$

of graded $\mathrm{gr}_{(u,v)}(D^{(h)})$-modules which is homogeneous with respect to both the (u,v)-grading and the total degree, such that $\varphi_1 = \mathrm{in}_{(u,v)}[\mathbf{m}_0](\psi_1)$ holds and no row vector of each φ_i is a zero vector. Under these assumptions, there exists a free resolution

$$\cdots \xrightarrow{\psi_3} (D^{(h)})^{r_2} \xrightarrow{\psi_2} (D^{(h)})^{r_1} \xrightarrow{\psi_1} (D^{(h)})^{r_0} \xrightarrow{\psi_0} M' \to 0 \tag{14}$$

of M' that is properly adapted to (u,v) with respect to the shift vectors $\mathbf{m}_i$, such that $\mathrm{in}_{(u,v)}[\mathbf{m}_{i-1}](\psi_i) = \varphi_i$ holds for any $i \geq 1$.

(2) Let M be a left D-module of finite type and let $(u,v) \in \mathbb{Z}^{2n}$ be a weight vector such that $u, v \geq 0$. In a presentation

$$D^{r_1} \xrightarrow{\psi_1} D^{r_0} \xrightarrow{\psi_0} M \to 0$$

of M, suppose that ψ_1 is (u,v)-involutive with respect to $\mathbf{m}_0 \in \mathbb{Z}^{r_0}$. Moreover, suppose that there exist $\mathbf{m}_i \in \mathbb{Z}^{r_i}$ $(i \geq 1)$ and an exact sequence

$$\cdots \xrightarrow{\varphi_2} \mathrm{gr}_{(u,v)}[\mathbf{m}_1](D^{r_1}) \xrightarrow{\varphi_1} \mathrm{gr}_{(u,v)}[\mathbf{m}_0](D^{r_0})$$

of $\mathrm{gr}_{(u,v)}(D)$*-modules that is homogeneous with respect to* (u,v)*, such that* $\varphi_1 = \mathrm{in}_{(u,v)}[\mathbf{m}_0](\psi_1)$ *and no row vectors of* φ_i *are zero vectors. Under these assumptions, there exists a free resolution*

$$\cdots \xrightarrow{\psi_3} D^{r_2} \xrightarrow{\psi_2} D^{r_1} \xrightarrow{\psi_1} D^{r_0} \xrightarrow{\psi_0} M \to 0$$

of M *that is properly adapted to* (u,v)*, such that* $\mathrm{in}_{(u,v)}[\mathbf{m}_{i-1}](\psi_i) = \varphi_i$ *holds for any* $i \geq 1$*.*

Proof: We denote the (j,k)-components of ψ_i and of φ_i as matrices by $(\psi_i)_{jk}$ and $(\varphi_i)_{jk}$ respectively, and the j-th row vectors of ψ_i and of φ_i by $(\psi_i)_j$ and $(\varphi_i)_j$ respectively. By the assumption, we have $\mathrm{ord}_{(u,v)}[\mathbf{m}_0]((\psi_1)_j) = m_{1j}$. Choosing arbitrary homogeneous elements $(\psi_2')_j$ $(j = 1, \ldots, r_2)$ of $(D^{(h)})^{r_1}$ such that $\mathrm{in}_{(u,v)}[\mathbf{m}_1]((\psi_2')_j) = (\varphi_2)_j$, we define an $r_2 \times r_1$ matrix ψ_2' with $(\psi_2')_1, \ldots, (\psi_2')_{r_2}$ as its row vectors. Then we have $\mathrm{ord}_{(u,v)}((\psi_2')_{ij}) \leq m_{2i} - m_{1j}$ with the equality holding at least for one j for each i. Since $\varphi_2 \circ \varphi_1 = 0$, we have

$$\sum_{j=1}^{r_1} (\psi_2')_{ij}(\psi_1)_j \in F_{(u,v)}^{m_{2i}-1}[\mathbf{m}_0]((D^{(h)})^{r_0}) \cap \mathrm{Im}\,\psi_1 \qquad (1 \leq i \leq r_2).$$

Here $N = \mathrm{Im}\,\psi_1 = \mathrm{Ker}\,\psi_0$ and $\{(\psi_1)_1, \ldots, (\psi_1)_{r_1}\}$ satisfy the assumptions of Lemma 3.2 since ψ_1 is (u,v)-involutive. Hence there exist $Q_{ij} \in D^{(h)}$ homogeneous of the same degree as $(\psi_2')_{ij}$ such that $\mathrm{ord}_{(u,v)}(Q_{ij}) \leq m_{2i} - m_{1j} - 1$ and

$$\sum_{j=1}^{r_1} (\psi_2')_{ij}(\psi_1)_j = \sum_{j=1}^{r_1} Q_{ij}(\psi_1)_j \qquad (1 \leq i \leq r_2).$$

Let ψ_2 be the matrix whose (i,j)-component is $(\psi_2')_{ij} - Q_{ij}$. Then the homomorphism induced by ψ_2 in the graded modules coincides with φ_2, and $\psi_1 \circ \psi_2 = 0$ holds.

Now we show $\mathrm{Im}\,\psi_2 = \mathrm{Ker}\,\psi_1$. Suppose $Q \in \mathrm{Ker}\,\psi_1$. Put $\overline{Q} := \mathrm{in}_{(u,v)}[\mathbf{m}_1](Q)$ with $k = \mathrm{ord}_{(u,v)}[\mathbf{m}_1](Q)$. Since $\overline{Q} \in \mathrm{Ker}\,\varphi_1 = \mathrm{Im}\,\varphi_2$, we know that

$$Q = \sum_{j=1}^{r_2} U_j(\psi_2)_j \ \in \mathrm{Im}\,\psi_2$$

holds with some $U = (U_1, \ldots, U_{r_2}) \in F_{(u,v)}^{k}[\mathbf{m}_2]((D^{(h)})^{r_2})$ applying Lemma 3.2 to $N = \mathrm{Ker}\,\psi_1$ and $\{(\psi_2)_1, \ldots, (\psi_2)_{r_2}\}$. We have shown, at the same time, that ψ_2 is (u,v)-involutive with respect to $\mathbf{m}_1$. Thus we can construct ψ_3, ψ_4, ... successively and get an exact sequence (14). Since ψ_1 is (u,v)-involutive, (13) is a free resolution of $\mathrm{gr}_{(u,v)}[\mathbf{m}_0](M')$. Hence (14) is a free resolution properly adapted to (u,v). We can prove (2) in the same way. □

THEOREM 3.1: *Any graded* $D^{(h)}$*-module* M' *of finite type has a* (u,v)*-minimal free resolution.*

Proof: Take a presentation (12) of M' such that ψ_1 has no nonzero constants as components by the same method as that in Proposition 2.2. Adding row vectors if necessary, we may assume that ψ_1 is (u,v)-involutive with respect to $\mathbf{m}_0 \in \mathbb{Z}^{r_0}$. This process does not produce nonzero constants since the elements are all homogeneous with respect to the total degree. Put

$$\mathbf{m}_1 := (\mathrm{ord}_{(u,v)}[\mathbf{m}_0]((\psi_1)_1), \cdots, \mathrm{ord}_{(u,v)}[\mathbf{m}_0]((\psi_1)_{r_1}))$$

and $\varphi_1 := \mathrm{in}_{(u,v)}[\mathbf{m}_0](\psi_1)$. Then we get an exact sequence

$$\mathrm{gr}_{(u,v)}[\mathbf{m}_1]((D^{(h)})^{r_1}) \xrightarrow{\varphi_1} \mathrm{gr}_{(u,v)}[\mathbf{m}_0]((D^{(h)})^{r_0}) \xrightarrow{\varphi_0} \mathrm{gr}_{(u,v)}[\mathbf{m}_0](M') \to 0.$$

Since φ_1 does not have nonzero constants as its components, we can construct a minimal free resolution of $\mathrm{gr}_{(u,v)}[\mathbf{m}_0](M')$ starting from the above presentation. We may assume that this free resolution is homogeneous with respect to both the (u,v)-grading and the total degree. We have only to lift this free resolution by applying Proposition 3.2. □

THEOREM 3.2: *A (u,v)-minimal free resolution of a left graded $D^{(h)}$-module M' of finite type is computable.*

Proof: Take an arbitrary $\mathbf{m}_0 \in \mathbb{Z}^{r_0}$. Let $\prec$ be a term order for $(D^{(h)})^{r_0}$ which refines the (u,v)-order; i.e., $\mathrm{ord}_{(u,v)}[\mathbf{m}_0](P) < \mathrm{ord}_{(u,v)}[\mathbf{m}_0](Q)$ implies $\mathrm{in}_\prec(P) \prec \mathrm{in}_\prec(Q)$ for $P, Q \in (D^{(h)})^{r_1}$ homogeneous of the same degree; here $\mathrm{in}_\prec$ denotes the leading term with respect to $\prec$. Take a presentation

$$(D^{(h)})^{r_1} \xrightarrow{\psi_1} (D^{(h)})^{r_0} \xrightarrow{\psi_0} M' \to 0$$

of M' such that the row vectors of ψ_1 form a Gröbner base of $\mathrm{Im}\,\psi_1$ with respect to $\prec$. Moreover, applying the pre-process stated in the proof of Proposition 2.2 to this presentation, we may assume that ψ_1 has no nonzero constants. Then we apply Algorithm 4.1 of La Scala and Stillman (1998) to M' and to $\mathrm{gr}_{(u,v)}[\mathbf{m}_0](M')$ 'in parallel', and then apply Algorithm 4.6 of La Scala and Stillman (1998) so as to obtain a minimal free resolution of $\mathrm{gr}_{(u,v)}(M')$. Here we use the Schreyer order induced by the term order $\prec$ for reduction, and as the deg to determine the reduction strategy, we adopt the total degree. As to the output of this procedure, we know that the free resolution of M' is a lifting of the free resolution of $\mathrm{gr}_{(u,v)}[\mathbf{m}_0](M')$ since the Schreyer frames in the terminology of La Scala and Stillman (1998) for M' and for $\mathrm{gr}_{(u,v)}[\mathbf{m}_0](M')$ coincide in view of the definition of the term order $\prec$.

More concretely, we replace the if-condition $f = 0$ in Algorithm 4.1 of La Scala and Stillman (1998) by the condition $\mathrm{ord}_{(u,v)}[\mathbf{m}_{i-2}](f) < \mathrm{ord}_{(u,v)}[\mathbf{m}_{i-1}](m)$ for $m \in (D^{(h)})^{r_{i-1}}$ (hence $f \in (D^{(h)})^{r_{i-2}}$). Here $\mathbf{m}_i$ is the shift vector determined by the Schreyer frame (cf. also Oaku and Takayama (in press, Section 9)). This condition is equivalent to the reduction of the initial part of m in

$\mathrm{gr}_{(u,v)}[\mathbf{m}_{i-2}]((D^{(h)})^{r_{i-2}})$ being zero. Applying Algorithm 4.6 to the output $\mathcal{H}_i$ of Algorithm 4.1 modified as above, we obtain a (u,v)-minimal free resolution of M'. Note that the computation itself is performed in $D^{(h)}$ not in $\mathrm{gr}_{(u,v)}(D^{(h)})$; in other words, we only modify the above if-condition and keep the other part of Algorithms 4.1 and 4.6 unchanged.

Let (14) be the free resolution of M' obtained as the output of the above procedure. Then the complex (13) induced by (14) is exact and each φ_i does not contain nonzero constants. Since the $\mathrm{gr}_{(u,v)}(D)$-module generated by the row vectors of $\mathrm{in}_{(u,v)}[\mathbf{m}_0](\psi_1)$ stays unchanged during the execution of the algorithm, we know that $\mathrm{Im}\,\varphi_1 = \mathrm{gr}_{(u,v)}[\mathbf{m}_0](\mathrm{Im}\,\psi_1)$. Hence (14) is a (u,v)-minimal free resolution of M' in view of Proposition 3.1. □

4. Minimal resolution of a D-module and its homogenization

Here we define a minimal free resolution of a D-module without using $D^{(h)}$, and show that its homogenization gives a minimal free resolution of a $D^{(h)}$-module. By using this fact we can relate the length of the minimal resolution of a $D^{(h)}$-module to the length of the minimal resolution of a graded module over the polynomial ring.

We write $(\mathbf{1},\mathbf{1}) = (1,\cdots,1,1,\ldots,1) \in \mathbb{Z}^{2n}$. Then the graded ring $\mathrm{gr}_{(\mathbf{1},\mathbf{1})}(D)$ is isomorphic to the polynomial ring $K[x,\xi]$. Hence for a graded $\mathrm{gr}_{(\mathbf{1},\mathbf{1})}(D)$-module of finite type, we can define the notion of minimal free resolution and it is unique up to isomorphism.

Definition: Let M be a left D-module of finite type. A free resolution

$$\cdots \xrightarrow{\varphi_2} D^{r_1} \xrightarrow{\varphi_1} D^{r_0} \xrightarrow{\varphi_0} M \to 0 \qquad (15)$$

of M is said to be a $(\mathbf{1},\mathbf{1})$-*minimal free resolution* of M if there exist $\mathbf{n}_i \in \mathbb{Z}^{r_i}$ such that

$$\varphi_i(F^k_{(\mathbf{1},\mathbf{1})}[\mathbf{n}_i](D^{r_i})) \subset F^k_{(\mathbf{1},\mathbf{1})}[\mathbf{n}_{i-1}](D^{r_{i-1}}) \qquad (i \geq 1,\ k \in \mathbb{Z})$$

holds and the complex

$$\cdots \xrightarrow{\overline{\varphi}_2} \mathrm{gr}_{(\mathbf{1},\mathbf{1})}[\mathbf{n}_1](D^{r_1}) \xrightarrow{\overline{\varphi}_1} \mathrm{gr}_{(\mathbf{1},\mathbf{1})}[\mathbf{n}_0](D^{r_0}) \xrightarrow{\overline{\varphi}_0} \mathrm{gr}_{(\mathbf{1},\mathbf{1})}[\mathbf{n}_0](M) \to 0$$

of graded $\mathrm{gr}_{(\mathbf{1},\mathbf{1})}(D)$-modules induced by (15) is a minimal free resolution of $\mathrm{gr}_{(\mathbf{1},\mathbf{1})}[\mathbf{n}_0](M)$.

In particular, a minimal free resolution of M is properly adapted to $(\mathbf{1},\mathbf{1})$. We can show the following by the same argument as in the preceding section:

PROPOSITION 4.1: *For a left D-module M, its $(\mathbf{1},\mathbf{1})$-minimal free resolution exists and computable.*

Definition (homogenization): For an element

$$P = \sum_{i=1}^{r} \sum_{\alpha,\beta\in\mathbb{N}^n} a_{\alpha\beta i} x^\alpha \partial^\beta e_i$$

of D^r and $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{Z}^r$, where $e_1, \dots, e_r$ are the canonical generators of D^r, put

$$\begin{aligned} k = \deg[\mathbf{n}](P) &:= \max\{|\alpha| + |\beta| + n_i \mid a_{\alpha\beta i} \neq 0\}, \\ H[\mathbf{n}](P) &:= \sum_{i=1}^{r} \sum_{\alpha,\beta\in\mathbb{N}^n} a_{\alpha\beta i} h^{k-|\alpha|-|\beta|-n_i} x^\alpha \partial^\beta e_i. \end{aligned}$$

We call $H[\mathbf{n}](P)$ the *homogenization* of P with respect to $\mathbf{n}$. When $\mathbf{n}$ is a zero vector, we also denote $H[\mathbf{n}](P)$ by P^h and call it simply the homogenization of P. Then $H[\mathbf{n}](P)$ is a homogeneous element of $(D^{(h)})^r[\mathbf{n}]$ of degree k. Moreover, for a left D-submodule N of D^r, we denote by $H[\mathbf{n}](N)$ the left $D^{(h)}$-submodule of $(D^{(h)})^r$ generated by $\{H[\mathbf{n}](P) \mid P \in N\}$.

Let us give a sufficient condition for the homogenization of a free resolution of a D-module to be a free resolution of a $D^{(h)}$-module:

PROPOSITION 4.2: *Assume that a free resolution*

$$\cdots \xrightarrow{\varphi_2} D^{r_1} \xrightarrow{\varphi_1} D^{r_0} \xrightarrow{\varphi_0} M \to 0 \tag{16}$$

of a left D-module M is properly adapted to $(u, v) = (\mathbf{1}, \mathbf{1}) = (1, \cdots, 1, 1, \dots, 1)$ with the shift vectors $\mathbf{n}_0, \mathbf{n}_1, \dots$. Then there exists an exact sequence

$$\cdots \xrightarrow{\psi_2} (D^{(h)})^{r_1}[\mathbf{n}_1] \xrightarrow{\psi_1} (D^{(h)})^{r_0}[\mathbf{n}_0] \xrightarrow{\psi_0} M' \to 0$$

of graded $D^{(h)}$-modules the dehomogenization of which coincides with (16). Moreover, we have $\operatorname{Im}\psi_1 = H[\mathbf{n}_0](N)$, namely $M' = (D^{(h)})^{r_0}/H[\mathbf{n}_0](N)$ with $N := \operatorname{Im}\varphi_1$.

Proof: Let $e_1^{(i)}, \dots, e_{r_i}^{(i)}$ be the canonical generators of D^{r_i} and put

$$\psi_i = \begin{pmatrix} H[\mathbf{n}_{i-1}](\varphi_i(e_1^{(i)})) \\ \vdots \\ H[\mathbf{n}_{i-1}](\varphi_i(e_{r_i}^{(i)})) \end{pmatrix}.$$

This defines a homomorphism

$$\psi_i : (D^{(h)})^{r_i}[\mathbf{n}_i] \longrightarrow (D^{(h)})^{r_{i-1}}[\mathbf{n}_{i-1}].$$

Since (16) is properly adapted to $(\mathbf{1}, \mathbf{1})$, we have

$$\deg[\mathbf{n}_{i-1}](\varphi_i(e_j^{(i)})) = n_{ij} \quad (i \geq 1,\ 1 \leq j \leq r_i).$$

This implies that ψ_i is homogeneous and $\psi_{i-1}\circ\psi_i = 0$ $(i \geq 2)$.

Let Q be a homogeneous element of $\mathrm{Ker}\,\psi_i$ as a graded submodule of $(D^{(h)})^{r_i}[\mathbf{n}_i]$. Since φ_{i+1} is $(\mathbf{1},\mathbf{1})$-involutive by the assumption, there exist $U_1,\dots,U_{r_{i+1}} \in D$ such that

$$Q|_{h=1} = \sum_{j=1}^{r_{i+1}} U_j\varphi_{i+1}(e_j^{(i+1)}) \tag{17}$$

and

$$\deg[\mathbf{n}_i](Q|_{h=1}) \geq \deg[\mathbf{n}_i](U_j\varphi_{i+1}(e_j^{(i)})) \quad (j = 1,\dots,r_{i+1}). \tag{18}$$

The same inequality holds for the homogenization of $Q|_{h=1}$ and those of $U_j\varphi_{i+1}(e_j^{(i+1)})$. The total degree of Q is not less than that of the homogenization of $Q|_{h=1}$. Hence there exist non-negative integers ν_1, …, $\nu_{r_{i+1}}$ such that

$$Q = \sum_{j=1}^{r_{i+1}} h^{\nu_j}(U_j)^h H[\mathbf{n}_i](\varphi_{i+1}(e_j^{(i+1)})) = \sum_{j=1}^{r_{i+1}} h^{\nu_j}(U_j)^h\psi_{i+1}(e_j^{(i+1)}).$$

Hence Q belongs to the image of ψ_{i+1}. Thus we have shown that

$$\cdots \xrightarrow{\psi_3} (D^{(h)})^{r_2}[\mathbf{n}_2] \xrightarrow{\psi_2} (D^{(h)})^{r_1}[\mathbf{n}_1] \xrightarrow{\psi_1} (D^{(h)})^{r_0}[\mathbf{n}_0]$$

is an exact sequence. Finally, let us prove $\mathrm{Im}\,\psi_1 = H[\mathbf{n}_0](N)$. Since $\mathrm{Im}\,\psi_1 \subset H[\mathbf{n}_0](N)$ is obvious by the definition, we have only to prove $\mathrm{Im}\,\psi_1 \supset H[\mathbf{n}_0](N)$. Let Q be a homogeneous element of $H[\mathbf{n}_0](N)$. Then there exist $U_1,\dots,U_{r_1} \in D$ such that

$$Q|_{h=1} = \sum_{j=1}^{r_1} U_j\varphi_1(e_j^{(1)})$$

and

$$\deg[\mathbf{n}_0](Q|_{h=1}) \geq \deg[\mathbf{n}_0](U_j\varphi_1(e_j^{(1)})) \quad (j = 1,\dots,r_1).$$

This implies $Q \in \mathrm{Im}\,\psi_1$ in the same way as in the former part of the proof. □

Corollary 4.1: *Let N be a left D-submodule of D^r and $\mathbf{n} \in \mathbb{Z}^r$. Suppose that G is a $(\mathbf{1},\mathbf{1})$-involutive base of N with respect to $\mathbf{n}$; i.e., G is a subset of N and $\{\mathrm{in}_{(\mathbf{1},\mathbf{1})}[\mathbf{n}](P) \mid P \in G\}$ generate $\mathrm{gr}_{(\mathbf{1},\mathbf{1})}[\mathbf{n}](N)$. Then $H[\mathbf{n}](N)$ is generated by $\{H[\mathbf{n}](P) \mid P \in G\}$.*

Proposition 4.3: *Applying Proposition 4.2 to a $(\mathbf{1},\mathbf{1})$-minimal free resolution*

$$\cdots \xrightarrow{\varphi_2} D^{r_1} \xrightarrow{\varphi_1} D^{r_0} \xrightarrow{\varphi_0} M \to 0$$

of a D-module M, we obtain a minimal free resolution

$$\cdots \xrightarrow{\psi_2} (D^{(h)})^{r_1}[\mathbf{n}_1] \xrightarrow{\psi_1} (D^{(h)})^{r_0}[\mathbf{n}_0] \xrightarrow{\psi_0} M' \to 0$$

of M' with $M'|_{h=1} = M$.

Proof: By the definition, $\mathrm{in}_{(\mathbf{1},\mathbf{1})}[\mathbf{n}_{i-1}](\varphi_i)$ does not contain nonzero constants. In view of the definition of the homogenization, this implies that ψ_i does not contain nonzero constants. □

Finally let us remark on the length of the minimal resolution. Let us recall the following fundamental fact:

LEMMA 4.1: *(Schapira (1985, Lemma B.2.2)) Assume that A is a Noetherian ring such that any left A-module of finite type has a free resolution of finite length. Suppose that M is a left A-module of finite type and* $\mathrm{Ext}^j_A(M, A) = 0$ *holds for any $j > p$. Then M has a free resolution of length at most* $\max\{p, 1\}$.

If A is a polynomial ring, D, or $D^{(h)}$, then the assumption of the above lemma is satisfied in view of, e.g., the Schreyer algorithm for free resolutions (see Eisenbud (1994, Corollary 15.11), Oaku and Takayama (in press, Theorem 9.11)). Since the global dimension of D is n, it follows that any D-module of finite type has a free resolution of length at most n. However, it seems an open problem whether such a free resolution is computable. More weakly, Proposition 4.3 immediately implies the following:

COROLLARY 4.2: *In a presentation*

$$D^{r_1} \xrightarrow{\varphi_1} D^{r_0} \to M \to 0$$

of M, assume that φ_1 is $(\mathbf{1},\mathbf{1})$-involutive with respect to $\mathbf{n}_0 \in \mathbb{Z}^{r_0}$ and $\mathrm{in}_{(\mathbf{1},\mathbf{1})}[\mathbf{n}_0](\varphi_1)$ *does not have nonzero constants. Then the length of a minimal free resolution of the left $D^{(h)}$-module $M' := (D^{(h)})^{r_0}/H[\mathbf{n}_0](\mathrm{Im}\,\varphi_1)$ coincides with the length of a minimal free resolution of* $\mathrm{gr}_{(\mathbf{1},\mathbf{1})}[\mathbf{n}_0](M)$, *hence is at most $2n$.*

EXAMPLE 4.1: Put $n = 1$, $x = x_1$, $\partial = \partial_1$. Consider two D-modules $M_1 = D/D\partial$ and $M_2 = D/(D\partial^2 + D(x\partial - 1))$. The natural homogenizations of these are $M_1' = D^{(h)}/D^{(h)}\partial$ and $M_2' = D^{(h)}/(D^{(h)}\partial^2 + D^{(h)}(x\partial - h^2))$. Note that M_1 and M_2 are isomorphic as D-modules. In fact, $\varphi : M_2 \to M_1$ which sends the residue class of 1 in M_2 to that of x in M_1 gives an isomorphism. A minimal free resolution of M_1' is

$$0 \to D^{(h)} \xrightarrow{\partial} D^{(h)} \to M_1' \to 0,$$

and a minimal free resolution of M_2' is

$$0 \to D^{(h)} \xrightarrow{\psi_2} (D^{(h)})^2 \xrightarrow{\psi_1} D^{(h)} \xrightarrow{\psi_0} M_2' \to 0,$$

$$\psi_1 = \begin{pmatrix} \partial^2 \\ x\partial - h^2 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} x & -\partial \end{pmatrix}.$$

In particular, M_1' and M_2' are not isomorphic as $D^{(h)}$-modules. Note that the dehomogenizations of the above resolutions give $(\mathbf{1},\mathbf{1})$-minimal resolutions of M_1 and of M_2 respectively.

Not an arbitrary $D^{(h)}$-module of finite type has a free resolution of length at most $2n$.

EXAMPLE 4.2: Put $n = 1$, $x = x_1$, $\partial = \partial_1$ and define M' by

$$M' = D^{(h)}/(D^{(h)}\partial + D^{(h)}x) = D^{(h)}/(D^{(h)}\partial + D^{(h)}x + D^{(h)}h^2).$$

Then a minimal free resolution of M' is

$$0 \to D^{(h)} \xrightarrow{\psi_3} (D^{(h)})^2 \xrightarrow{\psi_2} (D^{(h)})^2 \xrightarrow{\psi_1} D^{(h)} \xrightarrow{\psi_0} M' \to 0,$$

$$\psi_1 = \begin{pmatrix} \partial \\ x \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} -x\partial - 2h^2 & \partial^2 \\ -x^2 & x\partial - h^2 \end{pmatrix}, \quad \psi_3 = \begin{pmatrix} x & -\partial \end{pmatrix},$$

the length of which is $3 = 2n + 1$. Note that its dehomogenization is not a $(\mathbf{1},\mathbf{1})$-minimal resolution of $M'|_{h=1} = 0$ since $\psi_1|_{h=1}$ is not $(\mathbf{1},\mathbf{1})$-involutive.

5. Examples

One of the reasons why we are interested in (u, v)-minimal free resolutions is to make computation of the restriction of D-modules efficient. In Oaku and Takayama (in press), we gave an algorithm to compute the cohomology groups of the restriction of a given holonomic D-module M. For example, to compute the restriction to the origin, our algorithm requires construction of a $(-w, w)$-adapted (strict) free resolution with a componentwise positive $w \in \mathbb{Z}^n$. We proved that the Schreyer free resolution by an appropriate term order is $(-w, w)$-adapted. Once an adapted free resolution is obtained, we have only to compute Ker/Im of a complex of vector spaces of dimensions

$$O\left(\sum_{j=1}^{r_i}(k_1 - m_{ij})_+^n\right),$$

where r_i is the i-th Betti number of the resolution, $(m_{i1}, \ldots, m_{ir_i})$ is the associated i-th shift vector, and k_1 is the maximal integral root of the b-function of M with respect to $(-w, w)$. Unfortunately, Betti numbers of Schreyer free resolutions are usually big and our method often caused memory exhaustion in computing Ker/Im.

We have implemented modified La Scala's algorithm to construct (u, v)-minimal free resolutions. The algorithm is described in the proof of Theorem 3.2. By using our implementation, we have observed that the Betti numbers of (u, v)-minimal free resolutions are much smaller than those of Schreyer free resolutions for many examples. Lengths are also shorter.

In the sequel, we will see some examples of $(-w, w)$-minimal free resolutions. Especially, we compare Betti numbers of $(-w, w)$-minimal free resolutions and Schreyer free resolutions. Before presenting examples, we introduce some notations and explain some background of examples.

1. The Betti numbers of a Schreyer resolution depend not only on $(-w, w)$ but also on the tie-breaking order. We use the graded reverse lexicographic order as the tie-breaking order.
2. We denote free resolutions by sets of matrices. For instance, we denote the free resolution $\{\psi_i\}$ of Example 3.1 by

```
[
 [
   [    Dx*h-x*Dx-y*Dy ]
   [    Dy*h-x*Dx-y*Dy ]
   [    x*Dx^2-x*Dx*Dy+y*Dx*Dy-y*Dy^2 ]
 ]
 [
   [    x*Dx-x*Dy+y*Dy+x*h , -y*Dy-x*h , -h+x ]
   [    -Dy+h , Dx-h , 1 ]
 ]
]
```

3. Assume that a left ideal I is generated by homogeneous elements. For a given (u, v)-minimal free resolution of $D^{(h)}/I$, there exists a unique set of shift vectors $\{\mathbf{m}_i\}$ such that $\mathbf{m}_0 = 0$ and the resolution is properly adapted to (u, v). The set of shift vectors are presented by the name `Degree Shifts`. For example,

```
Degree Shifts: [[ 0 ]  , [ 1 , 1 , 1 ]  , [  1 , 2 ]  ]
```

   is the shift vectors satisfying the condition for Example 3.1.
4. Let $-r$ be the minimal integral root of the Berndstein-Sato polynomial of a polynomial f. We denote by $\mathrm{Ann}(Df^{-1})$ the output of the function `Sannfs(f,v)`, which is a set of generators of the annihilating ideal of $1/f^r$.
5. For a set G of elements of D, $F(G)$ denotes the set of the formal Laplace transformations of the elements of G.
6. The homogenization of $F(G)$ is denoted by $F^h(G)$.
7. Put $I = F(\mathrm{Ann}(Df^{-1}))$. The cohomology groups of the restriction of D/I to the origin agree with the singular cohomology groups of the space $\mathbf{C}^n \setminus V(f)$ by the Grothendieck comparison theorem. See Oaku and Takayama (1999) for details.

In the following examples, we always define the filtration of $D^{(h)}/I$ by

$$F^k_{(-w,w)}(D^{(h)}/I) = F^k_{(-w,w)}(D^{(h)})/(F^k_{(-w,w)}(D^{(h)}) \cap I).$$

Hence the Betti numbers of a $(-w, w)$-minimal free resolution of $D^{(h)}/I$ are uniquely determined by I and w. Incidentally or not, the following $(-w, w)$-minimal free resolutions are all minimal (cf. Example 3.1).

EXAMPLE 5.1: Put $I = F^h\,[\mathrm{Ann}\,(D(x^3 - y^2)^{-1})]$. The ideal I is generated by

$$-2x\partial_x - 3y\partial_y + h^2, \quad -3y\partial_x^2 + 2x\partial_y h.$$

Resolution type	Betti numbers
Schreyer	1, 4, 4, 1
$(-\mathbf{1},\mathbf{1})$-minimal	1, 2, 1
minimal	1, 2, 1

$(-\mathbf{1},\mathbf{1})$-minimal resolution

```
 [
  [
    [    -2*x*Dx-3*y*Dy+h^2 ]
    [    -3*y*Dx^2+2*x*Dy*h ]
  ]
  [
    [    -3*y*Dx^2+2*x*Dy*h , 2*x*Dx+3*y*Dy ]
  ]
 ]
Degree shifts
[    [    0 ]  , [    0 , 1 ] , [ 1 ] ]
```

Schreyer Resolution

```
 [
  [
    [    -2*x*Dx-3*y*Dy+h^2 ]
    [    -3*y*Dx^2+2*x*Dy*h ]
    [    9*y^2*Dx*Dy+3*y*Dx*h^2+4*x^2*Dy*h ]
    [    27*y^3*Dy^2+27*y^2*Dy*h^2-3*y*h^4-8*x^3*Dy*h ]
  ]
  [
    [    9*y^2*Dy+3*y*h^2 , 0 , 2*x , 1 ]
    [    -4*x^2*Dy*h , 0 , -3*y*Dy+4*h^2 , Dx ]
    [    2*x*Dy*h , 3*y*Dy-2*h^2 , Dx , 0 ]
    [    3*y*Dx , -2*x , 1 , 0 ]
  ]
  [
    [    -Dx , 1 , 2*x , 3*y*Dy-2*h^2 ]
  ]
 ]
```

EXAMPLE 5.2: $I = F^h\,[\mathrm{Ann}\,(D(x^3+y^3+z^3)^{-1})]$

Resolution type	Betti numbers
Schreyer	1, 12, 44, 75, 70, 39, 13, 2
$(-1,-2,-3,1,2,3)$-minimal	1, 4, 5, 2
minimal	1, 4, 5, 2

$(-1,-2,-3,1,2,3)$-minimal resolution

```
 [
  [
    [    x*Dx+y*Dy+z*Dz-3*h^2 ]
    [    y*Dz^2-z*Dy^2 ]
    [    x*Dz^2-z*Dx^2 ]
    [    x*Dy^2-y*Dx^2 ]
```

```
  ]
  [
    [    0 , -x , y , -z ]
    [    -x*Dz^2+z*Dx^2 , x*Dy , x*Dx+z*Dz-3*h^2 , z*Dy ]
    [    -x*Dy^2+y*Dx^2 , -x*Dz , y*Dz , x*Dx+y*Dy-3*h^2 ]
    [    -y*Dz^2+z*Dy^2 , x*Dx+y*Dy+z*Dz-2*h^2 , 0 , 0 ]
    [    0 , Dx^2 , -Dy^2 , Dz^2 ]
  ]
  [
    [    -x*Dx+3*h^2 , y , -z , -x , 0 ]
    [    -Dz^3-Dy^3 , -Dy^2 , Dz^2 , Dx^2 , -x*Dx-y*Dy-z*Dz ]
  ]
 ]
Degree shifts
[ [ 0 ]  , [  0 , 4 , 5 , 3 ] , [  3 , 5 , 6 , 4 , 9 ] , [  3 , 12 ]  ]
```

EXAMPLE 5.3: $I = F^h\,[\mathrm{Ann}\,(D(x^3 - y^2z^2 + y^2 + z^2)^{-1})]$.

Resolution type	Betti numbers
Schreyer	1, 13, 43, 50, 21, 2
$(-\mathbf{1},\mathbf{1})$-minimal	1, 7, 10, 4
minimal	1, 7, 10, 4

```
Degree Shifts
 [[ 0 ] , [ 2 , 2 , 2 , 2 , 2 , 2 , 2 ] ,
  [ 1 , 2 , 2 , 2 , 2 , 3 , 4 , 4 , 4 , 4 ] , [ 1 , 3 , 4 , 6 ]];
```

Put $f = x^3 - y^2z^2 + y^2 + z^2$. Then the dimensions of the singular cohomology groups $H^i(\mathbf{C}^3 \setminus V(f), \mathbf{C})$ are $\dim H^0 = 1$, $\dim H^1 = 1$, $\dim H^2 = 0$, $\dim H^3 = 8$. They are evaluated by applying the method of Oaku and Takayama (1999) to the $(-\mathbf{1},\mathbf{1})$-minimal free resolution. The method reduces the evaluation of the dimensions of singular cohomology groups to that of the dimensions of cohomology groups of a complex of vector spaces. The maximal integral root of the b-function of $D/I|_{h=1}$ with respect to $(-\mathbf{1},\mathbf{1})$ is 3 and the dimensions of vector spaces of the complex are $20, 28, 27, 11$. If we use the Schreyer resolution, these dimensions are $20, 49, 87, 73, 28, 5$.

EXAMPLE 5.4: $I = D^{(h)}\cdot\{x_1\partial_1 + 2x_2\partial_2 + 3x_3\partial_3, \partial_1^2 - \partial_2 h, -\partial_1\partial_2 + \partial_3 h, \partial_2^2 - \partial_1\partial_3\}$. This is a homogenization of the GKZ hypergeometric system associated with $A = (1,2,3)$ and $\beta = 0$ (see Saito et al. (1999) on GKZ systems).

Resolution type	Betti numbers
Schreyer	1, 10, 25, 23, 8, 1
$(-\mathbf{1},\mathbf{1})$-minimal	1, 4, 5, 2
minimal	1, 4, 5, 2

$(-\mathbf{1},\mathbf{1})$-minimal resolution

```
 [
  [
    [    x1*Dx1+2*x2*Dx2+3*x3*Dx3 ]
```

```
    [     Dx1^2-Dx2*h ]
    [     -Dx1*Dx2+Dx3*h ]
    [     Dx2^2-Dx1*Dx3 ]
  ]
  [
    [     Dx1*Dx2-Dx3*h , -x1*Dx2 , 2*x2*Dx2+3*x3*Dx3+3*h^2 , -x1*h ]
    [     Dx1^2-Dx2*h , -x1*Dx1-3*x3*Dx3-2*h^2 , 2*x2*Dx1 , 2*x2*h ]
    [     Dx2^2-Dx1*Dx3 , x1*Dx3 , x1*Dx2 , -2*x2*Dx2-3*x3*Dx3-4*h^2 ]
    [     0 , Dx3 , Dx2 , Dx1 ]
    [     0 , -Dx2 , -Dx1 , -h ]
  ]
  [
    [     Dx2 , -Dx3 , -Dx1 , -2*x2*Dx2-3*x3*Dx3-4*h^2 , -x1*Dx2-2*x2*Dx3 ]
    [     -Dx1 , Dx2 , h , -x1*h , -3*x3*Dx3-h^2 ]
  ]
 ]
Degree shifts
[ [  0 ]  , [ 0 , 2 , 2 , 2 ] , [ 2 , 2 , 2 , 3 , 3 ] , [  3 , 3 ]  ]
```

On the other hand, the Koszul complex of the homogenization of the affine toric ideal associated with the matrix $(1,2,3)$ induces the double complex

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (D^{(h)})^2 & \xrightarrow{d_2} & (D^{(h)})^3 & \xrightarrow{d_1} & D^{(h)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u_2} & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_0} & & \\
0 & \longrightarrow & (D^{(h)})^2 & \xrightarrow{d_2} & (D^{(h)})^3 & \xrightarrow{d_1} & D^{(h)} & \longrightarrow & 0.
\end{array}$$

Here we denote by d_i the minimal free resolution of the homogenization of the affine toric ideal associated with the 1×3 matrix A:

$$d_1 = \begin{pmatrix} \partial_1^2 - \partial_2 h \\ -\partial_1\partial_2 + \partial_3 h \\ \partial_2^2 - \partial_1\partial_3 \end{pmatrix}, \quad d_2 = \begin{pmatrix} -\partial_2 & -\partial_1 & -h \\ \partial_3 & \partial_2 & \partial_1 \end{pmatrix}.$$

Put $\ell = x_1\partial_1 + 2x_2\partial_2 + 3x_3\partial_3$ and define u_i as

$$u_2 = \begin{pmatrix} \ell + 4h^2 & 0 \\ 0 & \ell + 5h^2 \end{pmatrix}, \quad u_1 = \begin{pmatrix} \ell + 2h^2 & 0 & 0 \\ 0 & \ell + 3h^2 & 0 \\ 0 & 0 & \ell + 4h^2 \end{pmatrix}, \quad u_0 = (\,\ell\,).$$

Then the associated single complex is

$$L^1 \ni f \longmapsto (-d_2(f), u_2(f)) \in L^2 \oplus L^1,$$
$$L^2 \oplus L^1 \ni (f,g) \longmapsto (-d_1(f), u_1(f) + d_2(g)) \in L^3 \oplus L^2,$$
$$L^3 \oplus L^2 \ni (f,g) \longmapsto u_0(f) + d_1(g) \in L^3$$

with $L^1 = (D^{(h)})^2$, $L^2 = (D^{(h)})^3$, $L^3 = D^{(h)}$. This is also a $(-\mathbf{1},\mathbf{1})$-minimal free resolution of $D^{(h)}/I$. It can be checked by using `Kan/k0` and Proposition 3.1.

It would be an interesting problem to consider minimal free resolutions of GKZ hypergeometric systems systematically.

EXAMPLE 5.5: We consider the GKZ hypergeometric system associated with

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

and $\beta = (0, 0, 0, 0)$. The Betti numbers of a $(u, v) = (-\mathbf{1}, \mathbf{1})$-minimal free resolution of this system are as follows.

Resolution type	Betti numbers
Schreyer	1, 21, 132, 331, 431, 319, 134, 30, 3
$(-\mathbf{1}, \mathbf{1})$-minimal	1, 7, 20, 30, 25, 11, 2
minimal	1, 7, 20, 30, 25, 11, 2

```
Degree Shifts:
 [
   [ 0 ]
   [ 0,0,0,2,2,2,0 ]
   [ 0,0,0,2,0,0,2,0,2,2,2,3,3,2,2,2,2,2,2,2 ]
   [ 2,2,2,0,2,0,2,3,2,3,2,2,3,2,2,0,2,2,2,2,2,3,2,0,3,2,2,3,3,3 ]
   [ 3,3,3,3,2,2,2,3,2,3,2,3,0,2,3,2,3,2,2,2,3,3,2,2,3 ]
   [ 3,2,3,2,3,2,3,3,3,3,3 ]
   [ 3,3 ]
 ]
```

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