Algorithms for D-modules and Numerical Analysis

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What is D?

- Let D be the ring of differential operators of n variables

$$D = \mathbf{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

where

$$x_i x_j = x_j x_i, \partial_i \partial_j = \partial_j \partial_i, \partial_i x_j = x_j \partial_i + \delta_{ij}.$$

D acts on spaces of functions by

$$x_i \bullet F(x) = x_i F(x), \partial_i \bullet F(x) = \frac{\partial F}{\partial x_i}.$$

Example: $\partial_1 x_1 = x_1 \partial_1 + 1$, $\partial_1 x_1^2 = x_1^2 \partial_1 + 2x_1$.

- A system of linear partial differential equations is a left ideal I in D;

$L_1 \bullet f = \cdots = L_m \bullet f = 0 \Rightarrow I = DL_1 + \cdots + DL_m$ (a left ideal).

Algorithms for D-modules

$\overset{Applications}{\Longrightarrow}$

- Algebraic Geometry (de Rham cohomology, b-functions, ...)

  Saito,Sturmfels, Takayama : Gröbner Deformations

  of Hypergeometric Differential Equations, 2000, Chapter 5.

  Oaku, D-modules and Computational Mathematics, 2002

R.Baloul, A.Leykin, Y.Nakamura, H.Tsai, U.Walther, ...

- Hypergeometric Functions

  [SST], Chapters 1, 2, 3, and 4.

(E.Miller), M.Hertillo, L.Matsusevich, T.Tsushima, ...

- Automatic theorem proving of special function identities

  Petkovsek, M., Wilf, H.S., Zeilberger, D., $A = B$ 1996

- Numerical Analysis (New!): Preprocessing for numerical analysis of systems of linear differential equations

  $\ni$ hypergeometric differential equations, holonomic system

Q. What is hypergeometric function?

We call a definite integral of a product of exponential functions and power functions (with parameters) a hypergeometric function.

Example:

$$F(\alpha, \beta, \gamma; x) = \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha} dt$$

Example:

$$F(\beta_1, \dots, \beta_d; x_1, \dots, x_n) = \int_C \exp(f(x, t_1, \dots, t_d)) t_1^{-\beta_1 - 1} \cdots t_d^{-\beta_d - 1} dt_1 \cdots dt_d$$

Here, f is a polynomial in x and t. For example, we consider $f(x,t) = x_1 t_1^3 + x_2$.

Parameters in exponents are usually written in Greek letters. Parameters appearing as coefficients of a polynomial are usually written in x, y, ...

Differential equations for hypergeometric functions

Th. (Well-known) Hypergeometric functions satisfy holonomic systems for the variable $x = (x_1, \ldots, x_n)$. In other words, any hypergeometric function is a solution of a system of linear partial differential equations in x and

$$\text{dim(the ideal generated by the principal symbols)} = n.$$

Proof. The integral kernel has a structure of holonomic D-module. The general theory of D-modules says that "integral of a holonomic D-module in the sense of D-module is again a holonomic D-module." Q.E.D.

Reference for "...". Björk, Rings of differential operators, Chapter 1. "If M is holonomic, then $M/\partial_n M$ is holonomic."

Example:

$$F(\beta; x_1, x_2) = \int_C \exp(x_1 t + x_2 t^2) t^{-\beta-1} dt$$

$$(x_1\partial_1 + 2x_2\partial_2 - \beta) \bullet F = (\partial_1^2 - \partial_2) \bullet F = 0$$

Q. Why numerical analysis of hypergeometric functions?

Numerical evaluation of functions is an elementary question, but it is a fundamental problem and is related to a lot of advanced problems.

Applications: (1) Numerical check (verification?) of formulas in a digital formula book (with Y.Tamura et al). (2) Expecting to apply for determination of monodromy groups. (3) Expecting to apply for engineering problems like the digital signal processing (with Shiraki (NTT)).

Q. Explain the outline of your numerical evaluation method.

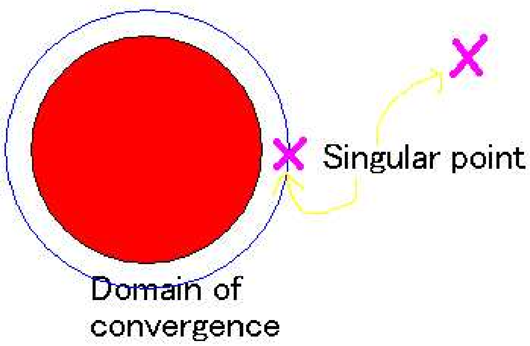
We can expect to obtain substantial information on hypergeometric functions by studying the differential equations which it satisfies, rather than dealing with the function itself.

Outline of our method

Step 1 Finding a system of partial differential equations for a given hypergeometric function

Step 2 Construction of series solutions around the singular locus.

Step 3 Finding ordinary differential equations from the system of partial differential equations to apply standard numerical methods such as the adaptive Runge-Kutta method.



We have a system of linear partial differential equations for a hypergeometric function. We want solve the system of differential equations numerically in the domain where series do not converge. However, we cannot apply standard methods of numerical analysis for the system. So, we need to translate the system into a system for which we can apply methods like the (adaptive) Runge-Kutta method.

↓

The translation can be done by Gröbner basis.

Step 3

Finding ordinary differential equations from a system of partial differential equations to apply standard numerical methods such as the adaptive Runge-Kutta method.

Q. What are initial term and initial ideal?

• For $\ell \in D$ and $u, v \in \mathbf{R}^n$ such that $u_i + v_i \geq 0$, $\mathrm{in}_{(u,v)}(\ell)$ is the sum of the heighest weight terms of ℓ with respect to (u, v). Here $u = (u_1, \ldots, u_n)$ stands for $x = (x_1, \ldots, x_n)$ and v stands for ∂.

Example: initial term

$$\mathrm{in}_{(0,0,1,1)}(\partial_1^2 + \partial_2 + \partial_1 + x_1^2 + x_2^2) = \xi_1^2 \in \mathbf{C}[x, \xi]$$

$$\mathrm{in}_{(-1,-1,1,1)}(x_1\partial_1 + 2x_2\partial_2 - 3) = x_1\partial_1 + x_2\partial_2 - 3 \in D \;\; (u + v = 0).$$

• The principal symbol $\sigma(\ell)$ is nothing but $\mathrm{in}_{(\mathbf{0},\mathbf{1})}(\ell)$

$$\mathrm{in}_{(u,v)}(I) = \mathbf{C} \cdot \{\mathrm{in}_{(u,v)}(\ell) \mid \ell \in I\} \quad \text{initial ideal}$$

D/I is called holonomic when $\dim \mathrm{in}_{(\mathbf{0},\mathbf{1})}(I) = n$.

Q. What is Gröbner basis (involutive basis)?

Example:

$$\sigma(I) = \text{in}_{(0,0,1,1)}(I) = \langle x_1\xi_1 + 2x_2\xi_2, \xi_1^2 \rangle$$

$L_1 = x_1\partial_1 + 2x_2\partial_2 - 3,\ L_2 = \partial_1^2 - \partial_2$

$L_1 + 2x_2 L_2 = x_1\partial_1 - 3 + 2x_2\partial_1^2 \to 0$

$G = \{g_1, \ldots, g_p\}$ is called Gröbner basis of I w.r.t. the weight (u, v) when

(1) I is generated by G. (2) $\text{in}_{(u,v)}(I)$ is generated by $\text{in}_{(u,v)}(G)$.

How to obtain the basis? $\to$ (0) Generate a "new element". (1) Reduce the element by a division algorithm. (2) The Buchberger criterion gives a condition to stop to generate a new element. (Buchgerger algorithm).

If we change the weight (u, v), then we can translate a set of generators into a different set of generators, which may have a nice property for a given purpose.

How to numerically solve in case of a system of polynomials?

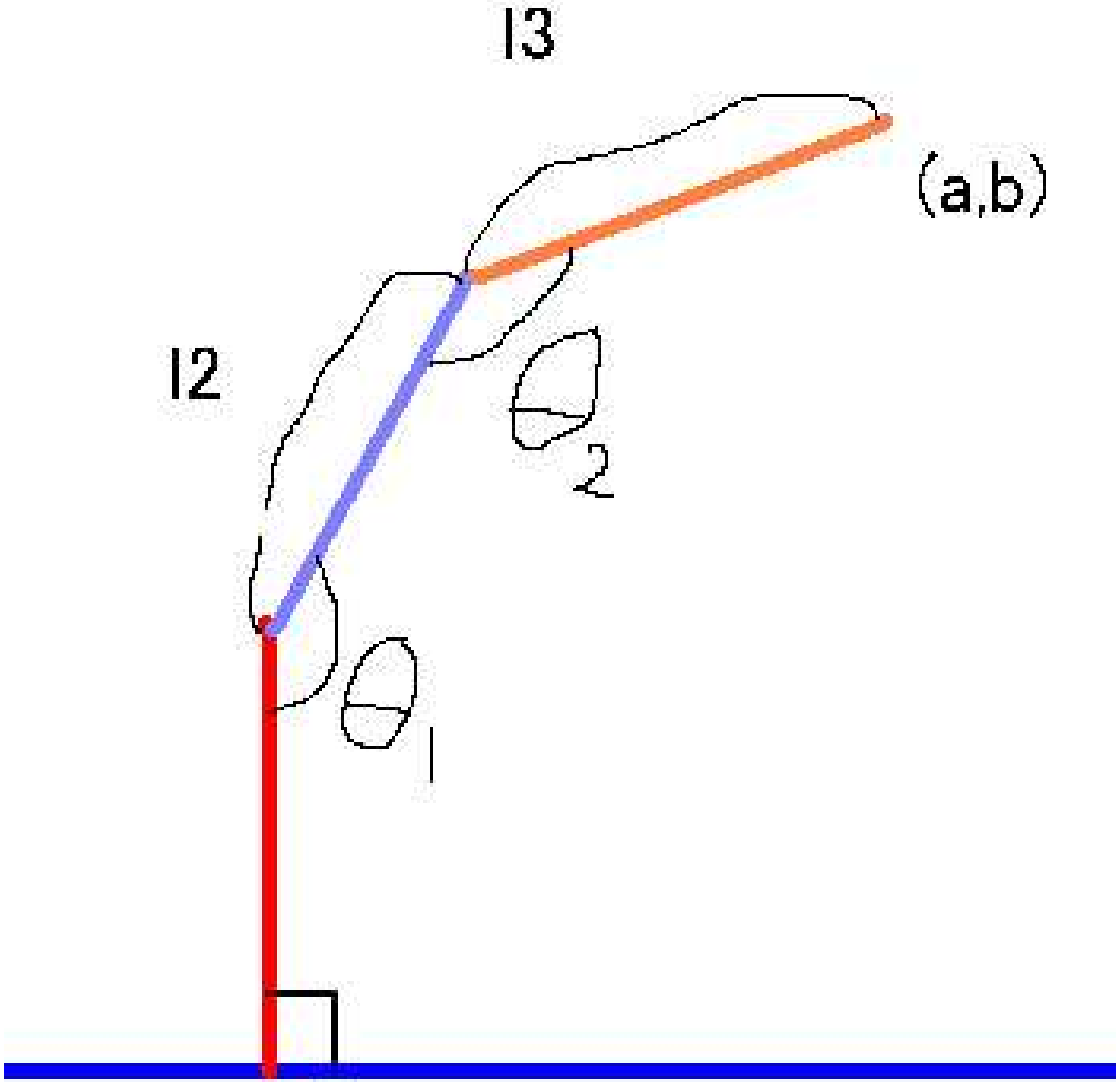
$$\begin{aligned} f_1(y_1,\ldots,y_n) &= 0 \\ &\cdots \\ f_p(y_1,\ldots,y_n) &= 0 \end{aligned}$$

J is the ideal generated by f_1, . . . , $f_p \in S_n = \mathbf{Q}[y_1,\ldots,y_n]$.

Th. (WN) If J is zero-dimensional in S_n, in other words, $\dim_{\mathbf{Q}} S_n/J < +\infty$, then $J \cap \mathbf{Q}[y_i]$ contains a non-zero element, in other words, J contains an algebraic equation in one variable $f(y_i)$. The element can be obtained by computing a Gröbner basis of J with the weight vector such that the weights for variables except y_i is 1 and the weight for y_i is 0.

$\Rightarrow$ Numerically solve $f(y_i) = 0$ (relatively easy). Determine other y_j from this solution. [Triangulation, cf. CS method].

Example: Robot arm with two joints. $c_i = \cos\theta_i, s_i = \sin\theta_i$



Example: Robot arm with two joints. $c_i = \cos\theta_i, s_i = \sin\theta_i$

```
def foo0(A,B) {
  F = [l3*(c1*c2-s1*s2)+l2*c1 - a,
       l3*(s1*c2+s2*c1)+l2*s1 - b,
       c1^2+s1^2-1,
       c2^2+s2^2-1];
  F = base_replace(F,[[l2,1],[l3,2],[a,A],[b,B]]);
  V = [c2,s2,c1,s1];
  G0 = hgr(F,V,2);
  return G0;
/* G0 = hgr(F,V,0); U = minipoly(G0,V,0,s1,s1); */
}
```

```
G2=foo0(11/10,21/10);
[-56200*s1^2+55020*s1-5061,-110*c1-210*s1+131,
  2200*s2+5620*s1-2751,-200*c2+31]
 pari(roots,G2[0]);
[ 0.10277369502480339860 0.87622986369405781120 ]
```

Q. Is there a D-analogy of this method?

- $R = \mathbf{C}(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \rangle$.

Example: $\partial_1 \left(\frac{x_1}{1-x_2} \right) \partial_1 = \left(\frac{x_1}{1-x_2} \right) \partial_1^2 + \frac{1}{1-x_2} \partial_1$.

Let I be a left ideal in D. Consider $J = RI$.

Th. (WN=well-known) If I is holonomic, then RI is zero-dimensional, i.e.,
$\dim_{\mathbf{Q}(x_1,\ldots,x_n)} R/(RI) < +\infty$.

Th. (WN) If J is zero-dimensional in R, then $J \cap \mathbf{C}\langle x, \partial_i \rangle$ contains a non-zero element (ordinary differential equation with respect to x_i),

which can be found, for example, by the Buchberger algorithm to obtain Gröbner basis with the weight vector
$(0, \ldots, 0; 1, 1, \ldots, 1, 0, 1, \ldots, 1)$

Q. Show me an example of numerical evaluation?

Example: (Bessel function in two variables, Okamoto-Kimura)

$$f(a; x, y) = \int_C \mathrm{exp}(-\frac{1}{4}t^2 - xt - y/t)t^{-a-1}dt$$

where $C = \vec{01} + \{e^{2\pi\sqrt{-1}\theta} \,|\, \theta \in [0, 2\pi]\} + \vec{10}$. The function $f(a; x, y)$ satisfies the holonomic system

$$\partial_x\partial_y + 1, \partial_x^2 - 2x\partial_x + 2y\partial_y + 2a, 2y\partial_y^2 + 2(a+1)\partial_y - \partial_x + 2x$$

The rank of the system is 3. Take $a = 1/2$. It admits a unique solution of the form $y^{-a}g(x, y)$ such that g is holomorphic at the origin and $g(0, 0) = 1$.

```
def bess2_ode_y(A) {
  F = [dx*dy+1 , dx^2-2*x*dx+2*y*dy+2*a , 2*y*dy^2+2*(a+1)*dy-dx+2*x];
  F = base_replace(F,[[a,A]]);
  G = sm1_gb([F,[x,y],[[dx,1]]]);
  return G[0];
}
```

```
bess2_ode_y(1/2);
```

```
[-dx+2*y*dy^2+3*dy+2*x,-2*y*dy^3-5*dy^2-2*x*dy-1]
```

Singular locus is $y = 0$.

Equations for g where $f = y^{-a} g$ and f satisfies the Bessel differential equations above :

```
4*y^2*dy^3+4*y*dy^2+(4*y*x-1)*dy-2*x+2*y, ...
```

Solve this ordinary differential equation numerically.

Series solution g such that $g(0,0) = 1$:

```
(1)+(-1/3*y^2)+(-2*y*x)+(1/210*y^4)+(2/15*y^3*x)+(2/3*y^2*x^2)+ ...
```

How to construct this solution? $\Longrightarrow$ disscuss later.

Q. Is it the ultimate method?

No. This method has several disadvantages. For example, finding the generator of

$$\mathbf{Q}(x_1, \ldots, x_n)\langle \partial_i \rangle \cap I$$

is time consuming and the generator often has an "apparant singularity".

Q. Is there a FGLM-like method?

Yes.

Case of polynomials

Th. Let $y^{\alpha(1)}, \ldots, y^{\alpha(r)}$ be the basis of $\mathbf{Q}[y_1, \ldots, y_n]/I$ as a vector space over $\mathbf{Q}$. There exist matrices $M_1, \ldots, M_n \in M(r, r, \mathbf{Q})$ such that

$$y_1 \begin{pmatrix} y^{\alpha(1)} \\ . \\ . \\ y^{\alpha(r)} \end{pmatrix} = M_1 \begin{pmatrix} y^{\alpha(1)} \\ . \\ . \\ y^{\alpha(r)} \end{pmatrix} \quad \text{mod } I$$

$$\cdots \qquad \cdots$$

$$\cdots \qquad \cdots$$

$$\cdots \qquad \cdots$$

$$y_n \begin{pmatrix} y^{\alpha(1)} \\ . \\ . \\ y^{\alpha(r)} \end{pmatrix} = M_n \begin{pmatrix} y^{\alpha(1)} \\ . \\ . \\ y^{\alpha(r)} \end{pmatrix} \quad \text{mod } I$$

D-analogy

Let $\partial_{\alpha(1)}, \ldots, \partial_{\alpha(r)}$ be the basis of $\mathbf{Q}(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \rangle / I$ as a vector space over $\mathbf{Q}(x_1, \ldots, x_n)$. There exist matrices $P_1, \ldots, P_n \in M(r, r, \mathbf{Q}(x))$ such that

$$\partial_1 \begin{pmatrix} \partial_{\alpha(1)} \\ \cdot \\ \cdot \\ \partial_{\alpha(r)} \end{pmatrix} = M_1 \begin{pmatrix} \partial_{\alpha(1)} \\ \cdot \\ \cdot \\ \partial_{\alpha(r)} \end{pmatrix} \mod I$$

$$\cdots \qquad \cdots$$

$$\cdots \qquad \cdots$$

$$\cdots \qquad \cdots$$

$$\partial_n \begin{pmatrix} \partial_{\alpha(1)} \\ \cdot \\ \cdot \\ \partial_{\alpha(r)} \end{pmatrix} = M_n \begin{pmatrix} \partial_{\alpha(1)} \\ \cdot \\ \cdot \\ \partial_{\alpha(r)} \end{pmatrix} \mod I$$

Put $F = (\partial^{\alpha(1)} \bullet f, \ldots, \partial^{\alpha(r)} \bullet f)^T$. Solve the ordinary differential equations

$$\begin{array}{rcl} \partial_1 \bullet F & = & P_1 F \\ \cdots & & \cdots \\ \cdots & & \cdots \\ \cdots & & \cdots \\ \partial_n \bullet F & = & P_n F \end{array}$$

$\partial_{\alpha(i)} \Rightarrow \theta^{\alpha(i)}$, $\theta_i = x_i \partial_i$ (sometimes more simpler)

Example: (Bessel differential equation in two variables, $a = 1/2$)
$F = (f, \partial_y \bullet f, \partial_x \bullet f)^T$

$$P_1 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ -1 & -2y & 2x \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{-x}{y} & \frac{-3}{2y} & \frac{1}{2y} \\ -1 & 0 & 0 \end{pmatrix}$$

$x \in [0.0, 1.4]$ (step size : 0.1), $y \in [0.2, 9.0]$,

```
2.037sec + gc : 0.4651sec
```

```
Risa/Asir, Version 20021209 (Kobe Distribution).
FreeBSD 3.4-STABLE
CPU: Pentium III/Pentium III Xeon/Celeron (1129.43-MHz 686-class CPU)
real memory  = 2147418112 (2097088K bytes)
```

Systems for D?

D-module algorithms for algebraic geometry ,
kan/sm1 (http://www.math.kobe-u.ac.jp/KAN),
Macaulay2 (http://www.math.uiuc.edu/Macaulay2).

Other packages

Ore algebra packages (F.Chyzak) in Maple (general, merged in Maple).
bfct package (M.Noro) in Risa/Asir (Efficient computation of b function).
yang package (K.Ohara) in Risa/Asir (general, simple, merged in Risa/Asir).
Plural (general, inherits Singular functions).

Step 2

Construction of series solutions around the singular locus.

Q. Why do we need to find series solutions?

Numerical recipes for hypergeometric functions

- Series solutions provide the best way for numerical evaluation out of a neighborhood of the border of the domain of convergence.
- Solving hypergeometric differential equation by the adaptive Runge-Kutta method is a reasonably nice method for numerical evaluation.
- Numerical integration (e.g. Barns integral representation) is sometimes useful.

- Use of series solution is necessary around the singular locus.

(Conditions for solutions are often given as an asymptotic properties.)

J.Van der Hoeven : Journal of Symbolic Computation (2001), 717–743. Numerical evaluation of holonomic functions in one variable. Complexity of numerical analytic continuation by the binary splitting algorithm.

Q. Show me an example in which series is better than numerical integratio

Example: Evaluate numerical value of the left hand side

$$F(\frac{1}{12},\frac{5}{12},\frac{1}{2};\frac{1323}{1331}) = \frac{3}{4}\sqrt[4]{11} \quad \text{(Beukers, 1993)}$$

where

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt$$

- Numerical Integration:

$$\begin{aligned} & F(\frac{1}{12},\frac{5}{12},\frac{1}{2};\frac{1323}{1331}) \\ = \quad & -\frac{5551469346902918931708093 21}{77265229938688}F(-\frac{31}{12},\frac{37}{12},\frac{13}{2};\frac{1323}{1331}) \\ & +\frac{2300849705553019085468253 1919}{401779195681 1776}F(-\frac{31}{12},\frac{37}{12},\frac{15}{2};\frac{1323}{1331}) \end{aligned}$$

It takes about 9 seconds to get the value in the accuracy 10^{-4}.

• Solving differential equation:
$F(a,b,c;z)$ satisfies the differential equation:

$$z(1-z)f'' + (c-(a+b+1)z)f' - abf = 0$$

$f(0) = 1$.
It takes about 2 seconds to get the value in the accuracy 10^{-4} by the adaptive Ruge-Kutta method (4th order).

• Evaluating series which is obtained from the differential equation:

$$F(a,b,c;x) = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(1)_n (c)_n} x^n,$$

$$(a)_n = a(a+1)\cdots(a+n-1).$$

It takes less than 1 second to get the value in the accuracy 10^{-4}

Q. How to construct series solutions around singularities?

In case of 1 variable, the Frobenius method and the Hukuhara-Turritin reduction are explained in text books (cf. DEtools in Maple). How about several variable case?

Th. (Hosono, Lian, S.T.Yau, 1996, 97) Construction of series solutions of GKZ systems for reflexive polytopes and a degenerate β. The main idea of the construction is the use of $\mathrm{in}_{(-w,w)}(I_A)$.
Th. (SST, 1998. cf. vanishing cycle sheaf: Kashiwara, Laurent). Suppose that D/I is regular holonomic. The rank of I is equal to the rank of $\mathrm{in}_{(-w,w)}(I)$.
Th. (SST, 2000). Suppose that D/I is regular holonomic. A construction algorithm of series solution basis of I at the boundary of $(\mathbf{C}^*)^n$. The main idea of the construction is the use of $\mathrm{in}_{(-w,w)}(I)$ (theorem above).

- (Open Question) Construct series solutions around the irregular singular point of holonomic system with the help of Villamayor's algorithm for resolution of singularities (cf. Majima, Bodnar, Schicho).

Step 1

Finding a system of partial differential equations for a given hypergeometric function (integral)

Q. What is holonomic function?

• A muli-valued analytic function f defined on (the universal covering of) $\mathbf{C}^n \setminus S$ with an algebraic set S of $\mathbf{C}^n$ is called a holonomic function if there exists a left ideal I of D so that $M = D/I$ is a holonomic system and $Pf = 0$ holds on $\mathbf{C}^n \setminus S$ for any $P \in I$.

Example: Hypergeometric function is a holonomic function.

Operations for holonomic functions

Problems:

1. Given two holonomic functions f, g and two differential operators P, Q, find a holonomic system which the function $Pf + Qg$ satisfies;
2. Given two holonomic functions f, g, find a holonomic system which the function fg satisfies;
3. Given a holonomic function $f(t,x)$, find a holonimic system which the integral $\int_C f(t,x)\,dt$ satisfies.

- Partial algorithms for computing differential equations for sums, products, and integrals of holonomic functions. cf. The book $A = B$ by Petkovsek, M., Wilf, H.S., Zeilberger, D.. A lot of interesting and mathematically important examples.

Th. (Oaku-T) These constructions can be algorithmically done. If the inputs are holonomic, then the output is holonomic.

The algorithm is based on a restriction/integration algorithm of D-modules by Oaku (1997) and a localization algorithm by Oaku, Takayama, Walther (2001). We use the Buchberger algorithm, especially the initial ideal $\mathrm{in}_{(-w,w)}(I)$ plays an inportant role.

Proof sketch is written in the proceedings. The core of this algorithm is the integration algorithm.

Q. What is the integral of a D-module?

Let I be a left ideal of D_{n+m}. $M = D_{n+m}/I$ is a left D_{n+m}-module.

Definition (integral of M, J.Bernstein (1971), ...):

$$\int M dx_{n+1} \cdots dx_{n+m} := D_{n+m}/(I + \partial_{n+1} D_{n+m} + \cdots + \partial_{n+m} D_{n+m})$$

Th. (Oaku) If $M = D_{n+m}/I$ is holonomic and $(J, k_1) = \mathtt{integral0}(I)$, then

$$\int M dx_{n+1} \cdots dx_{n+m} \simeq \left(\sum_{\alpha_{n+1}+\cdots+\alpha_{n+m} \leq k_1} D_n x^\alpha \right) / J$$

Cor. Suppose that $f(x_1, \ldots, x_{n+m})$ is a holonomic function and $I \bullet f = 0$. Put $J_0 = J \cap D_n$ where $(J, k_1) = \mathtt{integral0}(I)$. Then, D_n/J_0 is holonomic and

$$J_0 \bullet \int_C f(x_1, \ldots, x_{n+m}) dx_{n+1} \cdots dx_{n+m} = 0$$

for a suitable cycle C.

Q. Show me an example of the integral of a D-module?

Put $t = x_3$, $f(x_1, x_2, t) = \exp(x_1 t^2 + x_2 t)$,
$I = \langle \partial_t - (2tx_1 + x_2), \partial_1 - t^2, \partial_2 - t \rangle$.
Gröbner basis for $(0, 0, 1, 0, 0, -1)$:

```
[ -t +Dx2, -2*t*x1 -x2*h+Dt*h, -t^2 +Dx1*h, 2*x1*Dx2+x2*h -Dt*h,
  -Dx2^2+Dx1*h, -2*x1*Dx1*h-x2*Dx2*h-h^3 +Dt*Dx2*h ]
```

$b(s) = s$
integral0(I) $= (\{2x_1\partial_2 + x_2, -\partial_2^2 + \partial_1, -2x_1\partial_1 - x_2\partial_2 - 1\}, 0)$

$$F(x_1, x_2) = \int_C \exp(x_1 t^2 + x_2 t) dt$$

is annihilated by $\{2x_1\partial_2 + x_2, -\partial_2^2 + \partial_1, -2x_1\partial_1 - x_2\partial_2 - 1\}$.

Notations for `integral0`

- Weight $(w, -w)$

$x_1, \ldots, x_n$:0 , $x_{n+1}, \ldots, x_{n+m}$: 1,
$\partial_1, \ldots, \partial_n$:0 , $\partial_{n+1}, \ldots, \partial_{n+m}$: -1

- Left normally ordered expression

$$\mathrm{lno}(\ell) = \sum a_{pq} x^p \partial^q, \quad \ell \in D, \ell = \sum a_{pq} x^p \partial^q \text{ in } D$$

- Right normally ordered expression

$$\mathrm{rno}(\ell) = \sum a_{pq} \partial^p x^q, \quad \ell \in D, \ell = \sum a_{pq} \partial^p x^q \text{ in } D$$

- (u, v)-order

$$\mathrm{ord}_{(u,v)}(a x^\alpha \partial^\beta) = u \cdot \alpha + v \cdot \beta$$

Q. What is the algorithm `integral0`?

- Input: I, a left ideal in D_{n+m}.

Step 1: Compute a Gröbner basis G of I for the weight vector $(w, -w)$.

Step 2: Find the generator $b(s)$ of

$$\mathrm{in}_{(w,-w)}(I) \cap K[s], \quad s = -(\partial_{n+1}x_{n+1} + \cdots + \partial_{n+m}x_{n+m})$$

Step 3: Let k_1 be the maximal non-negative integral root of $b(s) = 0$. If there is no such k_1, then return $(\{1\}, 0)$, else

$$J = \{\mathrm{rno}(x^\alpha g_i) \,|\, \mathrm{ord}_{(w,-w)}(x^\alpha g_i) \leq k_1, \; g_i \in G\}_{|\partial_{n+1}=\cdots=\partial_{n+m}=0}$$

where $x^\alpha = x_{n+1}^{\alpha_{n+1}} \cdots x_{n+m}^{\alpha_{n+m}}$.
return (J, k_1).

Note that J is a submodule in D_n free module of finite rank

$$\sum_{\alpha_{n+1}+\cdots+\alpha_{n+m} \leq k_1} D_n x^\alpha$$

References for integral0

- H.Wilf, D.Zeilberger, An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. Invent. Math. 108 (1992), 575–633. (yielded a lot of interesting examples, but heuristic method)
- T.Oaku, Algorithms for b-functions, restrictions, and algebraic local cohomology groups of D-modules. Advances in Applied Mathematics **19** (1997), 61–105. (0-th integral (integral0))
- M.Saito, B.Sturmfels, N.Takayama, Gröbner Deformations of Hypergeometric Differential Equations. (Book), Algorithms and Computation in Mathematics **6**, Springer, 2000. (General weight vector, Speed up of Step 1)
- T.Oaku, N.Takayama, Algorithms for D-modules – restriction, tensor product, localization, and algebraic local cohomology groups. Journal of Pure and Applied Algebra 156 (2001), 267–308. (k-th integral, cohomological integration)
- M.Noro, An efficient modular algorithm for computing the global b-function, proceedings ICMS 2002, 147–157. (Speed up of Step 2)

Q. Why localization?

Th. (WN) If I is holonomic, then RI is zero-dimensional.

Zero-dimensionality does not imply holonomy.

Example: Find a holonomic annihilating ideal of $(x_1^3 - x_2^2)^{-1}$.
Trivila annihilating operators $(x_1^3 - x_2^2)\partial_1 - 3x_1^2, (x_1^3 - x_2^2)\partial_2 + 2x_2$ is not holonomic, but $\dim_{K(x_1,x_2)} R/RI = 1$ (zero-dimensional).
$\langle (x_1^3 - x_2^2)\xi_1, (x_1^3 - x_2^2)\xi_2 \rangle$ (characteristic ideal. Dimension is 3. It is not holonomic.)

- For given zero-dimensional I, can we construct $\tilde{I}$ such that $\tilde{I} \subseteq I$ and $\tilde{I}$ is holonomic? $\Rightarrow$ Yes.

Example: $(x_1^3 - x_2^2)\partial_1 - 3x_1^2, \quad (x_1^3 - x_2^2)\partial_2 + 2x_2,$
$2x_2\partial_1 + 3x_1^2\partial_2, \quad 2x_1\partial_1 + 3x_2\partial_2 + 6 \qquad$ (Missing equations)

Q. What is the localization algorithm?

Input: $\{\ell_1, \ldots, \ell_p\}$ generators of a zero-dimensional ideal and annihilates a function g.
Output: an ideal $\tilde{I}$ such that $D_n/\tilde{I}$ is holonomic.
x_{n+1} : a new variable.
f : $f = 0$ contains the non-holonomic point (,i.e., point where the local dimension of the characteristic variety $> n$).

$$\phi \ : \ D \ni P \mapsto P_{|\partial_i \mapsto \partial_i - x_{n+1}^2 f_i \partial_{n+1}}, \quad f_i = \frac{\partial f}{\partial x_i}$$

Step 1: Put $I_{n+1} = \{\phi(\ell_1), \ldots, \phi(\ell_m), 1 - f x_{n+1}\}$. Call `integral0`$(I_{n+1})$ with respect to x_{n+1}.
Step 2: Let (J, k_1) be the output of Step 1. Compute $\tilde{J} = J \cap D \cdot x_{n+1}^{k_1}$. ($\tilde{J}$ annihilates g/f^{k_1+2}.)
Step 3: Compute $\tilde{I}$ from $\tilde{J}$.

O-T-Walther (2000), cf. H.Tsai (2001).

Example of localization

Let $n = 3$, and consider the ideal J generated by the system

$$
\begin{aligned}
(x^3 - y^2z^2)^2\partial_x &+ 3x^2, \\
(x^3 - y^2z^2)^2\partial_y &- 2yz^2, \\
(x^3 - y^2z^2)^2\partial_z &- 2y^2z.
\end{aligned}
$$

These operators are annihilators of the exponential function $e^{1/f}$ where $f(x, y, z) = x^3 - y^2z^2$, but it is not holonomic.
Applying the localization algorithm, we obtain

$$
\begin{aligned}
&-3y\partial_y + 3z\partial_z,\ -2xyz^2\partial_x - 3x^3\partial_y - 4yz^2,\ -2xy^2z\partial_x - 3x^3\partial_z - 4y^2z, \\
&6xz^3\partial_x\partial_z + 9x^3\partial_y^2 + 6xz^2\partial_x + 6yz^2\partial_y + 6z^3\partial_z + 12z^2, \\
&-6y^2z^3\partial_z + 4x^4\partial_x + 12x^3z\partial_z + 8x^3 + 12, \\
&6yz^4\partial_z^2 - 4x^4\partial_x\partial_y - 12x^3z\partial_y\partial_z + 18yz^3\partial_z - 8x^3\partial_y - 12\partial_y, \\
&8x^5\partial_x^2 + 24x^4z\partial_x\partial_z + 18x^3z^2\partial_z^2 + 64x^4\partial_x + 102x^3z\partial_z + 80x^3 + 24x\partial_x + 48 \\
&-6z^5\partial_z^3 + 4x^4\partial_x\partial_y^2 + 12x^3z\partial_y^2\partial_z - 36z^4\partial_z^2 + 8x^3\partial_y^2 - 36z^3\partial_z + 12\partial_y^2
\end{aligned}
$$

which annihilate the function $x^{-2}e^{1/f}$ and holonomic.

Application to Digital Formula Book

Q. Are there mathematical formula books on computer?

(1) http://functions.wolfram.com
(2) Sasaki's formula book project (1980's) based on the formula book from Iwanami publ. co. (cont. Morinaga, Murakami 2003)
(3) ...

What is OpenMath?

OpenMath:

The OpenMath is an XML application to describe mathematical objects.

Emphasis on semantics.

Started about 1997. EU project.

http://www.openmath.org/

Example: $1 + x$

In TEX, `$1+x$`

In presentation MathML (http://www.w3c.org),

```
<mrow>
  <mn>1</mn>
  <mo>+</mo>
  <mi>x</mi>
</mrow>
```

These expressions lose some semantic information.

OpenMath keeps semantic information and it is extensible by using content dictionaries.

OpenXM/fb project based on OpenMath

(Thesis by Y.Tamura (2003))

Digital formula book project OpenXM/fb.

http://www.openxm.org

The project is editing a formula book for hypergeometric functions by utilizing OpenMath XML.

```
<?xml version="1.0" encoding="ISO-2022-JP"?>
<formula>
  <title>
    Quadratic transformation of an independent variable
  </title>
  <author> E. Goursat </author>
  <editor> Yasushi Tamura </editor>

  <tfb macroset="http://www.openxm.org/fb/hfb.txt">
    2 * arith1.root(nums1.pi,2)
      * hypergeo0.gamma(a + b + (1 / 2))
      / hypergeo0.gamma(a + (1 / 2))
      / hypergeo0.gamma(b + (1 / 2))
      * hypergeo1.hypergeometric2F1(a,b,1 / 2,x)
    ~relation1.eq~
    (hypergeo1.hypergeometric2F1(2 * a, 2 * b, a + b + (1 / 2),
```

```
        1 + arith1.root(x,2) / 2)
      + hypergeo1.hypergeometric2F1(2 * a, 2 * b, a + b + (1 / 2),
          1 ~arith1.minus~ arith1.root(x,2) / 2));
  </tfb>

  <description>
    Quadratic transformation of independent variable
  </description>
  <reference>
    <xref uri="http://www.openxm.org/fb/bib-goursat1.xml"
      linkend="goursat1" page="118"/>
  </reference>
  <evidence checker="Mathematica">
    @@ /. {a->1/2,b->3/5,c->-2/11,x->0.2}
  </evidence>
</formula>
```

This code describes the formula

$$\frac{2\sqrt{\pi}\,\Gamma(a+b+\frac{1}{2})\,{}_2F_1(a,b,\frac{1}{2},x)}{\Gamma(a+\frac{1}{2})\,\Gamma(b+\frac{1}{2})} = {}_2F_1(2\,a,2\,b,a+b+\frac{1}{2},\frac{1-\sqrt{x}}{2})+{}_2F_1(2\,a,2\,b,a+b+\frac{1}{2},\frac{1+\sqrt{x}}{2})$$

which was found by E.Goursat more than a century ago.

Summary

Outline of our evaluation method for HG functions

- Finding a system of partial differential equations for a given hypergeometric function
- Construction of series solutions around the singular locus.
- Finding ordinary differential equations from the system of paritial differential equations to apply standard numerical methods such as the adaptive Runge-Kutta method.

Todo:
(1) Efficiency both in algebraic part and numerical part should be studied more.
(2) Estimation of numerical error.
(3) Series expansion at irregular singular points.
(4) Examples in signal processing.