

DISTRIBUTIONS OF DISCRIMINANTS OF CUBIC ALGEBRAS

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ABSTRACT. We study the space of binary cubic and quadratic forms over the ring of integers \mathcal{O} of an algebraic number field k . By applying the theory of prehomogeneous vector spaces founded by M. Sato and T. Shintani, we can associate the zeta functions to these spaces. Applying these zeta functions, we derive some density theorems on the distributions of discriminants of cubic algebras over \mathcal{O} . In the case k is a quadratic field, we give a correction term as well as the main term. These are generalizations of Shintani's asymptotic formulas of the mean values of class numbers of binary cubic forms over \mathbb{Z} .

1. INTRODUCTION

Let k be a number field and \mathcal{O} the ring of integers of k . Let r_1 and r_2 be the number of real and complex places of k . We denote by Δ_k , h_k and $\zeta_k(s)$ the absolute discriminant, the class number and the Dedekind zeta function of k , respectively. We put

$$\mathfrak{A}_k := (\text{Res}_{s=1} \zeta_k(s)) \cdot \frac{\zeta_k(2)}{2^{r_1+r_2+1}}, \quad \mathfrak{B}_k := (\text{Res}_{s=1} \zeta_k(s)) \cdot \frac{3^{r_1+r_2/2} \zeta_k(1/3)}{5 \cdot 2^{r_1+r_2} \Delta_k^{1/2}} \left(\frac{\Gamma(1/3)^3}{2\pi} \right)^{[k:\mathbb{Q}]}$$

For $0 \leq i \leq r_1$, let $h_i(n)$ be the numbers of the following set:

$$h_i(n) := \# \left\{ (R, F) \left| \begin{array}{l} F \text{ is a cubic extension of } k \text{ with } r_1 + 2i \text{ real places,} \\ R \text{ is an order of } F \text{ containing } \mathcal{O}, \text{ and } N(\Delta_{R/\mathcal{O}}) = n. \end{array} \right. \right\}.$$

Here $\Delta_{R/\mathcal{O}}$ is the relative discriminant of R/\mathcal{O} (which is an integral ideal of \mathcal{O}) and $N(\Delta_{R/\mathcal{O}})$ is its ideal norm. Note that we count pairs (R, F) up to isomorphism. One of the primary purpose of this paper is to investigate the function $\sum_{n < X} h_i(n)$ as $X \rightarrow \infty$. Here we state our result when k is a quadratic field.

Theorem 1.1 (Theorem 7.20). *Let k be a quadratic field. For any $\varepsilon > 0$,*

$$\binom{r_1}{i}^{-1} \sum_{n < X} h_i(n) = 3^{-i-r_2} \mathfrak{A}_k X + 3^{-i/2} \mathfrak{B}_k X^{5/6} + O(X^{9/11+\varepsilon}) \quad (X \rightarrow \infty),$$

where $\binom{r_1}{i}$ is the binomial coefficient.

The case $k = \mathbb{Q}$ is known by Shintani [Sh75]. For the case $[k : \mathbb{Q}] \geq 3$, see Theorem 7.20.

We explain one more theorem we prove in this paper. We call a finite \mathcal{O} -algebra a *cubic algebra* if it is projective of rank 3 as an \mathcal{O} -module. We denote by $\mathcal{C}(\mathcal{O})$ the set of isomorphism classes of cubic algebras of \mathcal{O} . For a fractional ideal \mathfrak{a} of k , we put $\mathcal{C}(\mathcal{O}, \mathfrak{a}) = \{R \in \mathcal{C}(\mathcal{O}) \mid \bigwedge^3 R \cong \mathfrak{a}\}$. It is known that $\mathcal{C}(\mathcal{O}, \mathfrak{a})$ depends only on the ideal class of \mathfrak{a} and that $\mathcal{C}(\mathcal{O}) = \coprod_{\mathfrak{a} \in \text{Cl}(k)} \mathcal{C}(\mathcal{O}, \mathfrak{a})$ (we use the same symbol \mathfrak{a} to denote its ideal class.) In general for a projective \mathcal{O} -module M of rank m , the class of the ideal isomorphic to $\bigwedge^m M$ is called the *Steinitz class* of M .

We count the number of $\mathcal{C}(\mathcal{O}, \mathfrak{a})$ for each \mathfrak{a} . More precisely, for $0 \leq i \leq r_1$ we count $\mathcal{C}(\mathcal{O}, \mathfrak{a})_i = \{R \in \mathcal{C}(\mathcal{O}, \mathfrak{a}) \mid R \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{r_1+2i} \times \mathbb{C}^{3r_2+r_1-i}\}$. An interesting phenomenon we prove in the case k is a quadratic field is that the Steinitz class is *not uniformly distributed* in the $X^{5/6}$ -term if $\text{Cl}(k)$ contains a non-trivial 3-torsion element.

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Theorem 1.2 (Theorem 7.8). *Let k be a quadratic field. For any $\varepsilon > 0$,*

$$\binom{r_1}{i}^{-1} \sum_{\substack{R \in \mathcal{C}(\mathcal{O}, \mathfrak{a})_i \\ N(\Delta_{R/\mathcal{O}}) \leq X}} \frac{1}{\#(\text{Aut}(R))} = \left(1 + \frac{1}{3^{i+r_2}}\right) \frac{\mathfrak{A}_k}{h_k} X + \tau(\mathfrak{a}) \frac{\mathfrak{B}_k h_k^{(3)}}{3^{i/2} h_k} X^{5/6} + O(X^{7/9+\varepsilon}) \quad (X \rightarrow \infty).$$

Here $\#(\text{Aut}(R))$ denotes the cardinality of the automorphisms of R as an \mathcal{O} -algebra and $h_k^{(3)}$ the number of 3-torsion elements of $\text{Cl}(k)$ (which is a power of 3). Also for $\mathfrak{a} \in \text{Cl}(k)$, we put $\tau(\mathfrak{a}) = 1$ if there exists $\mathfrak{b} \in \text{Cl}(k)$ such that $\mathfrak{a} = \mathfrak{b}^3$ and $\tau(\mathfrak{a}) = 0$ otherwise.

Before explaining the contents of this paper, we include a brief historical overview of the area in order to clarify the background of our results. The study of distributions of discriminants or mean values of class numbers of rings or fields extensions is a classical topic in algebraic number theory and has a long history. The roots of this topic trace to Gauss, who is the first mathematician introducing a group theoretical approach to number theory. In [Gauss], he found that the set of orbits $\text{GL}(2)_{\mathbb{Z}} \backslash \text{Sym}^2 \mathbb{Z}^2$ of integral binary quadratic forms corresponds bijectively to the set of ideal classes of quadratic rings. Using this he gave conjectures for the asymptotic property of the average number of class numbers of quadratic rings. This conjecture was first proved by Lipschitz for the imaginary case, and by Siegel for the real case. Siegel [Si44] also proved a density theorem for $\text{GL}(n)_{\mathbb{Z}} \backslash \text{Sym}^2 \mathbb{Z}^n$ in general.

If one considers cubic objects as we shall do in this paper, the most basic representation is the space of binary cubic forms $(\text{GL}(2), \text{Sym}^3 \text{Aff}^2)$. An interpretation of the set of orbits of the space of integral binary cubic forms $\text{GL}(2)_{\mathbb{Z}} \backslash \text{Sym}^3 \mathbb{Z}^2$ in terms of cubic rings was discovered by Delone-Faddeev [DF64], and many applications to number theory or representation theory have been obtained to the present. For example, this interpretation was used by Davenport and Heilbronn [DH71] to prove the density of discriminants of cubic fields $\sum_{[F:\mathbb{Q}]=3, |\Delta_F| < X} 1 \sim 3^{-1} \zeta(3)^{-1} X$ ($X \rightarrow \infty$).

In 1972, Shintani [Sh72] made a significant contribution to the study of the class numbers of $\text{GL}(2)_{\mathbb{Z}} \backslash \text{Sym}^3 \mathbb{Z}^2$ by applying the zeta function theory of prehomogeneous vector spaces founded by M. Sato and Shintani [SS74]. He gave the analytic continuations, functional equations and residue formulas of the Dirichlet series. Combined with the results of zeta functions of binary quadratic forms [Sh75], he gave a correction term [Sh75, Theorem 4] to the main term of Davenport's asymptotic formula of distributions of discriminants of irreducible cubic rings over \mathbb{Z} .

His works [Sh72, Sh75] are the starting point of our work. The zeta functions of prehomogeneous vector spaces can be defined over any algebraic number fields, and by the use of adelic language numerous contributions to number theory has been obtained. For the case of the space of binary cubic forms, a series of work by Wright [Wr85] and Datskovsky-Wright [DW86, DW88] gave the generalization of the Davenport-Heilbronn's density theorem over \mathbb{Q} above to over a general number field k with finite number of local conditions. This was recently improved by Kable-Wright [KW05] to prove an equidistribution result for the Steinitz classes of cubic extensions. In both cases what is called filtering process was used to count cubic field extensions of k which corresponds a set of rational equivalence classes, rather than the cubic algebras of \mathcal{O} which to a set of integral equivalence classes. The density problems of integral equivalence classes are not well considered other than over \mathbb{Z} , and this is what we focus in this paper. In the integral equivalence case, Landau's Tauberian theorem [L15] modified by Sato-Shintani [SS74] gives a sharp error estimate since our zeta function satisfies a functional equation. (Note that a part of Theorem 7.8 is obtained by Wright in his thesis [Wr82] as the mean value of class numbers.)

We first prove Theorem 1.2 (Theorem 7.8) and after that Theorem 1.1 (Theorem 7.20). The step is to separate the reducible algebras of $\mathcal{C}(\mathcal{O})$, i.e., those $R \in \mathcal{C}(\mathcal{O})$ with $R \otimes_{\mathcal{O}} k$ not a field. As in the case of \mathbb{Z} treated in [Sh75], these are parameterized essentially by

$(B(2), \text{Sym}^2 \text{Aff}^2)$ where $B(2)$ is the Borel subgroup of $GL(2)$ consisting of lower triangular matrices. We generalize the argument of [Sh75] to general number fields both algebraically and analytically.

To the author's best knowledge, Theorem 1.2 is the first unequidistribution result of the Steinitz classes. We mention that the idea that the character in the zeta function segregates cubic algebras by their Steinitz classes is due to Kable-Wright [KW05]. For a conjecture of distributions of Steinitz classes for fields, see [KW05, Introduction].

After the summary of the theory of binary cubic forms, a brief illustration of some recent progress and arising problems of related topics is appropriate. The parameterizations of extensions degree 2, 3, 4 and 5 was first systematically established in the celebrated work of Wright-Yukie [WY92] via 8 prehomogeneous representations containing $(GL(2), \text{Sym}^2 \text{Aff}^2)$ and $(GL(2), \text{Sym}^3 \text{Aff}^2)$ as basic cases. A program to prove a series of density theorems was proposed, and besides, deep contributions to algebraic number theory were indicated. After Wright-Yukie, the arithmetic theory of \mathbb{Z} -orbits was first handled by Kable [K00] for one special case, and recently being developed greatly by Bhargava [B04a, B04b, B04c, B05]. In his published works, he gave generalizations of Gauss' and Delone-Faddeev's orbit ring maps in the degree 2, 3 and 4 case and also obtained the main term of the density of discriminants of quartic fields and rings. He also announced some results for the quintic case. On the other hand the quintic case is also handled slightly earlier by Kable-Yukie [KY04b, KY04a, KY05] and an upper bound of the number of quintic fields ordered by the size of discriminants is obtained.

All of the 8 prehomogeneous vector spaces treated in [WY92] are what we call as *parabolic type* classified by Rubenthaler in his thesis [Ru82]. These are obtained by choosing a semisimple group and a maximal parabolic subgroup. For details, see [WY92], [Y93] or their references also. The case of binary cubic form corresponds to the exceptional group G_2 and its Heisenberg parabolic subgroup, and our results might contribute to the theory of G_2 such as [GGS02]. In addition, much progress of the theory of zeta function of prehomogeneous vector space should be done both globally and locally. One more problem we would like to mention due to its number theoretical interest is various kinds of generalizations of Ohno [O97] and Nakagawa's [N98] extra functional equation of the global zeta function for the space of binary cubic forms over \mathbb{Q} to general number fields or other prehomogeneous representations. We hope these theory will be developed in the future.

We conclude this section with a brief review of contents of this paper. In Section 2 we define the representations we consider in this paper. Main objects we consider are the space of binary cubic forms $(G, V) = (GL(2), \text{Sym}^3 \text{Aff}^2)$ and the space of binary quadratic forms (B, W) , where $W = \text{Sym}^2 \text{Aff}^2$ and B is close to the Borel subgroup of $GL(2)$ consisting of lower triangular matrices. An "embedding" of (B, W) into (G, V) described in Definition 2.1 plays an essential role when removing the contributions of reducible algebras. The algebraic part of this paper is developed in Section 3. We consider the group theoretical parameterization of cubic algebras by means of (G, V) and (B, W) . Since arithmetic plays no roles here, we consider it over general Dedekind domains. For the space of binary cubic forms (G, V) , this is regarded as a generalization of Delone-Faddeev's orbit ring map [DF64].

The rest of this paper is devoted to the analytic theory. After we introduce notations for number fields and give normalizations of invariant measure in Section 4, we concentrate on the analysis of the zeta functions. Both (G, V) and (B, W) are typical examples of prehomogeneous vector spaces and the associated zeta functions have been studied by many mathematicians after Shintani's pioneering works [Sh72, Sh75]. As we desire to work over general number fields, we need to rewrite his work into adelic language. For (G, V) , this is done by Wright [Wr85]. In Section 5 we give the adelic version of the zeta function of (B, W) in [Sh75, Chapter 1]. We choose F. Sato's modified approach [Sat81] where the "enlarged representation"

$(H, U) = (\mathrm{GL}(1) \times \mathrm{GL}(2), \mathrm{Sym}^2 \mathrm{Aff}^2 \oplus \mathrm{Aff}^2)$ is well used. We note that the adelic zeta function for (H, U) is handled by Yukie [Y93, Chapter 7] in a slightly different formulation.

In Section 6 we deal with the archimedean local theory. Since this determines the gamma factor of the functional equations, this is important for our purposes. Fortunately this is well established and we briefly recall it. In Section 7 we study the density theorems. We define our target Dirichlet series and describe them by means of the integral expressions of the global zeta functions. Combined with Sato-Shintani's Tauberian theorem [SS74, Theorem 3], we find the asymptotic formulas.

In Section 8, we study some zeta integrals for the space of binary cubic forms (G, V) . In particular, we obtain the residue of the second pole as well as the meromorphic continuation of the zeta integral corresponding to irreducible cubic forms. We hope this result will be useful when attacking the conjecture of Datskovsky-Wright [DW02] or Roberts [Ro01] on the second term of the asymptotics of cubic field discriminants. We note that a part of our result is obtained by Yukie in an unpublished note [Y03] by a fairly different method.

In Appendices we give some supplements to Section 6 that are logically independent of our main theorems. In particular, the explicit formula of the local zeta function for the standard test function at finite places is given. This result is closely related to Ibukiyama-Saito's work [IS95].

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Notation. For a finite set X we denote by $\#X$ its cardinality. The standard symbols \mathbb{Q} , \mathbb{R} , \mathbb{C} and \mathbb{Z} will denote respectively the set of rational, real and complex numbers and the rational integers. If V is a scheme defined over a ring R and S is an R -algebra then V_S denotes its S -rational points (we do *not* use the notation V_S in the sense of the base change). If an abstract group G acts on a set X , then for $x \in X$ we set $\mathrm{Stab}(G; x) = \{g \in G \mid gx = x\}$. If $\mathfrak{x} \in G \backslash X$ is the class of $x \in X$, we also denote the group by $\mathrm{Stab}(G; \mathfrak{x})$, which is well defined up to isomorphism. We always regard the affine n -space Aff^n the set of row vectors on which $\mathrm{GL}(n)$ acts from the right.

2. PREHOMOGENEOUS VECTOR SPACES

In this section, we introduce representations (G, V) , (B, W) , (H, W) and (H, U) we consider in this paper and discuss their basic properties. The first one is the space of binary cubic forms. The remaining three are concerned with the space of binary quadratic forms, and they are closely related. All of these are what we call prehomogeneous vector spaces in the sense of [SS74].

2.1. The space of binary cubic forms. Let $V = \mathrm{Sym}^3 \mathrm{Aff}^2$. We regard V as the space of binary cubic forms of variables $v = (v_1, v_2)$. Elements of V are expressed in the form $x = x_0 v_1^3 + x_1 v_1^2 v_2 + x_2 v_1 v_2^2 + x_3 v_2^3$. We choose $x = (x_0, x_1, x_2, x_3)$ as the coordinate system of V . We define the action of $G = \mathrm{GL}(2)$ on V by

$$(gx)(v) = (\det g)^{-1} x(vg) = \frac{1}{ad - bc} x(av_1 + cv_2, bv_1 + dv_2), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

Representation	Group	Vector space	R.I. & Characters
(G, V)	$\mathrm{GL}(2)$	$\mathrm{Sym}^3 \mathrm{Aff}^2$	$P \leftrightarrow \chi^2$
(B, W)	as in (2.1)	$\mathrm{Sym}^2 \mathrm{Aff}^2$	$Q_1 \leftrightarrow \chi_1, Q_2 \leftrightarrow \chi_2^2$
(H, W)	$\mathrm{GL}(1) \times \mathrm{GL}(2)$	$\mathrm{Sym}^2 \mathrm{Aff}^2$	$Q_2 \leftrightarrow \chi_2^2$
$(H, U) = (H, W \oplus S)$	$\mathrm{GL}(1) \times \mathrm{GL}(2)$	$\mathrm{Sym}^2 \mathrm{Aff}^2 \oplus \mathrm{Aff}^2$	$R_1 \leftrightarrow \chi_1, R_2 \leftrightarrow \chi_2^2$

TABLE 1. Prehomogeneous vector spaces

This action is slightly different from the usual one. Note that this twisted representation is faithful, whereas the usual action has kernel μ_3 . Let $P(x)$ be the discriminant of x ;

$$P(x) = x_1^2 x_2^2 - 4x_0 x_2^3 - 4x_1^3 x_3 + 18x_0 x_1 x_2 x_3 - 27x_0^2 x_3^2.$$

We put $\chi(g) = \det(g)$. Then we have $P(gx) = \chi(g)^2 P(x)$. We put $V^{\mathrm{ss}} = \{x \in V \mid P(x) \neq 0\}$. Then V^{ss} is a single G -orbit over any algebraically closed field.

2.2. The spaces of binary quadratic forms and an embedding. Let $W = \mathrm{Sym}^2 \mathrm{Aff}^2$, regard as the space of binary quadratic forms of variables $v = (v_1, v_2)$, and express elements as $y = y(v) = y_1 v_1^2 + y_2 v_1 v_2 + y_3 v_2^2$. We choose $y = (y_1, y_2, y_3)$ as the coordinate system of W . Let $H = \mathrm{GL}(1) \times \mathrm{GL}(2)$ and express elements of H as $h = (t, g)$ where $t \in \mathrm{GL}(1)$ and $g \in \mathrm{GL}(2)$. We define characters χ_1 and χ_2 on H by $\chi_1(h) = t$ and $\chi_2(h) = t \det(g)$. We define the action of H on W by

$$(hy)(v) = ty(vg) = ty(av_1 + cv_2, bv_1 + dv_2), \quad h = (t, g) = \left(t, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \in H.$$

We define a subgroup B of H by

$$(2.1) \quad B = \left\{ b = \left(t, \begin{pmatrix} 1 & 0 \\ u & p \end{pmatrix} \right) \mid t, p \in \mathbb{G}_m, u \in \mathbb{G}_a \right\} \cong \mathbb{G}_m^2 \times \mathbb{G}_a,$$

and consider the representation (B, W) . We put $Q_1(y) = y_1$ and $Q_2(y) = y_2^2 - 4y_1 y_3$. Then we have $Q_1(by) = \chi_1(b)Q_1(y)$, $Q_2(by) = \chi_2(b)^2 Q_2(y)$ for $b \in B$, $y \in W$. We put $W^{\mathrm{ss}} = \{y \in W \mid Q_1(y)Q_2(y) \neq 0\}$, which is a single B -orbit over any algebraically closed field. We also consider the representation (H, W) . This representation has a single basic relative invariant polynomial $Q_2(y)$ with the character χ_2^2 . We put $W^\sim = \{y \in W \mid Q_2(y) \neq 0\}$, which is a single H -orbit over any algebraically closed field.

Let $S = \mathrm{Aff}^2$. We express elements of S as $\bar{y} = (\bar{y}_1, \bar{y}_2)$, and choose this as the coordinate system of S . We put $U = W \oplus S$ and express elements of U as $\tilde{y} = (y, \bar{y}) = (y_0, y_1, y_2, \bar{y}_1, \bar{y}_2)$. Then we can define the action of H on U as follows:

$$h\tilde{y} = h(y, \bar{y}) = (hy, \bar{y}g^{-1}), \quad h = (t, g) \in H.$$

We put $R_1(\tilde{y}) = y_0 \bar{y}_1^2 + y_1 \bar{y}_1 \bar{y}_2 + y_2 \bar{y}_2^2$ and $R_2(\tilde{y}) = Q_2(y) = y_2^2 - 4y_1 y_3$. Then we have $R_1(h\tilde{y}) = \chi_1(h)R_1(\tilde{y})$ and $R_2(h\tilde{y}) = \chi_2(h)^2 R_2(\tilde{y})$. We put $U^{\mathrm{ss}} = \{\tilde{y} \in U \mid R_1(\tilde{y})R_2(\tilde{y}) \neq 0\}$, which is a single H -orbit over any algebraically closed field.

In later sections, we use an ‘‘embedding’’ (B, W) into (G, V) which will play a significant role in this paper. We define the embedding here.

Definition 2.1. For a binary quadratic form $y \in W$, we regard the binary cubic form $y^* = v_2 y$ as an element of V . Then $P(y^*) = Q_1(y)^2 Q_2(y)$. We embed B into G via the map

$$B \ni b = \left(t, \begin{pmatrix} 1 & 0 \\ u & s \end{pmatrix} \right) \mapsto b^* = \begin{pmatrix} t & 0 \\ tu & ts \end{pmatrix} \in G.$$

Then for $y \in W$ and $b \in B$, we have $(by)^* = b^* y^* \in V$.

3. PARAMETERIZATION OF CUBIC ALGEBRAS OVER A DEDEKIND DOMAIN

Throughout this section, we assume \mathcal{O} a Dedekind domain and k its quotient field. In this section we give group theoretical parameterizations of \mathcal{O} -algebras which are projective of rank 3 as \mathcal{O} -modules by means of the representations (G, V) and (B, W) . The results are Propositions 3.6 and 3.12. The notion of Steinitz class naturally arises in the process. A review of geometric interpretations of orbits over fields are included.

3.1. Projective modules over a Dedekind domain. For the convenience of reader, we review the basic properties of projective modules and fractional ideals of Dedekind domains. For details, see Milnor's book [M71, §1] for example. We assume all the \mathcal{O} -modules to be finitely generated.

Definition 3.1. (1) An element m of an \mathcal{O} -module M is called a *torsion element* if there exists a non-zero element $a \in \mathcal{O}$ such that $am = 0$. The set of torsion elements are called the *torsion submodule* of M . If the torsion submodule of M is trivial, M is called *torsion free*.

(2) For an \mathcal{O} -module M , $M \otimes k$ is a vector space over k . The dimension is called the *rank* of M .

Proposition 3.2. (1) *If M is torsion free, then the map $M \rightarrow M \otimes k$ is injective.*

(2) *An \mathcal{O} -module M is torsion free if and only if M is projective. In particular, any fractional ideal of \mathcal{O} is projective, and any submodule of a projective module is projective.*

(3) *Let M be a projective \mathcal{O} -module of rank n . Then there exists a non-zero ideal \mathfrak{a} of \mathcal{O} such that $M \cong \mathcal{O}^{n-1} \oplus \mathfrak{a}$ as \mathcal{O} -modules. The ideal class of \mathfrak{a} is uniquely determined by M and called the Steinitz class of M , which we denote by $\text{St}(M)$.*

(4) *For projective modules M_1 and M_2 , we have $\text{St}(M_1 \oplus M_2) = \text{St}(M_1 \otimes M_2) = \text{St}(M_1) \cdot \text{St}(M_2)$ where the last product is of the ideal class group.*

(5) *Let \mathfrak{a} be a fractional ideal of \mathcal{O} . Then the inverse ideal \mathfrak{a}^{-1} of \mathfrak{a} is given by $\{x \in k \mid x\mathfrak{a} \subset \mathcal{O}\}$. Moreover, $\text{Hom}_{\mathcal{O}\text{-module}}(\mathfrak{a}, \mathcal{O}) \cong \mathfrak{a}^{-1}$ as \mathcal{O} -modules.*

(6) *For any non-zero fractional ideals $\mathfrak{a}, \mathfrak{b}$ and \mathfrak{c} , there exist elements $x, y \in k$ such that $x\mathfrak{a} + y\mathfrak{b} = \mathfrak{c}$.*

3.2. A generalization of Delone-Faddeev's orbit ring map. Let \mathfrak{a} be a fractional ideal of \mathcal{O} . We use the same symbol \mathfrak{a} to denote its ideal class if there is no confusion. We denote by $\text{Cl}(k)$ the ideal class group of k .

Definition 3.3. Let $\mathcal{C}(\mathcal{O})$ be the set of isomorphism classes of finite \mathcal{O} -algebras which are projective of rank 3 as \mathcal{O} -modules. Elements of $\mathcal{C}(\mathcal{O})$ are called *cubic algebras*. For any fractional ideal \mathfrak{a} , we define $\mathcal{C}(\mathcal{O}, \mathfrak{a}) = \{R \in \mathcal{C}(\mathcal{O}) \mid \text{St}(R) = \mathfrak{a}\}$.

Proposition 3.4. (1) *We have $\mathcal{C}(\mathcal{O}) = \coprod_{\mathfrak{a} \in \text{Cl}(k)} \mathcal{C}(\mathcal{O}, \mathfrak{a})$.*

(2) *Let $R \in \mathcal{C}(\mathcal{O}, \mathfrak{a})$. There is an isomorphism $R/\mathcal{O} \cong \mathcal{O} \oplus \mathfrak{a}$ of \mathcal{O} -modules.*

Proof. (1) immediately follows from Proposition 3.2, and we only need to consider (2). We prove R/\mathcal{O} is torsion free. Let $r \in R$ and non-zero $x \in \mathcal{O}$ satisfy $rx \in \mathcal{O}$. Then $r \in k$. Since R is a finite \mathcal{O} -algebra, r is integral over \mathcal{O} . Hence we have $r \in \mathcal{O}$ because \mathcal{O} is integrally closed in k . This shows that R/\mathcal{O} is torsion free and hence projective. We have $R \cong \mathcal{O} \oplus (R/\mathcal{O})$ and therefore $R/\mathcal{O} \cong \mathcal{O} \oplus \mathfrak{a}$ by Proposition 3.2 (3). \square

We consider a parameterization of $\mathcal{C}(\mathcal{O}, \mathfrak{a})$ using the representation (G, V) , which is a generalization of Delone-Faddeev's orbit ring map [DF64, Section 15] over \mathbb{Z} .

Definition 3.5. We put

$$G_k \supset G_{\mathfrak{a}} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \in \mathcal{O}, b \in \mathfrak{a}, c \in \mathfrak{a}^{-1}, d \in \mathcal{O}, ad - bc \in \mathcal{O}^\times \right\},$$

$$V_k \supset V_{\mathfrak{a}} = \{x \mid x_0 \in \mathfrak{a}, x_1 \in \mathcal{O}, x_2 \in \mathfrak{a}^{-1}, x_3 \in \mathfrak{a}^{-2}\}.$$

Then $G_{\mathfrak{a}} \cdot V_{\mathfrak{a}} \subset V_{\mathfrak{a}}$.

We could naturally construct isomorphisms $G_{\mathfrak{a}} \cong \text{Aut}(\mathcal{O} \oplus \mathfrak{a})$ and $V_{\mathfrak{a}} \cong \text{Sym}^3(\mathcal{O} \oplus \mathfrak{a}) \otimes \wedge^2(\mathcal{O} \oplus \mathfrak{a})$, such that the canonical action compatible.

Proposition 3.6. (1) *There exists the canonical bijection between $\mathcal{C}(\mathcal{O}, \mathfrak{a})$ and $G_{\mathfrak{a}} \backslash V_{\mathfrak{a}}$ making the following diagram commutative:*

$$\begin{array}{ccc} G_{\mathfrak{a}} \backslash V_{\mathfrak{a}} & \longrightarrow & \mathcal{C}(\mathcal{O}, \mathfrak{a}) \\ \downarrow P & & \downarrow \text{discriminant} \\ (\mathcal{O}^\times)^2 \backslash \mathfrak{a}^{-2} & \xrightarrow{\times \mathfrak{a}^2} & \{\text{integral ideals of } \mathcal{O}\}. \end{array}$$

Here, the right vertical arrow is to take the discriminant, and the low horizontal arrow is given by multiplying \mathfrak{a}^2 . Moreover, this diagram is functorial with respect to ring homomorphisms of Dedekind domains.

(2) *For each $R \in \mathcal{C}(\mathcal{O}, \mathfrak{a})$, let x_R be the corresponding element in $G_{\mathfrak{a}} \backslash V_{\mathfrak{a}}$. Then $\text{Aut}_{\mathcal{O}\text{-alg}}(R) \cong \text{Stab}(G_{\mathfrak{a}}; x_R)$.*

Proof. Since the proof is similar to [DF64, Section 15] or [GGS02, Proposition 4.2], we shall be brief. For each $R \in \mathcal{C}(\mathcal{O}, \mathfrak{a})$, the binary cubic form

$$x_R: R/\mathcal{O} \longrightarrow \wedge^2(R/\mathcal{O}), \quad \xi \longmapsto \xi \wedge \xi^2$$

can be regarded as an element of $G_{\mathfrak{a}} \backslash V_{\mathfrak{a}}$ since $R/\mathcal{O} \cong \mathcal{O} \oplus \mathfrak{a}$. This map $R \mapsto x_R$ gives the desired bijection. To see that this map is in fact bijective, we will write down this correspondence explicitly.

Let $R \in \mathcal{C}(\mathcal{O}, \mathfrak{a})$. By Proposition 3.4 we fix an \mathcal{O} -module isomorphism $R \cong \mathcal{O} \oplus \mathcal{O} \oplus \mathfrak{a}$ such that $(1, 0, 0)$ is the multiplicative identity 1 of R . We regard R as a subalgebra of $R \otimes k \cong k \oplus k \oplus k$. Let $\omega_1 = (0, 1, 0), \omega_2 = (0, 0, 1)$. Then $R = \{p + q\omega_1 + r\omega_2 \mid p, q \in \mathcal{O}, r \in \mathfrak{a}\}$. Let $\omega_1\omega_2 = \alpha + \beta\omega_1 + \gamma\omega_2$. Then since $\omega_1(\mathfrak{a}\omega_2) \subset R$, we have $\alpha, \beta \in \mathfrak{a}^{-1}, \gamma \in \mathcal{O}$. Hence by replacing ω_1, ω_2 to $\omega_1 - \gamma, \omega_2 - \beta$ if necessary, we may assume $\omega_1\omega_2 \in \mathfrak{a}^{-1}$. Let

$$(3.1) \quad \omega_1^2 = j - b\omega_1 + a\omega_2, \quad \omega_2^2 = l - d\omega_1 + c\omega_2, \quad \omega_1\omega_2 = m.$$

Then the associativity of the product of the algebra $R \otimes k$ requires

$$(3.2) \quad j = -ac, \quad l = -bd, \quad m = -ad.$$

Also since $\mathcal{O}\omega_1^2, \mathfrak{a}^2\omega_2^2 \subset R$, we have

$$(3.3) \quad a \in \mathfrak{a}, \quad b \in \mathcal{O}, \quad c \in \mathfrak{a}^{-1}, \quad d \in \mathfrak{a}^{-2}.$$

On the other side, For any a, b, c, d satisfying (3.3), the \mathcal{O} -module $\mathcal{O} \oplus \mathcal{O}\omega_1 \oplus \mathfrak{a}\omega_2$ becomes an \mathcal{O} -algebra if we take $(1, 0, 0)$ as its multiplicative identity and define the multiplication law by (3.1) with (3.2). Let $\xi(v) = v_1\omega_1 + v_2\omega_2$. Then by computation we have

$$(3.4) \quad 1 \wedge \xi(v) \wedge \xi(v)^2 = (av_1^3 + bv_1^2v_2 + cv_1v_2^2 + dv_2^3) \cdot 1 \wedge \omega_1 \wedge \omega_2.$$

This shows that x_R is a class of $av_1^3 + bv_1^2v_2 + cv_1v_2^2 + dv_2^3 \in V_{\mathfrak{a}}$. These show that the upper horizontal arrow bijective.

To see the commutativity of the diagram, it is enough to check it locally, i.e., we may assume \mathcal{O} a discrete valuation ring. Since this compatibility is known for a PID in [DF64] (or by a simple computation), we have (1).

We consider (2). Any $\tilde{\psi} \in \text{Aut}_{\mathcal{O}\text{-alg}}(R)$ obviously induces $\psi \in \text{Aut}_{\mathcal{O}\text{-module}}(R/\mathcal{O})$. We fix an isomorphism $R/\mathcal{O} \cong \mathcal{O} \oplus \mathfrak{a}$ and consider $\psi \in G_{\mathfrak{a}}$, $x_R(v) \in V_{\mathfrak{a}}$. We also define $\omega_1, \omega_2, a, b, c, d$ as above by using this isomorphism, so that $x_R(v) = av_1^3 + bv_1^2v_2 + cv_1v_2^2 + dv_2^3 \in V_{\mathfrak{a}}$. Let

$$\omega'_1 = \tilde{\psi}(\omega_1), \quad \omega'_2 = \tilde{\psi}(\omega_2), \quad \text{and} \quad \xi'(v) = v_1\omega'_1 + v_2\omega'_2.$$

Then since $\tilde{\psi} \in \text{Aut}_{\mathcal{O}\text{-alg}}(R)$, equation (3.1) also holds for the pair (ω'_1, ω'_2) and hence we have

$$(3.5) \quad 1 \wedge \xi'(v) \wedge \xi'(v)^2 = (av_1^3 + bv_1^2v_2 + cv_1v_2^2 + dv_2^3) \cdot 1 \wedge \omega'_1 \wedge \omega'_2.$$

By (3.4), (3.5) and $1 \wedge \omega'_1 \wedge \omega'_2 = (\det \psi) \cdot 1 \wedge \omega_1 \wedge \omega_2$, we see $\psi(x_R) = x_R$ by

$$\begin{aligned} \psi(x_R(v)) \cdot 1 \wedge \omega'_1 \wedge \omega'_2 &= x_R(v\psi) \cdot 1 \wedge \omega_1 \wedge \omega_2 = 1 \wedge \xi(v\psi) \wedge \xi(v\psi)^2 \\ &= 1 \wedge \xi'(v) \wedge \xi'(v)^2 = x_R(v) \cdot 1 \wedge \omega'_1 \wedge \omega'_2. \end{aligned}$$

Hence we have a map $\text{Aut}_{\mathcal{O}\text{-alg}}(R) \rightarrow \text{Stab}(G_{\mathfrak{a}}; x_R)$. To see this map an isomorphism is a routine task and we omit the details here. \square

3.3. Parameterization of reducible algebras. We next consider the parameterization of the *reducible algebras*, that is, those $R \in \mathcal{C}(\mathcal{O}, \mathfrak{a})$ with $R \otimes k$ not fields.

Definition 3.7. We define subsets of $\mathcal{C}(\mathcal{O}, \mathfrak{a})$ as follows:

$$\begin{aligned} \mathcal{C}(\mathcal{O}, \mathfrak{a})^3 &= \{R \mid R \otimes k \text{ is a cubic field extension of } k\}, \\ \mathcal{C}(\mathcal{O}, \mathfrak{a})^2 &= \{R \mid R \otimes k \cong k \times F \text{ where } F \text{ is a quadratic field extension of } k\}, \\ \mathcal{C}(\mathcal{O}, \mathfrak{a})^1 &= \{R \mid R \otimes k \cong k \times k \times k\}, \\ \mathcal{C}(\mathcal{O}, \mathfrak{a})^0 &= \{R \mid R \otimes k \text{ is an inseparable algebra}\}. \end{aligned}$$

We also define subsets of $V_{\mathfrak{a}}$ as follows:

$$\begin{aligned} V_{\mathfrak{a}}^3 &= \{x \mid x(v) \text{ is irreducible over } k\}, \\ V_{\mathfrak{a}}^2 &= \{x \mid x(v) \text{ decomposes into degree 1 and 2 irreducible polynomials over } k\}, \\ V_{\mathfrak{a}}^1 &= \{x \mid x(v) \text{ has three distinct rational roots in } \mathbb{P}^1\}, \\ V_{\mathfrak{a}}^0 &= \{x \mid x(v) \text{ has a multiple root in } \mathbb{P}^1\}. \end{aligned}$$

Finally, we put $\mathcal{C}(\mathcal{O}, \mathfrak{a})^{\text{red}} = \coprod_{0 \leq i \leq 2} \mathcal{C}(\mathcal{O}, \mathfrak{a})^i$, $V_{\mathfrak{a}}^{\text{red}} = \coprod_{0 \leq i \leq 2} V_{\mathfrak{a}}^i$.

We omit the routine proof of the following lemma.

Lemma 3.8. (1) *We have $\mathcal{C}(\mathcal{O}, \mathfrak{a}) = \coprod_{0 \leq i \leq 3} \mathcal{C}(\mathcal{O}, \mathfrak{a})^i$, $V_{\mathfrak{a}} = \coprod_{0 \leq i \leq 3} V_{\mathfrak{a}}^i$.*

(2) *Each $V_{\mathfrak{a}}^i$ is a $G_{\mathfrak{a}}$ -invariant subset.*

(3) *The set $\mathcal{C}(\mathcal{O}, \mathfrak{a})^i$ corresponds bijectively to the set $G_{\mathfrak{a}} \backslash V_{\mathfrak{a}}^i$ via the map of Proposition 3.6.*

Hence, we consider $G_{\mathfrak{a}} \backslash V_{\mathfrak{a}}^{\text{red}}$. Let $\mathfrak{a}, \mathfrak{c}$ be fractional ideals of \mathcal{O} .

Definition 3.9. We put

$$\begin{aligned} B_k \supset B_{\mathfrak{a}, \mathfrak{c}} &= \left\{ \left(t, \begin{pmatrix} 1 & 0 \\ u & p \end{pmatrix} \right) \mid t, p \in \mathcal{O}^{\times}, u \in \mathfrak{a}^{-1}\mathfrak{c}^{-2} \right\}, \\ W_k \supset W_{\mathfrak{a}, \mathfrak{c}} &= \{y \mid y_1 \in \mathfrak{c}, y_2 \in \mathfrak{a}^{-1}\mathfrak{c}^{-1}, y_3 \in \mathfrak{a}^{-2}\mathfrak{c}^{-3}\}. \end{aligned}$$

Then $B_{\mathfrak{a}, \mathfrak{c}} \cdot W_{\mathfrak{a}, \mathfrak{c}} \subset W_{\mathfrak{a}, \mathfrak{c}}$.

Definition 3.10. We define subsets of $W_{\mathfrak{a}, \mathfrak{c}}$ as follows:

$$\begin{aligned} W_{\mathfrak{a}, \mathfrak{c}}^2 &= \{y \mid y(v) \text{ is irreducible over } k\}, \\ W_{\mathfrak{a}, \mathfrak{c}}^1 &= \{y \mid y(v) \text{ has two distinct rational roots in } \mathbb{P}^1\}, \\ W_{\mathfrak{a}, \mathfrak{c}}^0 &= \{y \mid y(v) \text{ has a multiple root in } \mathbb{P}^1\}. \end{aligned}$$

Then $W_{\mathfrak{a},\mathfrak{c}} = \coprod_{0 \leq i \leq 2} W_{\mathfrak{a},\mathfrak{c}}^i$ and each $W_{\mathfrak{a},\mathfrak{c}}^i$ is a $B_{\mathfrak{a},\mathfrak{c}}$ -invariant subset.

To construct the orbit ring map, we need a lemma.

Lemma 3.11. *Let $q, s, m, n \in k$ satisfy $qa^{-1} + s\mathcal{O} = ma^{-1} + n\mathcal{O} = \mathfrak{c}$. Then there exists $\gamma \in G_{\mathfrak{a}}$ such that*

$$\gamma \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} q \\ s \end{pmatrix}.$$

Proof. Let $\pi: \mathfrak{a}^{-1} \oplus \mathcal{O} \ni x = (x_1, x_2) \mapsto qx_1 + sx_2 \in \mathfrak{c}$. This is surjective, and since \mathfrak{c} is projective there is a section μ of π . Hence we have an isomorphism

$$\phi: \mathfrak{a}^{-1} \oplus \mathcal{O} \longrightarrow \mathfrak{c} \oplus \ker \pi, \quad x \longmapsto (\pi(x), x - \mu \circ \pi(x)).$$

For $\pi': \mathfrak{a}^{-1} \oplus \mathcal{O} \ni (x_1, x_2) \mapsto mx_1 + nx_2 \in \mathfrak{c}$ we also have an isomorphism

$$\phi': \mathfrak{a}^{-1} \oplus \mathcal{O} \longrightarrow \mathfrak{c} \oplus \ker \pi', \quad x \longmapsto (\pi'(x), x - \mu' \circ \pi'(x)).$$

By Proposition 3.2, we have an \mathcal{O} -module isomorphism $\varphi: \ker \pi \rightarrow \ker \pi'$. Let $\Phi = \phi'^{-1} \circ (\text{id} \oplus \varphi) \circ \phi$, which is an element of $\text{Aut}_{\mathcal{O}\text{-module}}(\mathfrak{a}^{-1} \oplus \mathcal{O})$;

$$\begin{array}{ccc} \mathfrak{a}^{-1} \oplus \mathcal{O} & \xrightarrow{\phi} & \mathfrak{c} \oplus \ker \pi \\ \Phi \downarrow & & \downarrow \text{id} \oplus \varphi \\ \mathfrak{a}^{-1} \oplus \mathcal{O} & \xrightarrow{\phi'} & \mathfrak{c} \oplus \ker \pi' \end{array}$$

Then Φ is represented by an element $\gamma \in G_{\mathfrak{a}}$ i.e., $\Phi(x) = x\gamma$ for $x \in \mathfrak{a}^{-1} \oplus \mathcal{O}$. By the commutativity of the diagram above, we have $\pi' \circ \Phi(x) = \pi(x)$, i.e., $x\gamma \begin{pmatrix} m \\ n \end{pmatrix} = x \begin{pmatrix} q \\ s \end{pmatrix}$. Since this holds for any x , we have the lemma. \square

Proposition 3.12. (1) *We fix \mathfrak{a} . For each \mathfrak{c} , there exists the canonical map $\psi_{\mathfrak{a},\mathfrak{c}}: B_{\mathfrak{a},\mathfrak{c}} \backslash W_{\mathfrak{a},\mathfrak{c}} \rightarrow G_{\mathfrak{a}} \backslash V_{\mathfrak{a}}^{\text{red}}$ making the following diagram commutative:*

$$\begin{array}{ccc} B_{\mathfrak{a},\mathfrak{c}} \backslash W_{\mathfrak{a},\mathfrak{c}} & \xrightarrow{\psi_{\mathfrak{a},\mathfrak{c}}} & G_{\mathfrak{a}} \backslash V_{\mathfrak{a}}^{\text{red}} \\ Q_1^2 Q_2 \downarrow & & \downarrow P \\ (\mathcal{O}^\times)^2 \backslash \mathfrak{a}^{-2} & \xlongequal{\quad} & (\mathcal{O}^\times)^2 \backslash \mathfrak{a}^{-2}. \end{array}$$

(2) *Moreover, collecting $\psi_{\mathfrak{a},\mathfrak{c}}$ for all $\mathfrak{c} \in \text{Cl}(k)$ we obtain a bijective map*

$$\coprod_{\mathfrak{c} \in \text{Cl}(k)} B_{\mathfrak{a},\mathfrak{c}} \backslash W_{\mathfrak{a},\mathfrak{c}}^2 \longrightarrow G_{\mathfrak{a}} \backslash V_{\mathfrak{a}}^2.$$

Also for $y \in W_{\mathfrak{a},\mathfrak{c}}^2$ we have $\text{Stab}(B_{\mathfrak{a},\mathfrak{c}}; y) \cong \text{Stab}(G_{\mathfrak{a}}; \psi_{\mathfrak{a},\mathfrak{c}}(y))$.

Proof. We fix \mathfrak{a} . Let

$$\begin{aligned} V_{\mathfrak{a},\mathfrak{c}}^{\text{red}} &= \{(mv_1 + nv_2)(lv_1^2 + l'v_1v_2 + l''v_2^2) \in V_{\mathfrak{a}}^{\text{red}} \mid m, n, l, l', l'' \in k, ma^{-1} + n\mathcal{O} = \mathfrak{c}\}, \\ V_{\mathfrak{a},\mathfrak{c}}^2 &= V_{\mathfrak{a},\mathfrak{c}}^{\text{red}} \cap V_{\mathfrak{a}}^2, \end{aligned}$$

which depend only on the ideal class of \mathfrak{c} . Then we have $V_{\mathfrak{a}}^{\text{red}} = \bigcup_{\mathfrak{c} \in \text{Cl}(k)} V_{\mathfrak{a},\mathfrak{c}}^{\text{red}}$ and $V_{\mathfrak{a}}^2 = \coprod_{\mathfrak{c} \in \text{Cl}(k)} V_{\mathfrak{a},\mathfrak{c}}^2$. (Note that the second union is disjoint while the first one is in general not.) We use the embedding of Definition 2.1 to construct $\psi_{\mathfrak{a},\mathfrak{c}}$. We fix $q, s \in k$ such that $qa^{-1} + s\mathcal{O} = \mathfrak{c}$. Then, $q \in \mathfrak{a}\mathfrak{c}$, $s \in \mathfrak{c}$, and also there exist $p \in \mathfrak{c}^{-1}, r \in \mathfrak{a}^{-1}\mathfrak{c}^{-1}$ such that $ps - qr \in \mathcal{O}^\times$. We put $g_{\mathfrak{a},\mathfrak{c}} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in G_k$. We define

$$\tilde{\psi}_{\mathfrak{a},\mathfrak{c}}: W_{\mathfrak{a},\mathfrak{c}} \longrightarrow V_{\mathfrak{a},\mathfrak{c}}^{\text{red}}, \quad y \longmapsto g_{\mathfrak{a},\mathfrak{c}}y^*.$$

Then $\tilde{\psi}_{\mathfrak{a},\mathfrak{c}}(W_{\mathfrak{a},\mathfrak{c}}^2) \subset V_{\mathfrak{a},\mathfrak{c}}^2$ and

$$P\left(\tilde{\psi}_{\mathfrak{a},\mathfrak{c}}(y)\right) = P(g_{\mathfrak{a},\mathfrak{c}}y^*) = \det(g_{\mathfrak{a},\mathfrak{c}})^2 P(y^*) = (ps - qr)^2 Q_1(y)^2 Q_2(y).$$

Also considering B as a subgroup of G via the embedding of Definition 2.1, we see

$$(3.6) \quad g_{\mathfrak{a},\mathfrak{c}}^{-1} G_{\mathfrak{a}} g_{\mathfrak{a},\mathfrak{c}} \cap B_k = B_{\mathfrak{a},\mathfrak{c}}.$$

This shows that the map $\tilde{\psi}_{\mathfrak{a},\mathfrak{c}}$ induces a well defined map $\psi_{\mathfrak{a},\mathfrak{c}}: B_{\mathfrak{a},\mathfrak{c}} \backslash W_{\mathfrak{a},\mathfrak{c}} \rightarrow G_{\mathfrak{a}} \backslash V_{\mathfrak{a},\mathfrak{c}}^{\text{red}}$ making the diagram in the proposition commutative. Let $g'_{\mathfrak{a},\mathfrak{c}} = \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix} \in G_k$ where $q'\mathfrak{a}^{-1} + s'\mathcal{O} = \mathfrak{c}$, $p' \in \mathfrak{c}^{-1}$, $r' \in \mathfrak{a}^{-1}\mathfrak{c}^{-1}$ and $p's' - q'r' \in \mathcal{O}^\times$. Then since

$$g'_{\mathfrak{a},\mathfrak{c}} g_{\mathfrak{a},\mathfrak{c}}^{-1} = \frac{1}{ps - qr} \begin{pmatrix} p's - q'r & q'p - p'q \\ sr' - rs' & s'p - r'q \end{pmatrix} \in G_{\mathfrak{a}},$$

we have $g'_{\mathfrak{a},\mathfrak{c}} y^* \in G_{\mathfrak{a}} g_{\mathfrak{a},\mathfrak{c}} y^*$. This shows that $\psi_{\mathfrak{a},\mathfrak{c}}$ does not depend on the choice of $g_{\mathfrak{a},\mathfrak{c}}$.

We claim that $\tilde{\psi}_{\mathfrak{a},\mathfrak{c}}$ is surjective. To see this, let

$$x = (mv_1 + nv_2)(lv_1^2 + l'v_1v_2 + l''v_2^2) \in V_{\mathfrak{a},\mathfrak{c}}^{\text{red}}, \quad m\mathfrak{a}^{-1} + n\mathcal{O} = \mathfrak{c}.$$

We take $\gamma \in G_{\mathfrak{a}}$ as in Lemma 3.11. By changing x to γx if necessary, we assume $m = q, n = s$. Also by a variation of Gauss' lemma, we have $l \in \mathfrak{c}^{-1}, l' \in \mathfrak{a}^{-1}\mathfrak{c}^{-1}$, and $l'' \in \mathfrak{a}^{-2}\mathfrak{c}^{-1}$. This shows $\tilde{\psi}_{\mathfrak{a},\mathfrak{c}}$ is surjective and hence so is $\psi_{\mathfrak{a},\mathfrak{c}}: B_{\mathfrak{a},\mathfrak{c}} \backslash W_{\mathfrak{a},\mathfrak{c}} \rightarrow G_{\mathfrak{a}} \backslash V_{\mathfrak{a},\mathfrak{c}}^{\text{red}}$. This shows (1) and the surjectivity of the map of (2).

Let $y, y' \in W_{\mathfrak{a},\mathfrak{c}}^2$. We assume $\psi_{\mathfrak{a},\mathfrak{c}}(y)$ and $\psi_{\mathfrak{a},\mathfrak{c}}(y')$ lie in the same $G_{\mathfrak{a}}$ -orbit. Then there exists $\gamma \in G_{\mathfrak{a}}$ such that $v_2 y' = (g_{\mathfrak{a},\mathfrak{c}}^{-1} \gamma g_{\mathfrak{a},\mathfrak{c}})(v_2 y) \in V_k$. Since y and y' are irreducible quadratic forms, $g_{\mathfrak{a},\mathfrak{c}}^{-1} \gamma g_{\mathfrak{a},\mathfrak{c}}$ must fix the linear form v_2 , i.e., $g_{\mathfrak{a},\mathfrak{c}}^{-1} \gamma g_{\mathfrak{a},\mathfrak{c}} \in B_k$. Combined with (3.6) we have $b = g_{\mathfrak{a},\mathfrak{c}}^{-1} \gamma g_{\mathfrak{a},\mathfrak{c}} \in B_{\mathfrak{a},\mathfrak{c}}$. Hence $y' = by$ and therefore y and y' lie in the same $B_{\mathfrak{a},\mathfrak{c}}$ -orbit. This shows that the map of (2) is injective. The second statement of (2) can be proved similarly. \square

Remark 3.13. In [Sh75], the corresponding statement for \mathbb{Z} is shown in the proof of Lemma 12.

3.4. Geometric interpretation of orbits over a field. The sets of non-singular orbits of representations we defined in Section 2 have well known geometric interpretations over a field. We recall those facts here. In this subsection let $\mathcal{O} = k$ be an arbitrary field.

Definition 3.14. We denote by $\mathcal{E}_2(k)$ and $\mathcal{E}_3(k)$ the sets of isomorphism classes of separable quadratic and cubic algebras of k , respectively.

We first consider the space of binary cubic forms (G, V) .

Definition 3.15. For $x = x(v_1, v_2) \in V_k^{\text{ss}}$, we define

$$\begin{aligned} Z_x &= \text{Proj } k[v_1, v_2]/(x(v_1, v_2)), \\ k(x) &= \Gamma(Z_x, \mathcal{O}_{Z_x}). \end{aligned}$$

We regard $k(x)$ as an element of $\mathcal{E}_3(k)$, which is possible because elements of V_k^{ss} are separable cubic polynomials.

Proposition 3.16. *The map $x \mapsto k(x)$ gives a bijection between $G_k \backslash V_k^{\text{ss}}$ and $\mathcal{E}_3(k)$. Moreover this map coincides with the Delone-Faddeev map we defined in Proposition 3.6 in the domain of the definition.*

We next consider the spaces of binary quadratic forms (B, W) , (H, W) and (H, U) .

Definition 3.17. For $\tilde{y} = (y, \bar{y}) \in U_k^{\text{ss}}$, we define

$$\begin{aligned} Z_{\tilde{y}} &= Z_y = \text{Proj } k[v_1, v_2]/(y(v_1, v_2)), \\ \tilde{k}(\tilde{y}) &= \tilde{k}(y) = \Gamma(Z_y, \mathcal{O}_{Z_y}). \end{aligned}$$

We regard $\tilde{k}(\tilde{y}) = \tilde{k}(y)$ as an element of $\mathcal{E}_2(k)$.

Proposition 3.18. (1) *The map $y \mapsto \tilde{k}(y)$ gives bijections between $B_k \backslash W_k^{\text{ss}}, H_k \backslash W_k$, and $\mathcal{E}_2(k)$. Also the map $\tilde{y} \mapsto \tilde{k}(\tilde{y})$ gives a bijection between $H_k \backslash U_k^{\text{ss}}$ and $\mathcal{E}_2(k)$.*

(2) *For $y \in W_k^{\text{ss}}$, we have $k(y^*) \cong \tilde{k}(y) \times k$ where $y^* \in V_k^{\text{ss}}$ is defined in Definition 2.1.*

4. NOTATIONS FOR NUMBER FIELDS AND INVARIANT MEASURES

For the rest of this paper, we assume k is a number field and \mathcal{O} the ring of integers of k . In this section we prepare notation for number fields, and fix various invariant measure both locally and globally. Let $\mathbb{R}_+ = \{t \in \mathbb{R}^\times \mid t > 0\}$. For an integral ideal I of \mathcal{O} the ideal norm of I is denoted by $N(I)$. We extend it to general fractional ideals in the obvious manner. Let $\mathfrak{M}, \mathfrak{M}_\infty, \mathfrak{M}_f, \mathfrak{M}_\mathbb{R}$ and $\mathfrak{M}_\mathbb{C}$ denote respectively the set of all places of k , all infinite places, all finite places, all real places and all complex places. For $v \in \mathfrak{M}$, k_v denotes the completion of k at v and $|\cdot|_v$ denotes the normalized absolute value on k_v . If $v \in \mathfrak{M}_f$ then \mathcal{O}_v denotes the ring of integers of k_v , \mathfrak{p}_v the maximal ideal of \mathcal{O}_v and q_v the cardinality of $\mathcal{O}_v/\mathfrak{p}_v$. For $t \in k_v^\times$, we define $\text{ord}_v(t)$ such that $|t|_v = q_v^{-\text{ord}_v(t)}$. For any separable quadratic algebra L_v of k_v , let \mathcal{O}_{L_v} denote the ring of integral elements of L_v . That is, if L_v is a quadratic extension then \mathcal{O}_{L_v} is the integer ring of L_v and if $L_v = k_v \times k_v$ then $\mathcal{O}_{L_v} = \mathcal{O}_v \times \mathcal{O}_v$. If k_1/k_2 is a finite extension of either number fields or non-Archimedean local fields then we shall write Δ_{k_1/k_2} for the relative discriminant of the extension; it is an ideal in the ring of integers of k_2 . By conventions, we let $\Delta_{k_2 \times k_2/k_2}$ be the integer ring of k_2 . To ease the notational burden we shall use the same symbol, Δ_{k_1} , for the classical absolute discriminant of k_1 over \mathbb{Q} . Since this number generates the ideal Δ_{k_1} , the resulting notational identification is harmless. We put $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_\mathbb{C}(s) = (2\pi)^{1-s}\Gamma(s)$.

Returning to k , we let r_1, r_2, h_k, R_k and e_k be respectively the number of real places, the number of complex places, the class number, the regulator and the number of roots of unity of k . It will be convenient to set $\mathfrak{C}_k = 2^{r_1}(2\pi)^{r_2}h_kR_k e_k^{-1}$. Let $\zeta_k(s)$ be the Dedekind zeta function of k .

We refer to [We74] as the basic reference for fundamental properties on adèles. The rings of adèles and finite adèles are denoted by \mathbb{A} and \mathbb{A}_f , and the groups of ideles and finite ideles of k are by \mathbb{A}^\times and \mathbb{A}_f^\times , respectively. We put $k_\infty = k \otimes_\mathbb{Q} \mathbb{R}$ and $\widehat{\mathcal{O}} = \mathcal{O} \otimes_\mathbb{Z} \widehat{\mathbb{Z}}$. Note that $\widehat{\mathcal{O}} = \prod_{v \in \mathfrak{M}_f} \mathcal{O}_v$, $\mathbb{A}_f = \widehat{\mathcal{O}} \otimes_\mathcal{O} k$, and $\mathbb{A} = k_\infty \times \mathbb{A}_f$. Following as usual, the k_∞ -rational points X_{k_∞} of a variety X of k is abbreviated to X_∞ . The adelic absolute value $|\cdot|$ on \mathbb{A}^\times is normalized so that, for $t \in \mathbb{A}^\times$, $|t|$ is the module of multiplication by t with respect to any Haar measure dx on \mathbb{A} , i.e., $|t| = d(tx)/dx$. We define $|\cdot|_\infty$ on k_∞^\times similarly. Let $\mathbb{A}^1 = \{t \in \mathbb{A}^\times \mid |t| = 1\}$. Let \mathbb{A}^0 be the unique maximal compact subgroup of \mathbb{A}^\times . Suppose $[k : \mathbb{Q}] = n$. For $\lambda \in \mathbb{R}_+$, $\underline{\lambda} \in \mathbb{A}^\times$ is the idele whose component at any infinite place is $\lambda^{1/n}$ and whose component at any finite place is 1. Then we have $|\underline{\lambda}| = \lambda$. Let $\mathfrak{d} \in \mathbb{A}_f^\times$ be a differential idele of k .

From now on we give normalizations of measures. We first prepare common notation for products of local measures. Let X be an algebraic group over k . Once we normalize a local measure dx_v on X_{k_v} for each $v \in \mathfrak{M}$, then we always let $dx_\infty = \prod_{v \in \mathfrak{M}_\infty} dx_v$, $dx_f = \prod_{v \in \mathfrak{M}_f} dx_v$ and $d_{\text{pr}}x = \prod_{v \in \mathfrak{M}} dx_v$, which are measures on $X_\infty, X_{\mathbb{A}_f}$, and $X_{\mathbb{A}}$, respectively. Hence $d_{\text{pr}}x = dx_\infty dx_f$. Below we will choose a different global measure that will be denoted by dx . We give the ratio of dx and $d_{\text{pr}}x$.

For any $v \in \mathfrak{M}_f$, we choose a Haar measure dx_v on k_v to satisfy $\int_{\mathcal{O}_v} dx_v = 1$. We write dx_v for the ordinary Lebesgue measure if v is real, and for twice the Lebesgue measure if v is imaginary. On the other hand, we choose a Haar measure dx on \mathbb{A} so that $\int_{\mathbb{A}/k} dx = 1$. Then $d_{\text{pr}}x = \Delta_k^{1/2} dx$ (see [We74], p. 91). For a vector space V , we always choose the Haar measure on $V_{\mathbb{A}}$ such that the volume of $V_{\mathbb{A}}/V_k$ is one. If an isomorphism $V \cong \text{Aff}^n$ is fixed, using this identification we choose the local measure dx_v on V_{k_v} as the n -product of the measure on k_v normalized above. Hence $d_{\text{pr}}x = \Delta_k^{n/2} dx$. For any $v \in \mathfrak{M}_f$, we normalize the Haar measure $d^\times t_v$ on k_v^\times such that $\int_{\mathcal{O}_v^\times} d^\times t_v = 1$. Let $d^\times t_v(x) = |x|_v^{-1} dx_v$ if $v \in \mathfrak{M}_\infty$. We choose a Haar measure $d^\times t^1$ on \mathbb{A}^1 such that $\int_{\mathbb{A}^1/k^\times} d^\times t^1 = 1$. Using this measure and the decomposition

$$\mathbb{R}_+ \times \mathbb{A}^1 \cong \mathbb{A}^\times, \quad (\lambda, t^1) \mapsto \lambda t^1,$$

we define an invariant measure $d^\times t$ on \mathbb{A}^\times by $d^\times t = d^\times \lambda d^\times t^1$ where $d^\times \lambda = d\lambda/\lambda$. Then $d_{\text{pr}}^\times t = \mathfrak{C}_k d^\times t$ (see [We74], p. 95). We also choose an invariant measure $d^\times t^0$ on \mathbb{A}^0 so that $\int_{\mathbb{A}^0} d^\times t^0 = 1$.

Let Ω^1 be the group of characters on \mathbb{A}^1/k^\times . For $\omega \in \Omega^1$, we put $\delta(\omega) = 1$ if ω is trivial and 0 otherwise. We extend ω to a character on $\mathbb{A}^\times/k^\times$ by assuming that $\omega(\lambda) = 1$ for any $\lambda \in \mathbb{R}_+$. Let Ω_v be the group of characters on k_v . If $v \in \mathfrak{M}_f$, we put $\delta(\omega_v) = 1$ if ω_v is trivial on \mathcal{O}_v^\times and 0 otherwise. For a vector space V , Let $\mathcal{S}(V_{\mathbb{A}})$, $\mathcal{S}(V_{k_v})$, $\mathcal{S}(V_\infty)$ and $\mathcal{S}(V_{\mathbb{A}_f})$ be the spaces of Schwartz–Bruhat functions on each of the indicated domains.

Let $T(2) \subset GL(2)$ be the set of diagonal matrices and $N(2) \subset GL(2)$ the set of the lower triangular matrices with diagonal entries 1. Then, $B(2) = T(2) \times N(2)$ is a Borel subgroup of $GL(2)$. (This should not be confused to the group B we defined in Section 2.) We express

$$\text{diag}(t_1, t_2) = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in T(2), \quad n(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \in N(2).$$

We briefly review the standard Haar measure on global and local measure on $GL(2)$. Let $\mathcal{K}(2) = \prod_{v \in \mathfrak{M}} \mathcal{K}_v(2)$ where $\mathcal{K}_v(2) = O(2), U(2), GL(2)_{\mathcal{O}_v}$ for $v \in \mathfrak{M}_{\mathbb{R}}, \mathfrak{M}_{\mathbb{C}}, \mathfrak{M}_f$, respectively. We choose an invariant measure $d\kappa, d\kappa_v$ on $\mathcal{K}(2), \mathcal{K}_v(2)$ such that $\int_{\mathcal{K}(2)} d\kappa = 1, \int_{\mathcal{K}_v(2)} d\kappa_v = 1$, respectively. Obviously, $d\kappa = d_{\text{pr}}\kappa$. We express element $b \in B(2)_{\mathbb{A}}$ as $b = tn(u) = \text{diag}(t_1, t_2)n(u)$ where $t_1, t_2 \in \mathbb{A}^\times$ and $u \in \mathbb{A}$. We put $d^\times t = d^\times t_1 d^\times t_2$ which is a Haar measure on $T(2)_{\mathbb{A}}$, and choose $db = |t_2/t_1| d^\times t du$ as a normalized right invariant measure on $B(2)_{\mathbb{A}}$. Note that if we write $b = n(u')t$ where $u' \in \mathbb{A}, t \in T(2)_{\mathbb{A}}$, then $db = d^\times t du'$. We define the local version of the right invariant measure db_v on $B(2)_{k_v}$ similarly. Then $d_{\text{pr}}b = \Delta_k^{1/2} \mathfrak{C}_k^2 db$. The group $GL(2)_{\mathbb{A}}$ has the decomposition $GL(2)_{\mathbb{A}} = \mathcal{K}(2)B(2)_{\mathbb{A}}$. We choose an invariant measure on $GL(2)_{\mathbb{A}}$ by $dg = d\kappa db$ for $g = \kappa b$. We define an invariant measure dg_v on $GL(2)_{k_v}$ similarly. Then $d_{\text{pr}}g = \Delta_k^{1/2} \mathfrak{C}_k^2 dg$.

Recall that we defined $H = GL(1) \times GL(2)$. We regard $GL(1)$ and $GL(2)$ as subgroups of H . We choose $dh = d^\times t dg$ as the normalized invariant measure on $H_{\mathbb{A}} = \mathbb{A}^\times \times GL(2)_{\mathbb{A}}$. We express elements of B as

$$a(t, p) = \left(t, \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \right), \quad n(u) = \left(1, \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \right).$$

We express element $b \in B_{\mathbb{A}}$ as $b = n(u)a(t, p)$ where $t, p \in \mathbb{A}^\times$ and $u \in \mathbb{A}$. We choose $db = d^\times t d^\times p du$ as a normalized right invariant measure on $B_{\mathbb{A}}$. The local measures dh_v, db_v on H_{k_v}, B_{k_v} are defined in the similar manner and we have $d_{\text{pr}}h = \Delta_k^{1/2} \mathfrak{C}_k^3 dh$ and $d_{\text{pr}}b = \Delta_k^{1/2} \mathfrak{C}_k^2 db$. Let $GL(2)_{\mathbb{A}}^1 = \{g \in GL(2)_{\mathbb{A}} \mid |\det(g)| = 1\}$. We choose a measure dg^1 on $GL(2)_{\mathbb{A}}^1$ such that

$$\int_{GL(2)_{\mathbb{A}}} f(g) dg = \int_{\mathbb{R}_+} \int_{GL(2)_{\mathbb{A}}^1} f(\text{diag}(\lambda^{1/2}, \lambda^{1/2})g^1) dg^1 d^\times \lambda.$$

It is well known that the volume of $GL(2)_{\mathbb{A}}^1/GL(2)_k$ with respect to dg^1 is $\Delta_k \zeta_k(2)/\mathfrak{C}_k$.

Let $\langle \cdot \rangle$ is a non-trivial additive character on \mathbb{A}/k and suppose $\langle x \rangle = \prod_{v \in \mathfrak{M}} \langle x_v \rangle_v$ for $x = (x_v) \in \mathbb{A}$. We can choose $\langle \cdot \rangle$ so that $\langle x_v \rangle_v = e^{2\pi i x_v}$ if $v \in \mathfrak{M}_{\mathbb{R}}$ and $\langle x_v \rangle_v = e^{2\pi i(x_v + \bar{x}_v)}$ if $v \in \mathfrak{M}_{\mathbb{C}}$ and fix such a character for all. We recall the Iwasawa-Tate zeta function. For $\Phi \in \mathcal{S}(\mathbb{A})$, $s \in \mathbb{C}$ and $\omega \in \Omega^1$, we define

$$(4.1) \quad \Sigma(\Phi, s, \omega) = \int_{\mathbb{A}^\times} |t|^s \omega(t) \Phi(t) d^\times t.$$

Let $\Sigma^+(\Phi, s, \omega)$ be the integral obtained from $\Sigma(\Phi, s, \omega)$ by restricting the domain of integration to $\{t \in \mathbb{A}^\times \mid |t| \geq 1\}$. The integral $\Sigma^+(\Phi, s, \omega)$ is an entire function of s . We define the Fourier transform Φ^* of Φ via $\Phi^*(x) = \int_{\mathbb{A}} \Phi(y) \langle xy \rangle dy$. Then by the Poisson summation formula, we have

$$\Sigma(\Phi, s, \omega) = \Sigma^+(\Phi, s, \omega) + \Sigma^+(\Phi^*, 1 - s, \omega^{-1}) + \delta(\omega) \left(\frac{\Phi^*(0)}{s-1} - \frac{\Phi(0)}{s} \right).$$

We put

$$(4.2) \quad \Sigma_{(0)}(\Phi, \omega) = \Sigma^+(\Phi, 0, \omega) + \Sigma^+(\Phi^*, 1, \omega^{-1}) - \delta(\omega) \Phi^*(0),$$

which is the constant term of the Laurent expansion of $\Sigma(\Phi, s, \omega)$ at $s = 0$.

5. GLOBAL THEORY FOR THE SPACES OF BINARY QUADRATIC FORMS

In this section, we define the global zeta functions for the spaces of binary quadratic forms described in Section 2 and give some principal part formulas as well as analytic continuations. These cases are typical example of prehomogeneous vector spaces, and after Shintani's pioneering work [Sh75, Chapter1] the zeta functions were investigated by many authors including [Sat81], [Sai93]. Since most of them are written in the classical language, we reconsidered it in the adelic settings along the line with F. Sato's formulation [Sat81]. Because of Proposition 3.12, our main interest is the zeta function associated to (B, W) . However as we will see in Lemma 5.7 (which is due to F. Sato), the zeta functions for (H, U) and (B, W) are essentially the same, and to give an analytic continuation of the zeta function it seems to be more natural to consider representations of reductive groups. Hence we mainly treat (H, U) in this section.

5.1. The zeta functions. We start with the definition of the zeta function for (H, U) .

Definition 5.1. For $\Phi \in \mathcal{S}(U_{\mathbb{A}})$, $s_1, s_2 \in \mathbb{C}$ and $\omega_1, \omega_2 \in \Omega^1$, we define

$$\begin{aligned} X(\Phi, s_1, s_2, \omega_1, \omega_2) &= \int_{H_{\mathbb{A}}/H_k} |t|^{s_1} |t \det g|^{2s_2} \omega_1(t) \omega_2(t \det g) \sum_{\tilde{y} \in U_k^{ss}} \Phi(h\tilde{y}) dh, \\ X^{(1+)}(\Phi, s_1, s_2, \omega_1, \omega_2) &= \int_{\substack{H_{\mathbb{A}}/H_k \\ |t \det g| \geq 1}} |t|^{s_1} |t \det g|^{2s_2} \omega_1(t) \omega_2(t \det g) \sum_{\tilde{y} \in U_k^{ss}} \Phi(h\tilde{y}) dh, \\ X^{(2+)}(\Phi, s_1, s_2, \omega_1, \omega_2) &= \int_{\substack{H_{\mathbb{A}}/H_k \\ |t \det g| \leq 1}} |t|^{s_1} |t \det g|^{2s_2} \omega_1(t) \omega_2(t \det g) \sum_{\tilde{y} \in U_k^{ss}} \Phi(h\tilde{y}) dh. \end{aligned}$$

For the rest of this section we consider the analytic continuation of $X(\Phi, s_1, s_2, \omega_1, \omega_2)$. When discussing the convergence of integrals we use the phrase *normally convergent* to mean absolutely and locally uniformly convergent. The convergence of zeta functions of prehomogeneous vectors space was investigated by many mathematicians and finally proved in full generality by H. Saito [Sai03]. For the convergence of the integrals above, the following lemma holds. In fact, (1) is contained in [Sai03] and (2) and (3) immediately follow from (1).

Lemma 5.2. *There exist $\delta_1 > 0, \delta_2 > 0$ such that the following hold.*

- (1) *The integral $X(\Phi, s_1, s_2, \omega_1, \omega_2)$ is normally convergent for $\Re(s_1) > \delta_1, \Re(s_2) > \delta_2$.*
- (2) *The integral $X^{(1+)}(\Phi, s_1, s_2, \omega_1, \omega_2)$ is normally convergent for $\Re(s_1) > \delta_1$.*

- (3) The integral $X^{(2+)}(\Phi, s_1, s_2, \omega_1, \omega_2)$ is normally convergent for $\Re(s_1 + 2s_2) > \delta_1 + 2\delta_2$, $\Re(s_2) > \delta_2$.

Remark 5.3. H. Saito's result is stronger than the above; it shows that the optimum domain of convergence is obtained by taking $\delta_1 = \delta_2 = 1$. In this section we give an alternative proof of this fact.

Before starting the analysis, we make two natural assumptions on Φ for practical purposes. The first one is:

Assumption 5.4. The test function $\Phi \in \mathcal{S}(U_{\mathbb{A}})$ is of the form $\Phi = \Psi \otimes \Upsilon$, where $\Psi \in \mathcal{S}(W_{\mathbb{A}})$ and $\Upsilon \in \mathcal{S}(S_{\mathbb{A}})$.

Since W and S are both H -invariant subspaces, this is enough for most of the applications. Let \mathcal{K} be the standard maximal compact subgroup of $H_{\mathbb{A}}$ i.e., $\mathcal{K} = \mathbb{A}^0 \times \mathcal{K}(2)$. Let $d\kappa = d^\times t^0 d\kappa_2$ be the measure on \mathcal{K} , so that the total volume of \mathcal{K} is 1. For $\Phi \in \mathcal{S}(U_{\mathbb{A}})$, we define $\mathcal{M}_{(\omega_1, \omega_2)}\Phi \in \mathcal{S}(U_{\mathbb{A}})$ by

$$\mathcal{M}_{(\omega_1, \omega_2)}\Phi(x) = \int_{\mathcal{K}} \omega_1(\chi_1(\kappa))\omega_2(\chi_2(\kappa))\Phi(\kappa x)d\kappa.$$

Then we have $X(\Phi, s_1, s_2, \omega_1, \omega_2) = X(\mathcal{M}_{(\omega_1, \omega_2)}\Phi, s_1, s_2, \omega_1, \omega_2)$ and $\mathcal{M}_{(\omega_1, \omega_2)}(\mathcal{M}_{(\omega_1, \omega_2)}\Phi) = \mathcal{M}_{(\omega_1, \omega_2)}\Phi$. Hence, without loss of generality, we will assume:

Assumption 5.5. The Schwartz-Bruhat function Φ satisfies $\mathcal{M}_{(\omega_1, \omega_2)}\Phi = \Phi$.

This assumption holds, for example, if Φ is \mathcal{K} -invariant and ω_1, ω_2 are trivial on \mathbb{A}^0 (cf. [KW05]). This assumption yields the similar assumptions for the components Ψ and Υ .

We now define the zeta function for (B, W) .

Definition 5.6. For $\Psi \in \mathcal{S}(W_{\mathbb{A}})$, $s_1, s_2 \in \mathbb{C}$ and $\omega_1, \omega_2 \in \Omega^1$, we define

$$Y(\Psi, s_1, s_2, \omega_1, \omega_2) = \int_{B_{\mathbb{A}}/B_k} |t|^{s_1} |tp|^{2s_2} \omega_1(t)\omega_2(tp) \sum_{y \in W_k^{\text{ss}}} \Psi(by)db,$$

$$Y^+(\Psi, s_1, s_2, \omega_1, \omega_2) = \int_{\substack{B_{\mathbb{A}}/B_k \\ |tp| \geq 1}} |t|^{s_1} |tp|^{2s_2} \omega_1(t)\omega_2(tp) \sum_{y \in W_k^{\text{ss}}} \Psi(by)db.$$

For the zeta functions $X(\Phi, s_1, s_2, \omega_1, \omega_2)$ and $Y(\Psi, s_1, s_2, \omega_1, \omega_2)$, the following lemma holds. Recall that the distribution Σ is the Iwasawa-Tate zeta function we defined in (4.1). (The local version of this lemma is also given in Lemma 7.10.)

Lemma 5.7. Let us define $\mathcal{R}_1\Upsilon \in \mathcal{S}(\mathbb{A})$ by $\mathcal{R}_1\Upsilon(x) = \Upsilon(x, 0)$. We have

$$X(\Phi, s_1, s_2, \omega_1, \omega_2) = Y(\Psi, s_1, s_2, \omega_1, \omega_2)\Sigma(\mathcal{R}_1\Upsilon, 2s_1, \omega_1^2),$$

$$X^{(1+)}(\Phi, s_1, s_2, \omega_1, \omega_2) = Y^+(\Psi, s_1, s_2, \omega_1, \omega_2)\Sigma(\mathcal{R}_1\Upsilon, 2s_1, \omega_1^2).$$

In particular, the integral $Y(\Psi, s_1, s_2, \omega_1, \omega_2)$ is normally convergent in the region $\Re(s_1) > \delta_1$, $\Re(s_2) > \delta_2$ and the integral $Y^+(\Psi, s_1, s_2, \omega_1, \omega_2)$ is normally convergent in the region $\Re(s_1) > \delta_1$.

Proof. We consider the first equation. The second one is proved exactly the same way. Let

$$Z' = \{\tilde{y} = (y, \bar{y}) \in U \mid Q_1(y)Q_2(y)\bar{y}_{21} \neq 0, \bar{y}_{22} = 0\}$$

and $B' = \text{GL}(1) \times \text{B}(2)$. Then it is easy to see that $U_k^{\text{ss}} = H_k \times_{B'_k} Z'_k$. We choose the measure db' on $B'_{\mathbb{A}}$ as the product measure of $d^\times t$ on \mathbb{A}^\times and db_2 on $\text{B}(2)_{\mathbb{A}}$. By denoting $\tilde{\omega}_1 = |\cdot|^{s_1}\omega_1$

and $\tilde{\omega}_2 = |\cdot|^{2s_2}\omega_2$ we have

$$\begin{aligned} X(\Phi, s_1, s_2, \omega_1, \omega_2) &= \int_{H_{\mathbb{A}}/B'_k} \tilde{\omega}_1(t)\tilde{\omega}_2(t \det g) \sum_{x \in Z'_k} \Phi(hx) dh \\ &= \int_{B'_{\mathbb{A}}/B'_k} \tilde{\omega}_1(t)\tilde{\omega}_2(t \det b_2) \sum_{x \in Z'_k} \Phi(b'x) db' \\ &= \int_{(\mathbb{A}^\times/k^\times)^3 \times \mathbb{A}/k} \tilde{\omega}_1(t)\tilde{\omega}_2(tt_1t_2) \sum_{y \in W_k^{ss}} \Psi((t, \text{diag}(t_1, t_2)n(u))y) \\ &\quad \times \sum_{\bar{y}_1 \in k^\times} R_1 \Upsilon(t_1^{-1}\bar{y}_1) \left| \frac{t_2}{t_1} \right| d^\times t d^\times t_1 d^\times t_2 du, \end{aligned}$$

by changing t_1 to t_1^{-1} and after that t to tt_1^2 and t_2 to t_2/t_1 , we have

$$\begin{aligned} &= \int_{(\mathbb{A}^\times/k^\times)^2 \times \mathbb{A}/k} \tilde{\omega}_1(t)\tilde{\omega}_2(tt_2) \sum_{y \in W_k^{ss}} \Psi((a(t, t_2)n(u))y) |t_2| d^\times t d^\times t_2 du \\ &\quad \times \int_{\mathbb{A}^\times/k^\times} \tilde{\omega}_1(t_1^2) \sum_{\bar{y}_1 \in k^\times} R_1 \Upsilon(t_1\bar{y}_1) d^\times t_1. \end{aligned}$$

Hence we have the lemma. \square

5.2. Principal part formula I. In this subsection, we give analytic continuations of functions $X(\Phi, s_1, s_2, \omega_1, \omega_2)$ and $Y(\Psi, s_1, s_2, \omega_1, \omega_2)$ to the region $\Re(s_1) > \delta_1$ and find the principal parts in this region. We first define a singular distribution, which arises as a principal part of $Y(\Psi, s_1, s_2, \omega_1, \omega_2)$.

Definition 5.8. For $\Psi \in \mathcal{S}(W_{\mathbb{A}})$, $s \in \mathbb{C}$ and $\omega \in \Omega^1$, we put

$$\Lambda(\Psi, s, \omega) = \int_{\mathbb{A}^\times \times \mathbb{A}} |t|^s \omega(t) \Psi(t, 2tu, tu^2) d^\times t du.$$

Lemma 5.9. *The integral $\Lambda(\Psi, s, \omega)$ is normally convergent for $\Re(s) > 1$.*

Proof. We define the local version of Λ by

$$\Lambda_v(\Psi_v, s, \omega_v) = \int_{k_v^\times \times k_v} |t|_v^s \omega_v(t) \Psi_v(t, 2t_u u_v, t_u u_v^2) d^\times t_u du_v.$$

For $v \in \mathfrak{M}_f$ we let $\Psi_{v,0}$ the characteristic function of $W_{\mathcal{O}_v}$. Then by computation we have

$$\Lambda_v(\Psi_{v,0}, s, \omega_v) = \delta_v(\omega_v) \frac{1 - q_v^{-2s}}{(1 - q_v^{-s})(1 - q_v^{-2s+1})}.$$

By considering the Euler product, we have the lemma. \square

We now consider (B, W) . We define the symmetric bilinear forms on W by $[y, z]_W = y_1 z_3 - 2^{-1} y_2 z_2 + y_3 z_1$. Let ι be the involution on B given by

$$\iota: B \longrightarrow B, \quad b = \left(t, \begin{pmatrix} 1 & 0 \\ u & p \end{pmatrix} \right) \longmapsto b^\iota = \left(t^{-1} p^{-2}, \begin{pmatrix} 1 & 0 \\ u & p \end{pmatrix} \right).$$

Then we have $[by, b'z]_W = [y, z]_W$ for all $y, z \in W$ and $b \in B$.

Definition 5.10. For $\Psi \in \mathcal{S}(W_{\mathbb{A}})$, we define its Fourier transform $\widehat{\Psi} \in \mathcal{S}(W_{\mathbb{A}})$ by

$$\widehat{\Psi}(y) = \int_{W_{\mathbb{A}}} \Psi(z) \langle [y, z]_W \rangle dz.$$

For $b \in B_{\mathbb{A}}$ and $\Psi \in \mathcal{S}(W_{\mathbb{A}})$, we define $\Psi_b \in \mathcal{S}(W_{\mathbb{A}})$ by $\Psi_b(y) = \Psi(by)$. Then we have $\widehat{\Psi}_b = |tp|^{-3}(\widehat{\Psi})_{b^v}$. Finally we define a operator \mathcal{R} as follows:

Definition 5.11. For $\Psi \in \mathcal{S}(W_{\mathbb{A}})$ we define $\mathcal{R}\Psi \in \mathcal{S}(\mathbb{A})$ by $\mathcal{R}\Psi(x) = \int_{\mathbb{A}^2} \Psi(x, u_2, u_3) du_2 du_3$.

The following proposition is an adelic version of [Sh75, Lemma 4].

Proposition 5.12. *We have*

$$\begin{aligned} Y(\Psi, s_1, s_2, \omega_1, \omega_2) &= Y^+(\Psi, s_1, s_2, \omega_1, \omega_2) + Y^+(\widehat{\Psi}, s_1, 3/2 - s_1 - s_2, \omega_1, \omega_1^{-2}\omega_2^{-1}) \\ &\quad + \frac{\delta(\omega_1^2\omega_2)}{2s_1 + 2s_2 - 3} \Lambda(\widehat{\Psi}, s_1, \omega_1) - \frac{\delta(\omega_2)}{2s_2} \Lambda(\Psi, s_1, \omega_1) \\ &\quad + \frac{\delta(\omega_2)}{2s_2 - 2} \Sigma(\mathcal{R}\Psi, s_1, \omega_1) - \frac{\delta(\omega_1^2\omega_2)}{2s_1 + 2s_2 - 1} \Sigma(\mathcal{R}\widehat{\Psi}, s_1, \omega_1). \end{aligned}$$

Proof. Let $\tilde{\omega}_1 = |\cdot|^{s_1}\omega_1$ and $\tilde{\omega}_2 = |\cdot|^{2s_2}\omega_2$. We set $Z = \{y \in W \mid y_1(y_2^2 - 4y_1y_3) = 0\}$ and put

$$I(\Psi, \tilde{\omega}_1, \tilde{\omega}_2) = \int_{\substack{B_{\mathbb{A}}/B_k \\ |tp| \leq 1}} \tilde{\omega}_1(t)\tilde{\omega}_2(tp) \left(\sum_{y \in Z_k} \widehat{\Psi}_b(y) - \sum_{y \in Z_k} \Psi_b(y) \right) db.$$

Then since $W_k = W_k^{\text{ss}} \amalg Z_k$, by the Poisson summation formula we have

$$\begin{aligned} Y(\Psi, s_1, s_2, \omega_1, \omega_2) &= Y^+(\Psi, s_1, s_2, \omega_1, \omega_2) \\ &\quad + Y^+(\widehat{\Psi}, s_1, 3/2 - s_1 - s_2, \omega_1, \omega_1^{-2}\omega_2^{-1}) + I(\Psi, \tilde{\omega}_1, \tilde{\omega}_2). \end{aligned}$$

We consider $I(\Psi, \tilde{\omega}_1, \tilde{\omega}_2)$. Let

$$Z_1 = \{y \in W \mid y_1 \neq 0, y_2^2 - 4y_1y_3 = 0\}, \quad Z_2 = \{y \in W \mid y_1 = 0\}.$$

Obviously we have $Z_k = Z_{1k} \amalg Z_{2k}$. We put $B_{\mathbb{A}}^- = \{b \in B_{\mathbb{A}} \mid |tp| \leq 1\}$ and define

$$\begin{aligned} I_1(\Psi, \tilde{\omega}_1, \tilde{\omega}_2) &= \int_{B_{\mathbb{A}}^-/B_k} \tilde{\omega}_1(t)\tilde{\omega}_2(tp) |tp|^{-3} \sum_{y \in Z_{1k}} \widehat{\Psi}(b^t y) db, \\ I_2(\Psi, \tilde{\omega}_1, \tilde{\omega}_2) &= \int_{B_{\mathbb{A}}^-/B_k} \tilde{\omega}_1(t)\tilde{\omega}_2(tp) \sum_{y \in Z_{1k}} \Psi(by) db, \\ I_3(\Psi, \tilde{\omega}_1, \tilde{\omega}_2) &= \int_{B_{\mathbb{A}}^-/B_k} \tilde{\omega}_1(t)\tilde{\omega}_2(tp) \left(\sum_{y \in Z_{2k}} \widehat{\Psi}_b(y) - \sum_{y \in Z_{2k}} \Psi_b(y) \right) db. \end{aligned}$$

Then $I(\Psi, \tilde{\omega}_1, \tilde{\omega}_2) = (I_1 - I_2 + I_3)(\Psi, \tilde{\omega}_1, \tilde{\omega}_2)$. Let us first consider I_1 . We let $w = (1, 0, 0) \in Z_{1k}$ and $B_w = \{a(1, p) \mid p \in \mathbb{G}_m\}$, which is the stabilizer of w in B . It is easy to see that $Z_{1k} = B_k w$. Hence we have

$$\begin{aligned} I_1(\Psi, \tilde{\omega}_1, \tilde{\omega}_2) &= \int_{B_{\mathbb{A}}^-/B_{wk}} \tilde{\omega}_1(t)\tilde{\omega}_2(tp) |tp|^{-3} \widehat{\Psi}(b^t w) db \\ &= \int_{\substack{(\mathbb{A}^\times \times \mathbb{A}^\times / k^\times) \times \mathbb{A} \\ |tp| \leq 1}} \tilde{\omega}_1(t)\tilde{\omega}_2(tp) |tp|^{-3} \widehat{\Psi}(1/tp^2, 2u/tp^2, u^2/tp^2) d^\times t d^\times pdu \end{aligned}$$

by changing t to $t^{-1}p^{-2}$,

$$= \int_{\substack{(\mathbb{A}^\times \times \mathbb{A}^\times / k^\times) \times \mathbb{A} \\ |tp| \geq 1}} (\tilde{\omega}_1\tilde{\omega}_2)(t^{-1}) |t|^3 \widehat{\Psi}(t, 2tu, tu^2) \cdot (\tilde{\omega}_1^2\tilde{\omega}_2)(p^{-1}) |p|^3 d^\times t d^\times pdu.$$

Since

$$\int_{\substack{\mathbb{A}^\times/k^\times \\ |tp| \geq 1}} (\tilde{\omega}_1^2 \tilde{\omega}_2)(p^{-1}) |p|^3 d^\times p = \frac{\delta(\omega_1^2 \omega_2)}{2s_1 + 2s_2 - 3} (\tilde{\omega}_1^2 \tilde{\omega}_2)(t) |t|^{-3},$$

we have $I_1(\Psi, \tilde{\omega}_1, \tilde{\omega}_2) = (2s_1 + 2s_2 - 3)^{-1} \delta(\omega_1^2 \omega_2) \Lambda(\widehat{\Psi}, s, \omega_1)$. Similarly we have $I_2(\Psi, \tilde{\omega}_1, \tilde{\omega}_2) = (2s_2)^{-1} \delta(\omega_2) \Lambda(\Psi, s_1, \omega_1)$. We finally consider $I_3(\Psi, \tilde{\omega}_1, \tilde{\omega}_2)$. We have

$$I_3(\Psi, \tilde{\omega}_1, \tilde{\omega}_2) = \int_{T_{\mathbb{A}}^-/T_k} \tilde{\omega}_1(t) \tilde{\omega}_2(tp) |p| J(\Psi_{a(t,p)}) d^\times t$$

where we put $T_{\mathbb{A}}^- = B_{\mathbb{A}}^- \cap T_{\mathbb{A}}$ and

$$J(\Psi) = \int_{N_{\mathbb{A}}/N_k} \left(\sum_{y \in S_{2k}} \widehat{\Psi}(n(u)y) - \sum_{y \in S_{2k}} \Psi(n(u)y) \right) du.$$

We consider $J(\Psi)$. We define $\mathcal{R}_3 \Psi \in \mathcal{S}(\mathbb{A})$ by $\mathcal{R}_3 \Psi(x) = \Psi(0, 0, x)$. By dividing the index set Z_{2k} into

$$Z_{2k} = \{(0, y_2, y_3) \mid y_2 \in k^\times, y_3 \in k\} \amalg \{(0, 0, y_3) \mid y_3 \in k\},$$

and performing integration separately, we have

$$J(\Psi) = \sum_{\alpha \in k} \mathcal{R}_3 \widehat{\Psi}(\alpha) - \sum_{\alpha \in k} \mathcal{R}_3 \Psi(\alpha) + \sum_{\alpha \in k^\times} \int_{\mathbb{A}} \widehat{\Psi}(0, \alpha, u) du - \sum_{\alpha \in k^\times} \int_{\mathbb{A}} \Psi(0, \alpha, u) du.$$

Since the equality $\sum_{\alpha \in k} \int_{\mathbb{A}} \widehat{\Psi}(0, \alpha, u) du = \sum_{\alpha \in k} \int_{\mathbb{A}} \Psi(0, \alpha, u) du$ holds by the Poisson summation formula, we have

$$J(\Psi) = \left(\sum_{\alpha \in k} \mathcal{R}_3 \widehat{\Psi}(\alpha) - \int_{\mathbb{A}} \mathcal{R}_3 \widehat{\Psi}(u) du \right) - \left(\sum_{\alpha \in k} \mathcal{R}_3 \Psi(\alpha) - \int_{\mathbb{A}} \mathcal{R}_3 \Psi(u) du \right).$$

Again by using the Poisson summation formula, we have

$$J(\Psi) = \sum_{\alpha \in k^\times} \mathcal{R} \Psi(\alpha) - \sum_{\alpha \in k^\times} \mathcal{R} \widehat{\Psi}(\alpha).$$

Note that we define the operator \mathcal{R} in Definition 5.11. Now we can easily see

$$J(\Psi_{a(t,p)}) = |t^{-2} p^{-3}| \sum_{\alpha \in k^\times} \mathcal{R} \Psi(t\alpha) - |t^{-1} p^{-2}| \sum_{\alpha \in k^\times} \mathcal{R} \widehat{\Psi}(t^{-1} p^{-2} \alpha)$$

and hence,

$$\begin{aligned} I_3(\Psi, \tilde{\omega}_1, \tilde{\omega}_2) &= \int_{T_{\mathbb{A}}^-/T_k} \tilde{\omega}_1(t) \tilde{\omega}_2(tp) |tp|^{-2} \sum_{\alpha \in k^\times} \mathcal{R} \Psi(t\alpha) d^\times t d^\times p \\ &\quad - \int_{T_{\mathbb{A}}^-/T_k} \tilde{\omega}_1(t) \tilde{\omega}_2(tp) |tp|^{-1} \sum_{\alpha \in k^\times} \mathcal{R} \widehat{\Psi}(t^{-1} p^{-2} \alpha) d^\times t d^\times p. \end{aligned}$$

Now straightforward calculations show that the integrals above equal to

$$(2s_2 - 2)^{-1} \delta(\omega_2) \Sigma(\mathcal{R} \Psi, s_1, \omega_1) \quad \text{and} \quad (2s_1 + 2s_2 - 1)^{-1} \delta(\omega_1^2 \omega_2) \Sigma(\mathcal{R} \widehat{\Psi}, s_1, \omega_1),$$

respectively. These give the desired description. \square

As a corollary to this proposition, we obtain the following.

Corollary 5.13. *The function $s_2(s_2 - 1)(2s_1 + 2s_2 - 1)(2s_1 + 2s_2 - 3)X(\Phi, s_1, \omega_1, s_2, \omega_2)$ can be holomorphically continued to the region $\Re(s_1) > \delta_1$. Also the following functional equation holds:*

$$X(\Psi \otimes \Upsilon, s_1, s_2, \omega_1, \omega_2) = X(\widehat{\Psi} \otimes \Upsilon, s_1, 3/2 - s_1 - s_2, \omega_1, \omega_1^{-2} \omega_2^{-1}).$$

5.3. Principal part formula II. In this subsection, we give analytic continuation of the function $X(\Phi, s_1, s_2, \omega_1, \omega_2)$ to the region $\Re(s_1 + 2s_2) > \delta_1 + 2\delta_2, \Re(s_2) > \delta_2$ and find the principal parts in this region.

Definition 5.14. Let $U_{(1)} = \{\tilde{y} \in U_k^{\text{ss}} \mid k(\tilde{y}) \cong k \times k\}$ and $U_{(2)} = U_k^{\text{ss}} \setminus U_{(1)}$. For $i = 1, 2$, we define

$$X_{(i)}(\Phi, s_1, s_2, \omega_1, \omega_2) = \int_{H_{\mathbb{A}}/H_k} |t|^{s_1} |t \det g|^{2s_2} \omega_1(t) \omega_2(t \det g) \sum_{x \in U_{(i)}} \Phi(hx) dh,$$

$$X_{(i)}^{(2+)}(\Phi, s_1, s_2, \omega_1, \omega_2) = \int_{\substack{H_{\mathbb{A}}/H_k \\ |\det g| \leq 1}} |t|^{s_1} |t \det g|^{2s_2} \omega_1(t) \omega_2(t \det g) \sum_{x \in U_{(i)}} \Phi(hx) dh.$$

Since $U_k^{\text{ss}} = U_{(1)} \amalg U_{(2)}$ we have $X = X_{(1)} + X_{(2)}$ and $X^{(2+)} = X_{(1)}^{(2+)} + X_{(2)}^{(2+)}$. By Lemma 5.2 (3), $X_{(i)}^{(2+)}(\Phi, s_1, s_2, \omega_1, \omega_2)$ is holomorphic for $\Re(s_1 + 2s_2) > \delta_1 + 2\delta_2, \Re(s_2) > \delta_2$. We consider $X_{(1)}$ and $X_{(2)}$ separately.

We define the bilinear forms on S by $[\bar{y}, \bar{z}]_S = \bar{y}_1 \bar{z}_2 - \bar{y}_2 \bar{z}_1$. Let ι be the involution on H given by $(t, g)^\iota = (t, -(\det g)^{-1}g)$. Then we have $[h\bar{y}, h'\bar{z}]_S = [\bar{y}, \bar{z}]_S$ for all $\bar{y}, \bar{z} \in S$ and $h \in H$.

Definition 5.15. For $\Upsilon \in \mathcal{S}(S_{\mathbb{A}})$, we define its Fourier transform $\widehat{\Upsilon}$ by

$$\widehat{\Upsilon}(\bar{y}) = \int_{S_{\mathbb{A}}} \Upsilon(\bar{z}) \langle [\bar{y}, \bar{z}]_S \rangle d\bar{z}.$$

We first consider $X_{(2)}(\Phi, s_1, s_2, \omega_1, \omega_2)$. Since the zeta function associated with (H, W) appears as a term of the residue at $s_1 = 1$ (this observation is due to Shintani [Sh75]), we recall the definition of the zeta function for the space. In this case, due to a convergence problem, we are not allowed to let the index set of the summation in the integral to be W_k^{\sim} . We use $W_{(2)}$ defined below instead. Let $dh^1 = d^\times t dg^1$ for $h^1 = (t, g^1), t \in \mathbb{A}^\times, g^1 \in \text{GL}(2)_{\mathbb{A}}^1$.

Definition 5.16. Let $W_{(2)} = \{y \in W_k^{\sim} \mid k(y) \not\cong k \times k\}$. For $\Psi \in \mathcal{S}(W_{\mathbb{A}})$, $s_1, s_2 \in \mathbb{C}$ and $\omega_1, \omega_2 \in \Omega^1$, we define

$$U(\Psi, s_1, s_2, \omega_1, \omega_2) = \int_{H_{\mathbb{A}}^1/H_k} |t|^{s_1+2s_2} \omega_1(t) \omega_2(t \det g^1) \sum_{y \in W_{(2)}} \Psi(h^1 y) dh^1.$$

The function $U(\Psi, s_1, s_2, \omega_1, \omega_2)$ is essentially a one variable function but we use the above definition for convenience. It is well known that the integral converges if $\Re(s_1 + 2s_2) > \delta_3$ for some δ_3 . By changing δ_1 and δ_2 if necessary we may assume $\delta_3 = \delta_1 + 2\delta_2$. For $p \in \mathbb{G}_m$, we have $(tp^2, p^{-1}g)y = (t, g)y$. Hence by computation we could see:

Lemma 5.17.

$$U(\Psi, s_1, s_2, \omega_1, \omega_2) = U(\Psi, 1 - s_1, s_1 + s_2 - 1/2, \omega_1^{-1}, \omega_1^2 \omega_2).$$

Proposition 5.18.

$$\begin{aligned} X_{(2)}(\Psi \otimes \Upsilon, s_1, s_2, \omega_1, \omega_2) &= X_{(2)}^{(2+)}(\Psi \otimes \Upsilon, s_1, s_2, \omega_1, \omega_2) + X_{(2)}^{(2+)}(\Psi \otimes \widehat{\Upsilon}, 1 - s_1, s_1 + s_2 - 1/2, \omega_1^{-1}, \omega_1^2 \omega_2) \\ &+ \frac{\widehat{\Upsilon}(0)}{s_1 - 1} U(\Psi, s_1, s_2, \omega_1, \omega_2) - \frac{\Upsilon(0)}{s_1} U(\Psi, s_1, s_2, \omega_1, \omega_2). \end{aligned}$$

Proof. Let $\tilde{\omega}_1 = |\cdot|^{s_1} \omega_1$ and $\tilde{\omega}_2 = |\cdot|^{2s_2} \omega_2$. Since $y(v)$ is a irreducible quadratic polynomial for $\tilde{y} = (y, \bar{y}) \in U_{(2)}$, $R_1(\tilde{y}) = 0$ implies $\bar{y} = 0$, i.e. , $U_{(2)} = W_{(2)} \times \{\tilde{y} \in S_k \mid \bar{y} \neq (0, 0)\}$. Hence

by the Poisson summation formula,

$$\begin{aligned} & X_{(2)}(\Phi, s_1, s_2, \omega_1, \omega_2) - X_{(2)}^{(2+)}(\Phi, s_1, s_2, \omega_1, \omega_2) \\ &= \int_{\substack{H_{\mathbb{A}}/H_k \\ |\det g| \geq 1}} \tilde{\omega}_1(t) \tilde{\omega}_2(t \det g) |\det g| \sum_{\tilde{y} \in U_{(2)}} \Psi(hy) \widehat{\Upsilon}(h^t \tilde{y}) dh \\ &+ \int_{\substack{H_{\mathbb{A}}/H_k \\ |\det g| \geq 1}} \tilde{\omega}_1(t) \tilde{\omega}_2(t \det g) \sum_{y \in W_{(2)}} \Psi(hy) \left(|\det g| \widehat{\Upsilon}(0) - \Upsilon(0) \right) dh. \end{aligned}$$

It is easy to see that the second integral in the formula above corresponds to the last second terms in the formula of the proposition. We consider the first integral in the formula, which is equal to

$$\int_{\substack{H_{\mathbb{A}}/H_k \\ |\det(g)| \leq 1}} \omega_1(t) \omega_2(t / \det g) |\det g|^{-1} \sum_{\tilde{y} \in U_{(2)}} \Psi(h^t y) \widehat{\Upsilon}(h \tilde{y}) dh.$$

For $y \in W$ we have $h^t y = (t, -(\det g)^{-1} g)y = (t(\det g)^{-2}, g)y$. With this relation in mind, by changing t to $t(\deg g)^2$ we could see that the integral is equal to $X_{(2)}^{(2+)}(\Psi \otimes \widehat{\Upsilon}, 1 - s_1, s_1 + s_2 - 1/2, \omega_1^{-1}, \omega_1^2 \omega_2)$. \square

We next consider $X_{(1)}^{(2+)}(\Phi, s_1, s_2, \omega_1, \omega_2)$. We introduce a function which plays an important role in our analysis.

Definition 5.19. For $\Upsilon \in \mathcal{A}(S_{\mathbb{A}})$, $s \in \mathbb{C}$ and $\omega \in \Omega^1$, let

$$\begin{aligned} \Sigma_2(\Upsilon, s, \omega) &= \int_{\mathbb{A}^\times \times \mathbb{A}^\times} |t_1 t_2|^s \omega(t_1 t_2) \Upsilon(t_1, t_2) d^\times t_1 d^\times t_2, \\ \Sigma_2^+(\Upsilon, s, \omega) &= \int_{\substack{\mathbb{A}^\times \times \mathbb{A}^\times \\ |t_1 t_2| \geq 1}} |t_1 t_2|^s \omega(t_1 t_2) \Upsilon(t_1, t_2) d^\times t_1 d^\times t_2. \end{aligned}$$

Note that the function $\Sigma_2^+(\Upsilon, s, \omega)$ is entire. We put $Z = \{(x_{21}, x_{22}) \in S \mid x_{21} x_{22} = 0\}$. Let

$$\begin{aligned} K(\Upsilon, t_1, t_2) &= |t_1 t_2|^{-1} \sum_{x_2 \in Z_k} \widehat{\Upsilon}(t_2^{-1} x_{21}, t_1^{-1} x_{22}) - \sum_{x_2 \in Z_k} \Upsilon(t_1 x_{21}, t_2 x_{22}), \\ J'(\Upsilon, s, \omega) &= \int_{\substack{(\mathbb{A}^\times/k^\times)^2 \\ |t_1 t_2| \leq 1}} |t_1 t_2|^s \omega(t_1 t_2) K(\Upsilon, t_1, t_2) d^\times t_1 d^\times t_2. \end{aligned}$$

Then by the Poisson summation formula, we have

$$\Sigma_2(\Upsilon, s, \omega) = \Sigma_2^+(\Upsilon, s, \omega) + \Sigma_2^+(\widehat{\Upsilon}, 1 - s, \omega^{-1}) + J'(\Upsilon, s, \omega).$$

For $\Upsilon \in \mathcal{A}(S_{\mathbb{A}})$, we define $\mathcal{R}_1 \Upsilon, \mathcal{R}_2 \Upsilon \in \mathcal{A}(\mathbb{A})$ by $\mathcal{R}_1 \Upsilon(x) = \Upsilon(x, 0)$, $\mathcal{R}_2 \Upsilon(x) = \Upsilon(0, x)$. Then the following holds. Note that we defined the distribution $\Sigma_{(0)}$ in (4.2).

Lemma 5.20. *We have $J'(\Upsilon, s, \omega) = \delta(\omega) J''(\Upsilon, s)$ where*

$$\begin{aligned} J''(\Upsilon, s) &= \frac{\widehat{\Upsilon}(0)}{(s-1)^2} + \frac{1}{s-1} \left(\Sigma_{(0)}(\mathcal{R}_1 \widehat{\Upsilon}, 0) + \Sigma_{(0)}(\mathcal{R}_2 \widehat{\Upsilon}, 0) \right) \\ &+ \frac{\Upsilon(0)}{s^2} - \frac{1}{s} \left(\Sigma_{(0)}(\mathcal{R}_1 \Upsilon, 0) + \Sigma_{(0)}(\mathcal{R}_2 \Upsilon, 0) \right). \end{aligned}$$

Proof. We shall calculate the integral J' by dividing it to the three integrals

$$J'_0 = \int_{|t_1| \leq 1, |t_2| \leq 1}, \quad J'_1 = \int_{1 \leq |t_1| \leq |t_2|^{-1}}, \quad J'_2 = \int_{1 \leq |t_2| \leq |t_1|^{-1}}.$$

Then by suitable uses of the Poisson summation formula for each integral, we have $J'_i(\Upsilon, \omega) = \delta(\omega)J''_i(\Upsilon, s)$ where

$$\begin{aligned} J''_0(\Upsilon, s) &= \frac{\widehat{\Upsilon}(0)}{(s-1)^2} + \frac{\Upsilon(0)}{s^2} - \frac{(\mathcal{R}_1\Upsilon)^*(0) + (\mathcal{R}_2\Upsilon)^*(0)}{s(s-1)} \\ &\quad + \frac{1}{s-1} \left(\Sigma_+(\mathcal{R}_1\widehat{\Upsilon}, 1-s) + \Sigma_+(\mathcal{R}_2\widehat{\Upsilon}, 1-s) \right) \\ &\quad - \frac{1}{s} \left(\Sigma_+(\mathcal{R}_1\Upsilon)^*(1-s) + \Sigma_+(\mathcal{R}_2\Upsilon)^*(1-s) \right), \\ J''_1(\Upsilon, s) &= \frac{1}{s-1} \left(\Sigma_+(\mathcal{R}_2\widehat{\Upsilon})^*(1) + \Sigma_+(\mathcal{R}_1\widehat{\Upsilon}, 0) - \Sigma_+(\mathcal{R}_1\widehat{\Upsilon}, 1-s) \right) \\ &\quad - \frac{1}{s} \left(\Sigma_+(\mathcal{R}_2\Upsilon)^*(1) + \Sigma_+(\mathcal{R}_1\Upsilon, 0) - \Sigma_+(\mathcal{R}_2\Upsilon)^*(1-s) \right), \\ J''_2(\Upsilon, s) &= \frac{1}{s-1} \left(\Sigma_+(\mathcal{R}_1\widehat{\Upsilon})^*(1) + \Sigma_+(\mathcal{R}_2\widehat{\Upsilon}, 0) - \Sigma_+(\mathcal{R}_2\widehat{\Upsilon}, 1-s) \right) \\ &\quad - \frac{1}{s} \left(\Sigma_+(\mathcal{R}_1\Upsilon)^*(1) + \Sigma_+(\mathcal{R}_2\Upsilon, 0) - \Sigma_+(\mathcal{R}_1\Upsilon)^*(1-s) \right). \end{aligned}$$

Since $(\mathcal{R}_1\widehat{\Upsilon})^*(0) = (\mathcal{R}_2\Upsilon)^*(0)$ and $(\mathcal{R}_2\widehat{\Upsilon})^*(0) = (\mathcal{R}_1\Upsilon)^*(0)$, by adding all them up, we have the formula. \square

Definition 5.21. We define $\Upsilon_{n(u)} \in \mathcal{A}(S_{\mathbb{A}})$ by $\Upsilon_{n(u)}(\bar{y}) = \Upsilon(n(u)\bar{y})$. We put

$$\begin{aligned} \Pi_1(\Psi, s, \omega) &= \int_{\mathbb{A}^\times \times \mathbb{A}} |t|^s \omega(t) \Psi(0, t, tu) du, \\ \Pi_2(\Phi, s, \omega) &= \int_{\mathbb{A}^\times \times \mathbb{A}} |t|^s \omega(t) \Psi(0, t, tu) \Sigma_{(0)}(\mathcal{R}_2(\Upsilon_{n(u)}), 0) d^\times t du. \end{aligned}$$

It is easy to see that these integrals are normally convergent for $\Re(s) > 2$.

Proposition 5.22. *We have*

$$\begin{aligned} X_{(1)}(\Psi \otimes \Upsilon, s_1, s_2, \omega_1, \omega_2) &= X_{(1)}^{(2+)}(\Psi \otimes \Upsilon, s_1, s_2, \omega_1, \omega_2) \\ &\quad + X_{(1)}^{(2+)}(\Psi \otimes \widehat{\Upsilon}, 1-s_1, s_1+s_2-1/2, \omega_1^{-1}, \omega_1^2 \omega_2) + 2^{-1} \delta(\omega_1) J(\Phi, s_1, s_2, \omega_1 \omega_2), \end{aligned}$$

where $J(\Phi, s_1, s_2, \omega_1 \omega_2)$ is given by

$$\begin{aligned} &\frac{1}{(s_1-1)^2} \widehat{\Upsilon}(0) \Pi_1(\Psi, s_1+2s_2, \omega_1 \omega_2) + \frac{1}{s_1^2} \Upsilon(0) \Pi_1(\Psi, s_1+2s_2, \omega_1 \omega_2) \\ &\quad + \frac{1}{s_1-1} \Sigma_{(0)}(\mathcal{R}_1\widehat{\Upsilon}, 0) \Pi_1(\Psi, s_1+2s_2, \omega_1 \omega_2) + \frac{1}{s_1-1} \Pi_2(\Psi \otimes \widehat{\Upsilon}, s_1+2s_2, \omega_1 \omega_2) \\ &\quad - \frac{1}{s_1} \Sigma_{(0)}(\mathcal{R}_1\Upsilon, 0) \Pi_1(\Psi, s_1+2s_2, \omega_1 \omega_2) - \frac{1}{s_1} \Pi_2(\Phi, s_1+2s_2, \omega_1 \omega_2). \end{aligned}$$

Proof. As before we put $\tilde{\omega}_1 = |\cdot|^{s_1} \omega_1$ and $\tilde{\omega}_2 = |\cdot|^{2s_2} \omega_2$. Let $U_{(1)} \ni w = (w_1, w_2)$, where $w_1 = (0, 1, 0)$, $w_2 = (1, 1)$. We put $B' = \mathrm{GL}(1) \times \mathrm{B}(2)$. We choose the measure db' on $B'_{\mathbb{A}}$ as the product measure of $d^\times t$ on \mathbb{A}^\times and db_2 on $\mathrm{B}(2)_{\mathbb{A}}$. Let $\Psi_{n(u)}(y) = \Psi(n(u)y)$. Then since

$U_{(1)} = H_k w$ and $\#(H_{wk}) = 2$, $X_{(1)}(\Phi, s_1, s_2, \omega_1, \omega_2)$ equals to

$$\begin{aligned} & \frac{1}{2} \int_{H_{\mathbb{A}}} \tilde{\omega}_1(t) \tilde{\omega}_2(t \det g) \Psi(hw_1) \Upsilon(hw_2) dh \\ &= \frac{1}{2} \int_{B'_{\mathbb{A}}} \tilde{\omega}_1(t) \tilde{\omega}_2(t \det b_2) \Psi(b'w_1) \Upsilon(b'w_2) db' \\ &= \frac{1}{2} \int_{(\mathbb{A}^{\times})^3 \times \mathbb{A}} \tilde{\omega}_1(t) \tilde{\omega}_2(tt_1 t_2) \Psi_{n(u)}(0, tt_1 t_2, 0) \Upsilon_{n(u)}\left(\frac{1}{t_1}, \frac{1}{t_2}\right) d^{\times} t d^{\times} t_1 d^{\times} t_2 du. \end{aligned}$$

By changing t_1, t_2 to $1/t_1, 1/t_2$ and after that t to $tt_1 t_2$, we have

$$X_{(1)}(\Phi, s_1, s_2, \omega_1, \omega_2) = \frac{1}{2} \int_{\mathbb{A}^{\times} \times \mathbb{A}} \tilde{\omega}_1 \tilde{\omega}_2(t) \Psi_{n(u)}(0, t, 0) \Sigma_2(\Upsilon_{n(u)}, s_1, \omega_1) d^{\times} t du.$$

Also the same modification yields

$$X_{(1)}^{(2+)}(\Phi, s_1, s_2, \omega_1, \omega_2) = \frac{1}{2} \int_{\mathbb{A}^{\times} \times \mathbb{A}} \tilde{\omega}_1 \tilde{\omega}_2(t) \Psi_{n(u)}(0, t, 0) \Sigma_2^+(\Upsilon_{n(u)}, s_1, \omega_1) d^{\times} t du.$$

Now by applying Lemma 5.20, we obtain the desired formula. \square

As a corollary to Propositions 5.18 and 5.22, we also obtain the following.

Corollary 5.23. *The function $s_1^2(s_1 - 1)^2 X(\Phi, s_1, s_2, \omega_1, \omega_2)$ can be holomorphically continued to the region $\Re(s_1 + 2s_2) > \delta_1 + 2\delta_2$, $\Re(s_2) > \delta_2$. Also the following functional equation holds.*

$$X(\Psi \otimes \Upsilon, s_1, s_2, \omega_1, \omega_2) = X(\Psi \otimes \hat{\Upsilon}, 1 - s_1, s_1 + s_2 - 1/2, \omega_1^{-1}, \omega_1^2 \omega_2).$$

Remark 5.24. This functional equation is nothing but the functional equation for the L -functions of quadratic extensions of k . For this fact, see Proposition B.9.

5.4. Analytic continuation. By putting together everything we have obtained in the two previous subsections, we obtain meromorphic continuation of the global zeta function.

Theorem 5.25. *Let $\tilde{X}(\Phi, s_1, s_2, \omega_1, \omega_2) = s_1^2(s_1 - 1)^2 s_2(s_2 - 1)(2s_1 + 2s_2 - 1)(2s_1 + 2s_2 - 3)X(\Phi, s_1, s_2, \omega_1, \omega_2)$. Then $\tilde{X}(\Phi, s_1, s_2, \omega_1, \omega_2)$ can be holomorphically continued to all \mathbb{C}^2 . Also the zeta function satisfies the following functional equations:*

$$\begin{aligned} X(\Psi \otimes \Upsilon, s_1, s_2, \omega_1, \omega_2) &= X(\hat{\Psi} \otimes \Upsilon, s_1, 3/2 - s_1 - s_2, \omega_1, \omega_1^{-2} \omega_2^{-1}) \\ &= X(\Psi \otimes \hat{\Upsilon}, 1 - s_1, s_1 + s_2 - 1/2, \omega_1^{-1}, \omega_1^2 \omega_2) = X(\hat{\Psi} \otimes \hat{\Upsilon}, 1 - s_1, 1 - s_2, \omega_1^{-1}, \omega_2^{-1}). \end{aligned}$$

Proof. By Corollaries 5.13 and 5.23, $\tilde{X}(\Phi, s_1, s_2, \omega_1, \omega_2)$ can be continued holomorphically to the tube domain

$$\mathcal{D} = \{(s_1, s_2) \in \mathbb{C}^2 \mid \Re(s_1) > \delta_1\} \cup \{(s_1, s_2) \in \mathbb{C}^2 \mid \Re(s_1 + 2s_2) > \delta_1 + 2\delta_2, \Re(s_2) > \delta_2\}.$$

Since the convex hull of \mathcal{D} coincides with \mathbb{C}^2 , $\tilde{X}(\Phi, s_1, s_2, \omega_1, \omega_2)$ can be continued holomorphically to the whole \mathbb{C}^2 (cf. [H73, Theorem 2.5.10]). The functional equation is obvious. \square

As a corollary to this theorem, combined with the Hartogs theorem [H73, Theorem 2.3.2], we can strength Lemmas 5.2 and 5.7 as follows (see Remark 5.3.)

Corollary 5.26. *The statements of Lemmas 5.2 and 5.7 hold with $\delta_1 = \delta_2 = 1$.*

6. ARCHIMEDEAN LOCAL THEORY FOR THE SPACES OF BINARY QUADRATIC FORMS

In this section we describe the gamma factor of the functional equation of the local zeta functions. This is used in the proof of Theorem 7.13 to determine the gamma factor of the functional equation of the Dirichlet series. We study some non-archimedean local theory in Appendix B.

6.1. The definition of unramified local zeta functions. In this subsection $v \in \mathfrak{M}$ is arbitrary. For convenience we introduce an index set of the orbits $H_{k_v} \backslash U_{k_v}^{\text{ss}}$.

Definition 6.1. For each $v \in \mathfrak{M}$ we let \mathcal{T}_v be the index set for the set of orbits $H_{k_v} \backslash U_{k_v}^{\text{ss}}$. By Proposition 3.18 \mathcal{T}_v corresponds bijectively to the set of isomorphism classes of separable quadratic algebras of k_v . For $j_v \in \mathcal{T}_v$, we denote by U_{k_v, j_v} or simply U_{j_v} the H_{k_v} -orbit in $U_{k_v}^{\text{ss}}$ corresponding to j_v . For $v \in \mathfrak{M}_{\mathbb{R}}$, we further let $\mathcal{T}_v = \{1, 2\}$ where the orbit corresponding to $\mathbb{R} \times \mathbb{R}$ is labeled 1 and \mathbb{C} labeled 2. For $v \in \mathfrak{M}_{\mathbb{C}}$ we let $\mathcal{T}_v = \{1\}$. Hence $U_{\mathbb{R}, 1} = \{\tilde{y} \in U_{\mathbb{R}}^{\text{ss}} \mid R_2(\tilde{y}) > 0\}$, $U_{\mathbb{R}, 2} = \{\tilde{y} \in U_{\mathbb{R}}^{\text{ss}} \mid R_2(\tilde{y}) < 0\}$ and $U_{\mathbb{C}, 1} = U_{\mathbb{C}}^{\text{ss}}$.

The unramified local zeta function is defined as follows:

Definition 6.2. Let $j_v \in \mathcal{T}_v$. Take an arbitrary element $\tilde{y} \in U_{j_v}$. For $\Phi_v \in \mathcal{A}(U_{k_v})$ and $s_1, s_2 \in \mathbb{C}$, we put

$$\begin{aligned} \mathcal{X}_{v, j_v}(\Phi_v, s_1, s_2) &= \int_{H_{k_v}} |R_1(h_v \tilde{y})|_v^{s_1} |R_2(h_v \tilde{y})|_v^{s_2} \Phi_v(h_v \tilde{y}) dh_v, \\ \mathcal{U}_{v, j_v}(\Phi_v, s_1, s_2) &= \int_{U_{j_v}} |R_1(\tilde{y}_v)|_v^{s_1-1} |R_2(\tilde{y}_v)|_v^{s_2-1} \Phi_v(\tilde{y}_v) d\tilde{y}_v. \end{aligned}$$

By the uniqueness of the invariant measure, these coincide up to a positive constant. We put $\mathcal{X}_{v, j_v} = b_{v, j_v} \mathcal{U}_{v, j_v}$.

These integrals converge at least for $\Re(s_1) > 1, \Re(s_2) > 1$. The next lemma follows from a straightforward computation of the Jacobian determinant of the double cover $H_{k_v} \ni h \mapsto h\tilde{y} \in U_{j_v}$ for suitable coordinate systems. Note that we defined $\Gamma_{\mathbb{R}}(s)$ and $\Gamma_{\mathbb{C}}(s)$ in Section 4.

Lemma 6.3. *Let $v \in \mathfrak{M}_{\infty}$. We have $b_{v, j_v} = 2\Gamma_{k_v}(2)$. In particular, it does not depend on j_v .*

We define the local Fourier transform exactly the same way as the global one.

Definition 6.4. Let $\Phi_v \in \mathcal{A}(U_{k_v})$. We define the Fourier transform of Φ_v by

$$\widehat{\Phi}_v(\tilde{y}_v) = \int_{U_{k_v}} \Phi_v(z_v) \langle [y_v, z_v]_W + [\bar{y}_v, \bar{z}_v]_S \rangle_v d\tilde{z}_v.$$

6.2. Functional equations at infinite places. In this subsection we give the functional equations of local zeta functions at \mathfrak{M}_{∞} . As in the global situation (Section 5), the local zeta functions satisfy two kinds of functional equation. However here we deal with only one kind which we need in order to prove density theorems. For other types, see Appendix A.

Proposition 6.5. *We assume $v \in \mathfrak{M}_{\infty}$. Let $\tilde{\Gamma}(s_1, s_2) = \Gamma(s_1)^2 \Gamma(s_2) \Gamma(s_1 + s_2 - 1/2)$.*

- (1) *The function $\tilde{\Gamma}(s_1, s_2)^{-1} \mathcal{U}_{v, j_v}(\Phi_v, s_1, s_2)$ is entire.*
- (2) *Let $\Phi \in \mathcal{A}(U_{\mathbb{R}})$. The functional equation for \mathbb{R} is:*

$$\begin{pmatrix} \mathcal{U}_{\mathbb{R}, 1}(\widehat{\Phi}, s_1, s_2) \\ \mathcal{U}_{\mathbb{R}, 2}(\widehat{\Phi}, s_1, s_2) \end{pmatrix} = \frac{2^{2s_2 - s_1}}{\pi^{3s_1 + 2s_2 - 1/2}} \tilde{\Gamma}(s_1, s_2) D_{\mathbb{R}}(s_1, s_2) \begin{pmatrix} \mathcal{U}_{\mathbb{R}, 1}(\Phi, 1 - s_1, 1 - s_2) \\ \mathcal{U}_{\mathbb{R}, 2}(\Phi, 1 - s_1, 1 - s_2) \end{pmatrix}$$

where the 2×2 matrix $D_{\mathbb{R}}(s_1, s_2) = (d_{\mathbb{R}, i, j}(s_1, s_2))$ is given by

$$\begin{pmatrix} 2 \cos^2(s_1 \pi / 2) \sin((s_1 + 2s_2) \pi / 2) & \sin(s_1 \pi) \cos(s_1 \pi / 2) \\ \sin(s_1 \pi) \cos(s_1 \pi / 2) & \sin(s_1 \pi) \cos((s_1 + 2s_2) \pi / 2) \end{pmatrix}.$$

- (3) *Let $\Phi \in \mathcal{A}(U_{\mathbb{C}})$. The functional equation for \mathbb{C} is:*

$$\mathcal{U}_{\mathbb{C}, 1}(\widehat{\Phi}, s_1, s_2) = \frac{2^{2(2s_2 - s_1)}}{\pi^{2(3s_1 + 2s_2 - 1/2)}} \tilde{\Gamma}(s_1, s_2)^2 D_{\mathbb{C}}(s_1, s_2) \mathcal{U}_{\mathbb{C}, 1}(\Phi, 1 - s_1, 1 - s_2)$$

where the 1×1 matrix $D_{\mathbb{C}}(s_1, s_2) = d_{\mathbb{C}, 1, 1}(s_1, s_2)$ is given by

$$- \sin^2(\pi s_1) \sin(\pi s_2) \cos(s_1 + s_2) \pi.$$

Proof. Let ∇_1, ∇_2 , respectively be the homogeneous linear differential operators of y of degree 3, 2 with constant coefficients defined by

$$\begin{aligned}\nabla_1 \exp\{[y, z]_W + [\bar{y}, \bar{z}]_S\} &= R_1(\tilde{z}) \exp\{[y, z]_W + [\bar{y}, \bar{z}]_S\}, \\ \nabla_2 \exp\{[y, z]_W\} &= R_2(\tilde{z}) \exp\{[y, z]_W\}.\end{aligned}$$

Then by computation we see

$$\begin{aligned}\nabla_1(R_1(\tilde{y})^{s_1} R_2(\tilde{y})^{s_2}) &= 4s_1^2(s_1 + s_2 + 1/2)R_1(\tilde{y})^{s_1-1} R_2(\tilde{y})^{s_2}, \\ \nabla_2(R_1(\tilde{y})^{s_1} R_2(\tilde{y})^{s_2}) &= 16s_2(s_1 + s_2 + 1/2)R_1(\tilde{y})^{s_1} R_2(\tilde{y})^{s_2-1}.\end{aligned}$$

Hence, by the repeated use of integration by parts we can show that $\tilde{\Gamma}(s_1, s_2)^{-1} \mathcal{U}_{v, j_v}(\Phi_v, s_1, s_2)$ is an entire function. This proves (1). The functional equation for the real place was derived by Shintani. The formula (2) can be found in [Sh75]. For the complex case, we choose $\Psi(\tilde{y}) = \exp\{-2\pi(|y_1|_{\mathbb{C}} + 2^{-1}|y_2|_{\mathbb{C}} + |y_3|_{\mathbb{C}} + |\bar{y}_1|_{\mathbb{C}} + |\bar{y}_2|_{\mathbb{C}})\}$ as the test function. Then as in [I00, Theorem 6.3.1] the local zeta function for Ψ is computable and we can see that $\mathcal{U}_{\mathbb{C}, 1}(\Psi, s_1, s_2) = 4\pi^{-1/2}(2\pi^3)^{1-s_1}(\pi^2/4)^{1-s_2}\tilde{\Gamma}(s_1, s_2)$. Now we can find $c_{\mathbb{C}, 1, 1}(s_1, s_2)$ as above since $\widehat{\Psi} = 2\Psi$. \square

We define the product objects for infinite places as follows.

Definition 6.6. (1) We put $\mathcal{T}_{\infty} = \prod_{v \in \mathfrak{M}_{\infty}} \mathcal{T}_v$, which we regard as the index set of orbits $H_{\infty} \backslash U_{\infty}^{\text{ss}}$. Elements of \mathcal{T}_{∞} are denoted by $j = (j_v)_{v \in \mathfrak{M}_{\infty}}$. We put $U_j = \prod_{v \in \mathfrak{M}_{\infty}} U_{j_v} \subset U_{\infty}^{\text{ss}}$, which is an H_{∞} -orbit corresponding to $j = (j_v) \in \mathcal{T}_{\infty}$.
(2) For $j \in \mathcal{T}_{\infty}$ and $\Phi_{\infty} \in \mathcal{S}(U_{\infty})$ we put

$$\mathcal{U}_{\infty, j}(\Phi_{\infty}, s_1, s_2) = \int_{U_j} |R_1(\tilde{y}_{\infty})|_{\infty}^{s_1-1} |R_2(\tilde{y}_{\infty})|_{\infty}^{s_2-1} \Phi_{\infty}(\tilde{y}_{\infty}) d\tilde{y}_{\infty}.$$

For $j = (j_v), l = (l_v) \in \mathcal{T}_{\infty}$, put $d_{\infty, j, l}(s_1, s_2) = \prod_{v \in \mathfrak{M}_{\infty}} d_{k_v, j_v, l_v}(s_1, s_2)$. If there is no confusion we drop the subscript ∞ and write $\mathcal{U}_j, d_{j, l}$ instead of $\mathcal{U}_{\infty, j}, d_{\infty, j, l}$, respectively.

(3) The Fourier transform $\widehat{\Phi}_{\infty}$ of $\Phi_{\infty} \in \mathcal{S}(U_{\infty})$ is defined similarly to Definition 6.4.

As a corollary to Proposition 6.5, we have the following:

Corollary 6.7. For $j, l \in \mathcal{T}_{\infty}$, we have

$$\mathcal{U}_j(\widehat{\Phi}_{\infty}, s_1, s_2) = \frac{2^{n(2s_2-s_1)}}{\pi^{n(3s_1+2s_2-1/2)}} \tilde{\Gamma}(s_1, s_2)^n \sum_{l \in \mathcal{T}_{\infty}} d_{j, l}(s_1, s_2) \mathcal{U}_l(\Phi_{\infty}, 1-s_1, 1-s_2).$$

7. DENSITY THEOREMS

In this section we consider the Dirichlet series associated with the prehomogeneous vector spaces (G, V) , (B, W) , and (H, U) . By the results of Section 3 these turn out in Lemmas 7.3, 7.10 to be counting functions of cubic algebras of \mathcal{O} . From the functional equations as well as the residue formulas of these Dirichlet series, we derive the asymptotic formulas on the distributions of discriminants of cubic algebras of \mathcal{O} . Our tool to find a density theorem is a modified version [SS74, Theorem 3] of Landau's Tauberian theorem [L15, Hauptsatz], which allows us to use the functional equation to derive some informations on the error term.

Before starting the analysis we prepare some notations concerning the ideal class group $\text{Cl}(k)$ of k . For a fractional ideal \mathfrak{a} , let $i(\mathfrak{a}) \in \mathbb{A}_f^{\times}(\subset \mathbb{A}^{\times})$ be the corresponding idele, which is well defined up to $\widehat{\mathcal{O}}^{\times}$ -multiple. That is, $i(\mathfrak{a}) \in \mathbb{A}_f^{\times}$ is characterized by the condition $\mathfrak{a} = k \cap i(\mathfrak{a})\widehat{\mathcal{O}}$. Then $|i(\mathfrak{a})| = N(\mathfrak{a})^{-1}$. Notice that the infinite component of $i(\mathfrak{a})$ is trivial. If there is no confusion we simply write \mathfrak{a} instead of $i(\mathfrak{a})$. The set of characters of $\text{Cl}(k)$ is denoted by $\text{Cl}(k)^*$. We regard $\omega \in \text{Cl}(k)^*$ as a character on the idele class group \mathbb{A}^1/k^{\times} via the standard composition of the maps

$$\mathbb{A}^1/k^{\times} \rightarrow \mathbb{A}^1/k_{\infty}^1 k^{\times} \widehat{\mathcal{O}}^{\times} \cong \mathbb{A}^{\times}/k_{\infty}^{\times} k^{\times} \widehat{\mathcal{O}}^{\times} \cong \mathbb{A}_f^{\times}/k^{\times} \widehat{\mathcal{O}}^{\times} \cong \text{Cl}(k) \rightarrow \mathbb{C}^{\times},$$

where we put $k_\infty^1 = \mathbb{A}^1 \cap k_\infty^\times$. Then $\omega(\mathfrak{a}) = \omega(i(\mathfrak{a}))$. Also ω is trivial as a character on $\text{Cl}(k)$ if and only if trivial as a character on \mathbb{A}^1/k^\times . Note that this character is trivial on \mathbb{A}^0 . For an affine space X defined over \mathcal{O} , let $\mathcal{S}_0(X_\mathbb{A}) \subset \mathcal{S}(X_\mathbb{A})$ be the set of elements of the form $\Phi_\infty \otimes \Phi_{f,0}$ where $\Phi_\infty \in \mathcal{S}(X_\infty)$ is arbitrary and $\Phi_{f,0}$ is the characteristic function on $X_{\widehat{\mathcal{O}}}$. Finally, for $\omega \in \text{Cl}(k)^*$ let $L(s, \omega)$ be the Hecke L -function with respect to ω ; $L(s, \omega) = \sum_{\mathfrak{a}} \omega(\mathfrak{a}) N(\mathfrak{a})^{-s}$ where \mathfrak{a} runs through the all integral ideals of \mathcal{O} .

7.1. The Dirichlet series for the space of binary cubic forms. We first give a remark on orbits $G_{k_v} \backslash V_{k_v}^{\text{ss}}$ for $v \in \mathfrak{M}_\infty$. Let K be either \mathbb{R} or \mathbb{C} . Then the map $\mathcal{E}_2(K) \ni F \mapsto F \times K \in \mathcal{E}_3(K)$ gives a bijection. Via this map, we can construct a bijection between two sets of orbits $H_K \backslash U_K^{\text{ss}}$ and $G_K \backslash V_K^{\text{ss}}$ which is useful for later purposes. More precisely:

Definition 7.1. Let $v \in \mathfrak{M}_\infty$. Then by Propositions 3.16 and 3.18, the map $B_{k_v} \backslash W_{k_v}^{\text{ss}} \rightarrow G_{k_v} \backslash V_{k_v}^{\text{ss}}$ induced by the map of Definition 2.1 is bijective. Hence we also use \mathcal{T}_v and \mathcal{T}_∞ as the index set for the sets of orbits of $G_{k_v} \backslash V_{k_v}^{\text{ss}}$ and $G_\infty \backslash V_\infty^{\text{ss}}$. As in Section 6, the orbit corresponding to $j \in \mathcal{T}_v$ in $V_{k_v}^{\text{ss}}$ is denoted by V_{k_v, j_v} or V_{j_v} , and to $j \in \mathcal{T}_\infty$ in V_∞^{ss} is denoted by V_j .

The Dirichlet series we are interestd in are as follows. We use the notations of Section 3.

Definition 7.2. (1) The index set \mathcal{T}_∞ parameterizes the separable cubic algebras of k_∞ . For $j \in \mathcal{T}_\infty$, let $k_\infty(j)$ the corresponding algebra and

$$\mathcal{C}(\mathcal{O}, \mathfrak{a})_j = \{R \in \mathcal{C}(\mathcal{O}, \mathfrak{a}) \mid R \otimes_{\mathcal{O}} k_\infty \cong k_\infty(j)\}.$$

(2) We regard $V_{\mathfrak{a}}$ as a lattice of V_∞ . For $s \in \mathbb{C}$ and $\omega \in \text{Cl}(k)^*$ we put

$$\begin{aligned} \vartheta_j(\mathfrak{a}; s) &= \sum_{x \in G_{\mathfrak{a}} \backslash (V_{\mathfrak{a}} \cap V_j)} \frac{(\#\text{Stab}(G_{\mathfrak{a}}; x))^{-1}}{N(\mathfrak{a})^{2s} |P(x)|_\infty^s} = \sum_{R \in \mathcal{C}(\mathcal{O}, \mathfrak{a})_j} \frac{(\#\text{Aut}(R))^{-1}}{N(\Delta_R/\mathcal{O})^s}, \\ \vartheta_j(s, \omega) &= \sum_{\mathfrak{a} \in \text{Cl}(k)} \omega(\mathfrak{a}) \vartheta_j(\mathfrak{a}; s), \end{aligned}$$

where the second equality of the upper formula follows from Proposition 3.6. Note that by this second equality, we see that $\vartheta_j(\mathfrak{a}; s)$ depends only on the ideal class of \mathfrak{a} .

(3) For $\Phi \in \mathcal{S}(V_\mathbb{A})$, $s \in \mathbb{C}$ and $\omega \in \Omega^1$, we define

$$Z^*(\Phi, s, \omega) = \int_{G_\mathbb{A}/G_k} |\det g|^{2s} \omega(\det g) \sum_{x \in V_k^{\text{ss}}} \Phi(gx) d_{\text{pr}g}.$$

(4) For each $j \in \mathcal{T}_\infty$ we define the local zeta function at \mathfrak{M}_∞ by

$$\mathcal{Z}_j(\Phi_\infty, s) = \int_{G_\infty} |P(g_\infty x)|_\infty^s \Phi_\infty(g_\infty x) dg_\infty.$$

where x is an arbitrary element of V_j .

Lemma 7.3. For $\Phi \in \mathcal{S}_0(V_\mathbb{A})$, $s \in \mathbb{C}$ and $\omega \in \text{Cl}(k)^*$,

$$Z^*(\Phi, s, \omega) = \sum_{j \in \mathcal{T}_\infty} \mathcal{Z}_j(\Phi_\infty, s) \vartheta_j(s, \omega),$$

Proof. Let $\mathcal{K}_f(2) = G_{\widehat{\mathcal{O}}}$. It is known that the double coset space $G_\infty \mathcal{K}_f(2) \backslash G_\mathbb{A}/G_k$ is represented by $\text{Cl}(k)$. More precisely, we have $G_\mathbb{A} = \coprod_{\mathfrak{a} \in \text{Cl}(k)} G_\infty \mathcal{K}_f(2) \cdot \text{diag}(1, \mathfrak{a}) \cdot G_k$. According to this decomposition, we define the partial zeta integral by

$$Z_{\mathfrak{a}}^*(\Phi, s, \omega) = \int_{G_\infty \mathcal{K}_f(2) \cdot \text{diag}(1, \mathfrak{a}) \cdot G_k / G_k} |\det g|^{2s} \omega(\det g) \sum_{x \in V_k^{\text{ss}}} \Phi(gx) d_{\text{pr}g}.$$

We put $\mathcal{K}_f(2)_\mathfrak{a} = \text{diag}(1, \mathfrak{a})^{-1} \cdot \mathcal{K}_f(2) \cdot \text{diag}(1, \mathfrak{a})$ and $\Phi_\mathfrak{a}(x) = \Phi(\text{diag}(1, \mathfrak{a})x)$. Then since $|\det(\mathcal{K}_f(2)_\mathfrak{a}G_k)| = \omega(\det(G_\infty\mathcal{K}_f(2)_\mathfrak{a}G_k)) = 1$, $\Phi_\mathfrak{a}$ is $\mathcal{K}_f(2)_\mathfrak{a}$ -invariant and $|\mathfrak{a}| = N(\mathfrak{a})^{-1}$, we have

$$Z_\mathfrak{a}^*(\Phi, s, \omega) = \frac{\omega(\mathfrak{a})}{N(\mathfrak{a})^{2s}} \int_{G_\infty\mathcal{K}_f(2)_\mathfrak{a}/G_k \cap G_\infty\mathcal{K}_f(2)_\mathfrak{a}} |\det g_\infty|_\infty^{2s} \sum_{x \in V_k^{ss}} \Phi_\mathfrak{a}(g_\infty x) dg_\infty dg_f.$$

We could easily see that $V_k \cap \text{diag}(1, \mathfrak{a})^{-1}V_\mathcal{O} = V_\mathfrak{a}$ as a subset of V_∞ and $G_k \cap \mathcal{K}_f(2)_\mathfrak{a} = G_\mathfrak{a}$ as a subset of G_∞ . Hence

$$Z_\mathfrak{a}^*(\Phi, s, \omega) = \frac{\omega(\mathfrak{a})}{N(\mathfrak{a})^{2s}} \int_{G_\infty/G_\mathfrak{a}} |\det g_\infty|_\infty^{2s} \sum_{x \in V_\mathfrak{a} \cap V_k^{ss}} \Phi_\infty(g_\infty x) dg_\infty \times \int_{\mathcal{K}_f(2)_\mathfrak{a}} dg_f.$$

Since $G_{\mathbb{A}_f}$ is unimodular, $\int_{\mathcal{K}_f(2)_\mathfrak{a}} dg_f = \int_{\mathcal{K}_f(2)} dg_f = 1$. Now by the usual modification we have

$$Z_\mathfrak{a}^*(\Phi, s, \omega) = \sum_{j \in \mathcal{T}_\infty} \mathcal{Z}_j(\Phi_\infty, s) \omega(\mathfrak{a}) \vartheta_j(\mathfrak{a}; s).$$

By summing over all $\mathfrak{a} \in \text{Cl}(k)$, we have the desired formula. \square

The Dirichlet series $\vartheta_j(s, \omega)$ is one of the series treated in [DW86], where its analytic continuation, functional equation, and residue formulas are described. We recall the results here.

Definition 7.4. For $j = (j_v) \in \mathcal{T}_\infty$, we define $r(j) = \#\{v \in \mathfrak{M}_\mathbb{R} \mid j_v = 1\}$. We put

$$\mathfrak{A}_k = \frac{\mathfrak{C}_k \zeta_k(2)}{2^{r_1+r_2+1} \Delta_k^{1/2}}, \quad \mathfrak{B}_k = \frac{3^{r_1+r_2/2} \mathfrak{C}_k \zeta_k(1/3)}{5 \cdot 2^{r_1+r_2} \Delta_k} \left(\frac{\Gamma(1/3)^3}{2\pi} \right)^n.$$

Definition 7.5. Let \hat{V} be the submodule of V defined by $\hat{V} = \{(x_0, 3x_1, 3x_2, x_3) \mid x_0, \dots, x_3 \in \text{Aff}\}$. Then G acts on \hat{V} also. For $\mathfrak{a} \in \text{Cl}(k)$ let $\hat{V}_\mathfrak{a} = V(k) \cap \text{diag}(\mathfrak{d}, \mathfrak{d}\mathfrak{a})^{-1} \hat{V}(\mathcal{O})$ (recall that \mathfrak{d} is a differential idele of k) and put

$$\hat{\vartheta}_j(s, \omega) = \sum_{\mathfrak{a} \in \text{Cl}(k)} \omega(\mathfrak{a}) \sum_{x \in G_\mathfrak{a} \setminus (\hat{V}_\mathfrak{a} \cap V_j)} \#(\text{Stab}(G_\mathfrak{a}; x))^{-1} N(\mathfrak{a})^{-2s} |P(x)|_\infty^{-s}.$$

For $j, l \in \mathcal{T}_\infty$, we define $c_{j,l}(s) = \prod_{v \in \mathfrak{M}} c_{k_v, j_v, l_v}(s)$ where

$$(c_{\mathbb{R}, i, j}(s)) = \frac{1}{2} \begin{pmatrix} \sin 2\pi s & 3 \sin \pi s \\ \sin \pi s & \sin 2\pi s \end{pmatrix}, \quad c_{\mathbb{C}, 1, 1}(s) = \sin^2 \pi s \sin(\pi s - \frac{\pi}{6}) \sin(\pi s + \frac{\pi}{6}).$$

Theorem 7.6 (Datskovsky-Wright [DW86]). *The Dirichlet series $\vartheta_j(s, \omega)$ can be holomorphically continued to the whole complex plane except for possible simple poles at $s = 1, 5/6$ with the residue $\delta(\omega) \mathfrak{A}_k(1 + 3^{-r(j)-r_2})$, $\delta(\omega^3) (5/6) \mathfrak{B}_k 3^{-r(j)/2}$, respectively. Moreover, they satisfy the functional equation*

$$\vartheta_j(1-s, \omega) = \frac{3^{n(6s-2)}}{\pi^{4ns}} \Gamma(s)^{2n} \Gamma(s-1/6)^n \Gamma(s+1/6)^n \sum_{l \in \mathcal{T}_\infty} c_{l,j}(s) \hat{\vartheta}_l(s, \omega^{-1}).$$

7.2. Distributions of discriminants of cubic algebras. We are now ready to prove a density theorem from the zeta function for (G, V) . Let $\mathfrak{a} \in \text{Cl}(k)$.

Definition 7.7. Let $h_k^{(3)}$ be the order of the subgroup of 3-torsion elements of $\text{Cl}(k)$. For $\mathfrak{a} \in \text{Cl}(k)$, we put $\tau(\mathfrak{a}) = 1$ if \mathfrak{a} is 3-divisible (that is, there exists $\mathfrak{b} \in \text{Cl}(k)$ such that $\mathfrak{a} = \mathfrak{b}^3$), and $\tau(\mathfrak{a}) = 0$ otherwise.

Theorem 7.8. (1) Let k be a quadratic field. For any $\varepsilon > 0$,

$$\sum_{\substack{R \in \mathcal{C}(\mathcal{O}, \mathfrak{a})_j \\ \mathcal{N}(\Delta_{R/\mathcal{O}}) \leq X}} \frac{1}{\#(\text{Aut}(R))} = \left(1 + \frac{1}{3^{r(j)+r_2}}\right) \frac{\mathfrak{A}_k}{h_k} X + \tau(\mathfrak{a}) \frac{\mathfrak{B}_k h_k^{(3)}}{3^{r(j)/2} h_k} X^{5/6} + O(X^{7/9+\varepsilon}) \quad (X \rightarrow \infty).$$

(2) Let k be a number field with $n = [k : \mathbb{Q}] \geq 3$. For any $\varepsilon > 0$,

$$\sum_{\substack{R \in \mathcal{C}(\mathcal{O}, \mathfrak{a})_j \\ \mathcal{N}(\Delta_{R/\mathcal{O}}) \leq X}} \frac{1}{\#(\text{Aut}(R))} = \left(1 + \frac{1}{3^{r(j)+r_2}}\right) \frac{\mathfrak{A}_k}{h_k} X + O(X^{(4n-1)/(4n+1)+\varepsilon}) \quad (X \rightarrow \infty).$$

Proof. By the orthogonality of the characters, $\vartheta_j(\mathfrak{a}; s) = h_k^{-1} \sum_{\omega \in \text{Cl}(k)^*} \omega(\mathfrak{a})^{-1} \vartheta(s, \omega)$. Hence by Theorem 7.6, we know the analytic properties of $\vartheta_j(\mathfrak{a}; s)$. In particular, the residues at $s = 1, 5/6$ are given by $h_k^{-1}(1 + 3^{-r(j)-r_2})\mathfrak{A}_k$, $\tau(\mathfrak{a})(5/6)\mathfrak{B}_k 3^{-r(j)/2} \cdot (h_k^{(3)}/h_k)$, respectively. Combined with [SS74, Theorem 3] we have the above formulas. (Note that in the notation of [SS74, Theorem 3], $\nu = 4n$, $\sum_{i=1}^{\nu} \alpha_i = 0$, $\sum_{i=1}^{\nu} \beta_i = 4n$, $\delta = 1$ and $\mu_1 = \mu_2 = 1 + \epsilon$ where $\epsilon > 0$ is arbitrary.) \square

7.3. The Dirichlet series for the spaces of binary quadratic forms. In this subsection we study Dirichlet series associated with the spaces (B, W) and (H, U) .

Definition 7.9. (1) For $j \in \mathcal{T}_{\infty}$ we define $W_j \subset W_{\infty}^{\text{ss}}$ similarly to U_j and put

$$\begin{aligned} \xi_j(\mathfrak{a}, \mathfrak{c}; s_1, s_2) &= \sum_{y \in B_{\mathfrak{a}, \mathfrak{c}} \setminus (W_{\mathfrak{a}, \mathfrak{c}} \cap W_j)} \frac{(\#(\text{Stab}(B_{\mathfrak{a}, \mathfrak{c}}; y)))^{-1}}{N(\mathfrak{c})^{-s_1} N(\mathfrak{a}\mathfrak{c})^{2s_2} |Q_1(y)|_{\infty}^{s_1} |Q_2(y)|_{\infty}^{s_2}}, \\ \xi_j(s_1, s_2, \omega_1, \omega_2) &= \sum_{\mathfrak{a}, \mathfrak{c} \in \text{Cl}(k)} \omega_1(\mathfrak{c})^{-1} \omega_2(\mathfrak{a}\mathfrak{c}) \xi_j(\mathfrak{a}, \mathfrak{c}; s_1, s_2), \\ \Xi_j(s_1, s_2, \omega_1, \omega_2) &= \xi_j(s_1, s_2, \omega_1, \omega_2) L(2s_1, \omega_1^2). \end{aligned}$$

(2) We define $Y^*(\Psi, s_1, s_2, \omega_1, \omega_2) = \Delta_k^{1/2} \mathfrak{C}_k^2 \cdot Y(\Psi, s_1, s_2, \omega_1, \omega_2)$. That is, we define Y^* as the integral exactly the same way as in Definition 5.6 except for replacing the measure db by $d_{\text{pr}}b$. Similarly, we put $X^*(\Phi, s_1, s_2, \omega_1, \omega_2) = \Delta_k^{1/2} \mathfrak{C}_k^3 \cdot X(\Phi, s_1, s_2, \omega_1, \omega_2)$ and $\Sigma^*(\mathcal{R}_1 \Upsilon, s, \omega) = \mathfrak{C}_k \cdot \Sigma(\mathcal{R}_1 \Upsilon, s, \omega)$.

(3) For $j \in \mathcal{T}_{\infty}$ we choose $y \in W_j$ and $\tilde{y} \in U_j$ arbitrary. We define

$$\begin{aligned} \mathcal{Y}_{\infty, j}(\Psi_{\infty}, s_1, s_2) &= \int_{B_{\infty}} |Q_1(b_{\infty} y)|_{\infty}^{s_1} |Q_2(b_{\infty} y)|_{\infty}^{s_2} \Psi_{\infty}(b_{\infty} y) db_{\infty}, \\ \mathcal{X}_{\infty, j}(\Phi_{\infty}, s_1, s_2) &= \int_{H_{\infty}} |R_1(h_{\infty} \tilde{y})|_{\infty}^{s_1} |R_2(h_{\infty} \tilde{y})|_{\infty}^{s_2} \Phi_{\infty}(h_{\infty} y) dh_{\infty}, \\ \Sigma_{\infty}(\mathcal{R}_1 \Upsilon_{\infty}, s) &= \int_{\mathbb{R}^{\times}} |t_{\infty}|_{\infty}^s \Upsilon_{\infty}(t_{\infty}, 0) d^{\times} t_{\infty}. \end{aligned}$$

Recall that we put $\mathcal{K} = \mathbb{A}^0 \times \mathcal{K}(2)$ which is a maximal compact subgroup of $H_{\mathbb{A}}$. Let \mathcal{K}_{∞} be the infinite component of this group.

Lemma 7.10. Assume $\Phi \in \mathcal{A}(U_{\mathbb{A}})$ is \mathcal{K} -invariant. We have

$$\begin{aligned} Y^*(\Psi, s_1, s_2, \omega_1, \omega_2) &= \sum_{j \in \mathcal{T}_{\infty}} \mathcal{Y}_{\infty, j}(\Psi_{\infty}, s_1, s_2) \xi_j(s_1, s_2, \omega_1, \omega_2), \\ X^*(\Phi, s_1, s_2, \omega_1, \omega_2) &= \sum_{j \in \mathcal{T}_{\infty}} \mathcal{X}_{\infty, j}(\Phi_{\infty}, s_1, s_2) \Xi_j(s_1, s_2, \omega_1, \omega_2). \end{aligned}$$

Proof. We put $\mathcal{K}_B = B_{\widehat{\mathcal{O}}}$. Then we have $B_{\mathbb{A}} = \coprod_{\mathfrak{a}, \mathfrak{c} \in \text{Cl}(k)} B_{\infty} \mathcal{K}_B \cdot a(\mathfrak{c}^{-1}, \mathfrak{a}\mathfrak{c}^2) \cdot B_k$. We could see that as a subset of W_{∞} or B_{∞} ,

$$W_k \cap a(\mathfrak{c}^{-1}, \mathfrak{a}\mathfrak{c}^2)^{-1} W_{\widehat{\mathcal{O}}} = W_{\mathfrak{a}, \mathfrak{c}}, \quad B_k \cap a(\mathfrak{c}^{-1}, \mathfrak{a}\mathfrak{c}^2)^{-1} \cdot \mathcal{K}_B \cdot a(\mathfrak{c}^{-1}, \mathfrak{a}\mathfrak{c}^2) = B_{\mathfrak{a}, \mathfrak{c}}.$$

Hence by the similar modification to Proposition 7.3 we obtain the first formula. We consider the second one. It is well known that $\Sigma^*(\Pi', s, \omega) = \Sigma_{\infty}(\Pi'_{\infty}, s) L(s, \omega)$ for $\Pi' \in \mathcal{S}_0(\mathbb{A})$. On the other hand, since Φ is \mathcal{K}_{∞} -invariant, we can prove that

$$\mathcal{X}_{\infty, j}(\Phi_{\infty}, s_1, s_2) = \mathcal{Y}_{\infty, j}(\Psi_{\infty}, s_1, s_2) \Sigma_{\infty}(\mathcal{R}_1 \Upsilon_{\infty}, 2s_1)$$

by the changes of variables similar to the proof of Lemma 5.7. Hence by Lemma 5.7 and the first formula we have the second one. \square

In order to determine the analytic properties of these Dirichlet series we recall the following well known result.

Lemma 7.11. *Let $j \in \mathcal{T}_{\infty}$ and $r_1, r_2 \in \mathbb{C}$. There exists a \mathcal{K}_{∞} -invariant function $\Phi_{\infty} \in \mathcal{S}(U_{\infty})$ such that the support of Φ_{∞} is contained in U_j , $\mathcal{X}_{\infty, j}(\Phi_{\infty}, s_1, s_2)$ is an entire function and $\mathcal{X}_{\infty, j}(\Phi_{\infty}, r_1, r_2) \neq 0$.*

Definition 7.12. We put $\widehat{U} = \{(y_0, 2y_1, y_2, \bar{y}_1, \bar{y}_2) \mid y_0, y_1, y_2, \bar{y}_1, \bar{y}_2 \in \text{Aff}\}$, which is a B -invariant submodule of U . We define the Dirichlet series $\widehat{\Xi}_j(s_1, s_2, \omega_1, \omega_2)$ for (B, \widehat{U}) similarly to Definition 7.9. (Also see Definition 7.5.)

Theorem 7.13. *The function $(s_1 - 1)^2 (s_2 - 1) (s_1 + s_2 - 3/2) \Xi_j(s_1, s_2, \omega_1, \omega_2)$ is entire. Moreover, the Dirichlet series $\Xi_j(s_1, s_2, \omega_1, \omega_2)$ satisfies a functional equation*

$$\Xi_j(1 - s_1, 1 - s_2, \omega_1, \omega_2) = \frac{2^{n(2s_2 - s_1)}}{\pi^{n(3s_1 + 2s_2 - 1/2)}} \widetilde{\Gamma}(s_1, s_2)^n \sum_{l \in \mathcal{T}_{\infty}} d_{l, j}(s_1, s_2) \widehat{\Xi}_j(s_1, s_2, \omega_1^{-1}, \omega_2^{-1}).$$

Proof. Take a \mathcal{K} -invariant function $\Phi \in \mathcal{S}_0(U_{\mathbb{A}})$ such that Φ_{∞} is as in Lemma 7.11. Then since the distributions $\Lambda(\Psi, s_1, \omega_1)$, $\Sigma(\mathcal{R}\widehat{\Psi}, s_1, \omega_1)$ in Proposition 5.12, $\Upsilon(0)$ in Proposition 5.18 and $\Sigma_{(0)}(\mathcal{R}_1 \widehat{\Upsilon}, 0) \Pi_1(\Psi, s_1 + 2s_2, \omega_1 \omega_2)$, $\Pi_2(\Psi \otimes \widehat{\Upsilon}, s_1 + 2s_2, \omega_1 \omega_2)$ in Proposition 5.22 have singular supports, $(s_1 - 1)^2 (s_2 - 1) (s_1 + s_2 - 3/2) X^*(\Phi, s_1, s_2, \omega_1, \omega_2)$ is entire. Also by Lemma 7.10,

$$X^*(\Phi, s_1, s_2, \omega_1, \omega_2) = \mathcal{X}_{\infty, j}(\Phi_{\infty}, s_1, s_2) \Xi_j(s_1, s_2, \omega_1, \omega_2).$$

This proves the first statement. On the other hand since $\widehat{\Phi}$ is also \mathcal{K} -invariant and $\widehat{\Phi}_{\widehat{\mathcal{O}}}$ is the characteristic function on $\widehat{U}_{\widehat{\mathcal{O}}}$, the same argument as in Lemma 7.10 shows that

$$X^*(\widehat{\Phi}, s_1, s_2, \omega_1, \omega_2) = \sum_{j \in \mathcal{T}_{\infty}} \mathcal{X}_j(\widehat{\Phi}_{\infty}, s_1, s_2) \widehat{\Xi}_j(s_1, s_2, \omega_1, \omega_2).$$

Hence by the two equations above, Theorem 5.25, Corollary 6.7 and Lemma 6.3 we have the functional equation. \square

7.4. Contributions of reducible algebras. Now we consider contributions of reducible algebras. We describe the analytic properties of the Dirichlet series in question for general characters of $\text{Cl}(k)$, but apparently we succeed in finding a density theorem with a good error estimate only from the Dirichlet series with trivial characters.

Definition 7.14. Let $\mathfrak{a}, \mathfrak{c}$ be fractional ideals. We put

$$\begin{aligned} \eta_j(\mathfrak{a}, \mathfrak{c}; s) &= \sum_{y \in B_{\mathfrak{a}, \mathfrak{c}} \setminus (W_{\mathfrak{a}, \mathfrak{c}} \cap W_j)} \frac{\#(\text{Stab}(B_{\mathfrak{a}, \mathfrak{c}}; y))^{-1}}{N(\mathfrak{a})^{2s} |Q_1(y)^2 Q_2(y)|_{\infty}^s}, \\ \eta_j(s, \omega_1, \omega_2) &= \sum_{\mathfrak{a}, \mathfrak{c} \in \text{Cl}(k)} \omega_1(\mathfrak{c}) \omega_2(\mathfrak{a}) \eta_j(\mathfrak{a}, \mathfrak{c}; s), \\ H_j(s, \omega_1, \omega_2) &= \eta_j(s, \omega_1, \omega_2) L(4s, \omega_2^2 / \omega_1^2). \end{aligned}$$

By definition we have $\eta_j(\mathbf{a}, \mathbf{c}; s) = \xi_j(\mathbf{a}, \mathbf{c}; 2s, s)$, $\eta_j(s, \omega_1, \omega_2) = \xi_j(2s, s, \omega_2/\omega_1, \omega_2)$ and $H_j(s, \omega_1, \omega_2) = \Xi_j(2s, s, \omega_2/\omega_1, \omega_2)$. The following lemma immediately follows from Theorem 7.13.

Lemma 7.15. *The function $(s - 1/2)^3(s - 1)H_j(s, \omega_1, \omega_2)$ is entire. Moreover, the series $H_j(s, \omega_1, \omega_2)$ satisfies a functional equation:*

$$H_j(1 - s, \omega_1, \omega_2) = 2^n \pi^{n(7/2-8s)} \Gamma(2s - 1)^{2n} \Gamma(s)^n \Gamma(3s - 3/2)^n \\ \times \sum_{l \in \mathcal{T}_\infty} d_{l,j}(2s - 1, s) \hat{\Xi}_j(2s - 1, s, \omega_1/\omega_2, 1/\omega_2).$$

We next consider the residue at $s = 1$.

Lemma 7.16. *The residue of $\eta_j(s, \omega_1, \omega_2)$ at $s = 1$ is $\delta(\omega_2) \mathfrak{A}_k L(2, \omega_1^{-1})/\zeta_k(2)$.*

Proof. Take $\Psi \in \mathcal{S}_0(W_{\mathbb{A}})$ such that $\text{supp}(\Psi_\infty) \in W_j$. By Lemma 7.10 we have

$$Y^*(\Psi, 2s, s, \omega_2/\omega_1, \omega_2) = \mathcal{Y}_{\infty,j}(\Psi_\infty, 2s, s) \eta_j(s, \omega_1, \omega_2)$$

We consider the residue at $s = 1$. By Proposition 5.12 we have

$$\text{Res}_{s=1} Y^*(\Psi, 2s, s, \omega_2/\omega_1, \omega_2) = \delta(\omega_2) 2^{-1} \Delta_k^{1/2} \mathfrak{E}_k^2 \Sigma(\mathcal{R}\Psi, 2, \omega_2/\omega_1) \\ = \delta(\omega_2) 2^{-1} \Delta_k^{1/2} \mathfrak{E}_k^2 \int_{\mathbb{A} \times \mathbb{A}^2} \omega_1^{-1}(t) |t|^2 \Psi(t, u_2, u_3) d^\times t du_1 du_2 \\ = \delta(\omega_2) 2^{-1} \Delta_k^{-1/2} \mathfrak{E}_k L(2, \omega_1^{-1}) \int_{k_\infty^3} |y_{1,\infty}|_\infty \Psi_\infty(y_\infty) dy_\infty.$$

On the other hand for $v \in \mathfrak{M}_\infty$ we could see

$$\int_{B_{k_v}} f(b_v y) db_v = 2 \int_{B_{k_v} y} f(z_v) |Q_1(z_v)|_v^{-1} |Q_2(z_v)|_v^{-1} dz_v$$

by computing the Jacobian determinant of the double cover $B_{k_v} \ni b_v \mapsto b_v y \in B_{k_v} y$ for their coordinate systems. This shows that

$$\mathcal{Y}_{\infty,j}(\Psi_\infty, 2, 1) = 2^{r_1+r_2} \int_{W_j} |Q_1(y_{1,\infty})|_\infty \Psi_\infty(y_\infty) dy_\infty = 2^{r_1+r_2} \int_{W_\infty} |y_{1,\infty}|_\infty \Psi_\infty(y_\infty) dy_\infty$$

since $\text{supp}(\Psi_\infty) \subset W_j$. Hence we have the proposition. \square

To prove the density theorem we prepare a lemma.

Lemma 7.17. *The Dirichlet series $1/\zeta_k(s) = \prod_{v \in \mathfrak{M}_f} (1 - N(\mathcal{P}_v)^{-s})$ is normally convergent for $\Re(s) > 1$, where we denote by \mathcal{P}_v the prime ideal of \mathcal{O} corresponding to $v \in \mathfrak{M}_f$.*

Proof. For two Dirichlet series $a(s) = \sum_{n \geq 1} a_n/n^s$ and $b(s) = \sum_{n \geq 1} b_n/n^s$ we write $a(s) \preceq b(s)$ to mean that $|a_n| \leq b_n$ for all n . Let p be a prime. For $k \in \mathbb{N}_{\geq 1}$ we have $1 - p^{-ks} \preceq (1 + p^{-s})^k \preceq (1 - p^{-s})^{-k}$. Hence $\prod_{v|p} (1 - N(\mathcal{P}_v)^{-s}) \preceq (1 - p^{-s})^{-n}$. Taking the product for all primes, we have $\zeta_k(s)^{-1} \preceq \zeta_{\mathcal{Q}}(s)^n$. Since the right hand side is also normally convergent for $\Re(s) > 1$, the left hand side also. \square

Proposition 7.18. *For any $\varepsilon > 0$,*

$$\sum_{\mathbf{a}, \mathbf{c} \in \text{Cl}(k)} \sum_{\substack{y \in B_{\mathbf{a}, \mathbf{c}} \setminus (W_{\mathbf{a}, \mathbf{c}} \cap W_j) \\ N(\mathbf{a}) | Q_1(y)^2 Q_2(y) |_\infty < X}} \frac{1}{\#(\text{Stab}(B_{\mathbf{a}, \mathbf{c}}; y))} = \mathfrak{A}_k X + O(X^{(5n-1)/(5n+1)+\varepsilon}) \quad (X \rightarrow \infty).$$

Proof. We write $\eta_j(s) = \eta_j(s, 1, 1)$ and $H_j(s) = H_j(s, 1, 1)$, where $1 \in \text{Cl}(k)^*$ is the trivial character. We fix $j \in \mathcal{T}_\infty$. Let $\eta_j(s) = \sum_{n \geq 1} a_n/n^s$, so that the left hand side of the proposition is $\sum_{n < X} a_n$. We denote this function by $A(X)$. We also put $H_j(s) = \sum_{n \geq 1} b_n/n^s$ and $B(X) = \sum_{n < X} b_n$. By Definition 7.14 and Lemma 7.16 $H_j(s)$ is holomorphic for $\Re(s) > 1/2$ except for a simple pole at $s = 1$ with the residue $\mathfrak{A}_k \zeta_k(4)$. Using the functional equation in Lemma 7.15, by [SS74, Theorem 3] we have $B(X) = \mathfrak{A}_k \zeta_k(4)X + O(X^{(5n-1)/(5n+1)+\varepsilon})$. Note that in the notation of [SS74, Theorem 3], $\nu = 4n$, $\sum_{i=1}^\nu \alpha_i = -7n/2$, $\sum_{i=1}^\nu \beta_i = 8n$, $\delta = 1$ and $\mu_1 = \mu_2 = 1 + \varepsilon$ where $\varepsilon > 0$ is arbitrary.

We put $1/\zeta_k(4s) = \sum_{n \geq 1} d_n/n^s$. Then by Definition 7.14, $a_n = \sum_{mk=n} d_m b_k$. We define $\rho(X) = B(X) - \mathfrak{A}_k \zeta_k(4)X$, $\mu = (5n-1)/(5n+1)$. Then for any $\varepsilon > 0$, there exists $M > 0$ such that $|\rho(X)| \leq MX^{\mu+\varepsilon}$. Hence we have

$$\begin{aligned} A(X) &= \sum_{mk < X} d_m b_k = \sum_{m < X} d_m \sum_{k < X/m} b_k = \sum_{m < X} d_m B(X/m) \\ &= \mathfrak{A}_k \zeta_k(4)X \sum_{m < X} \frac{d_m}{m} + \sum_{m < X} d_m \rho(X/m) \\ &= \mathfrak{A}_k \zeta_k(4)X \left(\frac{1}{\zeta_k(4)} - \sum_{m \geq X} \frac{d_m}{m} \right) + \sum_{m < X} d_m \rho(X/m). \end{aligned}$$

Since

$$\begin{aligned} \sum_{m \geq X} \frac{|d_m|}{m} &\leq \sum_{m \geq X} \frac{|d_m|}{m^{1/4+\varepsilon}} X^{-3/4+\varepsilon} = O(X^{-3/4+\varepsilon}), \\ \sum_{m < X} |d_m| |\rho(X/m)| &\leq \sum_{m < X} |d_m| \cdot M \frac{X^{\mu+\varepsilon}}{m^{\mu+\varepsilon}} = MX^{\mu+\varepsilon} \sum_{m < X} \frac{|d_m|}{m^{\mu+\varepsilon}} = O(X^{\mu+\varepsilon}), \end{aligned}$$

we have the proposition. \square

7.5. The distributions of irreducible algebras. We are now ready to prove a main theorem of this paper.

Definition 7.19. (1) We denote by L_∞ a separable cubic k_∞ -algebra. Let $r(L_\infty)$ be the integer such that $L_\infty \cong \mathbb{R}^{r_1+2r(L_\infty)} \times \mathbb{C}^{3r_2+r_1-r(L_\infty)}$ as \mathbb{R} -algebras. Notice that $0 \leq r(L_\infty) \leq r_1$.

(2) Let $h(n, L_\infty)$ be the numbers of the isomorphism classes of pairs (R, F) satisfying the following conditions: (a) F/k is a cubic (field) extension in $\overline{\mathbb{Q}}$ such that $F \otimes_k k_\infty \cong L_\infty$ as k_∞ -algebras, (b) R is an order of F containing \mathcal{O} , (c) $N(\Delta_{R/\mathcal{O}}) = n$.

Note that $N(\Delta_{R/\mathcal{O}})$ is the ideal norm of the relative discriminant of R/\mathcal{O} . The following is the main result of this paper. Note that we defined constants $\mathfrak{A}_k, \mathfrak{B}_k$ in Definition 7.4.

Theorem 7.20. (1) Let k be a quadratic field. For any $\varepsilon > 0$,

$$\sum_{n < X} h(n, L_\infty) = 3^{-r(L_\infty)-r_2} \mathfrak{A}_k X + 3^{-r(L_\infty)/2} \mathfrak{B}_k X^{5/6} + O(X^{9/11+\varepsilon}) \quad (X \rightarrow \infty).$$

(2) Let k be a number field with $n = [k : \mathbb{Q}] \geq 3$. For any $\varepsilon > 0$,

$$\sum_{n < X} h(n, L_\infty) = 3^{-r(L_\infty)-r_2} \mathfrak{A}_k X + O(X^{(5n-1)/(5n+1)+\varepsilon}) \quad (X \rightarrow \infty).$$

Proof. Let $\mathcal{C}(\mathcal{O})_j = \coprod_{\mathfrak{a} \in \text{Cl}(k)} \mathcal{C}(\mathcal{O}, \mathfrak{a})_j$. Since $\sum_{\mathfrak{a} \in \text{Cl}(k)} \tau(\mathfrak{a}) = h_k/h_k^{(3)}$, by summing over all $\mathfrak{a} \in \text{Cl}(k)$ for the formula of Theorem 7.8 we have

$$\sum_{\substack{R \in \mathcal{C}(\mathcal{O})_j \\ \mathcal{N}(\Delta_{R/\mathcal{O}}) \leq X}} \frac{1}{\#(\text{Aut}(R))} = \left(1 + \frac{1}{3^{r(j)+r_2}}\right) \mathfrak{A}_k X + \mathfrak{B}_k 3^{-r(j)/2} X^{5/6} + O(X^{(4n-1)/(4n+1)+\epsilon}).$$

We will compare the formula of Proposition 7.18 and the formula above. Since $\text{Aut}_{\mathcal{O}}(R)$ is a subgroup of $\text{Aut}_k(R \otimes_{\mathcal{O}} k)$, the order of $\text{Aut}_{\mathcal{O}}(R)$ is either 1, 2, 3 or 6. In particular, if $R \otimes_{\mathcal{O}} k$ is a non-cyclic cubic extension, then $\text{Aut}_{\mathcal{O}}(R)$ is trivial. We consider the algebras R such that $\text{Aut}_k(R \otimes_{\mathcal{O}} k)$ is isomorphic to either $\mathbb{Z}/3\mathbb{Z}$ or S_3 , the permutation group of degree 3. We denote by CG_k the set of cyclic cubic extensions of k . Let $p(n)$ be the number of orders R of type $k \times k \times k$ with $N(\Delta_{R/\mathcal{O}}) = n$, and $q(n)$ the the numbers of the pairs (R, F) such that $F \in \text{CG}_k$, R is an order of F , and $N(\Delta_{R/\mathcal{O}}) = n$. We claim that the contributions of these algebras to $\sum_{n < X} h(n, L_{\infty})$ can be ignored. To see this, we notice that for a positive sequence $\{a_n\}$ and a positive constant ρ , the series $\sum_{n \geq 1} a_n/n^s$ converges for $\Re(s) > \rho$ if and only if $\sum_{n < X} a_n = O(X^{\rho+\epsilon})$ for any $\epsilon > 0$. By [DW86, Theorem 6.1] we have

$$\begin{aligned} \sum_{n \geq 1} \frac{p(n)}{n^s} &= \zeta_k(2s)^3 \zeta_k(6s-1) / \zeta_k(4s)^2, \\ \sum_{n \geq 1} \frac{q(n)}{n^s} &= \sum_{F \in \text{CG}_k} \frac{1}{N(\Delta_{F/k})^s} \zeta_k(4s) \zeta_k(6s-1) \zeta_F(2s) / \zeta_F(4s). \end{aligned}$$

Let ϵ be an arbitrary positive number. From the first equality, we have $\sum_{n < X} p(n) = O(X^{1/3+\epsilon})$. Also by [Wr89, Theorem I.2] we know that $\sum_{F \in \text{CG}_k} N(\Delta_{F/k})^{-s}$ converges for $\Re(s) > 1/2$. Since the Dirichlet series $\zeta_F(2s)/\zeta_F(4s)$ is uniformly bounded by $\zeta_k(2s)^3/\zeta_k(4s)^3$, the Dirichlet series in the right hand side of the second formula converges for $\Re(s) > 1/2$, which implies that $\sum_{n < X} q(n) = O(X^{1/2+\epsilon})$. Now the theorem follows from Propositions 3.12 and 7.18. \square

Remark 7.21. Let us consider the case $k = \mathbb{Q}$. We see that $\mathfrak{A}_{\mathbb{Q}} = \pi^2/24$ and $\mathfrak{B}_{\mathbb{Q}} = r/10$ where we put $r = (2\pi)^{1/3} \zeta(2/3) \Gamma(1/3) \Gamma(2/3)^{-1}$. Let $h(n)$ be the numbers of isomorphism classes of orders of cubic fields with discriminant n . Then the proof of Theorem 7.20 also shows

$$\begin{aligned} \sum_{0 < n < X} h(n) &= (\pi^2/72)X + (\sqrt{3}r/30)X^{5/6} + O(X^{2/3+\epsilon}) \quad (X \rightarrow \infty), \\ \sum_{0 < n < X} h(-n) &= (\pi^2/24)X + (r/10)X^{5/6} + O(X^{2/3+\epsilon}) \quad (X \rightarrow \infty). \end{aligned}$$

This is what Shintani established in [Sh75, Theorem 4] and hence Theorem 7.20 is a generalization of this to an arbitrary number field. Note that Shintani used $\text{SL}(2)_{\mathbb{Z}}$ instead of $\text{GL}(2)_{\mathbb{Z}}$ and hence his result is twice to ours.

8. ON ZETA INTEGRALS FOR THE SPACE OF BINARY CUBIC FORMS

In this section we consider some zeta integrals associated with the space of binary cubic forms (G, V) . We give meromorphic continuations and describe some of the residues of these integrals in Proposition 8.6.

Recall that we associate $k(x) \in \mathcal{E}_3(k)$ for each $x \in V_k^{\text{ss}}$ in Definition 3.15. Let

$$\begin{aligned} V_{(1)} &= \{x \in V_k^{\text{ss}} \mid k(x) \cong k \times k \times k\}, \\ V_{(2)} &= \{x \in V_k^{\text{ss}} \mid k(x) \cong k \times E \text{ where } E/k \text{ is a quadratic field extension}\}, \\ V_{(3)} &= \{x \in V_k^{\text{ss}} \mid k(x)/k \text{ is a cubic field extension}\}. \end{aligned}$$

Then $V_k^{\text{ss}} = V_{(1)} \amalg V_{(2)} \amalg V_{(3)}$ and each of them is a G_k -invariant subset.

Definition 8.1. For $\Phi \in \mathcal{S}(V_{\mathbb{A}})$, $s \in \mathbb{C}$ and $\omega \in \Omega^1$, the global zeta function is defined by

$$Z(\Phi, s, \omega) = \int_{G_{\mathbb{A}}/G_k} |\det g|^{2s} \omega(\det g) \sum_{x \in V_k^{\text{ss}}} \Phi(gx) dg.$$

For $i = 1, 2, 3$, we also define the *zeta integrals* for $V_{(i)}$ by

$$Z_{(i)}(\Phi, s, \omega) = \int_{G_{\mathbb{A}}/G_k} |\det g|^{2s} \omega(\det g) \sum_{x \in V_{(i)}} \Phi(gx) dg.$$

By definition $Z(\Phi, s, \omega) = \sum_{1 \leq i \leq 3} Z_{(i)}(\Phi, s, \omega)$. We give some analytic properties of integrals $Z_{(i)}(\Phi, s, \omega)$. Note that $V_{(1)}$ is a single G_k -orbit and the results of $Z_{(1)}(\Phi, \omega, s)$ in this section is included in Datskovsky-Wright's analysis [DW86] of the orbital zeta functions. Before starting the analysis, without loss of generality we assume the following as in Section 5.

Assumption 8.2. The Schwartz-Bruhat function satisfies $\mathcal{M}_{\omega}\Phi = \Phi$, where the operator \mathcal{M}_{ω} is defined by $\mathcal{M}_{\omega}\Phi(x) = \int_{\mathcal{K}(2)} \omega(\det \kappa) \Phi(\kappa x) d\kappa$.

We will express $Z_{(i)}(\Phi, s, \omega)$ ($i = 1, 2$) by means of following zeta integrals of (B, W) .

Definition 8.3. Let $W_{(1)} = \{y \in W_k^{\text{ss}} \mid k(y) \cong k \times k\}$ and $W_{(2)} = W_k^{\text{ss}} \setminus W_{(1)}$. For $i = 1, 2$, we define the zeta integrals of (B, W) by

$$Y_{(i)}(\Psi, s_1, s_2, \omega_1, \omega_2) = \int_{B_{\mathbb{A}}/B_k} |t|^{s_1} |tp|^{2s_2} \omega_1(t) \omega_2(tp) \sum_{y \in W_{(i)}} \Psi(by) db.$$

Lemma 8.4. Let us define $\mathcal{R}_W\Phi \in \mathcal{S}(W_{\mathbb{A}})$ by $\mathcal{R}_W\Phi(y) = \Phi(y^*)$. We have

$$\begin{aligned} Z_{(2)}(\Phi, s, \omega) &= Y_{(2)}(\mathcal{R}_W\Phi, 2s, s, \omega, \omega), \\ Z_{(1)}(\Phi, s, \omega) &= 3^{-1} Y_{(1)}(\mathcal{R}_W\Phi, 2s, s, \omega, \omega). \end{aligned}$$

Proof. We can easily check that $V_{(2)} = G_k \times_{B(2)_k} (W_{(2)})^*$. Hence

$$\begin{aligned} Z_{(2)}(\Phi, s, \omega) &= \int_{G_{\mathbb{A}}/B(2)_k} |\det g|^{2s} \omega(\det g) \sum_{y \in W_{(2)}} \Phi(gy^*) dg \\ &= \int_{B(2)_{\mathbb{A}}/B(2)_k} |\det b_2|^{2s} \omega(\det b_2) \sum_{y \in W_{(2)}} \Phi(b_2 y^*) db_2 \\ &= \int_{B_{\mathbb{A}}/B_k} |t^2 p|^{2s} \omega(t^2 p) \sum_{y \in W_{(2)}} \Phi(b^* y^*) db \end{aligned}$$

which is equal to $Y_{(2)}(\mathcal{R}_W\Phi, 2s, s, \omega, \omega)$ since $\Phi(b^* y^*) = \mathcal{R}_W\Phi(by)$. This proves the first equality. Let $w = (1, 1, 0) \in W_{(1)}$. Then $w^* = (0, 1, 1, 0) \in V_{(1)}$. Since $V_{(1)}$ is a single G_k -orbit and $\text{Stab}(G_k; w^*)$ is of order 6, we have

$$Z_{(1)}(\Phi, s, \omega) = 6^{-1} \int_{B(2)_{\mathbb{A}}/B(2)_k} |\det b_2|^{2s} \omega(\det b_2) \Phi(b_2 w^*) db_2.$$

On the other hand, $W_{(1)}$ is also a single B_k -orbit and $\text{Stab}(B_k; w)$ is of order 2. Hence by a similar modification we obtain the second formula. \square

Definition 8.5. For $\Phi \in \mathcal{S}(V_{\mathbb{A}})$, we define $\mathcal{R}_a\Phi, \mathcal{R}_b\Phi \in \mathcal{S}(\mathbb{A})$ by

$$\mathcal{R}_a\Phi(x) = \int_{\mathbb{A}^2} \Phi(0, x, y_1, y_2) dy_1 dy_2, \quad \mathcal{R}_b\Phi(x) = \int_{\mathbb{A}^3} \Phi(x, y_1, y_2, y_3) dy_1 dy_2 dy_3.$$

- Proposition 8.6.** (1) *The zeta integral $Z_{(1)}(\Phi, s, \omega)$ is holomorphic for $\Re(s) > 1/3$.*
(2) *The zeta integral $Z_{(2)}(\Phi, s, \omega)$ is holomorphic for $\Re(s) > 1/2$ except for possible simple pole at $s = 1$ with the residue $\delta(\omega)2^{-1}\Sigma(\mathcal{R}_a\Phi, 2)$.*
(3) *The zeta integral $Z_{(3)}(\Phi, s, \omega)$ is holomorphic for $\Re(s) > 1/2$ except for possible simple poles at $s = 1, 5/6$ with the residue $\delta(\omega)2^{-1}\mathfrak{c}_k^{-1}\Delta_k\zeta_k(2) \int_{V_{\mathbb{A}}} \Phi(x) dx, \delta(\omega^3)6^{-1}\Sigma(\mathcal{R}_b\Phi, 1/3)$, respectively.*

Moreover each of the zeta integrals has meromorphic continuation to the whole complex plane.

Proof. (1) follows from either [DW86, Theorem 6.1] or Lemma 8.4 and Proposition B.9, and (2) follows from (1), Lemma 8.4, Proposition 5.12 and Theorem 5.25. On the other hand the analytic properties including meromorphic continuations and residue formulas for $Z(\Phi, s, \omega)$ are known by [Wr85, Theorem 6.4]. (3) follows from this and (1), (2). The meromorphic continuations are also proved one by one from (1) to (3). \square

APPENDIX A. REMAINING FUNCTIONAL EQUATIONS

Here, we collect 2 kinds of local functional equations for (H, U) . For the real case, the first equation was proved in [Sh72, Lemma 1.9] and the second one is proved in [Sat81, Lemma 2.9]. The complex case immediately follows from the proof of Proposition 6.5. We note that the functional equations in Proposition 6.5 follows from this table also.

Proposition A.1. (1) *The functional equations for \mathbb{R} are:*

$$\begin{aligned} \begin{pmatrix} \mathcal{U}_{\mathbb{R},1}(\widehat{\Psi} \otimes \Upsilon, s_1, s_2) \\ \mathcal{U}_{\mathbb{R},2}(\widehat{\Psi} \otimes \Upsilon, s_1, s_2) \end{pmatrix} &= \frac{2^{s_1+2s_2-1}}{\pi^{s_1+2s_2-1/2}} \Gamma(s_2) \Gamma(s_1 + s_2 - 1/2) \\ &\times \begin{pmatrix} \sin((s_1 + 2s_2)\pi/2) & \cos(s_1\pi/2) \\ \sin(s_1\pi/2) & \cos((s_1 + 2s_2)\pi/2) \end{pmatrix} \begin{pmatrix} \mathcal{U}_{\mathbb{R},1}(\Psi \otimes \Upsilon, s_1, 3/2 - s_1 - s_2) \\ \mathcal{U}_{\mathbb{R},2}(\Psi \otimes \Upsilon, s_1, 3/2 - s_1 - s_2) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} \mathcal{U}_{\mathbb{R},1}(\Psi \otimes \widehat{\Upsilon}, s_1, s_2) \\ \mathcal{U}_{\mathbb{R},2}(\Psi \otimes \widehat{\Upsilon}, s_1, s_2) \end{pmatrix} &= \frac{2}{(2\pi)^{2s_1}} \Gamma(s_1)^2 \\ &\times \begin{pmatrix} 2\cos^2(s_1\pi/2) & 0 \\ 0 & \sin(s_1\pi) \end{pmatrix} \begin{pmatrix} \mathcal{U}_{\mathbb{R},1}(\Psi \otimes \Upsilon, 1 - s_1, s_1 + s_2 - 1/2) \\ \mathcal{U}_{\mathbb{R},2}(\Psi \otimes \Upsilon, 1 - s_1, s_1 + s_2 - 1/2) \end{pmatrix}. \end{aligned}$$

(2) *The functional equations for \mathbb{C} are:*

$$\begin{aligned} \mathcal{U}_{\mathbb{C},1}(\widehat{\Psi} \otimes \Upsilon, s_1, s_2) &= \frac{2^{2s_1+4s_2-2}}{\pi^{2s_1+4s_2-1}} \Gamma(s_2)^2 \Gamma(s_1 + s_2 - 1/2)^2 \\ &\times \sin(s_2\pi) \cos(s_1\pi + s_2\pi) \mathcal{U}_{\mathbb{C},1}(\Psi \otimes \Upsilon, s_1, 3/2 - s_1 - s_2), \end{aligned}$$

$$\mathcal{U}_{\mathbb{C},1}(\Psi \otimes \widehat{\Upsilon}, s_1, s_2) = \frac{4}{(2\pi)^{4s_1}} \Gamma(s_1)^4 \sin^2(s_1\pi) \mathcal{U}_{\mathbb{C},1}(\Psi \otimes \Upsilon, 1 - s_1, s_1 + s_2 - 1/2).$$

By the same arguments of Section 7 we could find 2 more functional equations for Dirichlet series $\Xi_j(s_1, s_2, \omega_1, \omega_2)$.

APPENDIX B. NON-ARCHIMEDEAN LOCAL THEORY FOR THE SPACES OF BINARY QUADRATIC FORMS

Here we study some non-archimedean local theory. The explicit formula for the standard test function is obtained. An interesting corollary to this formula, we describe the orbital zeta functions by means of Dedekind zeta functions.

B.1. Explicit formula at finite places. In this subsection we give the explicit formula of the local zeta function for the standard test function at finite places.

For a while let $v \in \mathfrak{M}$ arbitrary (not assuming a finite place). To treat ramified characters also, we slightly generalize the notion of local zeta function.

Definition B.1. For $\tilde{y} \in U_{k_v}^{\text{ss}}$, $\Phi_v \in \mathcal{S}(U_{k_v})$, $\omega_{1v}, \omega_{2v} \in \Omega_v$ and $s_1, s_2 \in \mathbb{C}$, we define

$$\mathfrak{X}_{\tilde{y},v}(\Phi_v, s_1, s_2, \omega_{1v}, \omega_{2v}) = \int_{H_{k_v}} \omega_{1v}(t_v) \omega_{2v}(t_v \det g_v) |t_v|_v^{s_1} |t_v \det g_v|_v^{2s_2} \Phi_v(h_v \tilde{y}) dh_v$$

and call it *the local zeta function*.

If both ω_{1v}, ω_{2v} are trivial we often drop them and write $\mathfrak{X}_{\tilde{y},v}(\Phi_v, s_1, s_2)$. For $j_v \in \mathcal{T}_v$ such that $\tilde{y} \in U_{j_v}$, by definition $\mathfrak{X}_{v,j_v}(\Phi_v, s_1, s_2) = |R_1(\tilde{y})|_v^{s_1} |R_2(\tilde{y})|_v^{s_2} \mathfrak{X}_{\tilde{y},v}(\Phi_v, s_1, s_2)$.

For any $\tilde{y} \in U_{k_v}^{\text{ss}}$, The stabilizer of \tilde{y} in H_{k_v} consists of two elements and the non-trivial element is of the form $(1, g)$ with $\det(g) = -1$. This shows that $\mathfrak{X}_{\tilde{y},v}(\Phi_v, s_1, s_2, \omega_{1v}, \omega_{2v})$ is identically zero unless $\omega_{2v}(-1) = 1$. Hence we assume the following.

Assumption B.2. For any $v \in \mathfrak{M}$, $\omega_{2v}(-1) = 1$.

The analytic continuation of the local zeta function is known in more general settings than the prehomogeneous case. The meromorphic properties of complex powers of polynomials were studied by Bernstein and Gelfand [BG69] for infinite places and by Denef [D84, D85] for finite places. The following lemma is contained in their works. (We gave a proof for $v \in \mathfrak{M}_\infty$ in Proposition 6.5.)

Lemma B.3. *The local zeta function $\mathfrak{X}_{\tilde{y},v}(\Phi_v, s_1, s_2, \omega_1, \omega_2)$ has meromorphic continuation to \mathbb{C}^2 . Moreover, it is a rational function of $q_v^{s_1}, q_v^{2s_2}$ if $v \in \mathfrak{M}_f$.*

It will be convenient to attach to each orbit in $U_{k_v}^{\text{ss}}$ where $v \in \mathfrak{M}$, an index or type which records the arithmetic properties of v and the extension of k_v corresponding to the orbit. Recall that by Proposition 3.18, the orbit space $H_{k_v} \backslash U_{k_v}^{\text{ss}}$ corresponds bijectively to the set of isomorphism classes of separable quadratic algebra of k_v . The orbit corresponding to $k_v \times k_v$ will have the index (sp). The orbit corresponding to the unique unramified quadratic extension of k_v will have the index (ur). An orbit corresponding to a ramified quadratic extension of k_v will have the index (rm). Recall that the extension \mathbb{C}/\mathbb{R} is regarded as ramified.

Definition B.4. For each of H_{k_v} -orbits in $U_{k_v}^{\text{ss}}$, we choose and fix an element \tilde{z} which satisfies the following condition.

- (1) If the orbit is corresponding to $k_v \times k_v$, then $\tilde{z} = (0, 1, 0, 1, 1)$.
- (2) If $v \in \mathfrak{M}_\mathbb{R}$ and the orbit is corresponding to \mathbb{C} , then $\tilde{z} = (1/2, 0, 1/2, 1, 1)$.
- (3) Consider the case $v \in \mathfrak{M}_f$ and the orbit is corresponding to a quadratic field extension F/k_v . We choose and fix $z \in W_{k_v}$ such that $z_1 = 1$ and \mathcal{O}_F is generated over \mathcal{O}_v by roots of $z(v_1, 1)$. We put $\bar{z} = (1, 0)$ and choose $\tilde{z} = (z, \bar{z})$ as the orbital representative.

We call such fixed orbital representatives as the *standard orbital representatives*.

We note that for any standard representative \tilde{z} , we have $R_1(\tilde{z}) = 1$, and if $v \in \mathfrak{M}_f$ then the discriminant $R_2(\tilde{z})$ of $z(v)$ generates the ideal $\Delta_{\tilde{z},v}(\tilde{z})/k_v$.

We now assume $v \in \mathfrak{M}_f$. We give an explicit formula when Φ_0 is the characteristic function of $U_{\mathcal{O}_v}$. In this case the integral is 0 either ω_{1v} or ω_{2v} is ramified. Hence we consider the case both ω_{1v} and ω_{2v} are unramified.

Proposition B.5. *For any $v \in \mathfrak{M}_f$, let $\Phi_{v,0}$ be the characteristic function of $U_{\mathcal{O}_v}$. For a standard representative $\tilde{z} \in U_{k_v}^{\text{ss}}$,*

$$\mathfrak{X}_{\tilde{z},v}(\Phi_{v,0}, s_1, s_2) = (1 - q_v^{-2s_2})^{-1} (1 - q_v^{1-2s_1-2s_2})^{-1} (1 - q_v^{-s_1-2s_2})^{-1} R_{v,\tilde{z}}(s_1, s_2)$$

where

$$R_{v,\tilde{z}}(s_1, s_2) = \begin{cases} (1 - q_v^{-s_1-2s_2})^2 / (1 - q_v^{-s_1})^2 & \tilde{z} \text{ is of type (sp)}, \\ (1 - q_v^{-2s_1-4s_2}) / (1 - q_v^{-2s_1}) & \tilde{z} \text{ is of type (ur)}, \\ (1 - q_v^{-s_1-2s_2}) / (1 - q_v^{-s_1}) & \tilde{z} \text{ is of type (rm)}. \end{cases}$$

Proof. In the proof of this proposition, we drop the subscript v from various symbols such as $\mathfrak{X}_{\tilde{z},v}, \Phi_{0,v}, u_v, q_v$ if there is no confusion. We first consider the case \tilde{z} is of type (sp). We put $a = s_1 + 2s_2, b = s_1$. Then, by a standard modification, we have

$$\mathfrak{X}_{\tilde{z}}(\Phi_0, s_1, s_2) = \int_{(k_v^\times)^3 \times k_v} |t_1|_v^a |t_2 t_3|_v^b \Phi_0(0, t_1, t_1 u, t_2 - u t_3, t_3) d^\times t_1 d^\times t_2 d^\times t_3 du.$$

Let $\pi \in \mathcal{O}_v$ be a uniformizer. By changing t_1 to $\pi^{-1} t_1, t_3$ to $\pi^{-1} t_3$, and u to πu , we have

$$\mathfrak{X}_{\tilde{z}}(\Phi_0, s_1, s_2) = q^{a+b-1} \int_{(k_v^\times)^3 \times k_v} |t_1|_v^a |t_2 t_3|_v^b \Phi_0(0, t_1/\pi, t_1 u, t_2 - u t_3, t_3/\pi) d^\times t_1 d^\times t_2 d^\times t_3 du.$$

Hence if we let $\Phi'_0 \in \mathcal{S}(U_{k_v})$ be the characteristic function of $U_{\mathcal{O}_v} \setminus (\mathcal{O}_v \times \mathfrak{p}_v \times \mathcal{O}_v \times \mathcal{O}_v \times \mathfrak{p}_v)$, then $\mathfrak{X}_{\tilde{z}}(\Phi_0, s_1, s_2) = (1 - q^{1-a-b})^{-1} \mathfrak{X}_{\tilde{z}}(\Phi'_0, s_1, s_2)$. We consider the integral $\mathfrak{X}_{\tilde{z}}(\Phi'_0, s_1, s_2)$. We divide the domain of the integration into the following three subsets: (a) $t_1 \in \mathcal{O}_v^\times$, (b) $t_1 \in \pi^m \mathcal{O}_v^\times, t_3 \in \mathcal{O}_v^\times, u \in \mathcal{O}_v$ for $m \geq 1$, (c) $t_1 \in \pi^m \mathcal{O}_v^\times, t_3 \in \mathcal{O}_v^\times, u \in (\pi^{-m} \mathcal{O}_v \setminus \mathcal{O}_v)$ for $m \geq 1$. Then the value of the integral in each domain is found to be

$$(1 - q^{-b})^{-2}, \quad q^{-a} (1 - q^{-a})^{-1} (1 - q^{-b}), \quad \text{and} \quad q^{-a+b} (1 - q^{-a})^{-1} (1 - q^{-a+b}),$$

respectively. Adding all these up, we have $\mathfrak{X}_{\tilde{z}}(\Phi'_0, s_1, s_2) = (1 - q^{-a}) (1 - q^{-b})^{-2} (1 - q^{-a+b})^{-1}$, and this proves the formula for the case (sp).

We next consider the case x is of type (ur) or (rm). We put $F = \tilde{k}_v(\tilde{z})$. Let \mathcal{T}, \mathcal{N} be the trace and the norm of the quadratic extension F/k . We write $x_1(v) = \mathcal{N}(v_1 + \theta v_2)$ where $\theta \in \mathcal{O}_F$. Then by the definition of the standard orbital representative, θ generates \mathcal{O}_F over \mathcal{O}_v . Let Ψ_0 be the characteristic function of $W_{\mathcal{O}_v}$. In this case, we could see that $\mathfrak{X}_{\tilde{z}}(\Phi_0, s_1, s_2) = (1 - q_v^{-2s_1})^{-1} \mathfrak{X}'_{\tilde{z}}(\Psi_0, s_1 + s_2, s_2)$ where

$$\mathfrak{X}'_{\tilde{z}}(\Psi, a, b) = \int_{(k_v^\times)^2 \times k_v} |t_1|_v^a |t_2|_v^b \Psi(t_1, t_1 \mathcal{T}(u + t_2 \theta), t_1 \mathcal{N}(u + t_2 \theta)) d^\times t_1 d^\times t_2 du$$

for $\Psi \in \mathcal{S}(W_{\mathbb{A}})$. Let $\Psi'_0 \in \mathcal{S}(W_{\mathbb{A}})$ be the characteristic function of $W_{\mathcal{O}_v} \setminus (\mathfrak{p}_v^2 \times \mathfrak{p}_v \times \mathcal{O}_v)$. Then the similar observation as in the case of (sp), we have $\mathfrak{X}'_{\tilde{z}}(\Psi_0, a, b) = (1 - q^{1-2a+b})^{-1} \mathfrak{X}'_{\tilde{z}}(\Psi'_0, a, b)$. Let $j = 0$ if \tilde{z} is of type (ur) and $j = 1$ if \tilde{z} is of type (rm). Then it is easy to see that $\Psi'_0(t_1, t_1 \mathcal{T}(u + t_2 \theta), t_1 \mathcal{N}(u + t_2 \theta)) = 1$ if and only if

$$t_1 \in \mathcal{O}_v^\times, t_2 \in \mathcal{O}_v, u \in \mathcal{O}_v, \quad \text{or} \quad t_1 \in \pi \mathcal{O}_v^\times, t_2 \in \pi^{-j} \mathcal{O}_v, u \in \mathcal{O}_v.$$

Hence $\mathfrak{X}'_{\tilde{z}}(\Psi'_0, a, b) = (1 - q^{-b})^{-1} (1 + q^{-a+jb})$ and this finishes the proof. \square

As a corollary we obtain the following.

Lemma B.6. *For any $v \in \mathfrak{M}$, the local zeta function $\mathfrak{X}_{\tilde{y},v}(\Phi_v, s_1, s_2, \omega_{1v}, \omega_{2v})$ is holomorphic in the region $\Re(s_1) > 0, \Re(s_2) > 0, \Re(s_1 + s_2) > 1/2$.*

Proof. The statement for $v \in \mathfrak{M}_\infty$ follows from Proposition 6.5 (1). For $v \in \mathfrak{M}_f$, the support of Φ_v is contained in $\pi^{-m} U_{\mathcal{O}_v}$ for some integer m . Hence the result follows from Proposition B.5 and the relatively invariant property of the local zeta function under the action of H_{k_v} . \square

B.2. Orbital zeta functions. We now discuss the relation between global and local situations.

Definition B.7. For $\tilde{y} \in U_k^{\text{ss}}$, $\Phi \in \mathcal{A}(U_{\mathbb{A}})$ and $\omega_1, \omega_2 \in \Omega^1$, we define

$$X_{\tilde{y}}^*(\Phi, s_1, s_2, \omega_1, \omega_2) = \int_{H_{\mathbb{A}}} |t|^{s_1} |t \det g|^{2s_2} \omega_1(t) \omega_2(t \det g) \Phi(h\tilde{y}) d_{\text{pr}} h$$

and call it *the orbital zeta function*.

Note that this integral depends only on H_k -orbits. By the standard consideration the global zeta function decompose into as follows and this is the reason why we are interested in the orbital zeta functions.

Lemma B.8. We define $X^*(\Phi, s_1, s_2, \omega_1, \omega_2) = \Delta_k^{1/2} \mathfrak{C}_k^3 \cdot X(\Phi, s_1, s_2, \omega_1, \omega_2)$. Then

$$X^*(\Phi, s_1, s_2, \omega_1, \omega_2) = 2^{-1} \sum_{\tilde{y} \in H_k \backslash U_k^{\text{ss}}} X_{\tilde{y}}^*(\Phi, s_1, s_2, \omega_1, \omega_2).$$

The orbital zeta function has an Euler product. We consider this Euler product in more detail. For the rest of this subsection, we assume Φ is of the product form $\Phi = \prod_{v \in \mathfrak{M}} \Phi_v$. Let $\omega_i = \prod_{v \in \mathfrak{M}} \omega_{iv}$. Then by definition

$$X_{\tilde{y}}^*(\Phi, s_1, s_2, \omega_1, \omega_2) = \prod_{v \in \mathfrak{M}} \mathfrak{X}_{\tilde{y},v}(\Phi_v, s_1, s_2, \omega_{1v}, \omega_{2v}).$$

For each $v \in \mathfrak{M}$, let $\tilde{z}_{\tilde{y},v}$ be the standard representative lying in the orbit of \tilde{y} and put

$$\Theta_{\tilde{y},v}(\Phi_v, s_1, s_2, \omega_{1v}, \omega_{2v}) = \mathfrak{X}_{\tilde{z}_{\tilde{y},v}}(\Phi_v, s_1, s_2, \omega_{1v}, \omega_{2v}).$$

If we let $\tilde{y} = h\tilde{z}_{\tilde{y},v}$ for some $h = (t, g) \in H_{k_v}$, then $t = R_1(\tilde{y})/R_1(\tilde{z}_{\tilde{y},v})$ and $(t \det g)^2 = R_2(\tilde{y})/R_2(\tilde{z}_{\tilde{y},v})$. We put $\Delta_{\tilde{y},v} = R_2(\tilde{z}_{\tilde{y},v})/R_2(\tilde{y}) \in k_v^\times$. Then we have

$$\mathfrak{X}_{\tilde{y},v}(\Phi_v, s_1, s_2, \omega_{1v}, \omega_{2v}) = \tilde{\omega}_{1v} \left(\frac{R_1(\tilde{z}_{\tilde{y},v})}{R_1(\tilde{y})} \right) \tilde{\omega}_{2v}(\sqrt{\Delta_{\tilde{y},v}}) \Theta_{\tilde{y},v}(\Phi_v, s_1, s_2, \omega_{1v}, \omega_{2v}).$$

where we put $\tilde{\omega}_{1v} = |\cdot|^{s_1} \omega_{1v}$ and $\tilde{\omega}_{2v} = |\cdot|^{2s_2} \omega_{2v}$. Note that Assumption B.2 vanishes the ambiguity of the choice of the square root. We put $\Delta_{\tilde{y}} = (\Delta_{\tilde{y},v})_{v \in \mathfrak{M}} \in \mathbb{A}^\times$. Since $\Delta_{\tilde{y}} = (R_2(\tilde{z}_{\tilde{y},v}))_{v \in \mathfrak{M}}/R_2(\tilde{y})$, by the observation after Definition B.4, if we regard $\Delta_{\tilde{y}}$ and $\Delta_{\tilde{k}(\tilde{y})/k}$ as elements of $\mathbb{A}^\times/k^\times$ then they coincide. Consider the product of the above formula for $v \in \mathfrak{M}$. Since $R_1(\tilde{z}_{\tilde{y},v}) = 1$ for all $v \in \mathfrak{M}$ and $\omega_1(R_1(\tilde{y})) = 1$, we have

$$X_{\tilde{y}}^*(\Phi, s_1, s_2, \omega_1, \omega_2) = \tilde{\omega}_2(\sqrt{\Delta_{\tilde{y}}}) \prod_{v \in \mathfrak{M}} \Theta_{\tilde{y},v}(\Phi_v, s_1, s_2, \omega_{1v}, \omega_{2v}).$$

For a finite set T of places of k , we define the truncated zeta function $\zeta_{k,T}(s) = \prod_{v \in \mathfrak{M}_f \setminus T} (1 - q_v^{-s})^{-1}$. For an separable quadratic algebra F of k , we define $\zeta_{F,T}(s)$ similarly. By Lemmas B.3, B.6 and Proposition B.5, we have the meromorphic continuation of the orbital zeta functions as below.

Proposition B.9. Let $T \supset \mathfrak{M}_\infty$ be a finite set such that Φ_v is the characteristic function of $U_{\mathcal{O}_v}$ and ω_v is unramified unless $v \in T$. Then

$$\begin{aligned} X_{\tilde{y}}(\Phi, s_1, s_2, \omega_1, \omega_2) &= N(\Delta_{\tilde{k}(\tilde{y})/k})^{-s_2} \omega_2(\sqrt{\Delta_{\tilde{y}}}) \prod_{v \in T} \Theta_{\tilde{y},v}(\Phi_v, s_1, s_2, \omega_{1v}, \omega_{2v}) \\ &\quad \times \zeta_{k,T}(2s_2) \zeta_{k,T}(2s_1 + 2s_2 - 1) \zeta_{k,T}(s_1 + 2s_2) \zeta_{k(\tilde{y}),T}(s_1) \zeta_{k(\tilde{y}),T}(s_1 + 2s_2)^{-1}. \end{aligned}$$

This function is meromorphic on \mathbb{C}^2 and holomorphic in the region $\Re(s_1) > 1, \Re(s_2) > 1$.

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