CHAPTER IV

Solutions of Systems of Partial Differential Equations for Hypergeometric Functions of Two Variables

18. Solutions of the system for F_1. In Section 12 (Chapter II) we derived the systems (12.1), (12.2), (12.3) and (12.4) of partial differential equations satisfied by F_1, F_2, F_3 and F_4 respectively. Let us consider the system (12.1)

$$(12.1)\ \begin{cases} x(1-x)r + y(1-x)s + [\gamma-(\alpha+\beta+1)x]p - \beta yq - \alpha\beta z = 0\ , \\ y(1-y)t + x(1-y)s + [\gamma-(\alpha+\beta'+1)y]q - \beta' xp - \alpha\beta' z = 0\ . \end{cases}$$

In Section 5 (Chapter I) we proved that the Gauss differential equation has twenty four expressions of solutions of the form

$$x^{\rho_i}(1-x)^{\sigma_i} F(\alpha_i, \beta_i, \gamma_i, x_i)\ ,$$

where ρ_i, σ_i, α_i, β_i and γ_i are linear in α, β and γ, and x_i is one of the six transformations

$$x,\quad 1-x,\quad 1/x,\quad 1/(1-x),\quad (x-1)/x,\quad x/(x-1)\ :$$

(Cf. Theorem 5.2, p.31.) These solutions are divided into six groups and any two solutions of each group are linearly dependent. It is known that the system (12.1) has sixty expressions of solutions of the form

$$x^{\rho_i}(1-x)^{\sigma_i} y^{\rho'_i}(1-y)^{\sigma'_i}(x-y)^{\tau_i} F_1(\alpha_i, \beta_i, \beta'_i, \gamma_i, x_i, y_i)\ ,$$

where ρ_i, σ_i, ρ'_i, σ'_i, τ_i, α_i, β_i, β'_i and γ_i are linear

in α, β, β' and γ, and x_i and y_i are linear fractional expressions of x and y. Furthermore, these sixty solutions are divided into ten groups and any two solutions of each group are linearly dependent. In this section, we shall find these sixty expressions of solutions of (12.1).

To begin with, let us consider the simple integral representation of Euler for F_1 :

$$(10.4)\quad F_1(\alpha, \beta, \beta', \gamma, x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-xu)^{-\beta}(1-yu)^{-\beta'}\,du\ .$$

(Cf. Theorem 10.2, p.64.) In Section 10, we showed that the solution defined by the integral (10.4) has six expressions which are derived from (10.4) by changing variable u respectively by

$$(18.1)\ \begin{cases} u = v,\quad u = 1-v,\quad u = v/[(1-x)+vx],\quad u = v/[(1-y)+vy], \\ u = (1-v)/(1-vx),\quad u = (1-v)/(1-vy)\ . \end{cases}$$

(Cf. Theorem 10.3, p.65.)

The method of the Euler transform

$$\int_C (1-xu)^{\lambda-1}\varphi(u)\,du$$

which was explained in Chapter III suggested that, if we consider the integral of the form

$$(18.2)\qquad z(x, y) = \int_C (1-xu)^{\lambda-1}(1-yu)^{\mu-1}\varphi(u)\,du\ ,$$

we may determine λ, μ, φ and C so that (18.2) satisfies the system (12.1). Moreover the formula (10.4) suggestes that we may take

$$\lambda = -\beta+1, \quad \mu = -\beta'+1, \quad \varphi(u) = u^{\alpha-1}(1-u)^{\gamma-\alpha-1}$$

and that we may take curves joining any two points of 0, 1, 1/x, 1/y and ∞ as 0. This means that an integral

$$\text{(18.3)} \quad z(x, y) = \int_a^b u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-xu)^{-\beta}(1-yu)^{-\beta'}\,du$$

may satisfy the system (12.1), where a and b are any two values of 0, 1, 1/x, 1/y and ∞. We shall verify that (18.3) is a solution of (12.1).

Set

$$\varphi(u) = u^{\alpha-1}(1-u)^{\gamma-\alpha-1}$$

in (18.2). Suppose that the derivatives of $z(x, y)$ can be obtained by differentiation under the symbol of integration: i.e.

$$p = \int_C \beta u(1-xu)^{-\beta-1}(1-yu)^{-\beta'}\varphi(u)\,du,$$

$$q = \int_C \beta' u(1-xu)^{-\beta}(1-yu)^{-\beta'-1}\varphi(u)\,du,$$

$$r = \int_C \beta(\beta+1)u^2(1-xu)^{-\beta-2}(1-yu)^{-\beta'}\varphi(u)\,du,$$

$$s = \int_C \beta\beta' u^2(1-xu)^{-\beta-1}(1-yu)^{-\beta'-1}\varphi(u)\,du,$$

$$t = \int_C \beta'(\beta'+1)u^2(1-xu)^{-\beta}(1-yu)^{-\beta'-2}\varphi(u)\,du.$$

Then

$$L(z) = x(1-x)r + y(1-x)s + [\gamma-(\alpha+\beta+1)x]p - \beta yq - \alpha\beta z$$

$$= \int_C [\,\beta(\beta+1)x(1-x)u^2(1-xu)^{-\beta-2}(1-yu)^{-\beta'}$$

$$+ \beta\beta' y(1-x)u^2(1-xu)^{-\beta-1}(1-yu)^{-\beta'-1}$$

$$+ \beta(\gamma-(\alpha+\beta+1)x)u(1-xu)^{-\beta-1}(1-yu)^{-\beta'}$$

$$- \beta\beta' yu(1-xu)^{-\beta}(1-yu)^{-\beta'-1}$$

$$- \beta\alpha(1-xu)^{-\beta}(1-yu)^{-\beta'}\,]\,\varphi(u)\,du.$$

Now consider the function

$$\text{(18.4)} \quad G(x, y, u) = (1-xu)^{-\beta-1}(1-yu)^{-\beta'}u^{\alpha}(1-u)^{\gamma-\alpha}$$

$$= (1-xu)^{-\beta-1}(1-yu)^{-\beta'}u(1-u)\,\varphi(u).$$

Observe that

$$\frac{1}{G}\,\partial G/\partial u = x\,\frac{\beta+1}{1-xu} + y\,\frac{\beta'}{1-yu} + \frac{\alpha}{u} - \frac{\gamma-\alpha}{1-u}$$

and hence

$$\partial G/\partial u = (\beta+1)xu(1-u)(1-xu)^{-\beta-2}(1-yu)^{-\beta'}\varphi(u)$$

$$+ \beta' yu(1-u)(1-xu)^{-\beta-1}(1-yu)^{-\beta'-1}\varphi(u)$$

$$+ [\alpha(1-u)-(\gamma-\alpha)u](1-xu)^{-\beta-1}(1-yu)^{-\beta'}\varphi(u).$$

If we use two identities:

$$u(1-u) = -(1-x)u^2 + (1-xu)u$$

and

$$\alpha(1-u) - (\gamma-\alpha)u = \alpha - \gamma u = \alpha xu - \gamma u + \alpha(1-xu),$$

we obtain

$$\partial G/\partial u = [-(\beta+1)x(1-x)u^2(1-xu)^{-\beta-2}(1-yu)^{-\beta'}$$
$$+(\beta+1)xu(1-xu)^{-\beta-1}(1-yu)^{-\beta'}$$
$$-\beta' y(1-x)u^2(1-xu)^{-\beta-1}(1-yu)^{-\beta'-1}$$
$$+\beta' yu(1-xu)^{-\beta}(1-yu)^{-\beta'-1}$$
$$+(\alpha xu-\gamma u)(1-xu)^{-\beta-1}(1-yu)^{-\beta'}$$
$$+\alpha(1-xu)^{-\beta}(1-yu)^{-\beta'}]\varphi(u)$$
$$= -[(\beta+1)x(1-x)u^2(1-xu)^{-\beta-2}(1-yu)^{-\beta'}$$
$$+\beta' y(1-x)u^2(1-xu)^{-\beta-1}(1-yu)^{-\beta'-1}$$
$$+(\gamma-(\alpha+\beta+1)x)u(1-xu)^{-\beta-1}(1-yu)^{-\beta'}$$
$$-\beta' yu(1-xu)^{-\beta}(1-yu)^{-\beta'-1}$$
$$-\alpha(1-xu)^{-\beta}(1-yu)^{-\beta'}]\varphi(u) .$$

Thus we obtain

$$L(z) = -\beta\int_C \partial G/\partial u\, du = -\beta[G(x, y, u)]_C .$$

Interchanging x, y and β, β', we also get

$$M(z) = -\beta'[H(x, y, u)]_C ,$$

where

$$M(z) = y(1-y)t + x(1-y)s + [\gamma-(\alpha+\beta'+1)y]q - \beta' xp - \alpha\beta' z$$

and

(18.5) $$H(x, y, u) = (1-xu)^{-\beta}(1-yu)^{-\beta'-1}u^{\alpha}(1-u)^{\gamma-\alpha} .$$

Therefore, if we take C so that

(18.6) $$\begin{cases} [u^{\alpha}(1-u)^{\gamma-\alpha}(1-xu)^{-\beta-1}(1-yu)^{-\beta'}]_C = 0 , \\ [u^{\alpha}(1-u)^{\gamma-\alpha}(1-xu)^{-\beta}(1-yu)^{-\beta'-1}]_C = 0 , \end{cases}$$

then

$$L(z) = 0 \quad \text{and} \quad M(z) = 0 .$$

This means that (18.2) is a solution of (12.1). Thus we can take as C a path joining any two points of 0, 1, $1/x$, $1/y$ and ∞, if the real parts of the parameters α, β, β' and γ satisfy suitable conditions.

THEOREM 18.1: Let us set

(18.7) $$U(x, y, u) = u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-xu)^{-\beta}(1-yu)^{-\beta'} .$$

Then the system (12.1) has the following ten integrals as solutions:

(18.8-1) $\int_0^1 U(x, y, u)du$ if $0 < \operatorname{Re}\alpha < \operatorname{Re}\gamma$,

(18.8-2) $\int_\infty^0 U(x, y, u)du$ if $0 < \operatorname{Re}\alpha$, $\operatorname{Re}(\gamma-\beta-\beta') < 1$,

(18.8-3) $\int_1^\infty U(x, y, u)du$ if $\operatorname{Re}\alpha < \operatorname{Re}\gamma$, $\operatorname{Re}(\gamma-\beta-\beta') < 1$,

(18.8-4) $\int_0^{1/x} U(x, y, u)du$ if $0 < \operatorname{Re}\alpha$, $\operatorname{Re}\beta < 1$,

(18.8-5) $\int_0^{1/y} U(x, y, u)du$ if $0 < \operatorname{Re}\alpha$, $\operatorname{Re}\beta' < 1$,

(18.8-6) $\int_1^{1/x} U(x, y, u)du$ if $\operatorname{Re}\alpha < \operatorname{Re}\gamma$, $\operatorname{Re}\beta < 1$,

(18.8-7) $\int_1^{1/y} U(x, y, u)du$ if $\operatorname{Re}\alpha < \operatorname{Re}\gamma$, $\operatorname{Re}\beta' < 1$,

(18.8-8) $\int_{1/x}^{\infty} U(x, y, u)du$ if $\operatorname{Re}\beta < 1$, $\operatorname{Re}(\gamma-\beta-\beta') < 1$, (18.8-9) $\int_{1/y}^{\infty} U(x,y,u)du$ if $\mathrm{Re}\,\beta' < 1$, $\mathrm{Re}(\gamma-\beta-\beta') < 1$,

(18.8-10) $\int_{1/x}^{1/y} U(x,y,u)du$ if $\mathrm{Re}\,\beta < 1$, $\mathrm{Re}\,\beta' < 1$.

In particular, if

(18.9) $0 < \mathrm{Re}\,\alpha < \mathrm{Re}\,\gamma < \mathrm{Re}(\beta+\beta'+1) < 2$, $0 < \mathrm{Re}\,\beta$, $0 < \mathrm{Re}\,\beta'$,

then these ten integrals are solutions of (12.1) at the same time.

In order to fix paths of integrations and branches of $U(x,y,u)$ as functions of u, let us suppose that

$$\mathrm{Im}\,x > 0, \quad \mathrm{Im}\,y > 0 ,$$

or more precisely

(18.10) $0 < \arg x < \pi$, $0 < \arg y < \pi$.

Then $1/x$ and $1/y$ are in the lower half-plane. We assume that

(18.11) $-\pi < \arg(1/x) < 0$, $-\pi < \arg(1/y) < 0$.

On the other hand, $(1-1/x)$ and $(1-1/y)$ are in the upper half-plane. We assume that

(18.12) $0 < \arg(1-1/x) < \pi$, $0 < \arg(1-1/y) < \pi$.

Further assume that

(18.13) $-\pi < \arg(1/x) < \arg(1/y) < 0$

and

(18.14) $0 < \arg(1-1/x) < \arg(1-1/y) < \pi$.

Then $1/y$ lies in the domain bounded by three lines:

(a) $u = s$ $(1 \leqq s \leqq +\infty)$,

(b) $u = (1-s) + s/x$ $(0 \leqq s \leqq 1)$,

and

and

(c) $u = s\exp[i\arg(1/x)]$ $(1 \leqq s \leqq +\infty)$.

(See Fig. 18.1.)

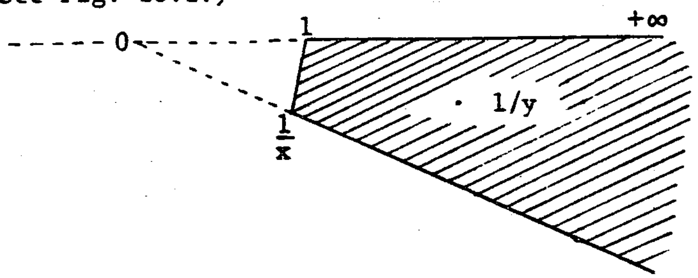


Fig. 18.1

This relation of the positions of $1/x$ and $1/y$ implies that y is in the sector:

$$0 < \arg y < \arg x$$

and that y is within the circle passing through three points 0, 1 and x. (See Fig. 18.2.) We shall fix the paths of integrations as indicated by Fig. 18.3.

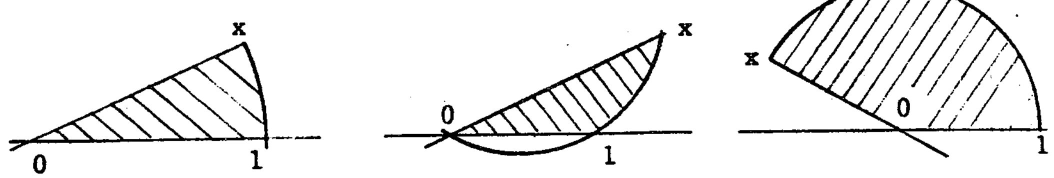


Fig. 18.2

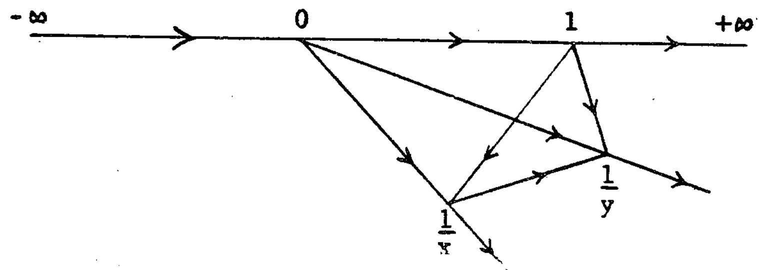


Fig. 18.3

We shall determine branches of $U(x, y, u)$ as indicated by Table 18.1. Note that it is sufficient to determine

$$\arg u, \quad \arg(1-u), \quad \arg(1-xu), \quad \arg(1-yu)$$

on each path of integration. As it is shown by (10.4), we have

$$\int_0^1 U(x, y, u)du = \text{const. } F_1(\alpha, \beta, \beta', \gamma, x, y) .$$

Other integrals of (18.8) are also expressed by means of F_1. To prove this, it is sufficient to make the change of variable u as indicated also by Table 18.1. For example,

$$\int_\infty^0 U(x, y, u)du = e^{-\pi i(\alpha-1)} \int_0^1 v^{\alpha-1}(1-v)^{\beta+\beta'-\gamma}(1-(1-x)v)^{-\beta}(1-(1-y)v)^{-\beta'}dv$$

$$= e^{-\pi i(\alpha-1)} \frac{\Gamma(\alpha)\Gamma(\beta+\beta'-\gamma+1)}{\Gamma(\alpha+\beta+\beta'-\gamma+1)} F_1(\alpha, \beta, \beta', \alpha+\beta+\beta'-\gamma+1, 1-x, 1-y) .$$

As the first integral (18.8-1) has six expressions, (Theorem 10.3, p.65), each integral has also six expressions. Therefore, altogether, we have sixty expressions of solutions of (12.1).

<u>Remark</u> 1: The integral (18.3) is transformed by the change of variable

$$u = 1/v$$

Table 18.1

integral	arg u	arg(1-u)	arg(1-xu)	arg(1-yu)	transformation
$\int_0^1$	0	0	$[-\pi,0]$	$[-\pi,0]$	
$\int_\infty^0$	π	0	$[0,\pi]$	$[0,\pi]$	$u=v/(v-1)$
$\int_1^\infty$	0	$-\pi$	$[-\pi,0]$	$[-\pi,0]$	$u=1/v$
$\int_0^{1/x}$	$[-\pi,0]$	$[0,\pi]$	0	$[0,\pi]$	$u=v/x$
$\int_0^{1/y}$	$[-\pi,0]$	$[0,\pi]$	$[-\pi,0]$	0	$u=v/y$
$\int_1^{1/x}$	$[-\pi,0]$	$[0,\pi]$	$[-\pi,0]$	$[0,\pi]$	$u=1/[x+(1-x)v]$
$\int_1^{1/y}$	$[-\pi,0]$	$[0,\pi]$	$[-\pi,0]$	$[-\pi,0]$	$u=1/[y+(1-y)v]$
$\int_{1/x}^\infty$	$[-\pi,0]$	$[0,\pi]$	$-\pi$	$[0,\pi]$	$u=1/(xv)$
$\int_{1/y}^\infty$	$[-\pi,0]$	$[0,\pi]$	$[-\pi,0]$	$-\pi$	$u=1/(yv)$
$\int_{1/x}^{1/y}$	$[-\pi,\pi]$	$[0,\pi]$	$[-\pi,0]$	$[0,\pi]$	$u=1/[y+(x-y)v]$

Table 18.2

integral	arg v	arg(v-1)	arg(v-x)	arg(v-y)	restrictions
$\int_0^1$	0	$-\pi$	$[-\pi,0]$	$[-\pi,0]$	$\mathrm{Re}(\beta+\beta'-\gamma) > -1$ $\mathrm{Re}(\gamma-\alpha) > 0$
$\int_\infty^0$	$-\pi$	$-\pi$	$[-\pi,0]$	$[-\pi,0]$	$\mathrm{Re}(\beta+\beta'-\gamma) > -1$ $\mathrm{Re}\ \alpha > 0$
$\int_1^\infty$	0	0	$[-\pi,0]$	$[-\pi,0]$	$\mathrm{Re}(\gamma-\alpha) > 0$ $\mathrm{Re}\ \alpha > 0$
$\int_0^x$	$[0,\pi]$	$[0,\pi]$	$[-\pi,0]$	$[0,2\pi]$	$\mathrm{Re}(\beta+\beta'-\gamma) > -1$ $\mathrm{Re}\ \beta < 1$
$\int_0^y$	$[0,\pi]$	$[0,\pi]$	$[-\pi,\pi]$	$[-\pi,0]$	$\mathrm{Re}(\beta+\beta'-\gamma) > -1$ $\mathrm{Re}\ \beta' < 1$
$\int_1^x$	$[0,\pi]$	$[0,\pi]$	$[-\pi,0]$	$[0,2\pi]$	$\mathrm{Re}(\gamma-\alpha) > 0$ $\mathrm{Re}\ \beta < 1$
$\int_1^y$	$[0,\pi]$	$[0,\pi]$	$[-\pi,\pi]$	$[-\pi,0]$	$\mathrm{Re}(\gamma-\alpha) > 0$ $\mathrm{Re}\ \beta' < 1$
$\int_x^\infty$	$[0,\pi]$	$[0,\pi]$	$[0,\pi]$	$[0,2\pi]$	$\mathrm{Re}\ \beta < 1$ $\mathrm{Re}\ \alpha > 0$
$\int_y^\infty$	$[0,\pi]$	$[0,\pi]$	$[-\pi,\pi]$	$[-\pi,0]$	$\mathrm{Re}\ \beta' < 1$ $\mathrm{Re}\ \alpha > 0$
$\int_x^y$	$[-\pi,\pi]$	$[0,\pi]$	$[0,2\pi]$	$[0,2\pi]$	$\mathrm{Re}\ \beta < 1$ $\mathrm{Re}\ \beta' < 1$

into

$$\int_c^d V(x, y, v)dv \ ,$$

where

$$V(x, y, v) = v^{\beta+\beta'-\gamma}(v-1)^{\gamma-\alpha-1}(v-x)^{-\beta}(v-y)^{-\beta'} \ ,$$

and

$$c = 1/b \ , \quad d = 1/a \ .$$

Thus we derive ten integrals of $V(x, y, v)$ from (18.8-1)$\sim$(18.8-10). These ten integrals are solutions of (F_1).

THEOREM 18.2: The integrals

$$\int_c^d v^{\beta+\beta'-\gamma}(v-1)^{\gamma-\alpha-1}(v-x)^{-\beta}(v-y)^{-\beta'}dv \tag{18.15}$$

are solutions of the system (12.2) under the respective restrictions given in Table 18.2, where c and d are any two points of $0, 1, x, y$ and ∞. The paths of integration are taken as indicated by Fig. 18.4 and branches of integrand are determined as indicated by Table 18.2.

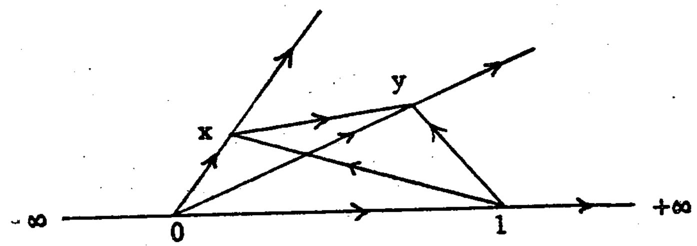


Fig. 18.4

In Fig. 18.4 and Table 18.2, we assumed that

$$\begin{cases} 0 < \arg y < \arg x < \pi \ , \\ 0 < \arg(y-1) < \arg(x-1) < \pi \ . \end{cases}$$

Remark 2. In order to guarantee the convergence of the integrals (18.3) and (18.15) we must assume certain conditions on the parameters α, β, β' and γ. There are two ways of weakening these conditions. One of them is to take closed paths of integration. The other is to introduce new idea, called a finite part of divergent integrals. The first method is as follows: For example, we replace the integrals

$$\int_0^1 U(x, y, u)du \quad \text{and} \quad \int_0^1 V(x, y, v)dv$$

respectively by the integrals

$$\int_C U(x, y, u)du\ , \qquad \int_C V(x, y, v)dv$$

along a closed curve C which is given by Fig. 18.5.

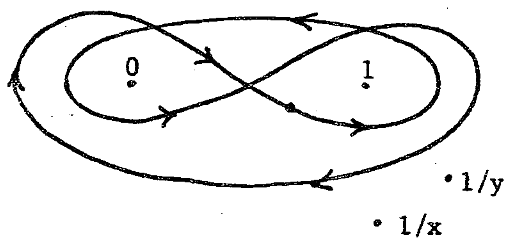


Fig. 18.5

Then

$$\int_C U(x,y,u)du = (1-e^{2\pi i(\gamma-\alpha)})(1-e^{2\pi i\alpha})\int_0^1 U(x,y,u)du$$

$$= (1-e^{2\pi i(\gamma-\alpha)})(1-e^{2\pi i\alpha})\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} F_1(\alpha,\beta,\beta',\gamma,x,y)$$

if none of α, $\gamma-\alpha$ and γ is an integer.

The second method, i.e. the method of the finite part of

divergent integrals is essentially based on the concept of analytic continuation. For example

$$\int_0^1 U(x, y, u)du$$

is originally defined for $0 < \mathrm{Re}\,\alpha < \mathrm{Re}\,\gamma$. However, for fixed (x, y), this integral can be analytically continuable with respect to the parameters α, β, β' and γ. Thus we can define this integral in a much larger domain. If we apply this idea to the integral

$$\frac{1}{\Gamma(\lambda)}\int_0^x (x-\xi)^{\lambda-1}\varphi(\xi)\,d\xi\ ,$$

then this integral becomes meaningful for every value of λ if $\varphi(x)$ is a C^∞-class function. We can regard

$$T : \varphi \rightarrow \frac{1}{\Gamma(\lambda)}\int_0^x (x-\xi)^{\lambda-1}\varphi(\xi)\,d\xi$$

as a linear functional. This functional is a distribution in the sense of L. Schwartz. In particular:

$$T\varphi = \begin{cases} \int_0^x \cdots \int_0^{\xi_1} \varphi(\xi)d\xi\, d\xi_1 \cdots d\xi_{n-1} & (\lambda = n), \\ \delta_x\,\varphi = \varphi(x) & (\lambda = 0), \\ \varphi^{(n)}(x)\ , & (\lambda = -n), \end{cases}$$

where n is a positive integer and δ_x is the Dirac-distribution.

19. Connection formula and monodromy representations for the system (12.1) satisfied by F_1. In this section, we shall briefly explain connection formulas among solutions of the system (12.1) and monodromy representations for the system (12.1).

To begin with, let us consider the ten integrals (18.15). We suppose that these ten integrals are well defined at the same time. This is possible under a suitable assumption on the parameters α, β, β' and γ. The paths of integration for these integrals are shown by Fig. 18.4. (Cf. Theorem 18.2, p.133.) We shall use the same idea as in Section 16 (Chapter III) to find connection formulas among solutions of (12.1). The basic idea is

(i) to take a closed curve consisting of parts of these paths of integration and small circular arcs and large circular arcs so that Cauchy's integral theorem can be applied, and then

(ii) to let the radius of small circular arcs tend to zero and to let the radius of large circular arcs tend to infinity.

By using various closed curves of this kind, we can find more than thirty relations among the ten integrals (18.15). For example, if we use the closed curve given by Fig. 19.1, we get

$$\int_{\infty}^{0} V\,dv + \int_{0}^{1} V\,dv + \int_{1}^{\infty} V\,dv = 0 .$$

It is known that the system (12.1) has only three linearly independent solutions. (Cf. Theorem 14.1, p.91.) This means that there are only seven independent relations. As these relations

show, any three of the ten integrals (18.15) are not necessarily linearly independent. It can be shown that three integrals

$$\int_{0}^{x} V\,dv , \quad \int_{0}^{y} V\,dv \quad \text{and} \quad \int_{0}^{1} V\,dv$$

are linearly independent.

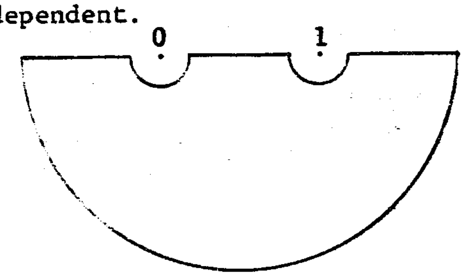


Fig. 19.1

We shall now proceed to the monodromy for the system (12.1). In Section 14, it was explained that the system (12.1) has the singular set which is the union of the five lines:

(19.1) $\{x=0\} \cup \{x=1\} \cup \{y=0\} \cup \{y=1\} \cup \{y=x\}$

and that any solution of the system (12.1) is regular analytic in the domain

(19.2) $\mathcal{D} = \mathbb{C}\times\mathbb{C} - \{x=0\}\cup\{x=1\}\cup\{y=0\}\cup\{y=1\}\cup\{y=x\}.$

(Cf. Section 14, p.94.) In the same way as we defined the fundamental group $\pi_1(D, x_0)$ of $D = \mathbb{C} - \{0, 1\}$ with the base point x_0 in Section 6 (Chapter I), we can define the fundamental group $\pi_1(\mathcal{D}, (x_0, y_0))$ of $\mathcal{D}$ with a base point (x_0, y_0). Then taking a fundamental system of solutions of (12.1), we can define the monodromy representation with respect to this fundamental system in the same way as we defined monodromy representations of the Gauss differential equation (6.1). (Cf. Section 6, Chapter I.) The monodromy representation thus defined is a homomorphism of $\pi_1(\mathcal{D}, (x_0, y_0))$ into $GL(3, \mathbb{C})$.

It is known that $\pi_1(D, x_0)$ is a free group generated by two elements. (See Section 17, Chapter III.) However, $\pi_1(\mathcal{D}, (x_0, y_0))$ is more complicated. It is true that $\pi_1(\mathcal{D}, (x_0, y_0))$ is generated by five elements. To find these five elements, consider a complex line in $\mathbb{C} \times \mathbb{C}$ which passes through the base point (x_0, y_0) and intersects with the five singular lines (19.1). This means that this line is not parallel to any of these five singular lines (19.1). Furthermore assume that this line does not go through four points $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$. These four points are intersection-points between the singular lines (19.1). This complex line can be identified with the complex plane $\mathbb{C}$. Then the intersection-points with five singular lines (19.1) are represented by five points A_1, A_2, A_3, A_4 and A_5 on this complex plane. The base point (x_0, y_0) is also represented by a point B on this complex plane. The intersection of $\mathcal{D}$ and this complex line is identified with $\mathbb{C} - \{A_1, \cdots, A_5\}$. The fundamental group $\pi_1(\mathbb{C} - \{A_1, \cdots, A_5\}, B)$ of $\mathbb{C} - \{A_1, \cdots, A_5\}$ is a free group generated by five elements. Those generators are represented by five loops surrounding $A_1, \cdots, A_5$ respectively. Let us denote these loops by $\ell_1, \cdots, \ell_5$. (See Fig. 19.2.) It can be proved

that $\pi_1(\mathcal{D}, (x_0, y_0))$ is generated by five elements corresponding to these five loops.

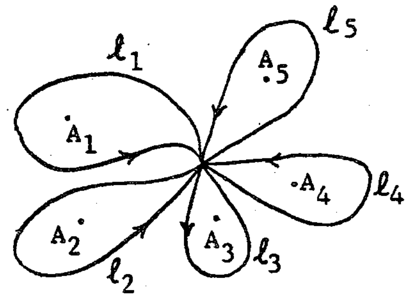
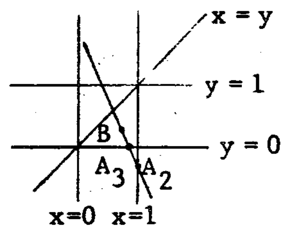


Fig. 19.2

Although $\pi_1(\mathbb{C} - \{A_1, \cdots, A_5\}, B)$ is a free group, $\pi_1(\mathcal{D}, (x_0, y_0))$ is not a free group. In other words, there are relations among the five generators. To see this, let us consider ℓ_2 and ℓ_3. Note that the complex line intersects with the singular lines $x = 1$ and $y = 0$ at A_2 and A_3 respectively. Suppose that the base point (x_0, y_0) is in a neighborhood of $(1, 0)$. The point $(1, 0)$ is the intersection-point of the two singular lines $x = 1$ and $y = 0$. More precisely speaking, suppose that (x_0, y_0) is in the domain

(19.3) $\quad 0 < |1 - x| < \delta, \quad 0 < |y| < \delta,$



where δ is a sufficiently small positive number. The domain (19.3) is homeomorphic to $\mathbb{C} \times \mathbb{C} - \{x = 1\} \cup \{y = 0\} = (\mathbb{C} - \{1\}) \times (\mathbb{C} - \{0\})$. The loop ℓ_2 can be deformed into a loop lying in the line $y = y_0$, and this deformed loop can be considered as a circle s_2 defined by

$$s_2 : \ x = 1 + (x_0 - 1)e^{i\theta} , \quad y = y_0 .$$

The loop ℓ_3 can be deformed into a loop lying in the line $x = x_0$, and this deformed loop can be considered as a circle s_3 defined by

$$s_3 : \ x = x_0 , \quad y = y_0 e^{i\theta} .$$

The product of s_2 and s_3 is a torus contained in $\mathcal{D}$, and s_2 and s_3 are two circles on this torus. It is well known that s_2 and s_3 are commutative on the torus. In other words, $s_2 s_3$ is homotopic to $s_3 s_2$ on the torus. Therefore, $s_2 s_3$ is homotopic to $s_3 s_2$ in $\mathcal{D}$. This means that $\ell_2 \ell_3$ is homotopic to $\ell_3 \ell_2$ in $\mathcal{D}$. Thus we conclude that $[\ell_2]$ and $[\ell_3]$ are commutative in the group $\pi_1(\mathcal{D}, (x_0, y_0))$. This shows that this group is not a free group. Similarly, $[\ell_1]$ and $[\ell_4]$ are commutative in $\pi_1(\mathcal{D}, (x_0, y_0))$. There are other relations. For example, there are relations among ℓ_1, ℓ_3 and ℓ_5. These relations arise from the fact that three singular lines $x = 0$, $y = 0$ and $x = y$ intersect at one point $(0, 0)$.

In order to compute the monodromy representation with respect to the fundamental system

$$\int_0^x V(x, y, v)dv , \qquad \int_0^y V(x, y, v)dv , \qquad \int_0^1 V(x, y, v)dv$$

of the system (12.1), let us first consider three loops ℓ_1, ℓ_2 and ℓ_5. The line $y = y_0$ intersects with three singular lines

$x = 0$, $x = 1$ and $y = x$ at $(0, y_0)$, $(1, y_0)$ and (y_0, y_0) respectively, but this line does not intersect with the other two singular lines. The base point (x_0, y_0) is on this line. Now we shall deform three loops ℓ_1, ℓ_2 and ℓ_5 into three loops ℓ'_1, ℓ'_2 and ℓ'_5 lying in the line $y = y_0$. If we identify the line $y = y_0$ with the complex plane, the loop ℓ'_1 can be considered as a loop starting and terminating at x_0 and surrounding the origin $x = 0$ in the positive sense. The loop ℓ'_2 can be considered as a loop starting and terminating at x_0 and surrounding the point $x = 1$ in the positive sense. The loop ℓ'_5 is a loop starting and terminating at x_0 and surrounding the point $x = y_0$ in the positive sense. (See Fig. 19.3.)

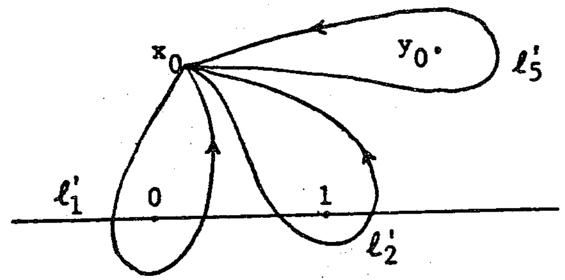


Fig. 19.3

If we continue

$$\int_0^x V(x, y, v)dv = \int_0^x v^{\beta+\beta'-\gamma}(v-1)^{\gamma-\alpha-1}(v-x)^{-\beta}(v-y)^{-\beta'}dv$$

analytically along ℓ'_1, then this integral becomes

$$e^{2\pi i(\beta'-\gamma)} \int_0^x V(x, y, v)dv .$$

This is due to the fact that arg v and arg(v-x) change to arg $v+2\pi i$ and $\arg(v-x)+2\pi i$, but other arguments do not change after the analytic continuation along ℓ_1'. Let us next continue the integral

$$\int_0^y V(x, y, v)dv \tag{19.4}$$

along the loop ℓ_1'. We must deform the path of integration so that x does not go across the path. Hence at the end of analytic continuation the path of integration becomes a path as shown by Fig. 19.4.

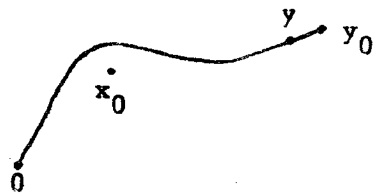


Fig. 19.4

Note that the path shown by 19.4 can be further deformed to a path as shown by Fig. 19.5.

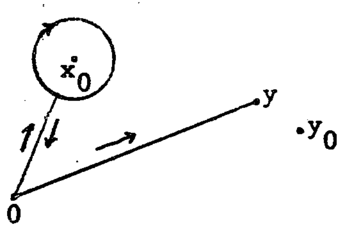


Fig. 19.5

Taking the change of arg(v-x) into consideration, we conclude that the integral (19.4) becomes

$$(e^{-2\pi i\beta} - 1)\int_0^x V\,dv + \int_0^y V\,dv .$$

Similarly the integral

$$\int_0^1 V(x, y, v)dv$$

becomes

$$(e^{-2\pi i\beta} - 1)\int_0^x V\,dv + \int_0^1 V\,dv$$

after the analytic continuation along ℓ_1'. Therefore the circuit matrix along ℓ_1' is

$$\begin{pmatrix} e^{2\pi i(\beta'-\gamma)} & 0 & 0 \\ e^{-2\pi i\beta}-1 & 1 & 0 \\ e^{-2\pi i\beta}-1 & 0 & 1 \end{pmatrix} .$$

In the same way we get other four matrices corresponding to the other four generators of $\pi_1(\mathcal{D}, (x_0, y_0))$. By these matrices the monodromy representation with respect to the fundamental system

$$\int_0^x V\,dv , \quad \int_0^y V\,dv , \quad \int_0^1 V\,dv$$

is completely determined.

20. General solutions of the systems (12.2), (12.3) and (12.4).
In this section, we shall derive general solutions of the systems of partial differential equations which are satisfied by F_2, F_3 and F_4 respectively. We shall first consider the system (12.2) which is satisfied by F_2:

$$(12.2)\quad \begin{cases} x(1-x)r - xys + [\gamma-(\alpha+\beta+1)x]p - \beta yq - \alpha\beta z = 0 , \\ y(1-y)t - xys + [\gamma'-(\alpha+\beta'+1)y]q - \beta' xp - \alpha\beta' z = 0 . \end{cases}$$

Let us make the change of variable

$$(20.1)\qquad z = x^{\lambda}y^{\mu}z' .$$

Then the system (12.2) is transformed into

$$(20.2)\quad \begin{cases} x(1-x)r'-xys'+[2\lambda+\gamma-(2\lambda+\mu+\alpha+\beta+1)x]p'-(\lambda+\beta)yq' \\ \qquad -[(\lambda+\mu+\alpha)(\lambda+\beta)-\lambda(\lambda+\gamma-1)x^{-1}]z' = 0 , \\ y(1-y)t'-xys'+[2\mu+\gamma'-(2\mu+\lambda+\alpha+\beta'+1)y]q'-(\mu+\beta')xp' \\ \qquad -[(\lambda+\mu+\alpha)(\mu+\beta')-\mu(\mu+\gamma'-1)y^{-1}]z' = 0 . \end{cases}$$

If we remove the two terms

$$\lambda(\lambda+\gamma-1)x^{-1} \quad \text{and} \quad \mu(\mu+\gamma'-1)y^{-1}$$

from (20.2), then the transformed system (20.2) has the same form as (12.2). For this purpose we shall take λ and μ so that

$$(20.3)\qquad \lambda(\lambda+\gamma-1) = 0 , \quad \mu(\mu+\gamma'-1) = 0.$$

The equations (20.3) yield the following four solutions:

$$(20.4)\quad \begin{cases} \lambda = 0 , \\ \mu = 0 , \end{cases} \begin{cases} \lambda = 1-\gamma , \\ \mu = 0 , \end{cases} \begin{cases} \lambda = 0 , \\ \mu = 1-\gamma' , \end{cases} \begin{cases} \lambda = 1-\gamma , \\ \mu = 1-\gamma' . \end{cases}$$

The first case of (20.4) corresponds to the identity transformation

$$z = z' .$$

The second case of (20.4) yields the transformation

$$(20.5)\qquad z = x^{1-\gamma}z'$$

and the transformed system (20.2) is obtained from (12.2) by replacing α, β, β', γ, γ' by $\alpha+1-\gamma$, $\beta+1-\gamma$, β', $2-\gamma$, γ' respectively. Therefore, this system admits a solution

$$(20.6)\qquad z' = F_2(\alpha+1-\gamma,\ \beta+1-\gamma,\ \beta',\ 2-\gamma,\ \gamma',\ x,\ y).$$

From this we obtain a solution of (12.2):

$$(20.7)\qquad z = x^{1-\gamma}F_2(\alpha+1-\gamma, \beta+1-\gamma, \beta', 2-\gamma, \gamma', x, y) .$$

Similarly, the third case of (20.4) yields a solution of (12.2):

$$(20.8)\qquad z = y^{1-\gamma'}F_2(\alpha+1-\gamma', \beta, \beta'+1-\gamma', \gamma, 2-\gamma', x, y)$$

and the four-th case of (20.4) yields a solution of (12.2):

$$(20.9)\qquad z = x^{1-\gamma}y^{1-\gamma'}F_2(\alpha+2-\gamma-\gamma', \beta+1-\gamma, \beta'+1-\gamma', 2-\gamma, 2-\gamma', x, y).$$

Thus we obtain the following theorem.

THEOREM 20.1: A general solution of the system (12.2) is given by

$$\begin{aligned}(20.10)\qquad z = {} & AF_2(\alpha, \beta, \beta', \gamma, \gamma', x, y) \\ & + Bx^{1-\gamma}F_2(\alpha+1-\gamma,\beta+1-\gamma,\beta',2-\gamma,\gamma',x,y) \\ & + Cy^{1-\gamma'}F_2(\alpha+1-\gamma',\beta,\beta'+1-\gamma',\gamma,2-\gamma',x,y) \\ & + Dx^{1-\gamma}y^{1-\gamma'}F_2(\alpha+2-\gamma-\gamma',\beta+1-\gamma,\beta'+1-\gamma',2-\gamma,2-\gamma',x,y),\end{aligned}$$

where A, B, C and D are arbitrary constants.

Let us consider the next the system (12.3) which is satisfied by F_3. We shall show that (12.3) is transformed into the system

(12.2) by the change of variables:

$$x = 1/\xi, \quad y = 1/\eta, \quad z = \xi^{\alpha}\eta^{\alpha'}z' . \tag{20.11}$$

In fact, by this transformation, the system

$$\begin{cases} x(1-x)r + ys + [\gamma-(\alpha+\beta+1)x]p - \alpha\beta z = 0 , \\ y(1-y)t + xs + [\gamma-(\alpha'+\beta'+1)y]q - \alpha'\beta'z = 0 \end{cases} \tag{12.3}$$

becomes

$$\begin{cases} \xi(1-\xi)\tilde{r} - \xi\eta\tilde{s} + [\alpha-\beta+1-(2\alpha+\alpha'+2-\gamma)\xi]\tilde{p} - \alpha\eta\tilde{q} - \alpha(\alpha+\alpha'-\gamma+1)z' = 0, \\ \eta(1-\eta)\tilde{t} - \xi\eta\tilde{s} + [\alpha'-\beta'+1-(2\alpha'+\alpha+2-\gamma)\eta]\tilde{q} - \alpha'\xi\tilde{p} - \alpha'(\alpha+\alpha'-\gamma+1)z' \\ \qquad = 0 , \end{cases} \tag{20.12}$$

where

$$\tilde{r} = \partial^2 z'/\partial\xi^2 , \quad \tilde{s} = \partial^2 z'/\partial\xi\partial\eta , \quad t = \partial^2 z'/\partial\eta^2 ,$$

$$\tilde{p} = \partial z'/\partial\xi , \quad \tilde{q} = \partial z'/\partial\eta .$$

The system (20.12) has the same form as (12.2) with parameters

$$\alpha+\alpha'-\gamma+1, \quad \alpha, \quad \alpha', \quad \alpha-\beta+1, \quad \alpha'-\beta'+1 .$$

Thus we can derive the following theorem from Theorem 20.1.

THEOREM 20.2: A general solution of the system (12.3) is given by

$$\begin{aligned} z = {} & Ax^{-\alpha}y^{-\alpha'}F_2(\alpha+\alpha'+1-\gamma, \alpha, \alpha', \alpha+1-\beta, \alpha'+1-\beta', 1/x, 1/y) \\ & + Bx^{-\beta}y^{-\alpha'}F_2(\beta+\alpha'+1-\gamma, \beta, \alpha', \beta+1-\alpha, \alpha'+1-\beta', 1/x, 1/y) \\ & + Cx^{-\alpha}y^{-\beta'}F_2(\alpha+\beta'+1-\gamma, \alpha, \beta', \alpha+1-\beta, \beta'+1-\alpha', 1/x, 1/y) \\ & + Dx^{-\beta}y^{-\beta'}F_2(\beta+\beta'+1-\gamma, \beta, \beta', \beta+1-\alpha, \beta'+1-\alpha', 1/x, 1/y) . \end{aligned} \tag{20.13}$$

In particular, for $z = F_3(\alpha, \alpha', \beta, \beta', \gamma, x, y)$, we determined the coefficients A, B, C and D by using the Barnes integral

representation in Section 11 (Chapter II). (See Theorem 11.3, p.74.)

Let us now consider the system (12.4) which is satisfied by F_4 :

$$\begin{cases} x(1-x)r - y^2t - 2xys + [\gamma-(\alpha+\beta+1)x]p - (\alpha+\beta+1)yq - \alpha\beta z = 0 , \\ y(1-y)t - x^2r - 2xys + [\gamma'-(\alpha+\beta+1)y]q - (\alpha+\beta+1)xp - \alpha\beta z = 0 . \end{cases} \tag{12.4}$$

This system is transformed into

$$\begin{cases} x(1-x)r' - y^2t' - 2xys' + [2\lambda+\gamma-(2\lambda+2\mu+\alpha+\beta+1)x]p' \\ -(2\lambda+2\mu+\alpha+\beta+1)yq' - [(\lambda+\mu+\alpha)(\lambda+\mu+\beta) - \lambda(\lambda+\gamma-1)x^{-1}]z' = 0 \\ y(1-y)t' - x^2r' - 2xys' + [2\mu+\gamma'-(2\lambda+2\mu+\alpha+\beta+1)y]q' \\ -(2\lambda+2\mu+\alpha+\beta+1)xp' - [(\lambda+\mu+\alpha)(\lambda+\mu+\beta) - \mu(\mu+\gamma'-1)y^{-1}]z' = 0 . \end{cases} \tag{20.14}$$

by the transformation

$$z = x^{\lambda}y^{\mu}z' . \tag{20.15}$$

As we did before, we shall choose λ and μ so that the terms

$$\lambda(\lambda+\gamma-1)x^{-1} \quad \text{and} \quad \mu(\mu+\gamma'-1)y'$$

may be removed from (20.14). Thus we determine λ and μ by

$$\lambda(\lambda+\gamma-1) = 0, \quad \mu(\mu+\gamma'-1) = 0 . \tag{20.16}$$

The equations (20.16) yield

$$\begin{cases} \lambda = 0 , \\ \mu = 0 , \end{cases} \quad \begin{cases} \lambda = 1-\gamma , \\ \mu = 0 , \end{cases} \quad \begin{cases} \lambda = 0 , \\ \mu = 1-\gamma' , \end{cases} \quad \begin{cases} \lambda = 1-\gamma , \\ \mu = 1-\gamma' . \end{cases}$$

Hence in the same way as we derived Theorem 20.1, we obtain the following theorem:

THEOREM 20.3: A general solution of the system (12.4) is given by

$$(20.17)\qquad z = AF_4(\alpha, \beta, \gamma, \gamma', x, y) + Bx^{1-\gamma}F_4(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, \gamma', x, y) + Cy^{1-\gamma'}F_4(\alpha-\gamma'+1, \beta-\gamma'+1, \gamma, 2-\gamma', x, y) + Dx^{1-\gamma}y^{1-\gamma'}F_4(\alpha-\gamma-\gamma'+2, \beta-\gamma-\gamma'+2, 2-\gamma, 2-\gamma', x, y) .$$

In the definition of the hypergeometric functions of two variables, we supposed that neither γ nor γ' is zero or a negative integer. Therefore, in deriving Theorems 20.1, 20.2 and 20.3, we must suppose corresponding conditions. For example, we must assume that

$$\gamma,\ \gamma' \neq \text{integer}$$

in Theorem 20.1.

21. Euler transform in double integral. In the previous section, we found four linearly independent solutions for each of the systems (12.2), (12.3), and (12.4). However, we would be unable to calculate the monodromy representations of those systems with respect to these fundamental systems of solutions. In Section 19, we explained how to use the simple integral representation of Euler for computing circuit matrices of the system (12.1) which is satisfied by F_1. A similar method based on the double integral representations of Euler may yield monodromy representations for the systems (12.2) and (12.3) which are satisfied respectively by F_2 and F_3, although, to the author's knowledge, nobody has ever tried to compute the monodromy representations for the system (12.2) and (12.3).

First of all, we must generalize Cauchy's integral theorem. Such a generalization was given by H. Poincaré as follows: Let $f(x, y)$ be a holomorphic function of x and y in a domain D. Let S $(\subset D)$ be a closed smooth surface of real dimension two. If there exists a set V of real dimension three such that

(i) $V \subset D$,

(ii) the boundary of V is S ,

then

$$\iint_S f(x, y)dx\,dy = 0 .$$

In Section 10 (Chapter II), we derived the double integral representations of Euler for F_1, F_2 and F_3. (See Theorem 10.1 on p.61.) We showed that the three integrals

(21.1) $$\iint_{u,v,1-u-v\geq 0} u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-xu-yv)^{-\alpha}\,du\,dv\ ,$$

(21.2) $$\int_0^1\int_0^1 u^{\beta-1}v^{\beta'-1}(1-u)^{\gamma-\beta-1}(1-v)^{\gamma'-\beta'-1}(1-xu-yv)^{-\alpha}\,du\,dv$$

and

(2.13) $$\iint_{u,v,1-u-v\geq 0} u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-xu)^{-\alpha}(1-yv)^{-\alpha'}\,du\,dv$$

are solutions of (12.1), (12.2) and (12.3) respectively, if these integrals are convergent. The method of the Euler transform which was explained in Chapter III suggests, as in Section 18, that we may replace the domains of integration

$$u,\ v,\ 1-u-v \geqq 0 \quad \text{and} \quad 0 \leqq u \leqq 1,\ \ 0 \leqq v \leqq 1$$

by other suitable domains of integrations so that the integrals thus obtained also satisfy the systems (12.1), (12.2) or (12.3). Actually we can find more than four solutions of (12.2) and (12.3) in this manner. However, any new solution of (12.1) can not be obtained by this method.

22. <u>Characterization of the systems of partial differential equations satisfied by</u> F_1, F_2, F_3 <u>and</u> F_4. Consider first a differential equation of the second order

(22.1) $$y'' + p(x)y' + q(x)y = 0\ .$$

Suppose that $x = a\ (\neq \infty)$ is a singular point of (22.1). This means that $x = a$ is a singular point of $p(x)$ or $q(x)$ or both. The point $x = a$ is called a regular singular point of (22.1) if $x = a$ is at most a simple pole of $p(x)$ and at most a double pole of $q(x)$. Suppose that $x = a$ is a regular singular point of (22.1). Then the equation (22.1) can be written in the form

(22.1') $$(x-a)^2y'' + (x-a)P(x)y' + Q(x)y = 0\ ,$$

where $P(x)$ and $Q(x)$ are holomorphic at $x = a$. As the general theory of linear differential equations guarantees, the differential equation (22.1') admits a solution of the form

$$y = (x-a)^{\rho}\varphi(x)\ ,$$

where ρ is a complex constant and $\varphi(x)$ is holomorphic at $x = a$ and $\varphi(a) \neq 0$. As it is easily checked, the quantity ρ is a root of the quadratic equation:

$$\rho(\rho-1) + P(a)\rho + Q(a) = 0\ .$$

Changing the letter, we call the equation

$$\lambda(\lambda-1) + P(a)\lambda + Q(a) = 0$$

the indicial equation of (22.1') at $x = a$. Two roots of the indicial equation are called the exponents of (22.1') at $x = a$.

Let ρ_1 and ρ_2 be the exponents of (22.1') at $x = a$. If $\rho_1 - \rho_2 \neq$ integer, then (22.1') admits two solutions

(22.2) $$(x-a)^{\rho_1}\varphi_1(x)\ , \quad (x-a)^{\rho_2}\varphi_2(x)\ ,$$

where φ_1 and φ_2 are holomorphic at $x = a$ and $\varphi_1(a) \neq 0$, $\varphi_2(a) \neq 0$. If $\rho_1 - \rho_2 =$ integer, we may suppose that

$$\rho_1 - \rho_2 = n\ ,$$

where n is a non-negative integer. Then (22.1') admits, in this case, two solutions

(22.3) $$(x-a)^{\rho_1}\varphi_1(x),\quad (x-a)^{\rho_2}\varphi_2(x) + \delta(x-a)^{\rho_1}\varphi_1(x)\log(x-a),$$

where φ_1 and φ_2 are holomorphic at $x = a$ and $\varphi_1(a) \neq 0$, $\varphi_2(a) \neq 0$ and δ is a constant which is either 0 or 1. In particular, if $\rho_1 - \rho_2 = 0$, then $\delta = 1$. However, if $\rho_1 - \rho_2 > 0$, the constant δ may be zero. A regular singular point $x = a$ is called logarithmic if $\delta = 1$. A system of solutions (22.2) or (22.3) forms a fundamental system of solutions of (22.1'). This fundamental system is called a canonical system of solutions of (22.1') at $x = a$.

Suppose next that $p(x)$ and $q(x)$ are holomorphic in a domain

$$r^{-1} < |x| < +\infty\ .$$

If we make the change of the independent variable:

$$x = 1/\xi\ ,$$

then the differential equation (22.1) becomes

(22.1") $$d^2y/d\xi^2 + \left(\frac{2}{\xi} - \frac{1}{\xi^2}\,p(1/\xi)\right)dy/d\xi + \frac{1}{\xi^4}\,q(1/\xi)y = 0\ .$$

We shall call the point at $x = \infty$ a regular singular point of (22.1) if $\xi = 0$ is a regular singular point of (22.1"). The indicial equation of (22.1") at $\xi = 0$ is called the indicial equation of (22.1) at $x = \infty$. The exponents of (22.1") at $\xi = 0$ are called the exponents of (22.1) at $x = \infty$. If $x = \infty$ is a regular singular point of (22.1), then $x = \infty$ is a zero of $p(x)$ of multiplicity at least one and a zero of $q(x)$ of multiplicity at least two. Therefore, the differential equation (22.1) can be written as

(22.1"') $$x^2y'' + xP(x)y' + Q(x)y = 0\ ,$$

where $P(x)$ and $Q(x)$ are holomorphic at $x = \infty$. The indicial equation of (22.1"') at $x = \infty$ is given by

$$\lambda(\lambda+1) - \lambda P(\infty) + Q(\infty) = 0\ .$$

The differential equation (22.1) is called an equation of Fuchsian type, if all singularities of (22.1) are regular singular points including the point at infinity. The Gauss differential equation:

(22.4) $$y'' + \left\{\frac{\gamma}{x} + \frac{\alpha+\beta-\gamma+1}{x-1}\right\}y' + \frac{\alpha\beta}{x(1-x)}\,y = 0$$

is an equation of Fuchsian type which has three regular singular points at $x = 0$, 1 and ∞. The indicial equations and the exponents at these three singular points are as follows:

singular point	indicial equation	exponents
0	$\lambda(\lambda-1)+\gamma\lambda = 0$	$0, \ 1-\gamma$
1	$\lambda(\lambda-1)+(\alpha+\beta-\gamma+1)\lambda = 0$	$0, \ \gamma-\alpha-\beta$
∞	$\lambda(\lambda+1)-(\alpha+\beta+1)\lambda+\alpha\beta = 0$	$\alpha, \ \beta$

If $\gamma \neq$ integer, then

$$F(\alpha, \beta, \gamma, x), \qquad x^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x)$$

form a canonical system of solutions of (22.4) at $x = 0$. If $\gamma-\alpha-\beta \neq$ integer, then

$$F(\alpha, \beta, \alpha+\beta+1-\gamma, 1-x), \quad (1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1, 1-x)$$

form a canonical system of solutions of (22.4) at $x = 1$. If $\alpha-\beta \neq$ integer, then

$$x^{-\alpha}F(\alpha, \alpha-\gamma+1, \alpha-\beta+1, 1/x), \quad x^{-\beta}F(\beta, \beta-\gamma+1, \beta-\alpha+1, 1/x)$$

form a canonical system of (22.4) at $x = \infty$.

A second order equation of Fuchsian type with three regular singular points at $x = a, b$ and ∞ can be written as

$$y'' + \frac{Ax+B}{(x-a)(x-b)}y' + \frac{Cx^2+Dx+E}{(x-a)^2(x-b)^2}y = 0, \tag{22.5}$$

where A, B, C, D and E are constants. The indicial equations of (22.5) at $x = a, b$ and ∞ are respectively given by

$$\begin{cases} \lambda(\lambda-1)+(Aa+B)(a-b)^{-1}\lambda+(Ca^2+Da+E)(a-b)^{-2} = 0, \\ \lambda(\lambda-1)+(Ab+B)(b-a)^{-1}\lambda+(Cb^2+Db+E)(a-b)^{-2} = 0, \\ \lambda(\lambda+1)-A\lambda+C = 0. \end{cases}$$

Let ρ_1, ρ_2 be the exponents at $x = a$, σ_1, σ_2 the exponents at $x = b$ and τ_1, τ_2 the exponents at $x = \infty$. Then the relations between roots and coefficients of a quadratic equation yield

$$\begin{cases} \rho_1+\rho_2 = 1-(Aa+B)(a-b)^{-1}, & \rho_1\rho_2 = (Ca^2+Da+E)(a-b)^{-2}, \\ \sigma_1+\sigma_2 = 1-(Ab+B)(b-a)^{-1}, & \sigma_1\sigma_2 = (Cb^2+Db+E)(b-a)^{-2}, \\ \tau_1+\tau_2 = -1+A, & \tau_1\tau_2 = C. \end{cases}$$

From these relations, we obtain

$$\rho_1+\rho_2+\sigma_1+\sigma_2+\tau_1+\tau_2 = 1. \tag{22.6}$$

This relation is called the Riemann or Fuchs relation.

Suppose that two points a and b ($a \neq b, a \neq \infty, b \neq \infty$) and six complex numbers $\rho_1, \rho_2, \sigma_1, \sigma_2, \tau_1$ and τ_2 satisfying (22.6) are given. Then a second order differential equation of Fuchsian type with three regular singular points at $x = a, b,$ and ∞ is uniquely determined in such a way that its exponents at $x = a, b,$ and ∞ are ρ_1, ρ_2 and σ_1, σ_2 and τ_1, τ_2 respectively. In fact, such a differential equation is given by

$$y'' + \left(\frac{1-\rho_1-\rho_2}{x-a}+\frac{1-\sigma_1-\sigma_2}{x-b}\right)y' + \left(\tau_1\tau_2+\frac{\rho_1\rho_2(a-b)}{x-a}-\frac{\sigma_1\sigma_2(a-b)}{x-b}\right)\frac{y}{(x-a)(x-b)} = 0. \tag{22.7}$$

The differential equation (22.7) is called Riemann's equation, and the set of all solutions of (22.7) are denoted by

$$(22.8) \qquad y = P\begin{Bmatrix} a & b & \infty & \\ \rho_1 & \sigma_1 & \tau_1 & x \\ \rho_2 & \sigma_2 & \tau_2 & \end{Bmatrix}.$$

The notation (22.8) is due to Riemann and it is called Riemann's P-function. The set of all solutions of the Gauss differential equation (22.4) is thus given by

$$(22.9) \qquad y = P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & x \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{Bmatrix}.$$

Now we shall explain the characterization of Riemann's differential equation due to Riemann. To do this, for simplicity, suppose that none of the singular points a, b and ∞ is logarithmic. Let $\mathfrak{F}$ be the set of all solutions of Riemann's differential equation (22.7). This means that $\mathfrak{F}$ denotes Riemann's P-function (22.8). Let D be the domain $D = S - \{a, b, \infty\}$, where S is the Riemann sphere. Then $\mathfrak{F}$ has the following properties:

(i) Every function in $\mathfrak{F}$ is analytic in D, but it may be multiple-valued.

(ii) For every point x_0 in D, there are two branches f_1 and f_2 of two functions or one in $\mathfrak{F}$ which are defined in a neighborhood of x_0 and which satisfy the condition

$$\begin{vmatrix} f_1(x_0) & f_2(x_0) \\ f_1'(x_0) & f_2'(x_0) \end{vmatrix} \neq 0 .$$

A function f defined in a neighborhood of x_0 is a branch of a function in $\mathfrak{F}$ if and only if f is a linear combination of f_1 and f_2.

(iii) In a neighborhood of a, there are two branches of functions in $\mathfrak{F}$ which are of the form

$$(x-a)^{\rho_1}\varphi_1(x), \qquad (x-a)^{\rho_2}\varphi_2(x),$$

where φ_1 and φ_2 are holomorphic at $x = a$ and $\varphi_1(a) \neq 0$, $\varphi_2(a) \neq 0$. In a neighborhood of $x = b$, there are two branches of functions in $\mathfrak{F}$ which are of the form

$$(x-a)^{\sigma_1}\psi_1(x), \qquad (x-b)^{\sigma_2}\psi_2(x),$$

where ψ_1 and ψ_2 are holomorphic at $x = b$ and $\psi_1(b) \neq 0$, $\psi_2(b) \neq 0$. In a neighborhood of $x = \infty$, there are two branches of functions in $\mathfrak{F}$ which are of the form

$$x^{-\tau_1}\chi_1(x), \qquad x^{-\tau_2}\chi_2(x),$$

where χ_1 and χ_2 are holomorphic at $x = \infty$ and $\chi_1(\infty) \neq 0$, $\chi_2(\infty) \neq 0$.

Riemann proved that the converse is true, i.e. Riemann's differential equation can be characterized by the properties (i), (ii) and (iii).

THEOREM 22.1: Suppose that a set of functions, $\mathfrak{F}$, satisfies the conditions (i), (ii) and (iii), where $\rho_1 + \rho_2 + \sigma_1 + \sigma_2 + \tau_1 + \tau_2 = 1$, $\rho_1 \neq \rho_2$, $\sigma_1 \neq \sigma_2$, $\tau_1 \neq \tau_2$. Then

$$\mathfrak{F} = P\begin{Bmatrix} a & b & \infty & \\ \rho_1 & \sigma_1 & \tau_1 & x \\ \rho_2 & \sigma_2 & \tau_2 & \end{Bmatrix}.$$

An outline of the proof is as follows: Let $y(x)$ be an arbitrary branch of a function in $\mathfrak{F}$ which is defined in a neighborhood of x_0. Then the condition (ii) implies that y, f_1 and f_2 are linearly dependent and hence

$$\begin{vmatrix} y(x) & f_1(x) & f_2(x) \\ y'(x) & f_1'(x) & f_2'(x) \\ y''(x) & f_1''(x) & f_2''(x) \end{vmatrix} = 0$$

in a neighborhood of x_0. This relation can be written as

$$y''(x) + p(x)y'(x) + q(x)y = 0$$

which is a linear differential equation. By using (ii), we can prove that $p(x)$ and $q(x)$ are holomorphic at x_0 and that they do not depend on the choice of branches f_1 and f_2. This means that $p(x)$ and $q(x)$ are uniquely determined by $\mathfrak{F}$. On the other hand, the condition (iii) assures that $x = a$, b and ∞ are regular singular points of this differential equation. This proves Theorem 22.1.

<u>Remark</u>. If some of a, b and ∞, say a, is logarithmic, then the first part of condition (iii) should be replaced by the following condition:

In a neighborhood of $x = a$ there are two branches of func-

tions in $\mathfrak{F}$ which are of the form:

$$(x-a)^{\rho_1}\varphi_1(x), \qquad (x-a)^{\rho_2}\varphi_2(x) + \delta(x-a)^{\rho_1}\varphi_1(x)\log(x-a),$$

where φ_1 and φ_2 are holomorphic at $x = a$ and $\varphi_1(a) \neq 0$, $\varphi_2(a) \neq 0$.

It was Picard who first generalized this result of Riemann to the system (12.1) which is satisfied by F_1. He considered a family of functions of x and y which satisfies three conditions corresponding to the three conditions given above, and he derived the system (12.1) from the family. More precisely speaking, let $\mathfrak{F}_1$ be a set of functions of two variables x and y which are defined in the domain:

$$\mathcal{D}_1 = S \times S - \{x=0\} \cup \{x=1\} \cup \{x=\infty\} \cup \{y=0\} \cup \{y=1\} \cup \{y=\infty\} \cup \{x=y\}.$$

As we explained before, the five lines $x = 0$, $x = 1$, $y = 0$, $y = 1$ and $x = y$ are singular lines of the system (12.1). In order to describe the behaviors of functions in $\mathfrak{F}_1$ as (x, y) tends to infinity, we must determine the set of points at infinity. We have remarked already that $\mathbb{C}\times\mathbb{C}$ has two natural compactifications, i.e. $\mathbb{P}^2$ and $S \times S$. Note that $S = \mathbb{P}^1$. (See Remark 1, Section 14, Chapter II, p.95.) Picard adopted $S \times S$ as a compactification of $\mathbb{C}\times\mathbb{C}$. Suppose that

(i) every function in $\mathfrak{F}_1$ is analytic in $\mathcal{D}_1$, but it may be multiple-valued;

(ii) for every point (x_0, y_0) in $\mathcal{D}_1$, there exist three branches

$f_1(x, y)$, $f_2(x, y)$ and $f_3(x, y)$ of functions in $\mathfrak{F}_1$ such that

$$\begin{vmatrix} f_1 & f_2 & f_3 \\ \partial f_1/\partial x & \partial f_2/\partial x & \partial f_3/\partial x \\ \partial f_1/\partial y & \partial f_2/\partial y & \partial f_3/\partial y \end{vmatrix} \neq 0 \quad \text{at} \quad (x, y) = (x_0, y_0)$$

and that a function $f(x, y)$ defined in a neighborhood of (x_0, y_0) is a branch of a function in $\mathfrak{F}_1$ if and only if f is a linear combination of f_1, f_2 and f_3 ;

(iii-1) for any point $(0, y_0)$, where $y_0 \neq 0, 1, \infty$, there are three branches of functions in $\mathfrak{F}_1$ which are of the form:

$$\varphi_1(x, y), \quad \varphi_2(x, y), \quad x^{\beta-\gamma+1}\varphi_3(x, y),$$

where φ_1, φ_2 and φ_3 are holomorphic at $(0, y_0)$;

(iii-2) for any point $(1, y_0)$, where $y_0 \neq 0, 1, \infty$, there are three branches of functions in $\mathfrak{F}_1$ which are of the form:

$$\psi_1(x, y), \quad \psi_2(x, y), \quad (x-1)^{\gamma-\alpha-\beta}\psi_3(x, y),$$

where ψ_1, ψ_2 and ψ_3 are holomorphic at $(1, y_0)$;

(iii-3) for any point (∞, y_0), where $y_0 \neq 0, 1, \infty$, there are three branches of functions in $\mathfrak{F}_1$ which are of the form:

$$x^{-\beta}\chi_1(x, y), \quad x^{-\beta}\chi_2(x, y), \quad x^{-\alpha}\chi_3(x, y),$$

where χ_1, χ_2, χ_3 are holomorphic at (∞, y_0);

(iii-4) for any point $(x_0, 0)$, where $x_0 \neq 0, 1, \infty$, there are three branches of functions in $\mathfrak{F}_1$ which are of the form:

$$\tilde{\varphi}_1(x, y), \quad \tilde{\varphi}_2(x, y), \quad y^{\beta'-\gamma+1}\tilde{\varphi}_3(x, y),$$

where $\tilde{\varphi}_1$, $\tilde{\varphi}_2$, $\tilde{\varphi}_3$ are holomorphic at $(x_0, 0)$;

(iii-5) for any point $(x_0, 1)$, where $x_0 \neq 0, 1, \infty$, there are three branches of functions in $\mathfrak{F}_1$ which are of the form:

$$\tilde{\psi}_1(x, y), \quad \tilde{\psi}_2(x, y), \quad (y-1)^{\gamma-\alpha-\beta'}\tilde{\psi}_3(x, y),$$

where $\tilde{\psi}_1$, $\tilde{\psi}_2$ and $\tilde{\psi}_3$ are holomorphic at $(x_0, 1)$;

(iii-6) for any point (x_0, ∞), where $x_0 \neq 0, 1, \infty$, there are three branches of functions in $\mathfrak{F}_1$ which are of the form:

$$y^{-\beta'}\tilde{\chi}_1(x, y), \quad y^{-\beta'}\tilde{\chi}_2(x, y), \quad y^{-\alpha}\tilde{\chi}_3(x, y),$$

where $\tilde{\chi}_1$, $\tilde{\chi}_2$ and $\tilde{\chi}_3$ are holomorphic at (x_0, ∞) ;

(iii-7) for any point (x_0, y_0), where $x_0 = y_0 \neq 0, 1, \infty$, there are three branches of functions in $\mathfrak{F}_1$ which are of the form:

$$\phi_1(x, y), \quad \phi_2(x, y), \quad (x-y)^{\gamma-\beta-\beta'}\phi_3(x, y),$$

where ϕ_1, ϕ_2 and ϕ_3 are holomorphic at (x_0, y_0).

Picard proved that, if $\mathfrak{F}_1$ satisfies (i), (ii) and (iii) and some additional hypotheses, then $\mathfrak{F}_1$ is the set of all solutions of the system (12.1). We do not know clearly what additional hypotheses must be required. Picard as well as Appell and Kampé de Fériet gave the hypotheses in a very vague form in their works. The principle of the proof is the same as in the case of Riemann's equations. Let $z(x, y)$ be an arbitrary branch of a function in $\mathfrak{F}_1$ which is defined in a neighborhood of a point $(x_0, y_0) \in \mathcal{D}_1$. Then the condition (ii) yields

$$\begin{vmatrix} z & z_1 & z_2 & z_3 \\ p & p_1 & p_2 & p_3 \\ q & q_1 & q_2 & q_3 \\ r & r_1 & r_2 & r_3 \end{vmatrix} = 0, \quad \begin{vmatrix} z & z_1 & z_2 & z_3 \\ p & p_1 & p_2 & p_3 \\ q & q_1 & q_2 & q_3 \\ s & s_1 & s_2 & s_3 \end{vmatrix} = 0,$$

$$\begin{vmatrix} z & z_1 & z_2 & z_3 \\ p & p_1 & p_2 & p_3 \\ q & q_1 & q_2 & q_3 \\ t & t_1 & t_2 & t_3 \end{vmatrix} = 0 ,$$

where

$p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$,
$t = \partial^2 z/\partial y^2$

and

$$p_j = \partial z_j/\partial x , \cdots .$$

These three relations can be written as

$$\begin{cases} r = \alpha_1(x, y)p + \alpha_2(x, y)q + \alpha_3(x, y)z , \\ s = \beta_1(x, y)p + \beta_2(x, y)q + \beta_3(x, y)z , \\ t = \gamma_1(x, y)p + \gamma_2(x, y)q + \gamma_3(x, y)z . \end{cases}$$

After very long calculations, Picard showed that this system coincides with (12.1).

PROBLEM 1. Complete the work of Picard.

Riemann's point of view can be applied to the systems (12.2), (12.3) and (12.4). Goursat applied the same principle to the systems (12.2) and (12.3). However, the author does not know whether or not the same principle has ever been applied to th system (12.4).

PROBLEM 2. Complete the work of Goursat.

PROBLEM 3. Derive the system for F_4 (i.e. the system (12.4)) by using Riemann's point of view.

CHAPTER V

Automorphic Functions, Reducibility and Generalizations

23. Automorphic functions. It is well known that the elliptic function was discovered by Gauss, although he did not publish his discovery. Then Abel and Jacobi rediscovered the elliptic function independently. The discovery of the elliptic function opened an epoch in the history of mathematics. Indeed, this discovery had influences not only on the analysis, but also upon all fields of mathematics. Gauss also found a new function which is related to the elliptic function, quadratic forms, and arithmetico-geometric mean. This new function is called the elliptic modular function. It was Dedekind who rediscovered the elliptic modular function. This is the earliest example of automorphic functions.

In 1972, Schwarz derived, from the Gauss differential function, automorphic functions other than the elliptic modular function. Then Fuchs tried to generalize Schwarz's results to more general linear differential equations of the second order. Stimulated by the work of Fuchs, Poincaré founded the general theory of automorphic functions and called a kind of automorphic functions the Fuchsian function.

We shall now explain how we can derive automorphic functions from a linear ordinary equation of the second order

(23.1) $$y'' + p(x)y' + q(x)y = 0 .$$

Suppose that (23.1) is a Fuchsian equation with regular singular points at $a_1, a_2, \cdots, a_n$. Set $D = S - \{a_1, \cdots, a_n\}$. Let $\varphi_1(x)$ and $\varphi_2(x)$ be linearly independent solutions of (23.1) and let

$$\rho : \pi_1(D, x_0) \longrightarrow GL(2, \mathbb{C})$$

be the monodromy representation of (23.1) with respect to the fundamental system of solutions

(23.2) $$\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} .$$

We denote by T the set of linear fractional transformations

$$t(z) = \frac{az+b}{cz+d} \quad (ad - bc \neq 0) ,$$

and define a homomorphism $\tau : GL(2, \mathbb{C}) \longrightarrow T$ by

$$\tau\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = t(\cdot) .$$

Then the composite map $\tau \circ \rho$ is a homomorphism of $\pi_1(D, x_0)$ into T. Let us denote by G the image of $\pi_1(D, x_0)$ under the map $\tau \circ \rho$:

$$G = \tau \circ \rho\,(\pi_1(D, x_0)) .$$

G is a subgroup of T. Let ℓ be a loop in D at x_0. If

$$\rho[\ell] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} ,$$

then fundamental system (23.2) becomes

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$$

by analytic continuation along ℓ .

Consider the ratio

$$f(x) = \varphi_1(x)/\varphi_2(x) .$$

Then $f(x)$ becomes

$$\frac{a\varphi_1(x) + b\varphi_2(x)}{c\varphi_1(x) + d\varphi_2(x)} = \frac{af(x)+b}{cf(x)+d}$$

by analytic continuation along ℓ . We let Δ denote the image of D by f. Suppose that Δ is a univalent domain in S. Then the inverse function $x = g(z)$ of $z = f(x)$ is a single-valued function defined in Δ . When x moves along the loop ℓ and comes back to the original point, the value z of $f(x)$ at the starting point becomes

$$\frac{az+b}{cz+d}$$

at the terminating point. This means that

$$g\left(\frac{az+b}{cz+d}\right) = g(z) .$$

We shall now give the definition of automorphic functions. Let G be a subgroup of T and Δ be a domain in S. Suppose that each linear fractional transformation of G maps Δ onto Δ . A meromorphic function $g(z)$ is called an automorphic function relative to G if for any $t \in G$ and any $z \in \Delta$ we have

$$g(t(z)) = g(z) .$$

Example 1: Let $\Delta = \mathbb{C}$ and $G = \{t_n;\ t_n(z) = z + 2n\pi i, n \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of all integers. Then

$$g(z) = e^z$$

is an automorphic function relative to G.

Example 2: Let $\Delta = \mathbb{C}$ and $G = \{t_{m,n}(z) = z + m\omega_1 + n\omega_2, m, n \in \mathbb{Z}\}$, where ω_1 and ω_2 are complex numbers such that

$$\mathrm{Im}[\omega_2/\omega_1] > 0 .$$

Then automorphic functions relative to G are elliptic functions with periods ω_1 and ω_2. A typical example is the Weierstrass elliptic function $\wp(z, \omega_1, \omega_2)$ given by

$$\wp(z, \omega_1, \omega_2) = \frac{1}{z^2} + \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \left\{ \frac{1}{(z-m\omega_1-n\omega_2)^2} - \frac{1}{(m\omega_1+n\omega_2)^2} \right\}.$$

Schwarz considered the Gauss differential equation under the assumption that all of the parameters α, β and γ are real numbers. He determined all cases that the image Δ of $D = \mathbb{C} - \{0, 1\}$ under the map

$$z = f(x) = \frac{\varphi_1(x)}{\varphi_2(x)}$$

is a univalent domain contained in S and hence the inverse function

$$x = g(z)$$

of f is single-valued in Δ. We shall summarize his results.

THEOREM 23.1: Assume that α, β and γ are all real. Then

(i) $g(z)$ is single-valued if and only if

$$\lambda = |1-\gamma| = 1/\ell , \quad \ell = 1, 2, \cdots, \infty,$$

$$\mu = |\gamma-\alpha-\beta| = 1/m , \quad m = 1, 2, \cdots, \infty,$$

$$\nu = |\alpha-\beta| = 1/n , \quad n = 1, 2, \cdots, \infty,$$

where if one of λ, μ and ν is one, the others are equal;

(ii) if

$$\lambda+\mu+\nu = 1/\ell + 1/m + 1/n > 1 ,$$

then $g(z)$ is a rational function;

(iii) if

$$\lambda+\mu+\nu = 1/\ell + 1/m + 1/n = 1 ,$$

then there exists a linear fractional transformation $t(z)$ in T such that $g(t(z))$ is either a simply periodic function or a doubly periodic function;

(iv) if

$$\lambda+\mu+\nu = 1/\ell + 1/m + 1/n < 1 ,$$

then there exists a linear fractional transformation $t(z)$ in T such that $g(t(z))$ is an automorphic function defined in $|z| < 1$.

In case (ii), (ℓ, m, n) is a permutation of $(1, k, k)$, $(2, 2, k)$, $k = 1, 2, \cdots$, $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$. In this case G is a finite group. In case (iii), (ℓ, m, n) is a permutation of $(1, \infty, \infty)$, $(2, 2, \infty)$, $(2, 4, 4)$, $(2, 3, 6)$ or $(3, 3, 3)$ and G is an infinite group isomorphic to a group of translations. In case (iv), G is an infinite group. In particular, if $\lambda = \mu = \nu = 0$, i.e. $\alpha = \beta = \frac{1}{2}$, $\gamma = 1$, the Gauss differential equation has the fundamental system of solutions

$$\varphi_1 = \int_0^{-\infty} \frac{du}{\sqrt{u(u-1)(u-x)}}, \quad \varphi_2 = \int_1^{\infty} \frac{du}{\sqrt{u(u-1)(u-x)}}$$

and Δ is the upper half-plane $\operatorname{Im} z > 0$ and G is the group generated by

$$t_1(z) = z+2, \quad t_2(z) = \frac{z}{2z+1}.$$

We remark that

$$\varphi_1 = 2\int_{1/\sqrt{x}}^{1} \frac{dv}{\sqrt{(1-v^2)(1-xv^2)}}, \quad \varphi_2 = \int_{-1}^{1} \frac{dv}{\sqrt{(1-v^2)(1-xv^2)}}.$$

It is known that $f(x) = \varphi_1(x)/\varphi_2(x)$ satisfy a differential equation of the third order. We shall find it. Let y_1 and y_2 be any fundamental system of solution of (23.1) and set

$$z = y_1(x)/y_2(x).$$

Since y_1 and y_2 are linear combinations of φ_1 and φ_2, we have

$$z = \frac{\alpha\varphi_1 + \beta\varphi_2}{\gamma\varphi_1 + \delta\varphi_2} = \frac{\alpha f + \beta}{\gamma f + \delta}.$$

It follows that z depends essentially on three parameters. First note that

$$y_1 y_2' - y_1' y_2 = C \exp\left\{-\int^x p(t)\,dt\right\},$$

where C is a constant. Observing that

$$\frac{z'}{z} = \frac{y_1'}{y_1} - \frac{y_2'}{y_2} = \frac{y_1' y_2 - y_1 y_2'}{y_1 y_2},$$

we have

$$z' = -C y_2^{-2} \exp\left\{-\int^x p(t)\,dt\right\},$$

from which

$$\frac{z''}{z} = -2\frac{y_2'}{y_2} - p.$$

Thus we have

$$(23.3) \qquad y_2' = -\tfrac{1}{2} y_2(x)\{p(x) + z''/z'\}.$$

Differentiating both sides, we get

$$\begin{aligned} y_2'' &= -\tfrac{1}{2} y_2'\{p(x) + z''/z'\} - \tfrac{1}{2} y_2\{p'(x) + [z''/z']'\} \\ &= \tfrac{1}{4} y_2\{p(x) + z''/z'\}^2 - \tfrac{1}{2} y_2\{p'(x) + z'''/z' - [z''/z']^2\} \end{aligned}$$

and hence

$$(23.4) \quad y_2'' = \tfrac{1}{4} y_2\{p(x)^2 - 2p'(x) - 2z'''/z' + 3[z''/z']^2 + 2p(x)z''/z'\}.$$

Since y_2 is a solution of (23.1), substituting (23.3) and (23.4) into (23.1), we get

$$\begin{aligned} &\tfrac{1}{4} y_2\{p(x)^2 - 2p'(x) - 2z'''/z' + 3[z''/z']^2 + 2p(x)z''/z'\} \\ &- \tfrac{1}{2} y_2\{p(x)^2 + p(x)z''/z'\} + q(x)y_2 = 0. \end{aligned}$$

Dividing both sides by $\frac{1}{2} y_2$, we obtain

$$(23.5) \qquad z'''/z' - \frac{3}{2}[z''/z']^2 = -2\left\{\frac{1}{4}p(x)^2 + \frac{1}{2}p'(x) - q(x)\right\}.$$

The left-hand side of (23.5), i.e.

$$(23.6) \qquad z'''/z' - \frac{3}{2}[z''/z']^2$$

is called the Schwarzian derivative of z and often denoted by $\{z, x\}$. The right-hand side of (23.4) or the quantity

$$I(x) = \tfrac{1}{4} p(x)^2 + \tfrac{1}{2} p'(x) - q(x)$$

is an invariant of (23.1) under a transformation of the form

$$(23.7) \qquad y = a(x)u.$$

In other words, if (23.1) is reduced to

$$u'' + p_1(x)u' + q_1(x)u = 0$$

by (23.7), we have

$$\tfrac{1}{4}p(x)^2+\tfrac{1}{2}p'(x)-q(x) = \tfrac{1}{4}p_1(x)^2+\tfrac{1}{2}p_1(x)-q_1(x) .$$

In particular, if we take $a(x) = \exp\left\{-\tfrac{1}{2}\int^x p(t)dt\right\}$, then (23.1) becomes

$$u''-I(x)u = 0 .$$

For Riemann's equation with scheme

$$\begin{matrix} a & b & \infty \\ \rho_1 & \sigma_1 & \tau_1 \\ \rho_2 & \sigma_2 & \tau_2 \end{matrix}$$

we get

$$\{z, x\} = \tfrac{1}{2}\left[(1-\lambda^2)\frac{a-b}{x-a}+(1-\mu^2)\frac{b-a}{x-b}+1-\nu^2\right]\frac{1}{(x-a)(x-b)} ,$$

where

$$\lambda^2 = (\rho_1-\rho_2)^2 , \quad \mu^2 = (\sigma_1-\sigma_2)^2 , \quad \nu^2 = (\tau_1-\tau_2)^2 .$$

In particular, for the Gauss equation, we have

$$\{z, x\} = \frac{1}{2}\left\{\frac{1-\lambda^2}{x^2}+\frac{1-\mu^2}{(x-1)^2}-\frac{1-\lambda^2-\mu^2+\nu^2}{x(x-1)}\right\} ,$$

where

$$\lambda^2 = (1-\gamma)^2 , \quad \mu^2 = (\gamma-\alpha-\beta)^2 , \quad \nu^2 = (\alpha-\beta)^2 .$$

About at the same time when the general theory of automorphic functions was founded by Poincaré, Picard discovered an example of automorphic function of two variables by utilizing the system (12.1) which is satisfied by F_1. As we explained before, the system (12.1) has three linearly independent solutions. Let us denote them by

$$z_1(x, y) , \quad z_2(x, y) , \quad z_3(x, y)$$

and let ρ_1 be the monodromy representation with respect to the fundamental system

(23.8) $$\begin{bmatrix} z_1(x, y) \\ z_2(x, y) \\ z_3(x, y) \end{bmatrix} .$$

The representation ρ_1 is a homomorphism of $\pi_1(\mathcal{D}, x_0)$ into $GL(3, \mathbb{C})$. For a loop ℓ at x_0, the homotopy class $[\ell]$ is an element of $\pi_1(\mathcal{D}_1, x_0)$. Now if

$$\rho_1([\ell]) = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} ,$$

then the fundamental system (23.8) becomes

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} z_1(x, y) \\ z_2(x, y) \\ z_3(x, y) \end{bmatrix}$$

by the analytic continuation along ℓ. Set

$$s = f(x, y) = \frac{z_1(x, y)}{z_3(x, y)} , \quad t = g(x, y) = \frac{z_2(x, y)}{z_3(x, y)} .$$

Then the functions $f(x, y)$ and $g(x, y)$ become

$$\frac{a_1z_1+b_1z_2+c_1z_3}{a_3z_1+b_3z_2+c_3z_3} = \frac{a_1f+b_1g+c_1}{a_3f+b_3g+c_3}$$

and

$$\frac{a_2z_1+b_2z_2+c_2z_3}{a_3z_1+b_3z_2+c_3z_3} = \frac{a_2f+b_2g+c_2}{a_3f+b_3g+c_3}$$

respectively by the analytic continuation along ℓ . Let

$$x = \varphi(s,t), \qquad y = \psi(s,t)$$

be the inverse of the transformation

$$s = f(x,y), \qquad t = g(x,y).$$

Suppose that φ and ψ are single-valued functions. Then an argument similar to that which we used before yields

$$\begin{cases} \varphi\left(\dfrac{a_1 s + b_1 t + c_1}{a_3 s + b_3 t + c_3}, \dfrac{a_2 s + b_2 t + c_2}{a_3 s + b_3 t + c_3}\right) = \varphi(s,t), \\ \psi\left(\dfrac{a_1 s + b_1 t + c_1}{a_3 s + b_3 t + c_3}, \dfrac{a_2 s + b_2 t + c_2}{a_3 s + b_3 t + c_3}\right) = \psi(s,t). \end{cases}$$

This means that φ and ψ are automorphic functions relative to a subgroup of linear fractional transformations which are derived from the monodromy representation ρ_1.

Picard first considered a special case

$$\alpha = \beta = \beta' = 1/3, \qquad \gamma = 1.$$

He showed that, in this case, we can choose z_1, z_2 and z_3 so that $\varphi(s,t)$ and $\psi(s,t)$ are automorphic functions defined in $|s|^2 + |t|^2 < 1$. In this special case, the system (12.1) has solutions

$$\int_0^1 \frac{du}{\sqrt[3]{u(u-1)(u-x)(u-y)}}, \qquad \int_x^1 \frac{du}{\sqrt[3]{u(u-1)(u-x)(u-y)}}, \quad \cdots.$$

We have already mentioned that the Gauss differential equation has similar integrals as solutions when $\alpha = \beta = \frac{1}{2}$, $\gamma = 1$. These integrals are integrals of simple algebraic functions of u

containing a parameter x or two parameters x, y. An integral of algebraic function is called an Abelian integral which is very important in the theory of algebraic functions and in the theory of algebraic geometry. The theory of automorphic functions has become a branch of mathematics related to number theory and algebraic geometry.

Finally we shall state an extension of the Schwarzian derivative to the case of two variables. Consider a completely integrable system of partial differential equations

$$(23.9) \qquad \begin{cases} r = \alpha_1 p + \alpha_2 q + \alpha_3 z \\ s = \beta_1 p + \beta_2 q + \beta_3 z \\ t = \gamma_1 p + \gamma_2 q + \gamma_3 z. \end{cases}$$

Let z_1, z_2 and z_3 be linearly independent solutions of (23.9) and set

$$u = \frac{z_1}{z_3}, \qquad v = \frac{z_2}{z_3}.$$

Then

$$\frac{\dfrac{\partial u}{\partial x}\dfrac{\partial^2 v}{\partial x^2} - \dfrac{\partial^2 u}{\partial x^2}\dfrac{\partial v}{\partial x}}{\Delta} = \alpha_2,$$

$$\frac{\dfrac{\partial^2 u}{\partial x^2}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial^2 v}{\partial x^2} + 2\left(\dfrac{\partial^2 u}{\partial x \partial y}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial^2 v}{\partial x \partial y}\right)}{\Delta} = \alpha_1 - 2\beta_2,$$

$$\frac{\dfrac{\partial u}{\partial x}\dfrac{\partial^2 v}{\partial y^2} - \dfrac{\partial^2 u}{\partial y^2}\dfrac{\partial v}{\partial x} + 2\left(\dfrac{\partial u}{\partial y}\dfrac{\partial^2 v}{\partial x \partial y} - \dfrac{\partial^2 u}{\partial x \partial y}\dfrac{\partial v}{\partial y}\right)}{\Delta} = \gamma_2 - 2\beta_1,$$

$$\frac{\dfrac{\partial^2 u}{\partial y^2}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial^2 v}{\partial y^2}}{\Delta} = \gamma_1 ,$$

where

$$\Delta = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} .$$

The four differential expressions on the left-hand sides are a generalization of the Schwarzian derivative to a map of $\mathbb{C}^2$ to $\mathbb{C}^2$: $(x, y) \rightsquigarrow (u, v)$.

24. Reducibility. We shall consider again a linear differential equation

(24.1) $$y'' + p(x)y' + q(x)y = 0 .$$

We shall write the equation (24.1) in a form

$$\left(\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)\right) y = 0 .$$

The expression

(24.2) $$\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$$

is a differential operator. It happens that the operator (24.2) is decomposed into a product of two operators:

(24.3) $$\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x) = \left(\frac{d}{dx} + s(x)\right)\left(\frac{d}{dx} + r(x)\right) .$$

The right-hand member of (24.3) means that

$$\left(\frac{d}{dx} + s(x)\right)\left(\frac{d}{dx} + r(x)\right) = \frac{d^2}{dx^2} + [r(x) + s(x)]\frac{d}{dx} + [r'(x) + s(x)r(x)] .$$

A polynomial is said to be reducible if it can be decomposed into a product of two polynomials of lower degrees. In general, the reducibility of polynomials depends on the field to which the coefficients of polynomials belong. Similarly, for differential operators, the reducibility depends on the coefficient-field. In other words, it is necessary to prescribe a field to which the coefficients p, q, r and s belong, if we discuss the decomposition (24.3) of the operator (24.2). We shall restrict ourselves to the set of all rational functions $\mathbb{C}(x)$. Namely, we shall suppose that coefficients of differential operators belong to $\mathbb{C}(x)$.

Remark that $\mathbb{C}(x)$ is not only closed under the usual addition, subtraction, multiplication and division, but also closed under the differentiation with respect to x. Such a set is called a differential field. A precise definition of a differential field is as follows:

A set K is called a differential field if

(i) K is a field in a usual sense;

(ii) there exists a map D from K into K such that, for any $a, b \in K$, we have

(a) $D(a+b) = D(a)+D(b)$,

(b) $D(ab) = D(a)b+aD(b)$.

The map is called a differentiation. An element c of K is called a constant if $D(c) = 0$.

Let us return to the operator (24.2). The equation (24.1) or the operator (24.2) is said to be reducible if the operator (24.2) admits a decomposition of the form (24.3) in $\mathbb{C}(x)$. Suppose that (24.2) admits a decomposition (24.3). Then any solution of

(24.4) $$y' + r(x)y = 0$$

is a solution of (24.1). This implies that there exists a non-

trivial solution of (24.1) which satisfies a first order linear differential equation (24.4). We shall show that the converse is also true. Let $\varphi(x)$ be a non-trivial solution of (24.1) which satisfies (24.4). This means that

(24.5) $$\varphi'' + p\varphi' + q\varphi = 0$$

and

(24.6) $$\varphi' + r\varphi = 0 .$$

Differentiating (24.6), we obtain

(24.7) $$\varphi'' + r\varphi' + r'\varphi = 0 .$$

Subtracting (24.7) from (24.5), we have

(24.8) $$(p-r)\varphi' + (q-r')\varphi = 0 .$$

Eliminating φ and φ' from (24.6) and (24.8), we get

$$\begin{vmatrix} 1 & r \\ p-r & q-r' \end{vmatrix} = q - r' - (p-r)r = 0 .$$

Set

$$s = p - r$$

to derive

$$r+s = p ,$$
$$r'+rs = q .$$

This means that (24.2) admits the decomposition (24.3).

THEOREM 24.1: The operator (24.2) is reducible if and only if there exists a non-trivial solution of (24.1) which satisfies a first order linear differential equation (24.4).

Consider the Gauss differential equation

(24.9) $x(1-x)y'' + [\gamma-(\alpha+\beta+1)x]y' - \alpha\beta y = 0$.

Suppose that (24.9) is reducible. Then there exists a non-trivial solution of (24.9) which satisfies a linear differential equation (24.4). Such a solution is given by

(24.10) $$y = \varphi(x) = \text{const.}\ \exp[-\int^x r(t)\,dt] .$$

Since $r(x)$ is a rational function in x, we have

$$-\int^x r(t)dt = r_1(x) + \sum \mu_i \log(x-a_i) ,$$

where μ_i are non-zero constants. Hence

$$\varphi(x) = \text{const.}\ e^{r_1(x)} \prod (x-a_i)^{\mu_i} .$$

From the fact that (24.9) is an equation of Fuchsian type, it follows that $r_1(x)$ must be a constant. Therefore, we can write $\varphi(x)$ in the form

(24.10') $$y = \varphi(x) = \text{const.}\ \prod (x-a_i)^{\mu_i} .$$

On the other hand, the equation (24.9) has its singular points at $x = 0$, $x = 1$ and $x = \infty$. The exponents at $x = 0$ are 0 and $1-\gamma$, while the exponents at $x = 1$ are 0 and $\gamma-\alpha-\beta$. Therefore, if $a_i \neq 0$ and $\neq 1$, then $\mu_i = 1$. This implies that $\varphi(x)$ is expressed as one of the following forms:

(24.11) $$\varphi(x) = \begin{cases} P(x) \\ x^{1-\gamma}P(x) , \\ (x-1)^{\gamma-\alpha-\beta}P(x) , \\ x^{1-\gamma}(x-1)^{\gamma-\alpha-\beta}P(x) , \end{cases}$$

while $P(x)$ is a polynomial in x. Let P be of degree $n \geq 0$. Then $\varphi(x)$ is rewritten in one of the following forms:

$$\varphi(x) = \begin{cases} x^n \phi(x) , \\ x^{1-\gamma+n} \phi(x) , \\ x^{\gamma-\alpha-\beta+n} \phi(x) , \\ x^{1-\alpha-\beta+n} \phi(x) , \end{cases}$$

where ϕ is holomorphic at $x = \infty$ and $\phi(\infty) \neq 0$. Since the exponents of (24.9) at $x = \infty$ are α and β , we get

(24.12) α or $\beta = -n$ or $\gamma-1-n$ or $\alpha+\beta-\gamma-n$ or $\alpha+\beta-1-n$.

From (24.12) we conclude that one of four quantities

$$\alpha ,\ \beta ,\ \gamma-\alpha \text{ and } \gamma-\beta$$

must be an integer. We can also prove that the converse is true, by using one of the transformations:

$$y = x^{1-\gamma}z ,$$
$$y = (x-1)^{\gamma-\alpha-\beta}z$$

and

$$y = x^{1-\gamma}(x-1)^{\gamma-\alpha-\beta}z .$$

THEOREM 24.2: The Gauss differential equation (24.9) is reducible if and only if one of the four quantities α, β, $\gamma-\alpha$ and $\gamma-\beta$ is an integer.

Suppose again that (24.9) is reducible. Let $y = \psi(x)$ be a solution of (24.9) which is linearly independent of the solution (24.10). Then

$$\text{(24.13)} \qquad \begin{bmatrix} \varphi(x) \\ \psi(x) \end{bmatrix}$$

is a fundamental system of (24.9). Let ρ be the monodromy representation of (24.9) with respect to the fundamental system (24.13). If ℓ_0 and ℓ_1 are loops at x_0 surrounding $x = 0$ and $x = 1$ respectively, then the function $\varphi(x)$ becomes const. $\varphi(x)$ by the analytic continuation along ℓ_0 and ℓ_1. Hence

$$\rho([\ell_0]) = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}, \quad \rho([\ell_1]) = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}.$$

Consequently, the monodromy group consists of lower triangular matrices. Conversely, it is easily verified that, if the monodromy group consists of lower triangular matrices, then $\varphi(x)$ takes one of the forms (24.11). This means that (24.9) is reducible. Thus we proved the following theorem.

THEOREM 24.3: The Gauss differential equation (24.9) is reducible if and only if there exists a fundamental system of (24.9) with respect to which the monodromy group consists of lower triangular matrices.

We shall now explain briefly how to generalize the concept of reducibility to linear differential equations of higher orders. Consider an n-th order equation

$$\text{(24.14)} \qquad y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$$

or the corresponding differential operator

$$\text{(24.15)} \qquad \frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x)\,.$$

The equation (24.14) or the operator (24.15) is said to be reducible if the operator (24.15) admits a decomposition

$$\text{(24.16)} \qquad \frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x)$$

$$= \left(\frac{d^{n_1}}{dx^{n_1}} + \cdots + p_{n_1}(x)\right)\left(\frac{d^{n_2}}{dx^{n_2}} + \cdots + p_{n_2}(x)\right),$$

where $0 < n_1$, $n_2 < n$. The following theorem is a generalization of Theorem 24.1.

THEOREM 24.4: The operator (24.15) is reducible if and only if there exists a non-trivial solution of (24.14) which satisfies a linear differential equation of an order lower than n.

A monodromy representation of (24.14) is a homomorphism ρ of $\pi_1(D, x_0)$ into $GL(n, \mathbb{C})$, where D is the greatest domain in which the coefficients of (24.14) are holomorphic. A monodromy representation ρ of (24.14) is said to be reducible if $\rho(\pi_1(D, x_0))$ consists of matrices of the form

$$\begin{array}{c} \\ n_1\left\{\right. \\ \\ \end{array}\!\!\left[\begin{array}{c|c} \overbrace{\;*\;}^{n_1} & 0 \\ \hline * & \underbrace{\;*\;}_{n_2} \end{array}\right]\!\!\left.\right\}n_2\,,$$

where n_1 and n_2 are determined by ρ and they are independent of each matrix of $\rho(\pi_1(D, x_0))$. The following theorem is a generalization of Theorem 24.3.

THEOREM 24.5: Suppose that (24.14) is an equation of Fuchsian type. Then the equation (24.14) is reducible if and only if there exists a reducible monodromy representation of (24.14).

We shall now proceed to the systems (12.1), (12.2), (12.3) and (12.4). The system (12.1) has a form

$$(24.17) \qquad \begin{cases} r = \alpha_1 p + \alpha_2 q + \alpha_3 z , \\ s = \beta_1 p + \beta_2 q + \beta_3 z , \\ t = \gamma_1 p + \gamma_2 q + \gamma_3 z , \end{cases}$$

while the systems (12.2), (12.3) and (12.4) have the form

$$(24.18) \qquad \begin{cases} r = a_1 s + a_2 p + a_3 q + a_4 z , \\ t = b_1 s + b_2 p + b_3 q + b_4 z . \end{cases}$$

In their book, Appell and Kampé de Fériet gave the following definition of reducibility. The system (24.17) (or (24.18)) is reducible if there exists a non-trivial solution of (24.17) (or (24.18)) which satisfies a system of partial differential equations whose solutions form a vector space of dimension < 3 (or < 4). For example, (12.1) is reducible if and only if there exists a non-trivial solution of (12.1) satisfying a system of the form

$$\begin{cases} p = a_1 q + a_2 z , \\ q = b_1 q + b_2 z , \end{cases} \quad \text{or} \quad \begin{cases} p = az \\ q = bz \end{cases} \quad \text{etc.}$$

We do not know whether a systematic study had been already done for

reducibility. We do not know any necessary and sufficient conditions for reducibility which are given explicitly in terms of parameters. Appell and Kampé de Fériet gave only a few examples. The reducibility of monodromy representations of the systems (12.1) $\sim$(12.4) can be introduced in a natural way. However, it seems to us that there is no clear statement concerning the relation between reducibility of systems and that of monodromy representations.

PROBLEM 1: Find a necessary and sufficient condition that the systems (12.1) $\sim$ (12.4) be reducible. Find also relations between reducibility of the systems (12.1) $\sim$ (12.4) and that of their monodromy representations.

25. Generalizations. We defined the hypergeometric functions of two variables by utilizing the Gauss function. In a natural way, we can introduce hypergeometric functions of n variables. Lauricella introduced the following four functions:

$$F_A(\alpha, \beta_1, \cdots, \beta_n, \gamma_1, \cdots, \gamma_n, x_1, \cdots, x_n)$$
$$= \sum \frac{(\alpha, m_1+\cdots+m_n)(\beta_1, m_1)\cdots(\beta_n, m_n)}{(\gamma_1, m_1)\cdots(\gamma_n, m_n)(1, m_1)\cdots(1, m_n)} x_1^{m_1}\cdots x_n^{m_n} ,$$

$$F_B(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, \gamma, x_1, \cdots, x_n)$$
$$= \sum \frac{(\alpha_1, m_1)\cdots(\alpha_n, m_n)(\beta_1, m_1)\cdots(\beta_n, m_n)}{(\gamma, m_1+\cdots+m_n)(1, m_1)\cdots(1, m_n)} x_1^{m_1}\cdots x_n^{m_n} ,$$

$$F_C(\alpha, \beta, \gamma_1, \cdots, \gamma_n, x_1, \cdots, x_n)$$
$$= \sum \frac{(\alpha, m_1+\cdots+m_n)(\beta, m_1+\cdots+m_n)}{(\gamma_1, m_1)\cdots(\gamma_n, m_n)(1, m_1)\cdots(1, m_n)} x_1^{m_1}\cdots x_n^{m_n}$$

and

$$F_D(\alpha, \beta_1, \cdots, \beta_n, \gamma, x_1, \cdots, x_n)$$
$$= \sum \frac{(\alpha, m_1+\cdots+m_n)(\beta_1, m_1)\cdots(\beta_n, m_n)}{(\gamma, m_1+\cdots+m_n)(1, m_1)\cdots(1, m_n)} x_1^{m_1}\cdots x_n^{m_n} .$$

In case when $n = 2$, we have

$$F_A = F_2, \quad F_B = F_3, \quad F_C = F_4 \quad \text{and} \quad F_D = F_1 .$$

There are many functions of intermediate types.

On the other hand, many functions of one variable are defined as generalization of the Gauss function. The Gauss function has the three expressions:

(i) $$\sum_{m=0}^{\infty} \frac{(\alpha, m)(\beta, m)}{(\gamma, m)(1, m)} x^m ,$$

(ii) $$\text{const.} \int_1^{\infty} u^{\alpha-\gamma}(u-1)^{\gamma-\beta-1}(u-x)^{-\alpha}\, du ,$$

(iii) $$\text{const.} \int_{-i\infty}^{+i\infty} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)} (-x)^s\, ds .$$

The second expression can be written also in the form

(ii') $$\text{const.} \int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha}\, du .$$

The series

$$F\begin{pmatrix} \alpha_1, \cdots, \alpha_{p+1} & \\ & x \\ \beta_1, \cdots, \beta_p & \end{pmatrix} = \sum_{m=0}^{\infty} \frac{(\alpha_1, m)\cdots(\alpha_{p+1}, m)}{(\beta_1, m)\cdots(\beta_p, m)(1, m)} x^m$$

is derived from the expression (i) in a natural manner. More generally, we get the following series:

$$\sum_{m=0}^{\infty} \frac{(\alpha_1, m)\cdots(\alpha_p, m)}{(\beta_1, m)\cdots(\alpha_q, m)} \frac{x^m}{(1, m)} .$$

It is known that

$$F\begin{pmatrix} \alpha_1, \cdots, \beta_{p+1} & \\ & x \\ \beta_1, \cdots, \beta_p & \end{pmatrix}$$
$$= \text{const.} \int_0^1 \cdots \int_0^1 u_1^{\alpha_1-1}(1-u_1)^{\beta_1-\alpha_1-1} \cdots u_p^{\alpha_p-1}(1-u_p)^{\beta_p-\alpha_p-1} \times$$
$$(1-u_1\cdots u_p x)^{-\alpha_{p+1}} du_1\cdots du_p .$$

From the expression (ii), we can derive the following generalization:

$$\int_1^{\infty} (u-a_1)^{b_1-1}(u-a_2)^{b_2-1}\cdots(u-a_p)^{b_p-1}(u-x)^{\lambda-1}\, du .$$

From the expression (iii), we can derive the following function:

$$\int_{-i\infty}^{+i\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i+s)}{\prod_{i=1}^{q} \Gamma(\beta_i+s)} \Gamma(-s)(-x)^s \, ds .$$

These ideas of generalization of the Gauss function are also applicable to generalization of Appell's and Lauricella's functions.

Finally, we shall talk about the confluence-principle. To start with, let us consider the Gauss function $F(\alpha, \beta, \gamma, x)$. Introducing a new parameter ε to define a function by

$$F(\alpha, 1/\varepsilon, \gamma, \varepsilon x)$$

and then taking a limit as ε tends to zero, we obtain a new function

$$\lim_{\varepsilon\to 0} F(\alpha, 1/\varepsilon, \gamma, \varepsilon x) = \sum_{m=0}^{\infty} \frac{(\alpha,m)\, x^m}{(\gamma,m)(1,m)} = F(\alpha, \gamma, x) .$$

This function $F(\alpha, \gamma, x)$ is also denoted by $G(\alpha, \gamma, x)$ and is called the confluent hypergeometric function. Furthermore, the confluent hypergeometric function satisfies the differential equation

(25.1) $$xy'' + (\gamma - x)y' - \alpha y = 0 .$$

On the other hand, it can be shown that $F(\alpha, \gamma, x)$ admits an integral representation:

(25.2) $$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1} e^{ux} \, du .$$

Note that, in this expression, Laplace transform takes the place of Euler transform. The differential equation (25.1) admits a

regular singular point at $x = 0$ and an irregular singular point at $x = \infty$. Such equations form an important class of equations which contains Bessel equation:

(25.3) $$x^2y'' + xy' + (x^2 - n^2)y = 0 .$$

The equation (25.3) has the Bessel function of order n as a solution.

Humbert obtained from Appell's functions seven functions by the confluence-principle. For example,

$$\lim_{\varepsilon\to 0} F_1(1/\varepsilon, \beta, \beta', \gamma, \varepsilon x, \varepsilon y) = \sum_{m,n=0}^{\infty} \frac{(\beta,m)(\beta',n)x^m y^n}{(\gamma,m+n)(1,m)(1,n)}$$

and the function defined by this series admits an integral representation

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')} \iint_{\substack{u,v\geq 0 \\ 1-u-v\geq 0}} u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'} e^{ux+vy} \, du\, dv .$$

Furthermore, this new function satisfies the system of partial differential equations

(25.4) $$\begin{cases} xr + ys + (\gamma - x)p - \beta z = 0 , \\ yt + xs + (\gamma - y)q - \beta' z = 0 . \end{cases}$$

From (25.4) we derive

$$(x-y)s + \beta q - \beta' p = 0 .$$

The confluence-principle is applied also to hypergeometric functions of more than two variables.