# KZ 型方程式の特異点解消
# と
# middle convolution

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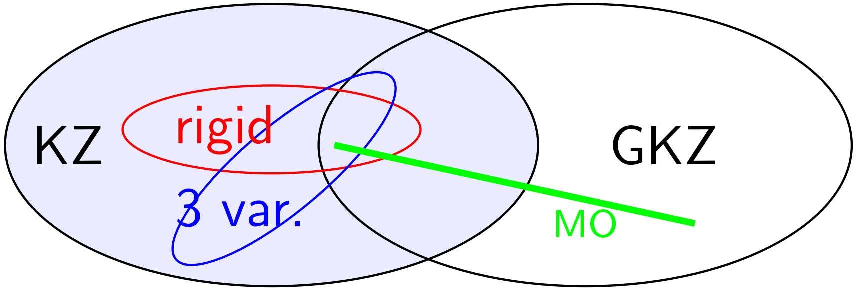
# KZ 型方程式




$$\mathcal{M}: \frac{\partial u}{\partial x_i} = \sum_{\substack{0\le\nu\le n-1\\ \nu\ne i}} \frac{A_{i\nu}}{x_i - x_\nu} u \quad (i \in L_n)$$

$$L_n := \{0,\dots,n-1\},\quad A_{ij} = A_{ji} \in M(N,\mathbb{C}),\ A_{ii} = 0,\ u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$$

積分可能条件 $[A_{ij}, A_{k\ell}] = [A_{ij}, A_{ik} + A_{jk}] = 0$

KZ 型方程式 $\hookleftarrow$ rigid フックス型常微分 (原岡) $\quad \frac{du}{dx} = \frac{A_{01}}{x-x_1}u + \cdots + \frac{A_{0n-1}}{x-x_{n-1}}u$

$\hookleftarrow$ 3 点の特異点を持つ $\mathbb{P}^1$ 上の Fuchs 型方程式

middle convolution と addition $\rightsquigarrow$ アクセサリー・パラメーターと局所構造

KZ 型方程式：$\mathrm{Sp}\,\mathcal{M} \leftarrow$ 常微分のスペクトル型 / (generalized) Riemann scheme

$$A_{i\infty} := -(A_{i0} + A_{i1} + \cdots + A_{in-1}) \qquad (i \in L_n)$$

$$\widetilde{L}_n := L_n \cup \{\infty\},\quad A_i = A_\emptyset = 0$$

$$A_{i_1\cdots i_k} := \sum_{1\le p<q\le k} A_{i_p i_q} \quad (\{i_1,\dots,i_k\} \subset \widetilde{L}_n)$$ (generalized) 留数行列

$$[A_I, A_J] = 0 \quad \bigl(I \subset J \text{ or } I \supset J \text{ or } I \cap J = \emptyset,\ I, J \subset \widetilde{L}_n\bigr)$$

$[A_I, A_{L_n}] = 0 \ \Rightarrow\ A_{L_n} = \kappa$ : scalar ($\Leftarrow \mathcal{M}$：既約) $\quad$ ($\kappa = 0$：斉次)

- $[A,B]=0 \;\Rightarrow\; \exists$ 同時 (一般) 固有空間分解：

$$A=\begin{pmatrix}0&&&\\&0&&\\&&0&\\&&&4\end{pmatrix},\quad B=\begin{pmatrix}1&&&\\&2&&\\&&2&\\&&&3\end{pmatrix}$$

$$\Rightarrow\; [A:B]=\{[0:1]_1,[0:2]_2,[4:3]_1\}=\{[0:1],[0:1]_2,[4:3]\}$$

- $[B_i,B_j]=0\;\;(i,j=1,..,r)\;\Rightarrow\;[B_1:\cdots:B_r]=\{[\lambda_{1,1}:\cdots:\lambda_{r,1}]_{m_1},\cdots\}$

**定義**. 有限集合 $L$ の可換部分集合族 $\mathcal{I}=\{I_\nu \mid \nu=1,\dots,r\}$

$$\overset{\text{def.}}{\Longleftrightarrow}\; I_\nu\subset L,\;|I_\nu|>1 \text{ and } (\,I_\nu\subset I_{\nu'} \text{ or } I_\nu\supset I_{\nu'} \text{ or } I_\nu\cap I_{\nu'}=\emptyset\,)$$

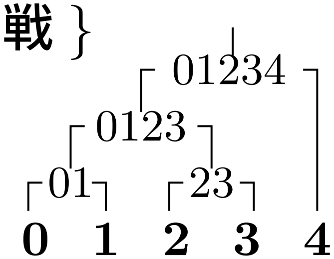
$\mathcal{I}$ が極大 $\overset{\text{def.}}{\Longleftrightarrow}$ $(\mathcal{I},\mathcal{I}'$：可換部分集合族で $\mathcal{I}\subset\mathcal{I}' \;\Rightarrow \mathcal{I}=\mathcal{I}'\,)$

**定義**. $\mathcal{L}_n := \{L_n=\{0,\dots,n-1\}$ の極大可換部分集合族 $\}$

$$\mathrm{Sp}\,\mathcal{M} := \Big\{[A_{I_1}:\cdots:A_{I_{n-1}}]\;\Big|\;\mathcal{I}=\{I_1,\dots,I_{n-1}\}\Big\}_{\mathcal{I}\in\mathcal{L}_n}$$

**Fact**. $\mathcal{L}_n\simeq\{L_n$の元でラベルづけられた $n$ チームのトーナメント戦 $\}$

$$|\mathcal{L}_n|=(2n-3)!!\qquad |\mathcal{I}|=n-1\;\;(\mathcal{I}\in\mathcal{L}_n)$$




**例**. $L_5=\{0,1,2,3,4\},\;|\mathcal{L}_5|=105$

$$\mathcal{I}=\Big\{\{0,1\},\{0,1,2,3\},\{2,3\},\{0,1,2,3,4\}\Big\}$$

$$\Big\{[A_{ij}:A_{k\ell}:A_{ijk\ell}],\,[A_{ij}:A_{ijk}:A_{\ell m}],\,[A_{ij}:A_{ijk}:A_{ijk\ell}]\;\Big|\;\{i,j,k,\ell,m\}=L_5\Big\}$$

# Blowing-up of singular points

KZ equation (Pfaffian form) : $du = \Omega u,\quad \Omega = \sum_{0\le i<j<n} A_{ij}\, d\log(x_i - x_j)$

$\mathcal{I} = \{I_1, \ldots, I_{n-2}, L_n\} \in \mathcal{L}_n$ : $L_n$ の極大可換部分集合族

$J,\ J' \in \tilde{\mathcal{I}} := \mathcal{I} \cup \bigcup_{\nu\in L_n}\{\{\nu\}\}$ with $L_n = J \sqcup J'$ : $\mathcal{I}$ の準決勝

$J = \{j_0, \ldots, j_k\},\ J' = \{j'_0, \ldots, j'_{k'}\} \quad (k + k' = n - 2)$

特異点 : $x_{j_0} = \cdots = x_{j_k}$ and $x_{j'_0} = \cdots = x_{j'_{k'}} \quad (k,\ k' \ge 0)$

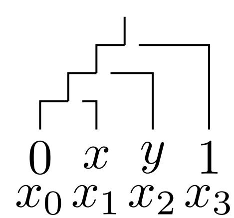
A local coordinate : $(x_{j_1}, \ldots, x_{j_k}, x_{j'_1}, \ldots, x_{j'_{k'}})$ with $x_{j_0} = 0$ and $x_{j'_0} = 1$

$\{n_i, n'_i\}$ : 試合 $I_i$ で対戦したチーム

**定義**. $X = (X_1, \ldots, X_{n-2})$ : 次で定まる局所座標系 $x_{n_i} - x_{n'_i} = \prod_{I_i\subset I_\nu \ne L_n} X_\nu$

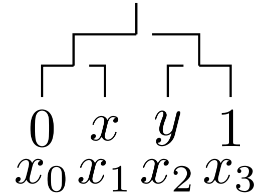
**定理 1**. $\Omega - \sum_{i=1}^{n-2} A_{I_i}\, d\log X_i$ は $X$ の原点の近傍で正則

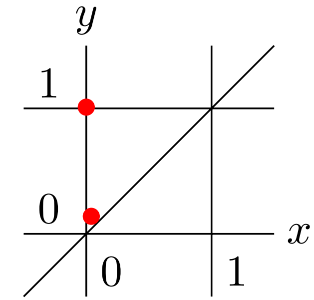
**Ex**. $(x, y) = (0, 0)$ : $\begin{cases} x_2 - x_0 = y = Y, \\ x_1 - x_0 = x = XY, \end{cases}$ $\begin{cases} x_1 - x_2 = (X - 1)Y, \\ (X, Y) = (\frac{x}{y}, y), \end{cases}$




$\frac{dx}{x} = \frac{dX}{X} + \frac{dY}{Y},\quad \frac{dy}{y} = \frac{dY}{Y},\quad \frac{d(x-y)}{x-y} = \frac{dY}{Y} + \frac{d(X-1)}{X-1},\quad \{I_1, I_2\} = \{\{0, 1\}, \{0, 1, 2\}\}$

$$(x,y)=(0,1):\begin{cases} x_1-x_0=x=X, \\ x_3-x_2=1-y=Y, \end{cases} \quad \begin{cases} X=x, \\ Y=y+1, \end{cases}$$



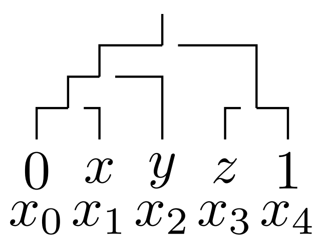

$$\Omega' = A_{x0}\frac{dX}{X} + A_{y1}\frac{dY}{Y} \quad (|x|,\ |y-1| \ll 1)$$

**補題**. $i, j \in L$ （$i \neq j$）に対して，$i$ と $j$ を共に含む $\mathcal{I}$ の元で極小なものを $I_{i,j}$ とする．このとき $x_{n_i} - x_{n'_i}$ （$1 \leq i \leq n-2$）は $\mathbb{C}$ 上一次独立で以下が成り立つ．

$$x_i - x_j = \sum_{\nu=1}^{n-2} \epsilon_\nu (x_{n_\nu} - x_{n'_\nu}) \text{ with } \begin{cases} \epsilon_\nu = 0 & (I_\nu \supsetneqq I_{i,j} \text{ or } I_\nu \cap I_{i,j} = \emptyset) \\ \epsilon_\nu \in \{1,-1\} & (I_\nu = I_{i,j}) \\ \epsilon_\nu \in \mathbb{Z} & (I_\nu \subsetneqq I_{i,j}) \end{cases}$$

$$\mathbf{n=5}$$

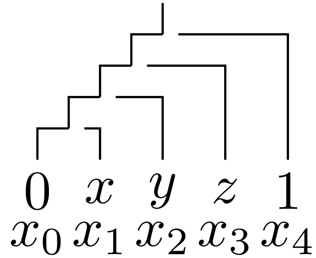
$$(x,y,z)=(0,0,1):$$




$$\begin{cases} x_2-x_0=y=Y, \\ x_1-x_0=x=XY, \\ x_3-x_4=z-1=Z, \\ (X,Y,Z)=(\frac{x}{y}, y, z-1) \end{cases}$$

$$\Omega' = A_{x0}\frac{dX}{X} + A_{xy0}\frac{dY}{Y} + A_{z1}\frac{dZ}{Z} \qquad (|x| \ll |y| \ll 1,\ |z-1| \ll 1)$$

$$(x,y,z)=(0,0,0):$$

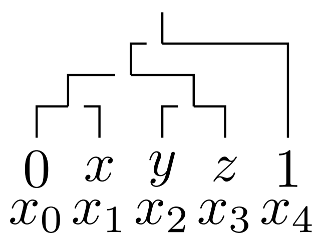



$$\begin{cases} x_3-x_0=z=Z,\\ x_2-x_0=y=YZ,\\ x_1-x_0=x=XYZ, \end{cases}$$

$$\begin{cases} X=\frac{x}{y},\\ Y=\frac{y}{z},\\ Z=z, \end{cases} \qquad \begin{cases} y-z=(Y-1)Z,\\ x-y=(X-1)YZ,\\ x-z=(XY-1)Z, \end{cases}$$

$$\Omega'=A_{x0}\frac{dX}{X}+A_{xy0}\frac{dY}{Y}+A_{xyz0}\frac{dZ}{Z} \qquad (|x|\ll|y|\ll|z|\ll 1)$$

$$(x,y,z)=(0,0,0):$$




$$\begin{cases} x_3-x_0=z=Z,\\ x_1-x_0=x=XZ,\\ x_3-x_2=z-y=YZ, \end{cases}$$

$$\begin{cases} X=\frac{z}{x},\\ Y=\frac{z-y}{z},\\ Z=z, \end{cases} \qquad \begin{cases} y=(1-Y)Z,\\ x-y=(X+Y-1)Z,\\ x-z=(X-1)Z, \end{cases}$$

$$\Omega'=A_{x0}\frac{dX}{X}+A_{yz}\frac{dY}{Y}+A_{xyz0}\frac{dZ}{Z} \qquad (|x|,\ |y-z|\ll|z|\ll 1).$$

# Middle convolution with respect to $x_0$

パラメータ $\mu \in \mathbb{C}$ をもつ $\mathcal{M}$ の convolution $\widetilde{\mathrm{mc}}_{x_0,\mu}\mathcal{M}$ の定義は

$$\tilde{\mathcal{M}}\ :\ \frac{\partial \tilde{u}}{\partial x_i} = \sum_{0\le\nu<n} \frac{\tilde{A}_{i\nu}}{x_i - x_\nu}\tilde{u} \quad (0 \le i < n)$$

$$\tilde{A}_{0k} = k \begin{pmatrix} 0 & \cdots & \overset{k}{0} & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ A_{01} & \cdots & A_{0k} + \mu & \cdots & A_{0n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \in M((n-1)N, \mathbb{C})$$

(Dettwiler-Reiter $\Leftarrow$ Katz),

$$\tilde{A}_{ij} = \begin{matrix} \\ \\ i \\ \\ j \\ \\ \\ \end{matrix} \begin{pmatrix} A_{ij} & & & & & & \\ & \ddots & & & & & \\ & & \overset{i}{A_{ij} + A_{0j}} & & \overset{j}{-A_{0j}} & & \\ & & & \ddots & & & \\ & & -A_{0i} & & A_{ij} + A_{0i} & & \\ & & & & & \ddots & \\ & & & & & & A_{ij} \end{pmatrix} \in M((n-1)N, \mathbb{C})$$

(Haraoka)

$$\iota_j(v) := (v)_j := j\begin{pmatrix}0\\ \vdots\\ v\\ \vdots\\ 0\end{pmatrix} \in \mathbb{C}^{(n-1)N} \quad (v \in \mathbb{C}^N), \quad \iota_j : \mathbb{C}^N \hookrightarrow \mathbb{C}^{(n-1)N}$$

$$\iota_I := \sum_{i\in I} \iota_i : \mathbb{C}^N \hookrightarrow \mathbb{C}^{(n-1)N}, \ (v)_I := \iota_I(v) \in \mathbb{C}^{(n-1)N} \quad (v \in \mathbb{C}^N, \ I \subset L_n^0)$$

$$\tilde{\mathcal{K}}_j := \iota_j(\mathrm{Ker}\, A_{0j}) = j\begin{pmatrix}0\\ \vdots\\ \mathrm{Ker}\, A_{0j}\\ \vdots\\ 0\end{pmatrix} \subset \mathbb{C}^{(n-1)N}, \quad L_n^0 := \{1, \dots, n-1\}$$

$$\tilde{\mathcal{K}}_\infty := \ker \tilde{A}_{0,\infty} \overset{\mu\neq 0}{=} \iota_{L_n}\big(\mathrm{Ker}\,(A_{0\infty} - \mu)\big) = \left\{\begin{pmatrix}v\\ \vdots\\ v\end{pmatrix} \mid A_{0\infty}v = \mu v\right\} \subset \mathbb{C}^{(n-1)N}$$

$$\tilde{\mathcal{K}} := \tilde{\mathcal{K}}_\infty + \bigoplus_{j=1}^{n} \tilde{\mathcal{K}}_j \quad (\text{direct sum} \ \Leftarrow \ \mu \neq 0) \ \Rightarrow \ \tilde{A}_I\text{-invariant} \quad (I \subset L_n)$$

$$\bar{A}_I := \tilde{A}_I|_{\mathbb{C}^{(n-1)N}/\tilde{\mathcal{K}}} \in M\big((n-1)N - \dim \tilde{\mathcal{K}}, \mathbb{C}\big)$$

middle convolution $\overline{\mathcal{M}} = \mathrm{mc}_{x_0,\mu}\,\mathcal{M} : \dfrac{\partial \bar{u}}{\partial x_i} = \displaystyle\sum_{0\le\nu<n} \frac{\bar{A}_{i\nu}}{x_i - x_\nu}\bar{u}$

$$\mathrm{mc}_{x_0,-\mu} \circ \mathrm{mc}_{x_0,\mu} = \mathrm{id} \quad (\Leftarrow \ \partial^\mu \circ \partial^{-\mu})$$

**Fact**. $A_{L_n} = \kappa, \ I \subset L_n \Rightarrow \ \bar{A}_I = \bar{A}_{\tilde{L}_n\setminus I} + \kappa + \mu \quad (\kappa = 0)$

**定義**. For $i \in L_n$ and $I,\ J \subset L_n$

$$\mathrm{md}_{i,J}(I) := \begin{cases} I \cup \{i\} & (I \supset J) \\ I \setminus \{i\} & (I \not\supset J) \end{cases} \qquad \mathrm{me}_{i,J}(I) := \begin{cases} 1 & (i \in I \supset J) \\ 0 & (i \notin I \text{ or } I \not\subset J) \end{cases}$$

$$i \in I \supset J \text{ or } i \notin I \not\supset J \ \Rightarrow \mathrm{md}_{i,J}(I) = I$$

**Fact**. $\mathrm{md}_{i,J}$ : { **可換族** } $\to$ { **可換族** }

**定理 2**. (i) For $\mathcal{I} = \{I^{(1)}, \ldots, I^{(n-1)}\} \in \mathcal{L}_n$ and $I \in \mathcal{I}$

$$[\tilde{A}_I] = [A_{I\cup\{0\}} + \mu]_{|I|-1} \sqcup [A_{I\setminus\{0\}}]_{n-|I|}$$

$$[\tilde{A}_{I^{(1)}} : \cdots : \tilde{A}_{I^{(n-1)}}] = \bigsqcup_{J\in\mathcal{I}} [A^J_{I^{(1)}} : \cdots : A^J_{I^{(n-1)}}]$$

$$[\tilde{A}_{I^{(1)}} : \cdots : \tilde{A}_{I^{(n-1)}}]|_{\mathcal{K}_j} = [A^{\{j\}}_{I^{(1)}} : \cdots : A^{\{j\}}_{I^{(n-1)}}]|_{\ker A_{0j}} \qquad (j \in L^0_n = L_n \setminus \{0\})$$

$$[\tilde{A}_{I^{(1)}} : \cdots : \tilde{A}_{I^{(n-1)}}]|_{\mathcal{K}_\infty} = [A^{L_n}_{I^{(1)}} : \cdots : A^{L_n}_{I^{(n-1)}}]|_{\ker(A_{0\infty}-\mu)} \quad (A_{0\infty} = A_{L^0_n} - A_{L_n})$$

$$A^K_I := A_{\mathrm{md}_{0,K}(I)} + \mathrm{me}_{0,K}(I)\cdot\mu$$

(ii) $U^{-1}_{b^0(\mathcal{I})} \tilde{A}_{I^{(i)}} U_{b^0(\mathcal{I})}$ : **同時ブロック上三角行列** $(i = 1, \ldots, n-1)$

**順序**. $\mathcal{I} = \{I^{(1)}, \dots, I^{(n-1)}\}$ において，以下を仮定してよい（$\mathcal{I}$ の元の順序づけ）.

$$\mathcal{I}_0 := \{I \mid 0 \in I \in \mathcal{I}\} = \{I^{(k_1)}, \dots, I^{(k_m)}\}$$

**1**. $k_p < k_q \quad \Leftrightarrow \quad I^{(k_p)} \subsetneq I^{(k_q)}$

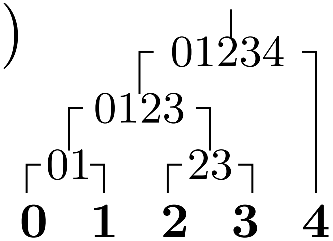
**2**. $k_p < i \leq j < k_{p+1} \quad \Leftarrow \quad I^{(k_p)} \setminus I^{(k_{p-1})} \supset I^{(i)} \supset I^{(j)} \qquad (I^{(k_0)} := \emptyset)$

**定義**. For $i \in L_n$ and $I \in \mathcal{I}$

$$b^i(I) \in \widetilde{\mathcal{I}} := \mathcal{I} \cup \bigsqcup_{\nu \in L_n} \{\{\nu\}\} \quad \text{so that} \quad i \notin b^i(I) \subsetneq I \text{ and } I \setminus b^i(I) \in \widetilde{\mathcal{I}}$$

$$GL((n-1)N, \mathbb{C}) \ni \left(U_{b^0(\mathcal{I})}\right)_{\substack{1 \leq i \leq n-1 \\ 1 \leq j \leq n-1}} := \begin{cases} \mathbf{1}_N & \left(i \in b^0(I^{(j)})\right) \\ 0 & \left(i \notin b^0(I^{(j)})\right) \end{cases}$$

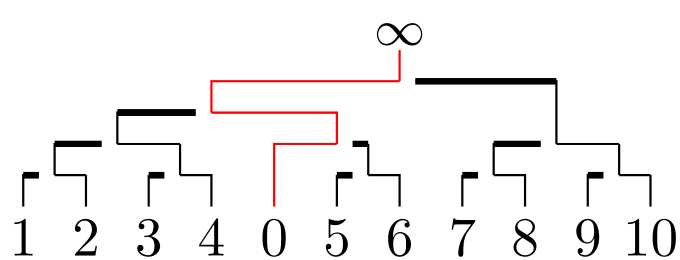
**例**. $L = \{0, 1, 2, 3, 4\}$




$$\mathcal{I} = \{\{0,1\}, \{0,1,2,3\}, \{2,3\}, \{0,1,2,3,4\}\}$$

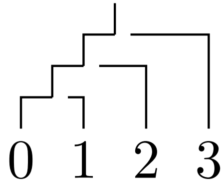
$$b^0(\{0,1\}) = \{1\},\ b^0(\{0,1,2,3\}) = \{2,3\},\ b^0(\{2,3\}) = \{2\},\ b^0(L_5) = \{4\}$$

$L = \{0, 1, \dots, 10\}$ :




$\mathcal{I}$ : $\{0,5,6\}, \{5,6\}, \{0,\dots,6\}, \{1,2,3,4\}, \{1,2\}, \{3,4\}, \{0,\dots,10\}, \{7,8,9,10\}, \{7,8\}, \{9,10\}$

$b^0$ : $\{5,6\},\ \{5\}, \{1,2,3,4\},\ \{1,2\},\ \{1\},\ \{3\}, \{7,8,9,10\},\ \{7,8\},\ \{7\},\ \{9\}$

**1.** 


$$\mathcal{I} = \big\{\{0,1\},\{0,1,2\},\{0,1,2,3\}\big\} \xrightarrow{b^0} \big\{\{1\},\{2\},\{3\}\big\}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_* \to V\tilde{A}_*U$$

$$\tilde{A}_{01} = \begin{pmatrix} A_{01}+\mu & A_{02} & A_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} A_{01}+\mu & A_{02} & A_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{A}_{012} = \begin{pmatrix} A_{012}+\mu & 0 & A_{03} \\ 0 & A_{012}+\mu & A_{03} \\ 0 & 0 & A_{12} \end{pmatrix} \to \begin{pmatrix} A_{012}+\mu & 0 & A_{03} \\ 0 & A_{012}+\mu & A_{03} \\ 0 & 0 & A_{12} \end{pmatrix}$$

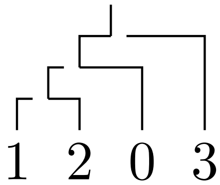
$$[\tilde{A}_{01} : \tilde{A}_{012}] = \{[A_{01}+\mu : A_{012}+\mu], [0 : A_{012}+\mu], [0 : A_{12}]\}$$

$$[\tilde{A}_{01} : \tilde{A}_{012}]|_{\mathcal{K}_1} = [A_{01}+\mu : A_{012}+\mu]|_{\ker A_{01}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012}]|_{\mathcal{K}_2} = [0 : A_{012}+\mu]|_{\ker A_{02}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012}]|_{\mathcal{K}_3} = [0 : A_{12}]|_{\ker A_{03}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012}]|_{\mathcal{K}_\infty} = [0 : A_{12}]|_{\ker (A_{0\infty}-\mu)}$$

**2.** 


$$\mathcal{I} = \big\{\{0,1,2\},\{1,2\},\{0,1,2,3\}\big\} \xrightarrow{b^0} \big\{\{1,2\},\{1\},\{3\}\big\}$$

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_* \to V\tilde{A}_* U$$

$$\tilde{A}_{012} = \begin{pmatrix} A_{012}+\mu & 0 & A_{03} \\ 0 & A_{012}+\mu & A_{03} \\ 0 & 0 & A_{12} \end{pmatrix} \to \begin{pmatrix} A_{012}+\mu & 0 & A_{03} \\ 0 & A_{012}+\mu & 0 \\ 0 & 0 & A_{12} \end{pmatrix}$$

$$\tilde{A}_{12} = \begin{pmatrix} A_{012}-A_{01} & -A_{02} & 0 \\ -A_{01} & A_{012}-A_{02} & 0 \\ 0 & 0 & A_{12} \end{pmatrix} \to \begin{pmatrix} A_{12} & -A_{01} & 0 \\ 0 & A_{012} & 0 \\ 0 & 0 & A_{12} \end{pmatrix}$$

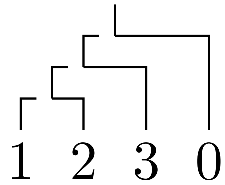
$$[\tilde{A}_{012} : \tilde{A}_{12}] = \{[A_{012}+\mu : A_{12}], [A_{012}+\mu : A_{012}], [A_{12} : A_{12}]\}$$

$$[\tilde{A}_{012} : \tilde{A}_{12}]|_{\mathcal{K}_1} = [A_{012} + \mu : A_{012}]|_{\ker A_{01}}$$

$$[\tilde{A}_{012} : \tilde{A}_{12}]|_{\mathcal{K}_2} = [A_{012} + \mu : A_{012}]|_{\ker A_{02}}$$

$$[\tilde{A}_{012} : \tilde{A}_{12}]|_{\mathcal{K}_3} = [A_{12} : A_{12}]|_{\ker A_{03}}$$

$$[\tilde{A}_{012} : \tilde{A}_{12}]|_{\mathcal{K}_\infty} = [A_{12} : A_{12}]|_{\ker (A_{0\infty}-\mu)}$$

**3.** 


$$\mathcal{I} = \{\{0,1,2,3\},\{1,2,3\},\{1,2\}\} \xrightarrow{b^0} \{\{1,2,3\},\{1,2\},\{1\}\}$$

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \quad \tilde{A}_* \to V\tilde{A}_* U$$

$$\tilde{A}_{123} = \begin{pmatrix} -A_{01} & -A_{02} & -A_{03} \\ -A_{01} & -A_{02} & -A_{03} \\ -A_{01} & -A_{02} & -A_{03} \end{pmatrix} \to \begin{pmatrix} A_{123} & -A_{01} - A_{02} & -A_{01} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{A}_{12} = \begin{pmatrix} A_{012} - A_{01} & -A_{02} & 0 \\ -A_{01} & A_{012} - A_{02} & 0 \\ 0 & 0 & A_{12} \end{pmatrix} \to \begin{pmatrix} A_{12} & 0 & 0 \\ 0 & A_{12} & -A_{01} \\ 0 & 0 & A_{012} \end{pmatrix}$$

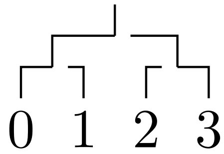
$$[\tilde{A}_{123} : \tilde{A}_{12}] = \{[A_{123} : A_{12}], [0 : A_{12}], [0 : A_{012}]\}$$

$$[\tilde{A}_{123} : \tilde{A}_{12}]|_{\mathcal{K}_1} = [0 : A_{012}]|_{\ker A_{01}}$$

$$[\tilde{A}_{123} : \tilde{A}_{12}]|_{\mathcal{K}_2} = [0 : A_{012}]|_{\ker A_{02}}$$

$$[\tilde{A}_{123} : \tilde{A}_{12}]|_{\mathcal{K}_3} = [0 : A_{12}]|_{\ker A_{03}}$$

$$[\tilde{A}_{123} : \tilde{A}_{12}]|_{\mathcal{K}_\infty} = [A_{123} : A_{12}]|_{\ker (A_{0\infty} - \mu)}$$

**4.** 


$$\mathcal{I} = \big\{\{0,1\},\{0,1,2,3\},\{2,3\}\big\} \xrightarrow{b^0} \big\{\{1\},\{2,3\},\{2\}\big\}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \tilde{A}_* \to V\tilde{A}_*U$$

$$\tilde{A}_{01} = \begin{pmatrix} A_{01}+\mu & A_{02} & A_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} A_{012}+\mu & A_{02}+A_{03} & A_{02} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{A}_{23} = \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{023}-A_{02} & -A_{03} \\ 0 & -A_{02} & A_{023}-A_{03} \end{pmatrix} \to \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{23} & -A_{02} \\ 0 & 0 & A_{023} \end{pmatrix}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}] = \{[A_{01}+\mu : A_{23}], [0 : A_{23}], [0 : A_{023}]\}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_1} = [A_{01} + \mu : A_{23}]|_{\ker A_{01}}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_2} = [0 : A_{023}]|_{\ker A_{02}}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_3} = [0 : A_{023}]|_{\ker A_{03}}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_\infty} = [0 : A_{23}]|_{\ker (A_{0\infty}-\mu)}$$

**1 の場合**

| $J\backslash\widetilde{A}$ | $\widetilde{01}$ | $\widetilde{012}$ |
|---|---|---|
| 01 | 01+$\mu$ | 012+$\mu$ |
| 012 | 0 | 012+$\mu$ |
| 0123 | 0 | 12 |
| 1 | 01+$\mu$ | 012+$\mu$ |
| 2 | 0 | 012+$\mu$ |
| 3 | 0 | 12 |
| $\infty$ | 0 | 12 |

**2 の場合**

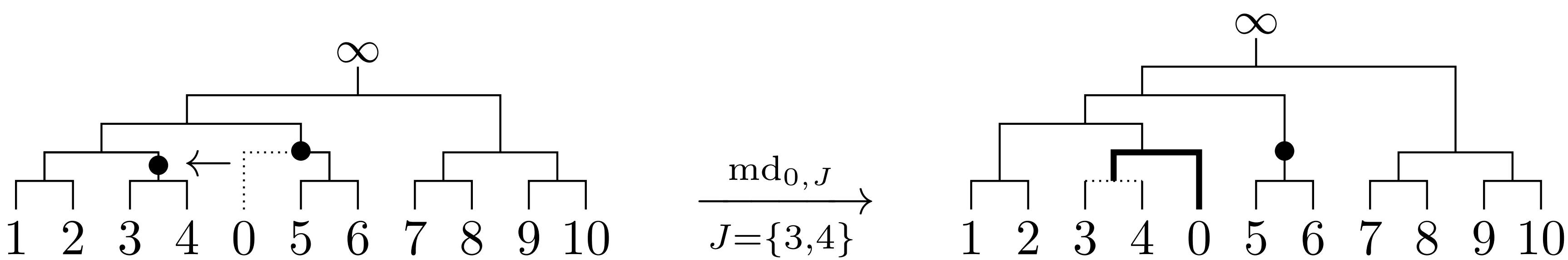
| $J\backslash\widetilde{A}$ | $\widetilde{012}$ | $\widetilde{12}$ |
|---|---|---|
| 012 | 012+$\mu$ | 12 |
| 12 | 012+$\mu$ | 012 |
| 0123 | 12 | 12 |
| 1 | 012+$\mu$ | 012 |
| 2 | 012+$\mu$ | 012 |
| 3 | 12 | 12 |
| $\infty$ | 12 | 12 |

**3 の場合**

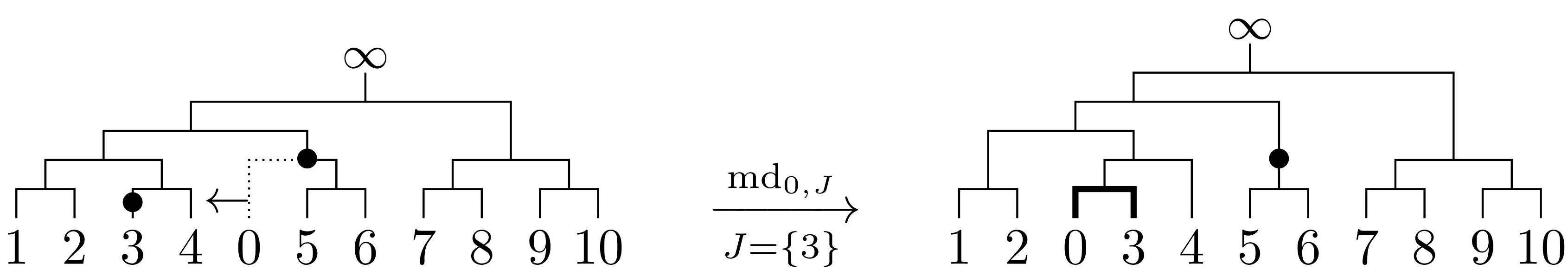
| $J\backslash\widetilde{A}$ | $\widetilde{123}$ | $\widetilde{12}$ |
|---|---|---|
| 0123 | 123 | 12 |
| 123 | 0 | 12 |
| 12 | 0 | 012 |
| 1 | 0 | 012 |
| 2 | 0 | 012 |
| 3 | 0 | 12 |
| $\infty$ | 123 | 12 |

**4 の場合**

| $J\backslash\widetilde{A}$ | $\widetilde{01}$ | $\widetilde{23}$ |
|---|---|---|
| 01 | 01+$\mu$ | 23 |
| 0123 | 0 | 23 |
| 23 | 0 | 023 |
| 1 | 01+$\mu$ | 23 |
| 2 | 0 | 023 |
| 3 | 0 | 023 |
| $\infty$ | 0 | 23 |



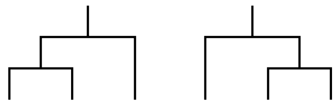

$$J=\{3,4\}:\{3,4\}\to\{0,3,4\},\ \{1,2,3,4\}\to\{0,1,2,3,4\},\ \{0,5,6\}\to\{5,6\}$$

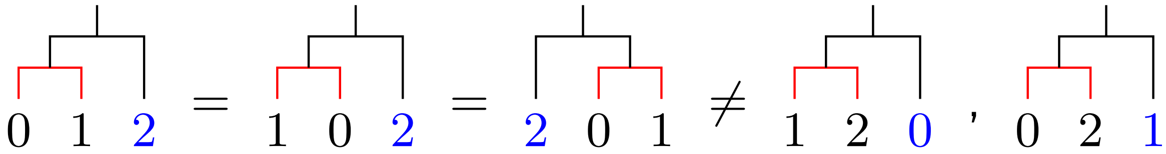


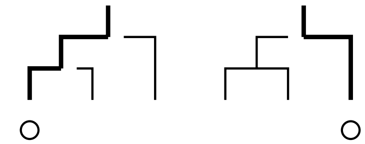

$$j=3:\{3,4\}\to\{0,3,4\},\ \{1,2,3,4\}\to\{0,1,2,3,4\},\ \{0,5,6\}\to\{5,6\},\ +\{0,3\}$$

# トーナメント戦

## 3 teams

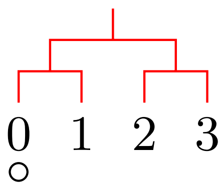
2 patterns : 



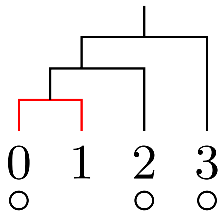
1 type, 2 patterns, 3 (cases of) tournaments, 2 win types

## 4 teams



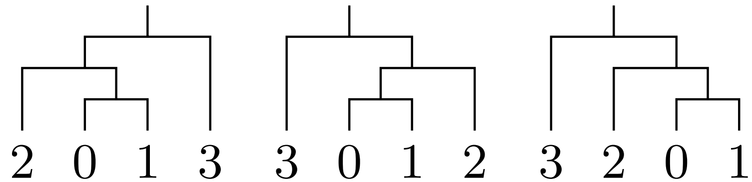

: 1 pattern

: $\frac{4!}{2^3} = 3$ cases



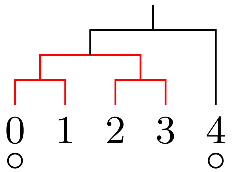

: 4 patterns

: $\frac{4!}{2} = 12$ cases



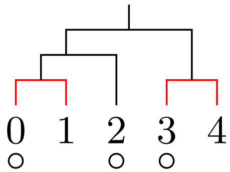

2 types, 5 patterns, 15 tournaments, 4 win types

## 5 teams



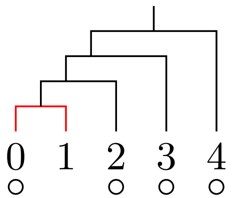

: 2 patterns

: $\frac{5!}{2^3} = 15$ cases,




: 4 patterns

: $\frac{5!}{2^2} = 30$ cases,




: 8 patterns

: $\frac{5!}{2} = 60$ cases

3 types, 14 patterns, 105 tournaments, 9 win types

# トーナメント戦に関わる場合の数

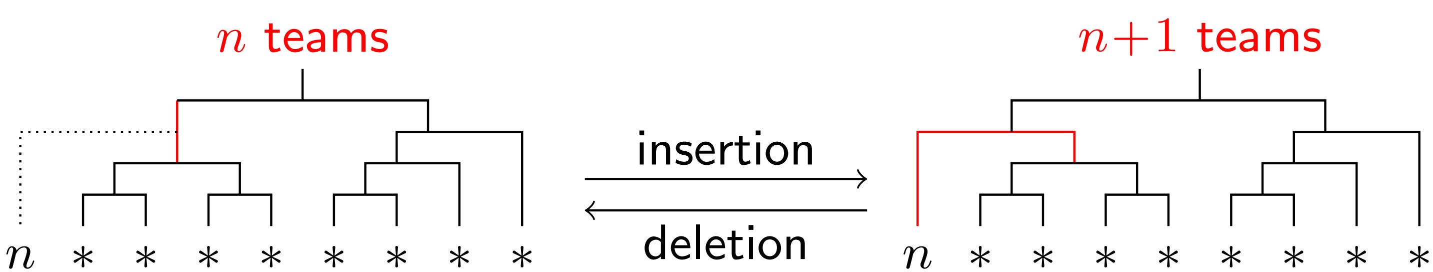
| teams | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $n$ |
|---|---|---|---|---|---|---|---|---|---|---|
| patterns | 1 | 2 | 5 | 14 | 42 | 132 | 429 | 1430 | 4862 | $T_n = \frac{(2n-2)!}{n!(n-1)!}$ |
| win types | 1 | 2 | 4 | 9 | 20 | 46 | 106 | 248 | 582 | $W_n$ |
| types | 1 | 1 | 2 | 3 | 6 | 11 | 23 | 46 | 98 | $U_n$ |
| tournaments | 1 | 3 | 15 | 105 | 945 | 10395 | 135135 | 2027025 | 34459425 | $K_n = (2n-3)!!$ |

$$T_n = \sum_{k=1}^{n-1} T_k \cdot T_{n-k}, \quad T_1 = 1, \qquad \text{(patterns)}$$

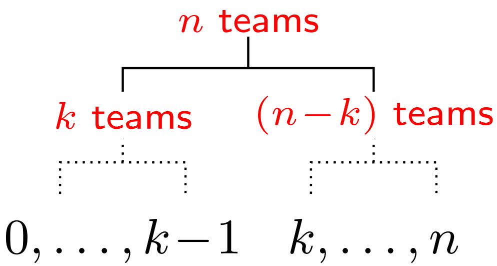
$$W_n = \sum_{1 \leq k \leq n-1} W_k \cdot U_{n-k}, \quad W_1 = 1, \qquad \text{(win types)}$$

$$U_n = \frac{1}{2}\Big( \sum_{1 \leq k \leq n-1} U_k \cdot U_{n-k} \ \big(+U_{\frac{n}{2}} \text{ if } n \text{ is even}\big) \Big), \quad U_1 = 1, \qquad \text{(types)}$$

$$K_n = \frac{1}{2} \sum_{k=1}^{n-1} {}_nC_k \cdot K_k \cdot K_{n-k}, \quad K_1 = 1. \qquad \text{(tournaments)}$$




$2n-1$ vertical line segments $\Rightarrow K_{n+1} = (2n-1)K_n \Rightarrow K_n = (2n-3)!!$.




$$1 \le k < n \;\Rightarrow\; \begin{cases} T_n \leftarrow T_k \cdot T_{n-k} \\ W_k \Leftarrow W_k \cdot U_{n-k} \end{cases}$$

$$k < n-k \;\Rightarrow\; \begin{cases} U_n \leftarrow U_k \cdot U_{n-k} \\ K_n \leftarrow {}_nC_k \cdot K_k \cdot K_{n-k} \end{cases}$$

$$k = n-k \;\Rightarrow\; \begin{cases} U_n \leftarrow \tfrac{1}{2} U_k (U_k - 1) + U_k, \\ K_n \leftarrow {}_nC_k \big(\tfrac{1}{2} K_k (K_k - 1)\big) + \big(\tfrac{1}{2}\, {}_nC_k\big) K_k \end{cases}$$

## Identities

$$K_n = (2n-3)!! = \frac{1}{2} \sum_{k=1}^{n-1} {}_nC_k \cdot (2k-1)!! \cdot (2n-2k-1)!!$$

$$T_n = C_{n-1} = \frac{(2n-2)!}{(n-1)!n!} = \sum_{k=1}^{n-k} \frac{(2k-2)!}{(k-1)!k!} \frac{(2(n-k)-2)!}{(n-k-1)!(n-k)!}$$

$$1 = \Big(1 - \sum_{k=1}^{\infty} U_n x^k\Big) \Big(\sum_{\ell=0}^{\infty} W_{\ell+1} x^{\ell}\Big),$$

$C_n$ : Catalan number

$U_n$ : Wedderburn-Etherington number

$$\begin{pmatrix} A_{12}+A_{02} & -A_{02} & 0 \\ -A_{01} & A_{12}+A_{01} & 0 \\ 0 & 0 & A_{12} \end{pmatrix} + \begin{pmatrix} A_{13}+A_{03} & 0 & -A_{03} \\ 0 & A_{13} & 0 \\ -A_{01} & 0 & A_{13}+A_{01} \end{pmatrix} + \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{23}+A_{03} & -A_{03} \\ 0 & -A_{02} & A_{23}+A_{02} \end{pmatrix}$$

$$= \begin{pmatrix} A_{0123}-A_{01} & -A_{02} & -A_{03} \\ -A_{01} & A_{0123}-A_{02} & -A_{03} \\ -A_{01} & -A_{02} & A_{0123}-A_{03} \end{pmatrix} \qquad (\Leftarrow A_{12}+A_{13}+A_{14}=A_{123})$$

$$\tilde{A}_{1\cdots k} = \begin{pmatrix} A_{0\cdots k}-A_{01} & \cdots & -A_{0k} & & & \\ \vdots & \cdots & \vdots & & & \\ -A_{01} & \cdots & A_{0\cdots k}-A_{0k} & & & \\ & & & A_{1\cdots k} & & \\ & & & & \ddots & \\ & & & & & A_{1\cdots k} \end{pmatrix}$$

$$I \subset L_n \; (= \{1, \ldots, n-1\}), \quad i \in I, \quad j \in L_n \setminus I$$

$$\tilde{A}_I(v)_J = (A_I v)_J \ \text{ in } \ V_J := \iota_J(\mathbb{C}^N) \qquad (I \subset \forall J \subset L_n)$$

$$\tilde{A}_I(v)_i = (A_{0I} v)_i - (A_{0i} v)_I \equiv (A_{0I} v)_i \mod V_I \qquad (i \in I)$$

$$\tilde{A}_I(v)_j = (A_I v)_j \qquad (j \notin I)$$

$$\tilde{A}_I(v)_{L_n} = (A_I v)_{L_n}$$

$$[\tilde{A}_I] = [A_I]_{n-|I|} \cup [A_{0I}]_{|I|-1}$$

$$\tilde{A}_I(v)_i = (A_{0I} v)_i \qquad (v \in \ker A_{0i})$$

$$\tilde{A}_{0\cdots k-1} = \begin{pmatrix} A_{01\cdots k-1}+\mu & & & A_{0k} & \cdots & A_{0n-1} \\ & \ddots & & \vdots & \cdots & \vdots \\ & & A_{01\cdots k-1}+\mu & A_{0k} & \cdots & A_{0n-1} \\ & & & A_{1\cdots k} & & \\ & & & & \ddots & \\ & & & & & A_{1\cdots k} \end{pmatrix}$$

$$I \subset L_n \quad (I = \{1, \ldots k-1\} \uparrow), \;\; i \in I, \;\; j \in L_n \setminus I$$

$$\tilde{A}_{0I}(v)_i = ((A_{0I}+\mu)v)_i$$

$$\tilde{A}_{0I}(v)_j = (A_{0j}v)_I + (A_I v)_j \equiv (A_I v)_j \mod V_I$$

$$(A_{0I}+\mu)v + \sum_{\nu \in L_n \setminus I} A_{0\nu} v = A_I v + \Big( \underbrace{\sum_{\nu=1}^{n-1} A_{0\nu}}_{: -A_{0\infty}} + \mu \Big) v$$

$$\equiv A_I v \mod \ker(A_{0\infty} - \mu)$$

$$[\tilde{A}_{0I}] = [A_I]_{n-1-|I|} + [A_{0I}+\mu]_{|I|}$$

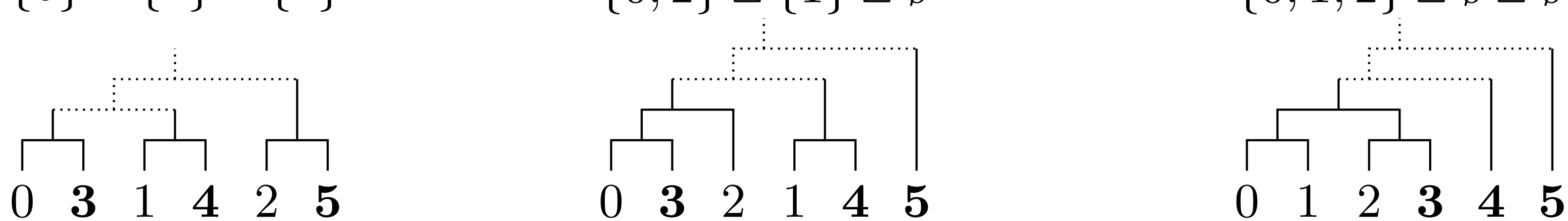
# KZ 型方程式とトーナメント戦の対応

| $n$ 変数の KZ 型方程式 | $n$ チームのトーナメント戦 |
|---|---|
| 極大可換留数行列族 | トーナメント戦 |
| 局所座標系 | トーナメント戦と各試合の勝敗結果 |
| KZ 型方程式のスペクトル | トーナメント戦の全体 |
| middle convolution を行う変数 | 優勝チームを決める |
| 極大可換留数行列族の上三角化の基底 | 優勝チームとトーナメント戦の結果 |
| middle convolution | 優勝チームの deletion と insertion |
| middle convolution の定義の kernel | 上で basic/top insertion |
| さらに $m$ 個の固定特異点がある場合 | $n$ チームを $m$ 個にグループ分け |

例：変数 $x_0, x_1, x_2$, 固定特異点 $y_3, y_4, y_5$ $\Rightarrow$ 105 個の極大可換留数行列族

$$\frac{\partial u}{\partial x_i} = \sum_{\substack{0\le\nu\le 2\\ \nu\ne i}} \frac{A_{i\nu}}{x_i - x_\nu} u + \sum_{\nu=3}^{5} \frac{A_{i\nu}}{x_i - y_\nu} u \quad (i = 0, 1, 2)$$

$\overset{3}{\{0\}} \sqcup \overset{4}{\{1\}} \sqcup \overset{5}{\{2\}}$ $\quad$ $\overset{3}{\{0,2\}} \sqcup \overset{4}{\{1\}} \sqcup \overset{5}{\emptyset}$ $\quad$ $\overset{3}{\{0,1,2\}} \sqcup \overset{4}{\emptyset} \sqcup \overset{5}{\emptyset}$




$\{\{0,3\},\{1,4\},\{2,5\}\}$ $\quad$ $\{\{0,3\},\{1,4\},\{0,2,3\}\}$ $\quad$ $\{\{0,1\},\{2,3\},\{0,1,2,3\}\}$

# 超平面に特異点をもつ Pfaff 系

$$du = \Omega u,\ \ \Omega = \sum_j A_j d\log f_j \quad (\Omega\wedge\Omega = 0)$$

$H_j := \{x \in \mathbb{C}^\infty \mid f_j(x) = 0\}$ : **超平面** ($f_j \in \mathbb{C}[x_1,\ldots,x_n]$ with $n \gg 1$)

$$f_j(x) = a_{j1}x_1 + \cdots + a_{jn}x_n + c_j$$

$\mathcal{H} := \bigcup_j H_j$

## Pfaff 系の変換

(1) addition : $\Omega \mapsto \Omega + \lambda\cdot d\log f$

(2) middle convolution : $\mathrm{mc}_{x_i,\mu}$ (Katz, Dettweiler-Reiter, Haraoka)

(3) **座標変換，代入** $x \mapsto R(x)$ **(制限も含む)**

**例：**

$(x_1,x_2,x_3,\ldots) \mapsto (x_{\sigma(1)},\ldots,x_{\sigma(n)},x_{n+1},\ldots) \quad (\sigma\in\mathfrak{S}_n)$

$(x_1,x_2,x_3,\ldots) \mapsto (ax_1{+}bx_2{+}c,\ dx_1{+}ex_2{+}f,\ x_3,\ldots) \quad (a{=}b{=}0$ **も可**$)$

$(x_1,x_2,x_3,\ldots) \mapsto (\frac{1}{x_1},\frac{x_2}{x_1},\frac{x_3}{x_1},\ldots)$

(4) **境界値写像 (特異超平面への制限)**

**Fact**. $\mathcal{H} \mapsto \bar{\mathcal{H}}$

**問**. 自明な方程式から出発してどのようなものが得られるか？　得られたものを解析せよ

(KZ 型のときは $\mathcal{H} = \sum_{1\le i<j\le n} \{x\in\mathbb{C}^\infty \mid x_i{-}x_j = 0\}$)

# Thank you for your attention!

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